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Higher Abelian Dijkgraaf–Witten Theory

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Abstract. Dijkgraaf–Witten theories are quantum field theories based on (form degree 1) gauge fields valued in finite groups. We describe their generalization based on p-form gauge fields valued in finite abelian groups, as field theories extended to codimension 2.

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1. Introduction and Summary

Dijkgraaf–Witten (DW) theories [1] are essentially Chern–Simons theories for gauge fields valued in a finite group Γ, and can be defined in any dimension. Their fields are connections on principal Γ-bundles. Due to the finiteness of Γ, there is only one connection on each principal bundle and it is necessarily flat. As a result, the space of fields is finite, and the path integral reduces to a finite instanton sum, making their exact quantization straightforward. For this reason, they are interesting toy models of quantum gauge field theories.

Abelian gauge fields have higher degree cousins, described locally by p-forms and globally by degree $p+1$ differential cohomology classes [2]. When the gauge group is $U(1)$, they can be thought of as connections on certain "higher circle bundles" that can be defined using higher category theory. We describe in the present paper generalizations of abelian Dijkgraaf–Witten theories whose fields are higher gauge fields valued in a finite abelian group Γ. Just as for ordinary DW theories, the path integrals are finite sums and we can describe the quantum theories exactly.

Two subtleties appear in the construction below. The first is about finding a good model for the higher gauge fields. We do not know a convenient higher generalization of principal bundles with connection valued in a finite group (but see [3–6] for the $p=2$ case). However, as Dijkgraaf–Witten theory is a gauge theory, only the set of isomorphism classes of fields matters. On a manifold M, the isomorphism classes of higher abelian gauge fields are given by $H^p(M; \Gamma)$, and we can take the fields to be cocycles valued in Γ. Indeed, ordinary Dijkgraaf–Witten

theories themselves can be reformulated in terms of 1-cocycles valued in Γ, instead of principal Γ-bundles.

The second subtlety is the determination of the measure (4.1) on the space of fields, which appears in the instanton sum defining the quantum theory. These factors crucially obey the relation (4.3), which ensures that the field theory functor is compatible with the gluing of manifolds with boundary, as we show in Section 6. Our restriction to abelian groups makes the measure constant across the space of fields, leaving only the dependence on the underlying manifold. The structure of the measure is nevertheless interesting, being given by an alternating product of orders of Γ-valued cohomology groups. It suggests an interpretation in terms of a tower of ghosts that is not made explicit in our construction. In more mathematical terms, it should coincide with the homotopy cardinality of a p-groupoid of fields, but we will not attempt to make this higher categorical structure manifest here.

Apart from the above, the proof of Freed and Quinn [7,8] showing that ordinary Dijkgraaf–Witten theory defines a field theory functor generalizes easily.

We define the higher abelian Dijkgraaf–Witten theories only as field theories extended to codimension 2, because we do not have a clear picture of the higher categorical objects assigned by the field theory functor to manifolds of codimension higher than 2. The heuristic arguments of [8] suggest, however, that there should be no problem defining these theories as fully extended field theories.

We should mention that closely related field theories have been constructed by Ševera in [9]. The idea of generalizing Dijkgraaf–Witten theory by replacing $B\Gamma$ by a more general classifying space was mentioned in [10], as well as the sketch of a general framework for finite path integration.

It would be interesting to construct Dijkgraaf–Witten theories of higher gauge fields valued in non-abelian finite groups. The quantum theory of non-abelian higher gauge fields is unknown,[1] and the latter appear in several physically interesting theories, such as (2,0) superconformal field theories in six dimensions or gauged supergravities. One may hope that the simple setting of Dijkgraaf–Witten theory will provide interesting insights. A possible avenue is to repeat the present construction in the framework of non-abelian cohomology (see for instance [11–13]). In the context of state sum models, results have been obtained by Yetter in [14] (see also [15]) in the case $p = 2$, and by Porter in [16] for generic p. We will not discuss further the non-abelian case here.

The paper is organized as follows. In Section 2, we explain that the isomorphism classes of fields in the higher DW theories are classified by the pth cohomology group of the underlying manifold with value in Γ. In Section 3, we describe the structures on space time manifolds required to define the theory. We describe in Section 4 the space of fields over a manifold, paying particular attention to the case where the latter has a boundary. We define there the measure factors crucial

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[1]See, however, [5] for an approach to quantization using the BV formalism.

for the definition of the theory in Section 5. In Section 6, we show that the field theory functor is compatible with the gluing of manifolds.

2. Degree p Γ-Valued Gauge Fields

Let Γ be a finite abelian group. We would like to construct a version of DW theory in which the fields on which the path integral is performed generalize principal Γ-bundles in the same way as p-form gauge fields generalize ordinary (i.e. 1-form) abelian gauge fields. While the case $p=2$ is rather well understood [3–6], we do not have a good picture for such objects for general p. However, we can make sense of their isomorphism classes as follows, which will turn out to be sufficient to formulate the DW theory.

We remark that the isomorphism classes of principal Γ-bundles over a manifold M are classified by $H^1(M;\Gamma)$, which is ultimately due to the fact that the classifying space $B\Gamma$ is an Eilenberg–MacLane space $K(\Gamma,1)$. The usual DW theory can be reformulated in terms of degree 1 Γ-valued cocycles instead of principal Γ-bundles. The precise model used for the cochains has no influence on what follows and we take singular cochains for definiteness. Of course, there is no bijection between principal Γ-bundles and Γ-valued 1-cocycles, but there is a bijection between the isomorphism classes of such objects. As the DW theory relies ultimately only on gauge invariant data, the two formulations are equivalent.[2] This is a concrete illustration of the fact, well known to physicists, that a "gauge symmetry" is only a redundancy in the description of the theory, and not a property of the theory itself.

In the cocycle formulation, the generalization to higher degree is obvious. The fields of the higher abelian DW theories are degree p Γ-valued cocycles. We declare that any two such cocycles are isomorphic if they differ by the differential of a degree $p-1$ Γ-valued cochain. The isomorphism classes of fields on a closed manifold M are, therefore, elements of the cohomology group $H^p(M;\Gamma)$. We will discuss the case of manifolds with boundary later.

As a further piece of evidence for the claim above, we remark that degree p $\mathbb{Z}_n$-valued gauge fields should be representable by flat degree $p+1$ differential cohomology classes that are also n-torsion, and that the latter are classified by $H^p(M;\mathbb{Z}_n)$.

3. Structures on Manifolds

We consider manifolds endowed with certain unspecified geometrical/topological structures, denoted by F (see Appendix A.4 of [17]). We assume that given a man-

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[2]To be more precise, as pointed out by the referee, we need the groupoids of fields to be equivalent as categories, i.e. the spaces of gauge transformations (morphisms) should be in bijection as well. In the case of interest to us, this condition is trivially satisfied. It is sufficient to check that in both models, the automorphism group of a field is given by $H^0(M;\Gamma)$.

ifold M, $\mathsf{F}(M)$ includes an orientation on M and a homotopy class of maps $[\gamma_P]$ from M to $K(\Gamma, p)$. $[\gamma_P]$ determines an element P of $H^p(M; \Gamma)$, hence an isomorphism class of gauge fields on M. We will call such manifolds *manifolds with F-structure*, or simply F-*manifolds*. We write $\bar{\mathsf{F}}$ for the structure encoding the same data as F, minus the homotopy class $[\gamma_P]$. We also assume that we are given a cohomology class $c_U \in H^d(K(\Gamma, p), U(1))$, that plays the role of the exponentiated action of the theory. The data $\mathsf{F}(M)$ then include a cohomology class $c := \gamma_P^* c_U \in H^d(M, U(1))$.

As explained in [7,8], there is a sense in which one can integrate c over the $d-k$-dimensional manifold M. For $k=0$, the integration map is the usual integration of cochains, yielding an element of $U(1)$. For $k=1$, one obtains a Hermitian line, i.e. a one-dimensional Hilbert space. For $k=2$, one obtain a two-Hermitian line, which is a category equivalent to the category $\mathcal{H}_1$ of finite dimensional Hilbert spaces (see for instance Appendix A.2 of [17]). For higher k, one obtains higher analogues of Hermitian lines [8]. We will write $\mathcal{I}_c$ for the integration map.

$\mathcal{I}_c$ is a field theory defined on manifolds with F-structure. It can be seen as a classical version of the DW theory [7,8]. More precisely, in the terminology of geometric quantization, it is the prequantum version of the DW theory determined by the exponentiated action c_U. The quantum DW theory $\mathcal{DW}_c$ is defined on manifolds with $\bar{\mathsf{F}}$-structure, via a sum of $\mathcal{I}_c$ over the space of isomorphism classes of degree p Γ-valued gauge fields. This sum should be interpreted as a path integral over the field space of the theory.

In the following, all the manifolds are assumed to be $\bar{\mathsf{F}}$-manifolds, and we denote F-manifolds by pairs (M, P), where M is a $\bar{\mathsf{F}}$-manifold and P is the gauge field isomorphism class encoded in $\mathsf{F}(M)$.

4. Fields

Let M be a $\bar{\mathsf{F}}$-manifold, possibly with boundary or corners. The fields on M are degree p Γ-valued cocycles, which we write hatted. A cocycle $\hat{P}_1$ is isomorphic to a cocycle $\hat{P}_2$ if they define the same cohomology class, i.e. if there is a degree $p-1$ cochain $\hat{\phi}$ such that $\hat{P}_2 = \hat{P}_1 + d\hat{\phi}$. As they have the same action on cocycles, we identify isomorphisms differing by the differential of a cochain, i.e. $\hat{\phi} \sim \hat{\phi} + d\hat{\rho}$. With these identifications, the automorphism group $\text{Aut}(\hat{P})$ is $H^{p-1}(M; \Gamma)$, which is a finite group. We write P for the cohomology class of $\hat{P}$.

We will also need the notion of relative cocycle. Let $\hat{Q}$ be a degree p Γ-valued cocycle over ∂M. A degree p Γ-valued cocycle on M relative to $\hat{Q}$ (in short a relative cocycle), is a pair $(\hat{P}, \hat{\theta})$ where $\hat{P}$ is a degree p Γ-valued cocycle on M and $\hat{\theta}$ is a degree $p-1$ Γ-valued cochain on ∂M such that $\hat{P}|_{\partial M} = \hat{Q} + d\hat{\theta}$. An isomorphism between two relative cocycles $(\hat{P}_1, \hat{\theta}_1)$ and $(\hat{P}_2, \hat{\theta}_2)$ is an equivalence class of degree $p-1$ Γ-valued cochain $\hat{\phi}$ on M such that $P_2 = P_1 + d\hat{\phi}$ and $\theta_2 = \theta_1 + \hat{\phi}|_{\partial M}$. Two such cochains are equivalent if they differ by the differential of a cochain vanishing on the boundary: $\hat{\phi} \sim \hat{\phi} + d\hat{\rho}$ with $\hat{\rho}|_{\partial M} = 0$. The automorphism

group $\mathrm{Aut}(\hat{P}, \hat{\theta})$ is $H^{p-1}(M, \partial M; \Gamma)$, the relative cohomology group with value in Γ, which is a finite group. We write (P, θ) for the cohomology class of $(\hat{P}, \hat{\theta})$.

We now define measure factors that play a crucial role in the definition of the theory, and prove a fundamental identity they satisfy. Let

$$\mu_M = \prod_{i=0}^{p-1} |H^i(M, \partial M; \Gamma)|^{(-1)^{p-i}}, \tag{4.1}$$

where $|G|$ denotes the order of the finite group G. Let us furthermore define for $N \subset M$, $N \cap \partial M = \emptyset$,

$$\mu_{(M,N)} = \prod_{i=0}^{p-1} |H^i(M, N \cup \partial M; \Gamma)|^{(-1)^{p-i}}. \tag{4.2}$$

Let K be the kernel of the map $H^p(M, N \cup \partial M; \Gamma) \to H^p(M, \partial M; \Gamma)$.

Lemma 4.1. *The following equality holds:*

$$\mu_M = |K| \mu_{(M,N)} \mu_N \tag{4.3}$$

Proof. This is an immediate consequence of the long exact sequence for relative cohomology:

$$\cdots \to H^{p-2}(N) \to H^{p-1}(M, N \cup \partial M) \to H^{p-1}(M, \partial M) \tag{4.4}$$
$$\to H^{p-1}(N) \to H^p(M, N \cup \partial M) \to H^p(M, \partial M),$$

where we suppressed the argument Γ in the cohomology groups. $\square$

Remark that in the ordinary Dijkgraaf–Witten theory, $\mu_{(M,\mathscr{P})} = 1/|\mathrm{Aut}(\mathscr{P})|$, where $\mathscr{P}$ is a principal Γ-bundle, and $\mathrm{Aut}(\mathscr{P})$ is the group of automorphisms of $\mathscr{P}$ leaving $\mathscr{P}|_{\partial M}$ fixed. When Γ is abelian, $|\mathrm{Aut}(\mathscr{P})| = |H^0(M, \partial M; \Gamma)|$, which is consistent with (4.1).

5. Definition of the Theory

In the following we use the following conventions. A 0-Hilbert space is a complex number. A 1-Hilbert space is a finite dimensional Hilbert space. The category of 1-Hilbert spaces is denoted by $\mathcal{H}_1$. A 2-Hilbert space [18] is a $\mathbb{C}$-linear category linearly equivalent to the nth Cartesian product of $\mathcal{H}_1$ with itself, endowed with extra structure, see also Appendix A.2 of [17]. In particular, a 2-Hilbert space H is endowed with a functor $(\bullet, \bullet)_H : H^{\mathrm{op}} \times H \to \mathcal{H}_1$, playing the role of the inner product. The 2-Hilbert spaces form a 2-category $\mathcal{H}_2$. $\mathcal{H}_2$ admits a dagger structure given by the complex conjugation and a symmetric monoidal structure described in Section 4.4 of [18].

We write $M^{d,p}$ for a generic $\bar{\mathsf{F}}$-manifold of dimension d with corners of dimension $d - p$ or higher. If $\mathsf{X} = \mathsf{F}, \bar{\mathsf{F}}$, $\mathcal{B}_{\mathsf{X}}^{d,p}$ is the bordism category consisting of X-manifolds of dimension $d - p, \ldots, d$ with corners of dimension $d - p$ or higher, see Appendix A.4 of [17]. The bordism category has a dagger structure given by the orientation reversal of manifolds, and a symmetric monoidal structure given by the disjoint union of manifolds.

We will define below the quantum DW theory as a 2-functor

$$\mathcal{DW}_c : \mathcal{B}_{\mathsf{F}}^{d,2} \to \mathcal{H}_2 \tag{5.1}$$

compatible with the dagger and the monoidal structures. We rely on the fact that the prequantum DW theory

$$\mathcal{I}_c : \mathcal{B}_{\mathsf{F}}^{d,2} \to \mathcal{H}_2 \tag{5.2}$$

is such a 2-functor [7,8]. (See also Section 4 of [17].)

5.1. CLOSED $d - k$-DIMENSIONAL MANIFOLDS

Here $k = 0, 1, 2$. The prequantum DW theory $\mathcal{I}_c$ associates a k-Hilbert space $\mathcal{I}_c(M^d, P)$ to a closed d-dimensional F-manifold (M^d, P). We define the value of the quantum DW theory on M^{d-k} by

$$\mathcal{DW}_c(M^{d-k}) = \sum_{P \in H^p(M^{d-k};\Gamma)} \mu_{M^{d-k}} \mathcal{I}_c(M^{d-k}, P). \tag{5.3}$$

The sum sign should be understood as an ordinary sum when $k = 0$, as a direct sum of Hilbert spaces when $k = 1$ and as the direct sum of 2-Hilbert spaces for $k = 2$ (see Appendix A.2 in [17]). The multiplication by $\mu_{M^{d-k}}$ also deserves an explanation. For $k = 0$ this is the ordinary multiplication of complex numbers by the rational number $\mu_{M^{d-k}}$. For $k = 1$, $\mu \in \mathbb{Q}_+$ and H a Hilbert space, μH is the vector space H, endowed with the inner product of H rescaled by μ: $(\bullet, \bullet)_{\mu H} = \mu(\bullet, \bullet)_H$. For $k = 2$, let H be a 2-Hilbert space, endowed with an inner product $(\bullet, \bullet)_H$ valued in $\mathcal{H}_1$. Then μH is the 2-vector space H, endowed with an inner product $(\bullet, \bullet)_{\mu H}$ defined as follows. For any $V_1, V_2 \in H$, $(V_1, V_2)_{\mu H} = \mu(V_1, V_2)_H$, where the multiplication on the right-hand side should be interpreted according to the $k = 1$ case we described above.

5.2. $d - k$-DIMENSIONAL MANIFOLDS WITH BOUNDARY

Here $k = 0, 1$. We define the value of the quantum DW theory on $M^{d-k,1}$ by an expression formally similar to (5.3):

$$\mathcal{DW}_c(M^{d-k,1}) = \sum_{P \in H^p(M^{d-k,1};\Gamma)} \mu_{M^{d-k,1}} \mathcal{I}_c(M^{d-k,1}, P). \tag{5.4}$$

Consistency requires that

$$\mathcal{DW}_c(M^{d-k,1}) \in \mathcal{DW}_c(\partial M^{d-k,1}). \tag{5.5}$$

But this is immediately implied by the corresponding relation for the prequantum DW theory: $\mathcal{I}_c(M^{d-k,1}, P) \in \mathcal{I}_c(\partial M^{d-k,1}, P|_{\partial M^{d-k,1}})$ [7,8]. Equation (5.5) implies in particular that given a bordism B^{d-k} between manifolds M_1^{d-k-1} and M_2^{d-k-1}, $\mathcal{DW}_c(B^{d-k})$ is a homomorphism ($k = 0$) or a $\mathbb{C}$-linear functor ($k = 1$) from $\mathcal{DW}_c(M_1^{d-k-1})$ to $\mathcal{DW}_c(M_2^{d-k-1})$.

5.3. d-DIMENSIONAL MANIFOLDS WITH CORNERS

Let $M^{d,2}$ be a d-dimensional manifold with $\partial M^{d,2} = -N_1 \cup N_2$, where $\partial N_1 = \partial N_2 = -M_1 \sqcup M_2$. We define

$$\mathcal{DW}_c(M^{d,2}) = \sum_{P \in H^p(M^{d,2};\Gamma)} \mu_{M^{d,2}} \mathcal{I}_c(M^{d,2}, P). \tag{5.6}$$

The fact that $\mathcal{DW}_c(M^{d,2})$ is a 2-morphism between the 1-morphisms $\mathcal{DW}_c(N_1)$ and $\mathcal{DW}_c(N_2)$ is directly inherited from the corresponding property of the prequantum DW theory [8].

5.4. HIGHER CODIMENSION

Formulas (5.3), (5.4) and (5.6) clearly have the same structure. Given a concrete construction of the prequantum DW field theory as a fully extended field theory, for instance along the lines proposed in [19], the same formulas should define the higher abelian DW theories as fully extended field theories. We expect the proof of the gluing law in the next section to be formally identical, see [8] for the case of ordinary DW theories.

5.5. COMPATIBILITY

The compatibility of $\mathcal{DW}_c$ with the † and monoidal structures of $\mathcal{B}_{\bar{\mathsf{F}}}^{d,2}$ and $\mathcal{H}_2$ comes from the compatibility of $\mathcal{I}_c$ with these structures [7,8,17], and the fact that $\mu(M_1 \cup M_2) = \mu(M_1)\mu(M_2)$ for M_1 and M_2 disjoint manifolds.

6. Gluing

The compatibility of the prequantum DW theory with gluing (i.e. the compatibility of the functor $\mathcal{I}_c$ with the composition of morphisms in $\mathcal{B}_{\mathsf{F}}^{d,2}$ and $\mathcal{H}_2$) is obvious from the locality of the integral. Because of the sums involved, the compatibility with gluing is not obvious in the DW theory and we check it here.

Let $M^{d-k,1}$ be as usual a $d - k$-dimensional $\bar{\mathsf{F}}$-manifold with boundary and let $N \subset M^{d-k,1}$ be a codimension 1 submanifold disjoint from the boundary. Let

$M_N^{d-k,1}$ be the manifold $M^{d-k,1}$ cut along N, whose boundary is $\partial M^{d-k,1} \cup N \cup -N$. The compatibility with gluing is equivalent to the following

Theorem 6.1. *We have:*

$$\mathcal{DW}_c(M^{d-k,1}) = \text{Tr}_{\mathcal{DW}_c(N)}(\mathcal{DW}_c(M_N^{d-k,1})), \tag{6.1}$$

where Tr *on the right-hand side denotes the contraction of*

$$\mathcal{DW}_c(M_N^{d-k,1}) \in \mathcal{DW}_c(\partial M^{d-k,1}) \otimes \mathcal{DW}_c(N) \otimes (\mathcal{DW}_c(N))^{\dagger} \tag{6.2}$$

using the canonical pairing between $\mathcal{DW}_c(N)$ and its dual.

Remark that the trace involves a scalar multiplication. For $k = 1$, the pairing is valued in $\mathcal{H}_1$ and the scalar multiplication is a tensor product-like operation between a Hilbert space and an element of the 2-Hilbert space $\mathcal{DW}_c(\partial M^{d-k,1})$, see for instance Appendix A.2 of [17].

Our proof of the gluing relation (6.1) is strongly inspired by the corresponding proof in [7,8], valid for the usual DW theory and its extended version. In the present proof, we write M for $M^{d-k,1}$ and M_N for $M_N^{d-k,1}$. We omit the argument Γ in all cohomology groups to simplify the notation. All cohomology groups are understood to be relative with respect to $\partial M \subset M, M_N$, and we suppress as well this information from the notation. Cocycles are always hatted and their cohomology classes are denoted by the same letter without a hat. Let us write $\pi : M_N \to M$ for the gluing map that identifies the components N and $-N$ of the boundary of M_N. It induces a map $\pi^* : H^p(M) \to H^p(M_N)$. Let $H_N^p(M)$ and $K(M, N)$ be the image and kernel of the inclusion $H^p(M, N) \to H^p(M)$. Similarly, let us write $H_{N \cup -N}^p(M_N)$ for the image of the inclusion $H^p(M_N, N \cup -N) \to H^p(M_N)$. We can realize any class in $H_{N \cup -N}^p(M_N)$ as a cocycle vanishing on $N \cup -N$, which we can push forward to M. We obtain in this way a map $\pi_* : H_{N \cup -N}^p(M_N) \to H_N^p(M)$. We need the following preliminary lemma.

Lemma 6.2. *π_* is surjective and the order of its kernel is $|K(M, N)|$.*

Proof. Let $P \in H_N^p(M)$. We can pick a representative cocycle $\hat{P}$ that vanishes on $N \subset M$. Let $\hat{R} = \pi^*(\hat{P})$ and let R be the corresponding cohomology class in $H_{N \cup -N}^p(M_N)$. Then $\pi_*(R) = P$, so π_* is surjective.

We now describe an action of $H^{p-1}(N)$ on $\pi_*^{-1}(S)$, for $S \in H_N^p(M)$. We show that it is transitive and compute its kernel, which allows us to deduce the order of the kernel of π_*.

The automorphism group of any cocycle on N is $H^{p-1}(N)$. An element $\psi \in H^{p-1}(N)$ acts on a cocycle representative $(\hat{P}, \hat{\theta}_N, \hat{\theta}_{-N}, \hat{\theta}_{\partial M})$ of a class P in $H^p(M_N, N \cup -N)$ by

$$\psi \cdot (\hat{P}, \hat{\theta}_N, \hat{\theta}_{-N}, \hat{\theta}_{\partial M}) = (\hat{P}, \hat{\theta}_N + \hat{\psi}, \hat{\theta}_{-N} + \hat{\psi}, \hat{\theta}_{\partial M}), \tag{6.3}$$

where $\hat{\psi}$ is a cocycle representative of ψ. This induces an action of $H^{p-1}(N)$ on $H^p(M_N, N \cup -N)$, which passes to $H^p_{N \cup -N}(M_N)$ (and which we still denote with ·). The kernel of this action at P is the image of the restriction $H^{p-1}(M) \to H^{p-1}(N)$.

The action of $H^{p-1}(N)$ on $H^p_{N \cup -N}(M_N)$ is transitive when restricted to $\pi_*^{-1}(S)$, for $S \in H^p_N(M)$, as the following argument shows. Let $P_1, P_2 \in H^p_{N \cup -N}(M_N)$, with cocycle representatives $\hat{P}_1$ and $\hat{P}_2$. By definition, we have $\hat{P}_i|_{\pm N} = d\hat{\theta}_{i, \pm N}$. Assume now that $\pi_*(P_1) = \pi_*(P_2) = S$. This means that there is an isomorphism ϕ such that $\hat{P}_2 = \hat{P}_1 + d\hat{\phi}$ and

$$\hat{\theta}_{1,N} - \hat{\theta}_{2,N} + \hat{\phi}|_N = \hat{\theta}_{1,-N} - \hat{\theta}_{2,-N} + \hat{\phi}|_{-N} = \hat{\psi}. \tag{6.4}$$

We have $d\hat{\psi} = 0$, so $\hat{\psi}$ is a cocycle on N. Therefore, $\psi \in H^{p-1}(N)$ and $\psi \cdot P_1 = P_2$.

From the knowledge of the action kernel and of the fact that the action is transitive, we deduce that the order of $\pi_*^{-1}(S)$ is $|H^{p-1}(N)|/|\mathrm{Im}(H^{p-1}(M) \to H^{p-1}(N)|$. The long exact sequence (4.4) shows that this is equal to $|K(M, N)|$. □

Proof of Theorem 6.1. We use the definition of the left-hand side to write

$$\mathcal{DW}_c(M) = \mu_M \sum_{P \in H^p(M)} \mathcal{I}_c(M, P). \tag{6.5}$$

Let $H^p_{\mathrm{ext}}(N)$ be the image of the restriction map $H^p(M) \to H^p(N)$, and let Q_{ext} be a choice of preimage for each $Q \in H^p_{\mathrm{ext}}(N)$. We decompose the sum over the classes in $H^p(M)$ as a sum over $H^p_{\mathrm{ext}}(N)$ and $H^p_N(M)$. The right-hand side of (6.5) becomes

$$\mu_M \sum_{Q \in H^p_{\mathrm{ext}}(N)} \sum_{P \in H^p_N(M)} \mathcal{I}_c(M, P + Q_{\mathrm{ext}}). \tag{6.6}$$

We use (4.3), the gluing relation for $\mathcal{I}_c$ and the linearity of the trace to rewrite (6.6) as

$$\mu_N \mu_{(M,N)} |K(M, N)| \sum_{Q \in H^p_{\mathrm{ext}}(N)} \mathrm{Tr}_{\mathcal{I}_c(N,Q)} \left(\sum_{P \in H^p_N(M)} \mathcal{I}_c(M_N, \pi^*(P + Q_{\mathrm{ext}})) \right). \tag{6.7}$$

Let $H^p_{\mathrm{ext}}(N \cup -N)$ be the image of the restriction map $H^p(M_N) \to H^p(N \cup -N)$ and let Q_{ext} be a choice of preimage for each $Q \in H^p_{\mathrm{ext}}(N \cup -N)$. Because the trace selects the diagonal component, we can replace the sum over $Q \in H^p_{\mathrm{ext}}(N)$ in (6.7) by a sum over $Q \in H^p_{\mathrm{ext}}(N \cup -N)$.

Lemma 6.2 says that the order of the kernel of the gluing map $H^p_{N \cup -N}(M_N) \to H^p_N(M)$ is $|K(M, N)|$, so we can also replace $|K(M, N)| \sum_{P \in H^p_N(M)}$ with $\sum_{P \in H^p_{N \cup -N}(M_N)}$. We obtain

$$\mu_N \mu_{(M,N)} \sum_{Q \in H^p_{\mathrm{ext}}(N \cup -N)} \mathrm{Tr}_{\mathcal{I}_c(N,Q)} \left(\sum_{P \in H^p_{N \cup -N}(M_N)} \mathcal{I}_c(M_N, P + Q_{\mathrm{ext}}) \right). \tag{6.8}$$

But now we use again the linearity of the trace, the fact that $\mu_{(M,N)} = \mu_{M_N}$ and remark that the full sum yields $\mathcal{DW}_c(M_N)$. Moreover, the trace over $\mathcal{DW}_c(N)$ is μ_N times the sum of the traces over $\mathcal{I}_c(N, Q)$, so we finally obtain (6.1).   □

This proves that $\mathcal{DW}_c$ is a 2-functor, hence defines a field theory.

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