NAVIER-STOKES SYSTEMS WITH QUASIMONOTONE VISCOSITY TENSOR

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Abstract: In [1] we investigated a class of Navier-Stokes systems which is motivated by models for electrorheological fluids. We obtained an existence result for a weak solution under mild monotonicity assumptions for the viscosity tensor. In this article, we continue the analysis of such systems, but with various notions of quasimonotonicity instead of classical pointwise monotonicity assumptions. Moreover we allow the external force to be of a more general form.

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1 Introduction

1.1 Retrospect of former results

In this paragraph we introduce some notations, and we recall the main results established in [1] in order to relate them later on with the new results which we derive below.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with Lipschitz boundary. In [1] we considered the following Navier-Stokes system for the velocity $u : \Omega \times [0, T) \to \mathbb{R}^n$ and the pressure $P : \Omega \times [0, T) \to \mathbb{R}$

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$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x, t, u, Du) + (u \cdot \nabla)u = f - \operatorname{grad} P \qquad \text{on } \Omega \times (0, T) \tag{1}$$

$$\operatorname{div} u = 0 \qquad \text{on } \Omega \times (0, T) \tag{2}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0, T) \tag{3}$$

$$u(\cdot, 0) = u_0 \qquad \text{on } \Omega \tag{4}$$

Here, $f \in L^{p'}(0, T; V')$ for some $p \in [1 + \frac{2n}{n+2}, \infty)$, where V consists of all functions in $W_0^{1,p}(\Omega, \mathbb{R}^n)$ with vanishing divergence. Moreover $u_0 \in L^2(\Omega; \mathbb{R}^n)$ is an arbitrary initial condition satisfying $\operatorname{div} u_0 = 0$, and σ satisfies the conditions (NS0)–(NS2) below. We allow the viscosity tensor σ to depend (non-linearly) on x, t, u and Du.

The problem (1)–(4) with the u-dependence of σ is motivated by the study of electrorheological fluid flows, as explained in the introduction of [1].

To fix some notation, let $\mathbb{M}^{m \times n}$ denote the real vector space of $m \times n$ matrices equipped with the inner product $M : N = M_{ij} N_{ij}$ (with the usual summation convention).

The following main assumptions are imposed on the viscosity tensor σ:

(NS0) (Continuity) $\sigma : \Omega \times (0, T) \times \mathbb{R}^n \times \mathbb{M}^{n \times n} \to \mathbb{M}^{n \times n}$ is a Carathéodory function, i.e. $(x, t) \mapsto \sigma(x, t, u, F)$ is measurable for every $(u, F) \in \mathbb{R}^n \times \mathbb{M}^{n \times n}$ and $(u, F) \mapsto \sigma(x, t, u, F)$ is continuous for almost every $(x, t) \in \Omega \times (0, T)$.

(NS1) (Growth and coercivity) There exist $c_1 \geqslant 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega \times (0, T))$, $\lambda_2 \in L^1(\Omega \times (0, T))$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega \times (0, T))$, $0 < \alpha < p$, such that

$$\begin{aligned} |\sigma(x, t, u, F)| &\leqslant \lambda_1(x, t) + c_1(|u|^{p-1} + |F|^{p-1}) \\ \sigma(x, t, u, F) : F &\geqslant -\lambda_2(x, t) - \lambda_3(x, t)|u|^\alpha + c_2|F|^p \end{aligned}$$

(NS2) (Monotonicity) σ satisfies one of the following conditions:

(a) For all $(x, t) \in \Omega \times (0, T)$ and all $u \in \mathbb{R}^n$, the map $F \mapsto \sigma(x, t, u, F)$ is a C^1-function and is monotone, i.e.

$$(\sigma(x, t, u, F) - \sigma(x, t, u, G)) : (F - G) \geqslant 0$$

for all $(x, t) \in \Omega \times (0, T)$, $u \in \mathbb{R}^m$ and $F, G \in \mathbb{M}^{n \times n}$.

(b) There exists a function $W : \Omega \times (0, T) \times \mathbb{R}^n \times \mathbb{M}^{n \times n} \to \mathbb{R}$ such that $\sigma(x, t, u, F) = \frac{\partial W}{\partial F}(x, t, u, F)$, and $F \mapsto W(x, t, u, F)$ is convex and C^1 for all $(x, t) \in \Omega \times (0, T)$ and all $u \in \mathbb{R}^n$.

(c) σ is strictly monotone, i.e. σ is monotone and $(\sigma(x, t, u, F) - \sigma(x, t, u, G)) : (F - G) = 0$ implies $F = G$.

We recall that the main point is that, in (a) and (b), it is not required that σ is strictly monotone or monotone in the variables (u, F) as it is usually assumed in previous work.

We will work with the following function spaces: Let

$$\mathscr{V} := \{\varphi \in C_0^\infty(\Omega; \mathbb{R}^n) : \operatorname{div} \varphi = 0\}.$$

Then, V denotes the closure of $\mathscr{V}$ in the space $W^{1,p}(\Omega; \mathbb{R}^n)$. A classical result of de Rham shows, that this space is

$$V = \{\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^n) : \operatorname{div} \varphi = 0\}.$$

In addition, we will have to work with $W^{s,2}(\Omega; \mathbb{R}^n)$, where $s > 1 + \frac{n}{2}$. Then, we denote by

$$V_s := \text{the closure of } \mathscr{V} \text{ in the space } W^{s,2}(\Omega)$$

and

$$\begin{aligned} H_q &:= \text{the closure of } \mathscr{V} \text{ in the space } L^q(\Omega), \text{ and} \\ H &:= H_2. \end{aligned}$$

Furthermore, let $\mathscr{W}$ denote the space defined by

$$\mathscr{W} := \{v \in L^p(0, T; V) : \partial_t v \in L^{p'}(0, T; V')\},$$

where the integrals are to be understood in the sense of Bochner and the time-derivative means here the vectorial distributional derivative. We recall that $\mathscr{W}$ is continuously embedded in $C^0([0, T]; H)$ and we always identify $v \in \mathscr{W}$ with its representative in $C^0([0, T]; H)$.

The main result we have proved in [1] is the following:

Theorem 1 *Assume that σ satisfies the conditions (NS0)–(NS2) for some $p \in [1 + \frac{2n}{n+2}, \infty)$. Then for every $f \in L^{p'}(0, T;\, V')$ and every $u_0 \in H$, the Navier-Stokes system (1)–(4) has a weak solution (u, P), with $u \in \mathscr{W}$, in the following sense: For every $v \in L^p(0, T; V)$ there holds*

$$\begin{aligned} \int_0^T \langle \partial_t u, v \rangle dt + \int_0^T \int_\Omega \sigma(x, t, u, Du) : Dv \, dx \, dt + \\ + \int_0^T \int_\Omega (u \cdot \nabla) u \cdot v \, dx \, dt = \int_0^T \langle f, v \rangle dt, \\ u(0, \cdot) = u_0. \end{aligned}$$

The weak solution in Theorem 1 is more than a classical weak solution and in particular the energy equality is satisfied (see Remark in [1, p. 245] for more details).

For the proof we used a Faedo-Galerkin method. By using the assumptions in (NS1) we constructed a Galerkin sequence (u_m) of approximating solutions. Several compactness properties were then established in [1] which allowed to extract a subsequence u_m converging weakly to some u in $L^p(0,T;V)$. But then, the monotonicity assumptions (NS2) (a) or (b) do not allow to use the classical monotonicity method (like in [9]) in order to pass to the limit in the Galerkin equations. To overcome this difficulty we then used a refinement of the monotone operator method (inspired by [3]) which involves the theory of Young measures. To this end, we studied the Young measure $\nu_{(x,t)}$ generated by the sequence of gradients (Du_m), and obtained a div-curl inequality which was the key ingredient to pass to the limit in the Galerkin equation. This inequality was formulated as follows:

Lemma 2 (A div-curl inequality) *The Young measure $\nu_{(x,t)}$ generated by the gradients Du_m of the Galerkin approximations u_m has the property, that for all $s \in [0,T]$:*

$$\int_0^s \int_\Omega \int_{\mathbb{M}^{n\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : \big(\lambda - Du\big) d\nu_{(x,t)}(\lambda) dx dt \leqslant 0. \quad (5)$$

In a final step we have shown that the existence result in Theorem 1 follows from the div-curl inequality whenever we use one of the monotonicity conditions in (NS2). Moreover we have derived some additional properties of the Galerkin approximations: In case (NS2) (a) there holds $\sigma(x,t,u_m,Du_m) \rightharpoonup \sigma(x,t,u,Du)$ in $L^{p'}(\Omega \times (0,T))$ (for a subsequence), in case (b) we have in addition $\sigma(x,t,u_m,Du_m) \to \sigma(x,t,u,Du)$ in $L^\beta(\Omega \times (0,T))$, for all $\beta \in [1,p')$, and in case (c), we even have $Du_m \to Du$ in $L^\alpha(\Omega \times (0,T))$ for all $\alpha \in [1,p)$.

1.2 Extension of the results to quasimonotone viscosity tensors

In this paper we intend to consider quasimonotonicity assumptions for σ rather than the classical pointwise monotonicity: Instead of (NS2) (a), (b) or (c) which are pointwise monotonicity conditions, we consider the assumptions (NS2) (d) and (e) below which represent monotonicity in an integrated form. Moreover we consider equation (1) with a source term which is allowed to be of a more general form. More precisely, we replace $f \in L^{p'}(0,T;V')$ by f satisfying the assumption (Hf):

(Hf) $f : \Omega \times (0,T) \times \mathbb{R}^n \times \mathbb{M}^{n\times n} \to \mathbb{R}^n$ is a Carathéodory function in the sense (NS0). Moreover we assume that one of the following additional conditions hold:

(i) For a constant $\beta < p-1$ and a function $\lambda_4 \in L^{p'}(\Omega \times (0,T))$ there holds
$$|f(x,t,u,F)| \leqslant \lambda_4(x,t) + C\left(|u|^\beta + |F|^\beta\right).$$

(ii) In addition to (i), the function f is independent of the fourth variable, or, for a.e. $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^n$, the mapping $F \to f(x,t,u,F)$ is linear.

We now introduce the definitions of quasimonotonicity which we are going to use:

Definition 3 *A function $\eta : \mathbb{M}^{n\times n} \to \mathbb{M}^{n\times n}$ is called* **strictly quasimonotone**, *if there exist constants $c > 0$ and $r > 0$ such that*

$$\int_\Omega (\eta(Du) - \eta(Dv)) : (Du - Dv)dx \geqslant c \int_\Omega |Du - Dv|^r dx$$

for all $u, v \in W_0^{1,p}(\Omega)$.

We say that η is **strictly p-quasimonotone**, *if*

$$\int_{\mathbb{M}^{n\times n}} (\eta(\lambda) - \eta(\bar{\lambda})) : (\lambda - \bar{\lambda})d\nu(\lambda) > 0$$

for all homogeneous $W^{1,p}$ gradient Young measures ν with center of mass $\bar{\lambda} = \langle \nu, \mathrm{id}\rangle$ which are not a single Dirac mass.

Remarks: (a) Note that the notion of p-quasimonotonicity is phrased in terms of gradient Young measures. Notice that although quasimonotonicity is "monotonicity in integrated form", the gradient of a quasiconvex function is not necessarily strictly p-quasimonotone. A simple example of a strictly p-quasimonotone function is the following: Assume that η satisfies the growth condition
$$|\eta(F)| \leqslant C\,|F|^{p-1}$$
with $p > 1$ and the structure condition

$$\int_\Omega (\eta(F + D\varphi) - \eta(F)) : D\varphi\, dx \geqslant c \int_\Omega |D\varphi|^p dx$$

for a constant $c > 0$ and for all $\varphi \in C_0^\infty(\Omega)$ and all $F \in \mathbb{M}^{n\times n}$. Then η is strictly p-quasimonotone. This follows easily from the definition if one uses that for every $W^{1,p}$ gradient Young measure ν there exists a sequence (Dv_k) generating ν for which $(|Dv_k|^p)$ is equiintegrable (see [5], [7]).

(b) In [10], Zhang introduced the following notion of quasimonotonicity: The continuous function $\eta : \mathbb{M}^{N\times n} \to \mathbb{M}^{N\times n}$ is quasimonotone (in the sense of Zhang) if for every $F \in \mathbb{M}^{N\times n}$, every open subset G of $\mathbb{R}^n$ and every $\varphi \in C_0^1(G;\mathbb{R}^N)$ there holds

$$\int_G \eta(F+D\varphi) : D\varphi dx \geqslant 0.$$

However, he uses a stronger notion of quasimonotonicity to prove his results, namely, that

$$\int_G \eta(F+D\varphi) : D\varphi dx \geqslant c \int_G |D\varphi|^p dx \tag{6}$$

for a fixed constant $c > 0$, along with the growth condition

$$|\eta(F)| \leqslant C|F|^{p-1} \tag{7}$$

for some constant $C > 0$. We would like to indicate that our definition of strict p-quasimonotonicity can be considered as a generalization of Zhang's notion of quasimonotonicity. In fact, a function which is quasimonotone in the sense of Zhang, i.e. which satisfies (6) and (7), is strictly p-quasimonotone. To see this, we consider a homogeneous $W^{1,p}$ gradient Young measure ν with center of mass $\bar\lambda := \langle \nu, \mathrm{id}\rangle$ and which is not a single Dirac mass. Then (for some fixed domain G), there exists a sequence (φ_k) in $C_0^\infty(G)$ such that the sequence of gradients $(D\varphi_k)$ generates ν and such that $(|D\varphi_k|)$ is equiintegrable (see Remark (a)). By (6) there holds

$$\int_G \eta(F+D\varphi_k) : D\varphi_k dx \geqslant c \int_G |D\varphi_k|^p dx. \tag{8}$$

The sequence $(|D\varphi_k|)$ is equiintegrable and, by (7), $(\eta(F+D\varphi_k) : D\varphi_k)$ is also equiintegrable. Hence by Ball's fundamental theorem on Young measures (see [2]), we obtain from (8), that

$$\int_G \int_{\mathbb{M}^{n\times n}} \eta(F+\lambda) : \lambda d\nu_x(\lambda) dx \geqslant c \int_G \int_{\mathbb{M}^{n\times n}} |\lambda|^p d\nu_x(\lambda) dx,$$

and since ν, by hypothesis, is homogeneous,

$$\int_{\mathbb{M}^{n\times n}} \eta(F+\lambda) : \lambda d\nu(\lambda) \geqslant c \int_{\mathbb{M}^{n\times n}} |\lambda|^p d\nu(\lambda).$$

A substitution $\lambda = \lambda' - \bar\lambda$, and the special choice $F = \bar\lambda$ yields

$$\int_{\mathbb{M}^{n\times n}} \eta(\lambda') : (\lambda' - \bar\lambda) d\nu(\lambda') \geqslant c \int_{\mathbb{M}^{n\times n}} |\lambda' - \bar\lambda|^p d\nu(\lambda') > 0 \tag{9}$$

since ν is not a single Dirac mass. Moreover,

$$\int_{\mathbb{M}^{n\times n}} \eta(\bar\lambda) : (\lambda' - \bar\lambda) d\nu(\lambda') = \eta(\bar\lambda) : \underbrace{\int_{\mathbb{M}^{n\times n}} \lambda' d\nu(\lambda')}_{=\bar\lambda} - \eta(\bar\lambda) : \bar\lambda \underbrace{\int_{\mathbb{M}^{n\times n}} d\nu(\lambda')}_{=1} = 0. \tag{10}$$

If we subtract (10) from (9), we arrive at

$$\int_{\mathbb{M}^{n\times n}} (\eta(\lambda') - \eta(\bar{\lambda})) : (\lambda' - \bar{\lambda}) d\nu(\lambda') > 0,$$

and hence η is strictly p-quasimonotone. We end this remark by noting that we can replace the power p on the right hand side of (6) by any power $r > 0$ and still have the same conclusion. $\diamondsuit$

So, in addition to (NS2) (a), (b) and (c), we will now consider the conditions:

(NS2) (Monotonicity) σ satisfies one of the following conditions:

(d) for a.e $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^n$, the function $F \to \sigma(x,t,u,F)$ is strictly quasimonotone.

(e) for a.e $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^n$, the function $F \to \sigma(x,t,u,F)$ is strictly p-quasimonotone.

The main result we prove in this paper is the following:

Theorem 4 *Assume that σ satisfies the conditions (NS0) and (NS1) for some $p \in [1 + \frac{2n}{n+2}, \infty)$. Let u_0 be given in H. Then we have:*

(i) If σ satisfies one of the condition (NS2) (c), (d) or (e) then for every f satisfying (Hf) (i), the Navier-Stokes system (1)–(4) has a weak solution (u, P), with $u \in \mathscr{W}$.

(ii) If σ satisfies one of the condition (NS2) (a) or (b) then for every f satisfying (Hf) (ii) the same conclusion holds.

Remark: The notion of a solution $u \in \mathscr{W}$ in Theorem 4 is the same as in Theorem 1: We have $u(0,\cdot) = u_0$ and there holds

$$\int_0^T \langle \partial_t u, v\rangle dt + \int_0^T \int_\Omega \sigma(x,t,u,Du) : Dv\, dx\, dt + \int_0^T \int_\Omega (u\cdot\nabla)u \cdot v\, dx\, dt =$$
$$= \int_0^T \int_\Omega f(x,t,u,Du)\cdot v\, dx\, dt \quad \forall v \in L^p(0,T;V).$$

1.3 Organization of the paper

In Section 2 we will resume the relevant results presented in [1, Section 1–6]. We will prove first that for every f satisfying (Hf) (i) we can construct a Galerkin sequence u_m of approximating solutions. In a second step we will show that the convergence properties for u_m established in [1] remain valid. We will then recover the same properties for the Young measure associated to (u_m, Du_m), and actually the div-curl inequality holds again.

In Section 3 we pass to the limit $m \to \infty$ in the Galerkin equations and prove Theorem 4. Like in [1], the key ingredient for identifying the weak limit $\sigma(x, t, u_m, Du_m) \rightharpoonup \sigma(x, t, u, Du)$ (where u is the weak limit in $L^p(0, T; V)$ of a relabeled subsequence of u_m) is the div-curl lemma (unless for the easier situation when (NS2) (d) holds). In the cases (NS2) (c), (d) or (e) we also get that $Du_m \to Du$ in measure, which allows to conclude the first part of Theorem 4. In the situation (NS2) (a) or (b), this additional property of convergence does not hold in general and we have to consider the stronger assumption (Hf) (ii) instead of (Hf) (i) in order to identify the weak limit in $f(x, t, u_m, Du_m) \rightharpoonup f(x, t, u, Du)$.

2 The Galerkin approximation

2.1 The Galerkin base

Let the functions $w_i \in V_s$ be a Galerkin base, as introduced in [1, Section 2]. We have shown that $W := \{w_1, w_2, \ldots\}$ is an orthonormal Hilbert base of H. In particular, the L^2-orthogonal projector $P_m : H \to H$ onto span$(w_1, w_2, \ldots, w_m)$, $m \in \mathbb{N}$ is defined by the formula

$$P_m u = \sum_{i=1}^{m} (w_i, u)_H w_i. \tag{11}$$

Of course, the operator norm $\|P_m\|_{\mathscr{L}(H,H)} = 1$. We have also shown that $\|P_m\|_{\mathscr{L}(V_s,V_s)} = 1$ and that P_m converges pointwise to the identity in $\mathscr{L}(V_s, V_s)$.

2.2 The Galerkin approximation

Let $m \in \mathbb{N}$. In the (Faedo-)Galerkin method one makes an Ansatz for approximating solutions u_m of the form

$$u_m(x, t) = \sum_{i=1}^{m} c_{mi}(t) w_i(x), \tag{12}$$

where $c_{mi} : [0, T) \to \mathbb{R}$ are supposed to be continuous bounded functions. We take care of the initial condition (4) by choosing the initial coefficients $c_{mi} := c_{mi}(0) = (u_0, w_i)_{L^2}$ such that

$$u_m(\cdot, 0) = \sum_{i=1}^{m} c_{mi} w_i(\cdot) \to u_0 \quad \text{in } L^2(\Omega) \text{ as } m \to \infty. \tag{13}$$

We try to determine the coefficients $c_{mi}(t)$ in such a way, that for every $m \in \mathbb{N}$ the system of ordinary differential equations

$$(\partial_t u_m, w_j)_H + \int_\Omega \sigma(x,t,u_m,Du_m) : Dw_j dx + b(u_m,u_m,w_j) = \\ = \int_\Omega f(x,t,u_m,Du_m) \cdot w_j dx \quad (14)$$

(with $j \in \{1, 2, \ldots, m\}$) is satisfied in the sense of distributions. In (14), we used the shorthand notation

$$b(u,v,w) := \int_\Omega ((u \cdot \nabla)v) \cdot w \, dx.$$

Let ε, J, r and K be the quantities introduced in [1, Section 3.1]. For any $j = 1, \ldots, m$ we can verify (by using the assumption (Hf)) that the function $\Theta : J \times K \to \mathbb{R}$ defined by

$$\Theta(t, c_1, \ldots, c_m) := \int_\Omega f(x, t, \sum_{i=1}^m c_i w_i, \sum_{i=1}^m c_i Dw_i) \cdot w_j dx$$

is a Carathéodory function. Moreover we obtain the estimate:

$$|\Theta(t, c_1, \ldots, c_m)| \leqslant C \int_\Omega \lambda_4(x,t) dx + C,$$

where C may depend on m and r but is independent of t. These results allow to conclude as in [1, Section 3.1] that there exists a local solution for equation (14). This solution u_m is on the form (12) and it verifies (14) in the sense $\mathscr{D}'(0, \varepsilon')$, for some $\varepsilon' > 0$ which, for the moment, may depend on m.

This local solution can be extended to the whole interval $[0,T)$ independent of m. To do this one can use the arguments presented in [1, Section 3.2]. More precisely, let τ be arbitrary in the existence interval. We have to replace the term $III = \int_0^\tau \langle f(t), u_m \rangle dt$ by $III = \int_0^\tau \int_\Omega f(x,t,u_m,Du_m) \cdot u_m dx dt$. By using the growth condition (Hf) (i) we then obtain

$$III \leqslant \|\lambda_4\|_{L^{p'}(\Omega \times (0,T))} \|u_m\|_{L^p(\Omega \times (0,T))} + C \|u_m\|_{L^p(0,T;V)}^{\beta+1}.$$

This inequality, in combination with the estimates for the terms I and II from [1, Section 3.2] which remain unchanged, permits again to obtain the estimate

$$|(c_{mi}(\tau))_{i=1,\ldots,m}|_{\mathbb{R}^m}^2 = \|u_m(\cdot, \tau)\|_{L^2(\Omega)}^2 \leqslant \bar{C}$$

for a constant $\bar{C}$ which is independent of τ (and of m). It follows that the functions c_{mj} can be extended to the whole interval $[0,T]$ and $u_m(x,t) = \sum_{j=1}^m c_{mj}(t) w_j(x)$ is a solution (not necessarily unique) of (14) in the sense $\mathscr{D}'(0,T)$. Moreover we obtain again

$$\|u_m\|_{C^0([0,T];L^2(\Omega))} \leqslant C. \quad (15)$$

2.3 Basic convergence properties

We easily recover the basic convergence properties presented in [1, Section 4.1], i.e. we may extract a subsequence, still denoted by u_m, verifying

$$u_m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H) \tag{16}$$
$$u_m \rightharpoonup u \quad \text{in } L^p(0,T;V) \tag{17}$$
$$-\operatorname{div}\sigma(x,t,u_m,Du_m) \rightharpoonup \chi \quad \text{in } L^{p'}(0,T;V') \tag{18}$$

for some $u \in L^p(0,T;V) \cap L^\infty(0,T;H)$ and $\chi \in L^{p'}(0,T;V')$. Moreover, by using (17) together with (Hf) (i) we may also assume that

$$f(x,t,u_m.Du_m) \rightharpoonup \xi \quad \text{in } L^{p'}(\Omega \times (0,T)) \tag{19}$$

for some $\xi \in L^{p'}(\Omega \times (0,T))$.

The principal difficulty will be, as in [1], to show that $\chi = -\operatorname{div}\sigma(x,t,u,Du)$. Moreover here, we will also have to identify ξ with $f(x,t,u,Du)$.

2.4 Convergence in measure

We recall first that for any q satisfying $2 < q < p^* := \frac{np}{n-p}$ we have the following chain of continuous injections:

$$V \overset{i}{\hookrightarrow} H_q \overset{i_0}{\hookrightarrow} H \overset{\gamma}{\cong} H' \overset{i_1}{\hookrightarrow} V'_s. \tag{20}$$

Here, $H \cong H'$ is the canonical isomorphism γ between the Hilbert space H and its dual. We take over from [1] the notation $j \circ i$ to denote the canonical injection of V into V'_s, that is, for $u \in V$:

$$\langle j \circ i \circ u, v\rangle = \int_\Omega uv\,dx \qquad \forall v \in V_s.$$

Next, we consider the time derivative of u_m in $\mathscr{D}'(0,T;V'_s)$, which is in fact a function in $L^{p'}(0,T;V'_s)$ given by the formula :

$$\langle \partial_t(j \circ i \circ u_m), v\rangle = \int_\Omega \partial_t u_m(x,t) P_m v(x) dx = -\int_\Omega \sigma(x,t,u_m,Du_m) : D(P_m v)dx - \\ - b(u_m,u_m,P_m v) + \int_\Omega f(x,t,u_m,Du_m) P_m v dx \qquad \forall v \in V_s, \text{ a.e. } t \in (0,T). \tag{21}$$

By using the fact that u_m is a bounded sequence in $L^p(0,T;V)$ together with the growth property (Hf) (i) we obtain

$$\left|\int_\Omega f(x,t,u_m,Du_m).P_m v dx\right| \leqslant C\|v\|_{V_s}\|f(x,t,u_m,Du_m)\|_{L^1(\Omega)} \leqslant \gamma_m(t)\|v\|_{Vs},$$

where (γ_m) is a bounded sequence in $L^{p'}(0,T)$.

Consequently we obtain :

$$|\langle \partial_t (j \circ i \circ u_m), v\rangle| \leqslant C_m(t)\|v\|_{V_s} \tag{22}$$

where (C_m) is a bounded sequence in $L^{p'}(0,T)$.

Remark: The estimate (25) in [1, p. 255] needs to be modified in the following way: $C \in L^{p'}(0,T)$ should be replaced by (C_m), a bounded sequence in $L^{p'}(0,T)$, like here in estimate (22). This modification has no consequence on the results presented as we will see in the sequel.

From (22) we conclude indeed, that $\{\partial_t j \circ i \circ u_m\}_m$ is a bounded sequence in $L^{p'}(0,T;V_s')$.

Consequently, by [1, Lemma 2], we may assume (for a further subsequence)

$$u_m \to u \quad \text{in } L^p(0,T;L^q(\Omega)) \text{ for all } q < p^* \text{ and in measure on } \Omega \times (0,T). \tag{23}$$

2.5 A regularity result for $\boldsymbol{u}$

The arguments developed in [1, Section 4.3] are easily carried over: It follows that the time derivative $\partial_t(j \circ i \circ u)$ is given by the formula

$$\langle \partial_t(j\circ i\circ u), v\rangle = \int_\Omega \xi v dx - \langle \chi, v\rangle - b(u,u,v), \quad \forall v \in V_s, \text{ a.e. } t \in (0,T). \tag{24}$$

This permits again to obtain the property $\partial_t(j \circ i \circ u) \in L^{p'}(0,T;V')$, which implies that $u \in \mathscr{W}$.

2.6 The limiting time values for u

We remark first that by the boundedness of the sequence $\partial_t(j \circ i \circ u_m)$ in $L^{p'}(0,T;V_s')$ we may extract a subsequence (not relabeled) such that

$$\partial_t(j \circ i \circ u_m) \rightharpoonup \partial_t(j \circ i \circ u) \quad \text{in } L^{p'}(0,T;V_s'). \tag{25}$$

Next, by the same manner as in [1, Section 4.4], we get

$$u_m(x,0) \to u_0(x) = u(x,0) \quad \text{in } H, \tag{26}$$

$$u_m(\cdot,T) \rightharpoonup u(\cdot,T), \quad \text{in } H. \tag{27}$$

The property (27) can be extended to all $t \in [0,T]$. Indeed, we have:

Lemma 5 *There exists a subsequence of $\{u_m\}$ (still denoted by u_m) with the property that for every $t \in [0,T]$*

$$u_m(\cdot,t) \rightharpoonup u(\cdot,t) \quad \text{in } H. \tag{28}$$

By (23) the convergence in (28) is actually strong for almost all $t \in [0,T]$.

Proof. We can take over the proof of [1, Lemma 4] with taking into account the remark after the relation (22). The estimate (25) in [1, p. 255] is not true and we have to replace $C(t)$ by $C_m(t)$, where $C_m(t)$ is a bounded sequence in $L^{p'}(0,T)$. In fact, after this correction the relation (25) in [1] is the same estimate as (22) in this paper. Recall next that we consider $p < \infty$ and thus the boundedness of C_m in $L^{p'}(0,T)$ implies the equiintegrability property. Consequently the estimate (43) in [1] holds true when we replace $C(t)$ by $C_m(t)$ and the rest of the proof follows. $\square$

2.7 The Young measure generated by the Galerkin approximation

The sequence (or at least a subsequence) of the gradients Du_m generates a Young measure $\nu_{(x,t)}$, and since u_m converges in measure to u on $\Omega \times (0,T)$, the sequence (u_m, Du_m) generates the Young measure $\delta_{u(x,t)} \otimes \nu_{(x,t)}$ (see, e.g., [6]). Now, we collect some facts about the Young measure ν in the following proposition:

Proposition 6 *The Young measure $\nu_{(x,t)}$ generated by the sequence $\{Du_m\}_m$ has the following properties:*

(i) $\nu_{(x,t)}$ *is a probability measure on* $\mathbb{M}^{n\times n}$ *for almost all* $(x,t) \in \Omega \times (0,T)$.

(ii) $\nu_{(x,t)}$ *satisfies* $Du(x,t) = \langle \nu_{(x,t)}, \mathrm{id}\rangle$ *for almost every* $(x,t) \in \Omega \times (0,T)$.

(iii) $\nu_{(x,t)}$ *has finite* p*-th moment for almost all* $(x,t) \in \Omega \times (0,T)$.

(iv) $\nu_{(x,t)}$ *is a homogeneous* $W^{1,p}$ *gradient Young measure for almost all* $(x,t) \in \Omega \times (0,T)$.

Proof. For (i), (ii) and (iii) see the proof of Proposition 5 in [1].

(iv) We have to show, that $\{\nu_{(x,t)}\}_{x\in\Omega}$ is for almost all $t \in (0,T)$ a $W^{1,p}$ gradient Young measure. To see this, we take a quasiconvex function q on $\mathbb{M}^{n\times n}$ with $q(F)/|F| \to 1$ as $F \to \infty$. Then, we fix $x \in \Omega$, $\delta \in (0,1)$ and use inequality (1.21) from [8, Lemma 1.6] with u replaced by $u_m(x,t)$, with $a := u(x,t) - Du(x,t)x$ and with $X := Du(x,t)$. Furthermore, we choose $r > 0$ such that $B_r(x) \subset \Omega$. Observe, that the singular part of the distributional gradient vanishes for u_m and, after integrating the inequality over the time interval $[t_0-\varepsilon, t_0+\varepsilon] \subset (0,T)$, we get

$$\begin{aligned}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)} q(Du_m(y,t))dydt+\\ +\frac{1}{(1-\delta)r}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)\setminus B_{\delta r}(x)} |u_m(y,t)-u(x,t)-Du(x,t)(y-x)|dydt \geqslant \\ \geqslant |B_{\delta r}(x)|\int_{t_0-\varepsilon}^{t_0+\varepsilon} q(Du(x,t))dt.\end{aligned}$$

Letting m tend to infinity in the inequality above, we obtain

$$\begin{aligned}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)}\int_{\mathbb{M}^{n\times n}} q(\lambda)d\nu_{(y,t)}(\lambda)dydt+\\ +\frac{1}{(1-\delta)r}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)\setminus B_{\delta r}(x)} |u(y,t)-u(x,t)+Du(x,t)(y-x)|dydt \geqslant \\ \geqslant |B_{\delta r}(x)|\int_{t_0-\varepsilon}^{t_0+\varepsilon} q(Du(x,t))dt.\end{aligned}$$

Now, we let $\varepsilon \to 0$ and $r \to 0$ and use the differentiability properties of Sobolev functions (see, e.g., [4]) and obtain, that for almost all $(x,t_0) \in \Omega \times (0,T)$

$$\int_{\mathbb{M}^{n\times n}} q(\lambda)d\nu_{(x,t_0)}(\lambda) \geqslant \frac{|B_{\delta r}(x)|}{|B_r(x)|}\, q(Du(x,t_0)).$$

Since $\delta \in (0,1)$ was arbitrary, we conclude that Jensen's inequality holds true for q and the measure $\nu_{(x,t)}$ for almost all $(x,t) \in \Omega \times (0,T)$. Using the characterization of $W^{1,p}$ gradient Young measures of [7] (e.g., in the form of [8, Theorem 8.1]), we conclude that in fact $\{\nu_{x,t}\}_{x\in\Omega}$ is a $W^{1,p}$ gradient Young measure on Ω for almost all $t \in (0,T)$. By the localization principle for gradient Young measures, we conclude then, that $\nu_{(x,t)}$ is a homogeneous $W^{1,p}$ gradient Young measure for almost all $(x,t) \in \Omega \times (0,T)$. □

2.8 A Navier-Stokes div-curl inequality

In this section, we prove a Navier-Stokes version of a "div-curl Lemma" which will be the key ingredient to obtain $\chi = -\operatorname{div}\sigma(x,t,u,Du)$. The results and the arguments presented in [1, Section 6] can be carried over with only minor modifications which we want to explain in the sequel.

In a first step we note that

$$\int_{s_1}^{s_2}\langle\chi,u\rangle dt+\frac{1}{2}\|u(\cdot,s_2)\|_H^2 = \int_{s_1}^{s_2}\int_\Omega \xi\cdot u dxdt+\frac{1}{2}\|u(\cdot,s_1)\|_H^2, \quad \forall\, 0\leqslant s_1\leqslant s_2\leqslant T. \tag{29}$$

This follows in the same way as the energy equality (46) in [1].

Next, we establish the following lemma:

Lemma 7 (A div-curl inequality) *The Young measure $\nu_{(x,t)}$ generated by the gradients Du_m of the Galerkin approximations u_m has the property that for all $s \in [0,T]$:*

$$\int_0^s \int_\Omega \int_{\mathbb{M}^{n\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : \big(\lambda - Du\big) d\nu_{(x,t)}(\lambda) dx dt \leqslant 0. \quad (30)$$

Proof. We follow the calculations in [1, Section 6.2]. The inequality (48) in [1] can be obtained without any change, and inequality (49) holds true if we replace $\langle f, u\rangle$ by $\int_\Omega \xi u dx$. By using the property (19) together with (23) we then obtain that

$$\lim_{m\to\infty} \int_0^s \int_\Omega f(x,t,u_m,Du_m) \cdot u_m dx dt = \int_0^s \int_\Omega \xi \cdot u dx dt.$$

Hence, by using (26) with (28) we obtain

$$\begin{aligned}\liminf_{m\to\infty} &\int_0^s \int_\Omega \sigma(x,t,u_m,Du_m) : Du_m dx dt \\ &\leqslant \int_0^s \int_\Omega \xi \cdot u dx dt - \frac{1}{2}\|u(\cdot,s)\|_H^2 + \frac{1}{2}\|u(\cdot,0)\|_H^2.\end{aligned}$$

The rest of the proof is identical to the proof in [1]. $\square$

Remarks:

(i) In the proof of the div-curl lemma of we do not use any monotonicity assumption.

(ii) An intermediary result is that, for all $s \in [0,T]$, there holds

$$\liminf_{m\to\infty} \int_0^s \int_\Omega (\sigma(x,t,u_m,Du_m) - \sigma(x,t,u,Du)) : (Du_m - Du) dx dt \leqslant 0.$$

To see this, repeat the proof of Lemma 6 in [1] with the modifications indicated above.

3 Passage to the limit

Here we will pass to the limit $m \to \infty$ in the Galerkin equations and prove Theorem 4. The first step is to identify the weak limit $\sigma(x,t,u_m,Du_m) \rightharpoonup \sigma(x,t,u,Du)$. This will follow from every monotonicity assumption listed in (NS2). In Subsection 3.1, we treat the special cases (NS2) (c), (d) and (e) for which we also obtain the convergence $Du_m \to Du$ in measure. The conclusion is then given in the last subsection.

3.1 The cases (NS2) (c), (d) and (e)

In these three cases we will prove that we may extract a subsequence with the property

$$Du_m \to Du \quad \text{in measure on } \Omega \times (0,T). \tag{31}$$

We consider first the case (NS2) (d). In this situation, elementary arguments are actually sufficient to prove (31), and we do not need the div-curl inequality: Observe that we have

$$\begin{aligned}
&\int_0^T \int_\Omega |Du_m - Du|^r \, dxdt \leqslant \\
&\quad \leqslant C \int_0^T \int_\Omega (\sigma(x,t,u_m,Du_m) - \sigma(x,t,u_m,Du)) : (Du_m - Du) dxdt \\
&\quad \leqslant C \int_0^T \int_\Omega (\sigma(x,t,u_m,Du_m) - \sigma(x,t,u,Du)) : (Du_m - Du) dxdt + \\
&\qquad + C \int_0^T \int_\Omega (\sigma(x,t,u,Du) - \sigma(x,t,u_m,Du)) : (Du_m - Du) dxdt.
\end{aligned} \tag{32}$$

We remark now that the limit inferior of the first term on the right hand side of (32) is less than or equal to zero (see remark (ii) after lemma 7). The second term vanishes when m tends to infinity because of (23). It follows that

$$\liminf_{m\to\infty} \int_0^T \int_\Omega |Du_m - Du|^r \, dxdt = 0,$$

and thus (31) holds for a subsequence.

In case (NS2) (c) we can take over the proof presented in [1]. It remains to consider the case (NS2) (e). We suppose that $\nu_{(x,t)}$ is not a Dirac mass on a set $(x,t) \in M \subset \Omega \times (0,T)$ of positive Lebesgue measure $|M| > 0$. Then, by the strict p-quasimonotonicity of $\sigma(x,t,u,\cdot)$, and the fact that $\nu_{(x,t)}$ is a homogeneous $W^{1,p}$ gradient Young measure (see Section 2.7) for almost all $(x,t) \in \Omega \times (0,T)$, we have for a.e. $(x,t) \in M$

$$\begin{aligned}
&\int_{\mathbb{M}^{n\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) > \\
&\qquad > \int_{\mathbb{M}^{n\times n}} \sigma(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) : \underbrace{\int_{\mathbb{M}^{n\times n}} \lambda d\nu_{(x,t)}(\lambda)}_{= Du(x,t)}.
\end{aligned}$$

Hence, by integrating over $\Omega \times (0,T)$, we get together with Lemma 7

$$\int_0^T \int_\Omega \int_{\mathbb{M}^{n\times n}} \sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) : Du(x,t)dxdt \geqslant$$
$$\geqslant \int_0^T \int_\Omega \int_{\mathbb{M}^{n\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda)dxdt >$$
$$> \int_0^T \int_\Omega \int_{\mathbb{M}^{n\times n}} \sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) : Du(x,t)dxdt$$

which is a contradiction. Hence, we have $\nu_{(x,t)} = \delta_{Du(x,t)}$ for almost every $(x,t) \in \Omega \times (0,T)$. From this, it follows that $Du_m \to Du$ on $\Omega \times (0,T)$ in measure for $m \to \infty$ (see, e.g., [6]).

3.2 Conclusion

For the cases (NS2) (c), (d) and (e) there holds

$$\sigma(x,t,u_m,Du_m) \to \sigma(x,t,u,Du) \quad \text{in } L^\beta(\Omega\times(0,T)), \forall\beta\in[1,p'), \tag{33}$$
$$f(x,t,u_m,Du_m) \to f(x,t,u,Du) \quad \text{in } L^\beta(\Omega\times(0,T)), \forall\beta\in[1,p'). \tag{34}$$

To see this, just use (31), the boundedness of the sequences $\sigma(x,t,u_m,Du_m)$ and $f(x,t,u_m,Du_m)$ in $L^{p'}(\Omega \times (0,T))$, and apply the Vitali convergence theorem. It then follows that

$$-\operatorname{div}\sigma(x,t,u_m,Du_m) \rightharpoonup \chi = -\operatorname{div}\sigma(x,t,u,Du) \quad \text{in } L^{p'}(0,T;V'), \tag{35}$$
$$f(x,t,u_m.Du_m) \rightharpoonup \xi = f(x,t,u,Du) \quad \text{in } L^{p'}(\Omega\times(0,T)), \tag{36}$$

These properties are sufficient to pass to the limit in the Galerkin equations and to conclude the proof of Theorem 4 in case (i) (see [1, p. 266]).

For the remaining cases (NS2) (a) and (b) the property (31) does not hold in general, but we however obtain $\sigma(x,t,u_m,Du_m) \rightharpoonup \sigma(x,t,u,Du)$ in $L^{p'}(\Omega \times (0,T))$ by using Lemma 7 in [1, Section 7]. This suffices to conclude the proof of Theorem 4 in case (ii): In fact, in this situation we have assumed that f satisfies the assumption (Hf) (ii), and it remains to show that the property (36) still holds without (31). By using (23) we easily verify that it is true for the particular situation in (Hf) (ii), when f is independent of the fourth variable. In the other situation we have that, for a.e $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^n$, the mapping $F \to f(x,t,u,F)$ is linear. Here we argue as follows to identify the weak limit ξ in (36):

$$f(x,t,u_m,Du_m) \rightharpoonup \int_{\mathbb{M}^{n\times n}} f(x,t,u,\lambda)d\nu_{(x,t)}(\lambda)$$
$$= f(x,t,u,\cdot) \circ \int_{\mathbb{M}^{n\times n}} \lambda d\nu_{(x,t)}(\lambda) = f(x,t,u,Du),$$

where for the last equality we have used the property (ii) of Proposition 6.

We can now again pass to the limit in the Galerkin equations and conclude the proof of Theorem 4 in case (ii). We end this discussion by noting, that the energy equality (29) holds true with ξ replaced by $f(x,t,u,Du)$ and with χ replaced by $-\operatorname{div}\sigma(x,t,u,Du)$.

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