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Rational Differential Systems, Loop Equations, and Application to the qth Reductions of KP

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Abstract. To any solution of a linear system of differential equations, we associate a matrix kernel, correlators satisfying a set of loop equations, and in the presence of isomonodromic parameters, a Tau function. We then study their semiclassical expansion (WKB type expansion in powers of the weight $\hbar$ per derivative) of these quantities. When this expansion is of topological type (TT), the coefficients of expansions are computed by the topological recursion with initial data given by the semiclassical spectral curve of the linear system. This provides an efficient algorithm to compute them at least when the semiclassical spectral curve is of genus 0. TT is a non-trivial property, and it is an open problem to find a criterion which guarantees it is satisfied. We prove TT and illustrate our construction for the linear systems associated to the qth reductions of KP—which contain the (p, q) models as a specialization.

1. Introduction

Let $\mathbf{L}(x)$ be a $d \times d$ matrix with entries being rational functions of x, and $\mathcal{P}$ the set of poles of $\mathbf{L}$. We consider matrix $\mathbf{\Psi}(x)$ (whose columns form a basis of solutions) of the differential system:

$$\hbar\,\partial_x \mathbf{\Psi}(x) = \mathbf{L}(x)\mathbf{\Psi}(x), \tag{1.1}$$

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i.e. $\mathbf{\Psi}(x)$ is a $d \times d$ invertible matrix solving (1.1). It is well known that $\mathbf{\Psi}(x)$ is locally holomorphic in $\widehat{\mathbb{C}}\backslash\mathcal{P}$. The matrix $\mathbf{L}$ (and thus $\mathbf{\Psi}$) may depend on $\hbar$ and on extra parameters t_α. The goal of this article is to establish a set of loop equations satisfied by some quantities built out of $\mathbf{\Psi}$ called "correlators", and analyze their consequences, especially for small $\hbar$ expansions—whether at the formal level, or at the level of asymptotics. Very often, if one wishes to study the asymptotic behavior in some parameter x or t_α of a differential system, one can introduce by hand a parameter $\hbar$ to put the system in the form (1.1), so that the asymptotic regime of interest corresponds to $\hbar \to 0$. We think of the loop equations as providing a new point of view on the study of the regime $\hbar \to 0$. Some aspects of the problem—in particular the possible connections to enumerative geometry—are hidden when studying merely the expansion for the solution $\mathbf{\Psi}$ itself. In a sense, the study of the correlators via the loop equations reorganize this expansion with a focus on the singularities at the turning points, and reveals a rich structure.

1.1. Outline

The paper is organized in three parts.

Notions and Properties. Firstly, in Sect. 2, we associate to any $d \times d$ invertible matrix $\mathbf{\Psi}(x)$ solution of a linear differential system:

- a $d \times d$ matrix $\mathbf{K}(x, y)$, called *matrix kernel*.
- an infinite family of functions $\mathcal{W}_n(\overset{a_1}{x_1}, \ldots, \overset{a_n}{x_n})$, indexed by a n-tuple of integers $a_1, \ldots, a_n \in [\![1, d]\!]$, called *n-point correlators*, or shortly, correlators.

We show that the n-point correlators satisfy a set of linear equations (Theorem 2.1) and a set of quadratic equations (Theorem 2.2). We use the name *loop equations* to refer collectively to those set of equations. We also introduce a notion of "insertion operator" (Definition 2.5) allowing the derivation of k-linear loop equations for $n \leq d$ (the size of the differential system) from the master ones. These results are of purely algebraic nature and hold for any system (1.1). When $\mathbf{L}$ depends on a set of parameters $\vec{t}$ preserving the monodromy of the solutions, we can also associate to $\mathbf{\Psi}(x, \vec{t})$ a Tau function $\mathcal{T}(\vec{t})$, defined up to a constant prefactor.

What we call "matrix kernel" can be thought as a parallel transport map of the connection $\hbar\partial_x - \mathbf{L}$ between the points x and y, in the basis provided by $\mathbf{\Psi}$. For the Zakharov–Shabat system, it is closely related to the soliton correlation matrix introduced in [50]. The n-point correlators are sums over n-cycles of cyclic products of the matrix kernel $\mathbf{K}$, that is the "connected part" of the $n \times n$ determinants built from $\mathbf{K}$. To our knowledge, the definition of correlators in the context of ODEs originally appeared in earlier work of two of the authors [8]. Though the definition is very simple, it is not part of classical textbooks on differential equations or integrable systems. We nevertheless think that the correlators are interesting objects. The main reason for us is that they satisfy loop equations in a universal form, and particular instances of those loop equations are relevant in random matrix theory and the enumerative geometry of surfaces. When specialized to integrable systems including a complete (infinite) set of times, the n-point correlators encode the nth order derivatives of the Tau function with respect to this family of times. In this regard, the sequence of correlators is a way to repackage information that is intermediate between the solution $\mathbf{\Psi}$ of the linear system and the corresponding Tau function. The relation between the correlators and the solution $\mathbf{\Psi}$ is sometimes called "boson–fermion correspondence".

Though we sometimes borrow the vocabulary of integrable systems or matrix models to make the reader feel more familiar, we insist that Theorem 2.2 is valid without any integrable property assumed for the differential system, and for differential systems which are not necessarily related to matrix models. Yet, for the differential systems appearing in the 1 and 2 hermitian matrix models, the matrix kernel and correlators can be realized as observables in the matrix model, as we review in "Appendix D". In this context, the insertion operator corresponds to the infinitesimal addition of a simple pole to the matrix model potential.

Semiclassical Expansions. In Sect. 3, we study the semiclassical expansion in powers of $\hbar$ and describe in detail the monodromy of its coefficients (Sects. 3.2–3.4). We introduce in Definition 3.3 the notion of "expansion of topological type"—also referred to as the TT property—and show that the expansion can be computed by the topological recursion of [41] when the TT property holds. In practice, the main consequence of our theory is Theorem 3.1, and in the presence of isomonodromic times, this also allows the computation of the semiclassical expansion of $\ln \mathcal{T}(\vec{t})$ (Corollary 4.2). In other words, we provide a method that can be applied—once the assumptions are checked—to establish a relation between the coefficients of the all-order WKB expansions and the geometric invariants computed by the topological recursion. Since those invariants can always be expressed in terms of intersection indices on the moduli space of curves [39], we learn that those WKB expansions have something to do with the enumerative geometry of surfaces, a fact which would be hidden at this level of generality if one did not consider correlators and the loop equations they satisfy.

Applications. Finally, in Sect. 5, we apply our theory to the linear system associated to the qth reduction of KP and illustrate it more specifically with examples of the (p, q) models (Sect. 6). As a motivation, those hierarchies are believed to describe the algebraic critical edge behavior that can be reached in the two-hermitian-matrix model, and universality classes of $2d$ quantum gravity coupled to conformal field theories [28,30,49,58]. In any qth reduction of KP, we show (Sects. 5.6–5.8) that the TT property holds, and that our Theorem 3.1 can be applied.

1.2. Comments

The earlier work [8] described the construction of Sect. 2 for general 2×2 rational systems, but implicitly assumed the TT property. It was illustrated for $(2m + 1, 2)$ systems in [10] and entails a rigorous proof—modulo checking the TT property, which had not been performed so far—of an equivalence between the three usual approaches of quantum gravity, namely topological gravity (in relation with intersection theory on the moduli space of curves), random maps, and $(2m + 1, 2)$ models (see [28] for a review on those equivalences in physics). Again taking the TT property as an assumption, [25] treated the models $(2m, 1)$, in relation with the merging of two cuts in random matrix theory. The TT property was made explicit and checked by integrability arguments in [17] for a 2×2 linear system associated to the Painlevé II equation [44], justifying the computation of asymptotics of the GUE Tracy–Widom law by the topological recursion. The same approach—with a justification of the TT property—was applied more recently [11] to the 2×2 linear system of associated to Painlevé V [52], relevant to get the GUE sine kernel law. So far, this concerned only 2×2 systems.

The present work aims at presenting a complete theory for general $d \times d$ rational systems and developing tools to study the TT property. Its application to the (p, q) models can then be used to establish rigorously the equivalence between the three quantum gravities for all (p, q) models. For clarity, this will appear in a separate work [16].

In [18], the two last authors have made a conjecture to construct an integrable system out of the topological recursion of a given spectral curve. The present work aims at the converse: showing that the semiclassical expansion of linear differential systems satisfying the TT property is computed by the topological recursion of their semiclassical spectral curve.

The TT property is neither expected to hold in general—even among integrable systems—nor obvious to establish for a given system. Our proof that it holds for the qth reduction of the KP hierarchy depends in an essential way on the integrability of the latter, i.e. on the existence of another system $\hbar\, \partial_t \mathbf{\Psi}(x, t) = \mathbf{M}(x, t)\mathbf{\Psi}(x, t)$ with rational coefficients in x, which is compatible with (1.1), and also on the specific form of $\mathbf{M}(x, t)$. This is clear from the technical results of Sects. 5.7 and 5.8. We can formulate the existence of TT property as Question 4.1.

Within the TT property, the structure of the asymptotic expansion of correlators is identified in Theorem 3.1, but when the semiclassical spectral curve has genus $\mathfrak{g} > 0$, it can contain "holomorphic parts" $H_n^{(g)}(z_1, \ldots, z_n)$, which form basically the moduli of the space of solutions of loop equations. A given solution $\mathbf{\Psi}(x)$ knows which $H_n^{(g)}(z_1, \ldots, z_n)$ is chosen. It thus remains to investigate which conditions have to be added to the loop equations to determine completely the holomorphic part. They probably should take the form of period conditions. Actually, for many interesting solutions $\mathbf{\Psi}(x)$, we expect the TT property to breakdown if the semiclassical spectral curve has genus $\mathfrak{g} > 0$.

We stress that, even when the TT property does not hold, the loop equations of Theorems 2.1 and 2.2 are still satisfied and provide some constraints on the asymptotic expansion of $\mathbf{\Psi}(x)$. In particular, the existence of a non-trivial moduli space of solutions of loop equations—which, in the context of expansion in powers of $\hbar$, can be parametrized by a "holomorphic part" at each order in $\hbar$—can be responsible for the breakdown of expansion in powers of $\hbar$, since the moduli may depend on $\hbar$ in a complicated way. This mechanism explains for instance the oscillatory asymptotics in matrix models which was first established in [26], and retrieved by other methods heuristically in [38] and rigorously in [21]. It is also implicit in [18], where the candidate Tau function is constructed as a sum over a lattice of step $\hbar$ in the moduli space of solutions of the loop equations: the interferences between the terms of the sum create in general an oscillatory $\hbar \to 0$, described by Theta functions evaluated at an argument proportional to $1/\hbar$. This suggests that in general when $\hbar \to 0$, the "fast variables" live in the moduli space of solutions of loop equations, whereas the dependence in the "adiabatic variables" is captured by the loop equations themselves.

An important open problem would be to show that the asymptotics of (bi)orthogonal polynomials are given by certain Baker–Akhiezer functions of an integrable system, which depend on the universality class. Around a point where the density of zeroes vanishes like a power p/q, the integrable system should be related to the (p, q) models. This remains beyond the scope of the present investigation.

2. Linear Differential Systems and Loop Equations

2.1. Kernel, Determinantal Formulae and Correlators

Definition 2.1. The *matrix kernel* is a $d \times d$ matrix depending on two variables $x_1, x_2 \in \widehat{\mathbb{C}} \backslash \mathcal{P}$, defined by:

$$\mathbf{K}(x_1, x_2) = \frac{\mathbf{\Psi}^{-1}(x_1)\mathbf{\Psi}(x_2)}{x_1 - x_2},$$

Since we have the relation:

$$\mathbf{\Psi}(x_2) = (x_1 - x_2)\, \mathbf{\Psi}(x_1)\, \mathbf{K}(x_1, x_2),$$

$\mathbf{K}(x_1, x_2)$ can be interpreted as the parallel transport of the connection $\hbar\partial_x - \mathbf{L}(x)$ from x_1 to x_2. In the context of integrable systems, it is closely related to the integrable kernel and to Baker–Akhiezer functions. Our matrix kernel obviously satisfies a replication formula:

$$\mathbf{K}(x_1, x_2)\mathbf{K}(x_2, x_3) = \frac{x_1 - x_3}{(x_1 - x_2)(x_2 - x_3)}\, \mathbf{K}(x_1, x_3), \tag{2.1}$$

and it has a simple pole at coinciding points:

$$\mathbf{K}(x_1, x_2) \underset{x_1 \to x_2}{\sim} \frac{\mathbf{1}_d}{x_1 - x_2}.$$

Definition 2.2. The *n-point correlators* are a family of symmetric functions in n variables, indexed by $a_1,\ldots,a_n \in [\![1,d]\!]$, defined as follows:

$$\mathcal{W}_1(\overset{a}{x}) = \lim_{x'\to x}\left(\mathbf{K}_{a,a}(x,x') - \frac{1}{x-x'}\right),$$

$$\forall n\geq 2, \qquad \mathcal{W}_n(\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n}) = (-1)^{n+1}\sum_{\sigma=n\text{-cycles}}\prod_{i=1}^{n}\mathbf{K}_{a_i,a_{\sigma(i)}}(x_i,x_{\sigma(i)}),$$

and the *"non-connected" n-point correlators* by:

$$\overline{\mathcal{W}}_n(\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n}) = \text{“det”}\,\mathbf{K}_{a_i,a_j}(x_i,x_j), \tag{2.2}$$

where "det" means that each occurrence of $\mathbf{K}_{a_i,a_i}(x_i,x_i)$ in the determinant should be replaced by $\mathcal{W}_1(\overset{a_i}{x_i})$.

In the context of integrable systems, the entries of $\mathbf{K}$ can be interpreted as fermionic observables—the sandwich of a vertex operator and a group element between two vacuum states in the infinite wedge formalism—while the correlators correspond to bosonic observables. The two are related by the "boson–fermion correspondence", and it is well known that bosonic observables are obtained from fermionic one by determinantal formulae. Here, we take the determinantal formula as a definition of the correlators $\mathcal{W}_n$.

For instance, we have for any $a,b\in[\![1,d]\!]$, with $a\neq b$:

$$\begin{aligned}\mathcal{W}_1(\overset{a}{x}) &= -\,\hbar^{-1}\big(\mathbf{\Psi}^{-1}(x)\mathbf{L}(x)\mathbf{\Psi}(x)\big)_{a,a},\\ \mathcal{W}_2(\overset{a_1}{x_1},\overset{a_2}{x_2}) &= -\,\mathbf{K}_{a_1,a_2}(x_1,x_2)\mathbf{K}_{a_2,a_1}(x_2,x_1),\\ \lim_{x_1\to x}\mathcal{W}_2(\overset{a}{x_1},\overset{b}{x}) &= -\,\hbar^{-2}\big(\mathbf{\Psi}^{-1}(x)\mathbf{L}(x)\mathbf{\Psi}(x)\big)_{a,b}\big(\mathbf{\Psi}^{-1}(x)\mathbf{L}(x)\mathbf{\Psi}(x)\big)_{b,a}.\end{aligned} \tag{2.3}$$

We may give another representation for the correlators, using:

Definition 2.3. We define the projectors on state a:

$$\mathbf{P}(\overset{a}{x}) = \mathbf{\Psi}(x)\,\mathbf{E}_a\,\mathbf{\Psi}^{-1}(x),$$

where $\mathbf{E}_a = \mathrm{diag}(0,\ldots,0,\overset{a}{1},0,\ldots,0)$ denotes the diagonal matrix with ath-entry 1, and zero entries elsewhere.

We observe that $\mathbf{P}(\overset{a}{x})$ form a basis of rank one projectors:

$$\mathbf{P}(\overset{a}{x})\mathbf{P}(\overset{b}{x}) = \delta_{a,b}\mathbf{P}(\overset{a}{x}), \qquad \mathrm{Tr}\;\mathbf{P}(\overset{a}{x}) = 1, \qquad \sum_{a=1}^{d}\mathbf{P}(\overset{a}{x}) = \mathbf{1}_d, \tag{2.4}$$

which satisfies a Lax equation

$$\hbar\,\partial_x\mathbf{P}(\overset{a}{x}) = [\mathbf{L}(x),\mathbf{P}(\overset{a}{x})]. \tag{2.5}$$

Proposition 2.1. *The correlators can be written:*

$$\mathcal{W}_1(\overset{a}{x}) = -\hbar^{-1}\mathrm{Tr}\,\mathbf{P}(\overset{a}{x})\mathbf{L}(x),$$

$$\mathcal{W}_2\left(\overset{a_1}{x_1},\overset{a_2}{x_2}\right) = \frac{\mathrm{Tr}\,\mathbf{P}(\overset{a_1}{x_1})\mathbf{P}(\overset{a_2}{x_2})}{(x_1-x_2)^2} = -\frac{\mathrm{Tr}\,\left(\mathbf{P}(\overset{a_1}{x_1})-\mathbf{P}(\overset{a_2}{x_2})\right)^2}{2\,(x_1-x_2)^2} + \frac{1}{(x_1-x_2)^2},$$

and for $n \geq 3$

$$\mathcal{W}_n(\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n}) = (-1)^{n+1}\sum_{\sigma=n\text{-cycles}} \frac{\mathrm{Tr}\,\mathbf{P}(\overset{a_1}{x_1})\mathbf{P}\left(\overset{a_{\sigma(1)}}{x_{\sigma(1)}}\right)\mathbf{P}\left(\overset{a_{\sigma^2(1)}}{x_{\sigma^2(1)}}\right)\cdots\mathbf{P}(\overset{a_{\sigma^{n-1}(1)}}{x_{\sigma^{n-1}(1)}})}{\prod_{i=1}^n(x_i - x_{\sigma(i)})}.$$

For instance, we can deduce if $a_1 \neq a_2$:

$$\lim_{x_1\to x_2} \mathcal{W}_2(\overset{a_1}{x_1},\overset{a_2}{x_2}) = -\hbar^{-2}\,\mathrm{Tr}\,\mathbf{P}(\overset{a_1}{x_2})\mathbf{L}(x_2)\mathbf{P}(\overset{a_2}{x_2})\mathbf{L}(x_2), \tag{2.6}$$

$$\mathcal{W}_3(\overset{a_1}{x_1},\overset{a_2}{x_2},\overset{a_3}{x_3}) = \frac{\mathrm{Tr}\,\mathbf{P}(\overset{a_1}{x_1})\left[\mathbf{P}(\overset{a_2}{x_2}),\mathbf{P}(\overset{a_3}{x_3})\right]}{(x_1-x_2)(x_2-x_3)(x_3-x_1)}. \tag{2.7}$$

Although it is not clear from the definition, the n-point correlators do not have poles at coinciding points when $n \geq 3$. If $I = [\![1,n]\!]$, $(x_i)_{i\in I}$ and $(a_i)_i \in [\![1,d]\!]^I$, we denote $\overset{a_I}{x_I}$ the family $(\overset{a_i}{x_i})_{i\in I}$.

Proposition 2.2. *For any* $n \geq 3$*, any* $a_1,\ldots,a_n \in [\![1,d]\!]$*, and* $1 \leq i \neq j \leq n$*,* $\mathcal{W}_n(\overset{a_I}{x_I})$ *is regular when* $x_i \to x_j$*.*

Proof. By symmetry, it is enough to consider $i = 1$ and $j = 2$. The definition of $\mathcal{W}_k(x_I^{a_I})$ implies that it can have at most simple poles when $x_1 \to x_2$. Let us compute its residue from Proposition 2.1:

$$\begin{aligned}
&\operatorname*{Res}_{x_1\to x_2} \mathcal{W}_n(\overset{a_I}{x_I}) = (-1)^{n+1} \\
&\left\{\sum_{\substack{\sigma=n-\text{cycle}\\ \sigma(1)=2}} \frac{\mathrm{Tr}\,\mathbf{P}(\overset{a_1}{x_2})\mathbf{P}(\overset{a_2}{x_2})\mathbf{P}\left(\overset{a_{\sigma(2)}}{x_{\sigma(2)}}\right)\cdots\mathbf{P}\left(\overset{a_{\sigma^{n-3}(2)}}{x_{\sigma^{n-3}(2)}}\right)\mathbf{P}\left(\overset{a_{\sigma^{n-2}(2)}}{x_{\sigma^{n-2}(2)}}\right)}{(x_{\sigma^{n-2}(2)}-x_2)(x_2-x_{\sigma(2)})\cdots(x_{\sigma^{n-3}(2)}-x_{\sigma^{n-2}(2)})}\right. \\
&\left. -\sum_{\substack{\sigma=n-\text{cycle}\\ \sigma^{-1}(1)=2}} \frac{\mathrm{Tr}\,\mathbf{P}(\overset{a_2}{x_2})\mathbf{P}(\overset{a_1}{x_2})\mathbf{P}\left(\overset{a_{\sigma^2(2)}}{x_{\sigma^2(2)}}\right)\cdots\mathbf{P}\left(\overset{a_{\sigma^{n-2}(2)}}{x_{\sigma^{n-2}(2)}}\right)\mathbf{P}\left(\overset{a_{\sigma^{n-1}(2)}}{x_{\sigma^{n-1}(2)}}\right)}{(x_{\sigma^{n-1}(2)}-x_2)(x_2-x_{\sigma^2(2)})\cdots(x_{\sigma^{n-2}(2)}-x_{\sigma^{n-1}(2)})}\right\}.
\end{aligned}$$

Using the relation $\mathbf{P}(\overset{a_1}{x_2})\mathbf{P}(\overset{a_2}{x_2}) = \delta_{a_1,a_2}\mathbf{P}(\overset{a_2}{x_2})$, we can rewrite:

$$\begin{aligned}
\operatorname*{Res}_{x_1\to x_2} \mathcal{W}_n(\overset{a_I}{x_I}) &= (-1)^{n+1}\delta_{a_1,a_2} \\
&\left\{ \sum_{\substack{\sigma=n\text{-cycle}\\ \sigma(1)=2}} \frac{\operatorname{Tr}\mathbf{P}(\overset{a_2}{x_2})\mathbf{P}\begin{pmatrix} a_{\sigma(2)} \\ x_{\sigma(2)}\end{pmatrix}\cdots\mathbf{P}\begin{pmatrix} a_{\sigma^{n-3}(2)} \\ x_{\sigma^{n-3}(2)}\end{pmatrix}\mathbf{P}\begin{pmatrix} a_{\sigma^{n-2}(2)} \\ x_{\sigma^{n-2}(2)}\end{pmatrix}}{(x_{\sigma^{n-2}(2)}-x_2)(x_2-x_{\sigma(2)})\cdots(x_{\sigma^{n-3}(2)}-x_{\sigma^{n-2}(2)})}\right. \\
&\left. - \sum_{\substack{\sigma=n\text{-cycles}\\ \sigma^{-1}(1)=2}} \frac{\operatorname{Tr}\mathbf{P}(\overset{a_2}{x_2})\mathbf{P}\begin{pmatrix} a_{\sigma^2(2)} \\ x_{\sigma^2(2)}\end{pmatrix}\cdots\mathbf{P}\begin{pmatrix} a_{\sigma^{n-2}(2)} \\ x_{\sigma^{n-2}(2)}\end{pmatrix}\mathbf{P}\begin{pmatrix} a_{\sigma^{n-1}(2)} \\ x_{\sigma^{n-1}(2)}\end{pmatrix}}{(x_{\sigma^{n-1}(2)}-x_2)(x_2-x_{\sigma^2(2)})\cdots(x_{\sigma^{n-2}(2)}-x_{\sigma^{n-1}(2)})}\right\}.
\end{aligned}$$

The two sums range over the set of $(n-1)$-cycles and are actually equal. We conclude that $\mathcal{W}_k(\overset{a_I}{x_I})$ is regular when $x_1 \to x_2$. □

2.2. Loop Equations

We show that correlators satisfy loop equations given in Theorems 2.1 and 2.2 below, and this is our main motivation to introduce correlators. These results are elementary algebraic consequences of the definitions of Sect. 2.1, but they are central to this article. Non-trivial consequences of loop equations will be obtained in Sect. 3.

Theorem 2.1 (Linear loop equation). *For any $n \geq 1$, any $c_2, \ldots, c_n \in [\![1, d]\!]$, we have:*

$$\sum_{a=1}^{d} \mathcal{W}_n(\overset{a}{x}, \overset{c_2}{y_2}, \ldots, \overset{c_n}{y_n}) = -\delta_{n,1}\hbar^{-1}\operatorname{Tr}\mathbf{L}(x) + \frac{\delta_{n,2}}{(x-y_2)^2}.$$

Proof. We first address the cases $n = 1, 2$ by direct computation starting from Proposition 2.1, and use the properties (2.4) of the projectors:

$$\begin{aligned}
\sum_{a=1}^{d} \mathcal{W}_1(\overset{a}{x}) &= -\hbar^{-1}\operatorname{Tr}\left(\sum_{a=1}^{d}\mathbf{P}(\overset{a}{x})\right)\mathbf{L}(x) = -\hbar^{-1}\operatorname{Tr}\mathbf{L}(x), \\
\sum_{a=1}^{d} \mathcal{W}_2(\overset{a}{x}, \overset{c}{y}) &= \frac{\operatorname{Tr}\left(\sum_{a=1}^{d}\mathbf{P}(\overset{a}{x})\right)\mathbf{P}(\overset{c}{y})}{(x-y)^2} = \frac{\operatorname{Tr}\mathbf{P}(\overset{c}{y})}{(x-y)^2} = \frac{1}{(x-y)^2}.
\end{aligned} \tag{2.8}$$

For $n \geq 3$, combining the representation of Proposition 2.1 and the fact that $\sum_{a=1}^{d}\mathbf{P}(\overset{a}{x}) = \mathbf{1}_d$, we find that:

$$\begin{aligned}
&\sum_{a=1}^{d} \mathcal{W}_n(\overset{a}{x}, \overset{c_I}{y_I}) \\
&= (-1)^{n+1}\sum_{\sigma=n\text{-cycle}} \frac{1}{(x-y_{\sigma(1)})(y_{\sigma^{-1}(1)}-x)} \frac{\operatorname{Tr}\mathbf{P}\begin{pmatrix} c_{\sigma(1)} \\ y_{\sigma(1)}\end{pmatrix}\cdots\mathbf{P}\begin{pmatrix} c_{\sigma^{n-1}(1)} \\ y_{\sigma^{n-1}(1)}\end{pmatrix}}{\prod_{i=1}^{n-2}\left(y_{\sigma^i(1)}-y_{\sigma^{i+1}(1)}\right)}
\end{aligned}$$

is a rational function of x, which vanishes in the limit $x \to \infty$. Singularities can only arise as simple poles at $x = y_i$ for $i \in I$, but their residue is 0 according to Proposition 2.2. Hence, the left-hand side vanishes identically. $\square$

Theorem 2.2 (Quadratic loop equations). *For any $n \geq 1$, any $c_2, \ldots, c_n \in [\![1, d]\!]$,*

$$\sum_{1 \leq a < b \leq d} \left(\mathcal{W}_{n+1}\left(\overset{a}{x}, \overset{b}{x}, \overset{c_I}{y_I}\right) + \sum_{J \subseteq I} \mathcal{W}_{|J|+1}\left(\overset{a}{x}, \overset{c_J}{y_J}\right) \mathcal{W}_{n-|J|}\left(\overset{b}{x}, \overset{c_{I \setminus J}}{y_{I \setminus J}}\right) \right)$$
$$= P_n\left(x; \overset{c_I}{y_I}\right)$$

is a rational function of x, with possible poles at $x = x_i$ for $i \in I$ and poles of $\mathbf{L}$.

As illustration, we give the formulae for P_n up to $n = 3$:

$$P_1(x) = \frac{1}{2\hbar^2}\big(- \operatorname{Tr} \mathbf{L}^2(x) + [\operatorname{Tr} \mathbf{L}(x)]^2 \big),$$
$$P_2(x; \overset{c}{y}) = \frac{1}{\hbar} \frac{\operatorname{Tr} \mathbf{L}(x)[\mathbf{P}(\overset{c}{y}) - \mathbf{1}_d]}{(x-y)^2},$$
$$P_3(x; \overset{c_1}{y_1}, \overset{c_2}{y_2}) = -\frac{1}{\hbar} \frac{\operatorname{Tr} [\mathbf{P}(\overset{c_1}{y_1})\mathbf{P}(\overset{c_2}{y_2}) + \mathbf{P}(\overset{c_2}{y_2})\mathbf{P}(\overset{c_1}{y_1})]\mathbf{L}(x)}{(x-y_1)(x-y_2)}$$
$$+ \frac{(y_1 - y_2)^2 \mathcal{W}_2(\overset{c_1}{y_1}, \overset{c_2}{y_2}) + 1}{(x-y_1)^2(x-y_2)^2}.$$

Proof. Notice that the left-hand side makes sense even if $n = 1$, because the function $\mathcal{W}_2\left(\overset{a}{x}, \overset{b}{x}\right) = \lim_{y \to x} \mathcal{W}_2(\overset{a}{y}, \overset{b}{x})$ is well defined when $a \neq b$, and given by (2.3). When $a \neq b$, $\mathcal{W}_n\left(\overset{a}{x}, \overset{b}{x}, \overset{c_I}{y_I}\right)$ can be computed from Definition 2.2, using $\mathbf{K}_{a,b}(x,x) = -\hbar^{-1}\left(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\right)_{a,b}(x)$. We introduce a new quantity $\widetilde{\mathcal{W}}_n \left(\overset{a}{x}, \overset{b}{x}, \overset{c_I}{y_I}\right)$, as follows:

- when $a = b$, it is computed from Definition 2.2 where each occurrence of $\mathbf{K}_{a,a}(x,x)$ is replaced by $-\hbar^{-1}(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi})_{a,b}(x)$ (which is also equal to $\mathcal{W}_1(\overset{a}{x})$),
- when $a \neq b$, it is equal to $\mathcal{W}_n\left(\overset{a}{x}, \overset{b}{x}, \overset{c_I}{y_I}\right)$.

We claim:

Lemma 2.1. *For all $n \geq 1$ and $a \in [\![1, d]\!]$:*

$$\widetilde{\mathcal{W}}_{n+1}\left(\overset{a}{x}, \overset{a}{x}, \overset{c_I}{y_I}\right) + \sum_{J \subseteq I} \mathcal{W}_{|J|+1}\left(\overset{a}{x}, \overset{c_J}{y_J}\right) \mathcal{W}_{n-|J|}\left(\overset{a}{x}, \overset{c_{I \setminus J}}{y_{I \setminus J}}\right) = 0.$$

The proof of the lemma will be given below. We deduce that:

$$P_k\left(x; \overset{c_I}{y_I}\right) = \frac{1}{2}\sum_{a,b=1}^{d} \widetilde{\mathcal{W}}_{n+1}\left(\overset{a}{x}, \overset{b}{x}, \overset{c_I}{y_I}\right)$$
$$+ \sum_{J\subseteq I}\frac{1}{2}\left(\sum_{a=1}^{d}\mathcal{W}_{|J|+1}(\overset{a}{x}, \overset{c_J}{y_J})\right)\left(\sum_{b=1}^{d}\mathcal{W}_{n-|J|}(\overset{b}{x}, \overset{c_{I\setminus J}}{y_{I\setminus J}})\right).$$

The last term is given by the linear loop equations (Theorem 2.1): it vanishes when $n \geq 5$ and is a rational function of x with poles at $x = x_i$ for some $i \in I$, or at poles of $\mathbf{L}$. We now focus on the first term, which is by definition:

$$Q_k := \frac{1}{2}\sum_{a,b=1}^{d} \widetilde{\mathcal{W}}_{n+1}\left(\overset{a}{x}, \overset{b}{x}, \overset{c_I}{y_I}\right) = \frac{(-1)^n}{2}\sum_{a,b=1}^{d}\hat{Q}_k^{a,b} \tag{2.9}$$

$$\hat{Q}_k^{a,b} = -\hbar^{-1}(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi})_{a,b}(x)\Bigg\{\sum_{\substack{\sigma=(n+1)\text{-cycle}\\ \sigma(1)=2}}\mathbf{K}_{c_{\sigma^{-1}(1)},a}(y_{\sigma^{-1}(1)}, x)\mathbf{K}_{b,c_{\sigma(2)}}(x, y_{\sigma(2)})$$
$$\times\prod_{i=1}^{n-2}\mathbf{K}_{c_{\sigma^i(2)},c_{\sigma^{i+1}(2)}}(y_{\sigma^i(2)}, y_{\sigma^{i+1}(2)})\Bigg\}$$
$$-\hbar^{-1}(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi})_{b,a}(x)\Bigg\{\sum_{\substack{\sigma=(n+1)\text{-cycle}\\ \sigma(2)=1}}\mathbf{K}_{a,c_{\sigma(1)}}(x, y_{\sigma(1)})\mathbf{K}_{c_{\sigma^{-1}(2)},b}(y_{\sigma^{-1}(2)}, x)$$
$$\times\prod_{i=1}^{n-2}\mathbf{K}_{c_{\sigma^i(1)},c_{\sigma^{i+1}(1)}}(y_{\sigma^i(1)}, y_{\sigma^{i+1}(1)})\Bigg\}$$
$$+\Bigg\{\sum_{\substack{\sigma=(n+1)\text{-cycle}\\ \sigma(1)\neq 2,\ \sigma(2)\neq 1}}\mathbf{K}_{a,c_{\sigma(1)}}(x, y_{\sigma(1)})\cdots\mathbf{K}_{c_{\sigma^{-1}(2)},b}(y_{\sigma^{-1}(2)}, x)$$
$$\times\mathbf{K}_{b,c_{\sigma(2)}}(x, y_{\sigma(2)})\cdots\mathbf{K}_{c_{\sigma^{-1}(1)},a}(y_{\sigma^{-1}(1)}, x)\Bigg\}.$$

The two first lines are equal by symmetry. Performing the sum over a and b, and replacing the matrix kernels involving the variable x by their definition, we find:

$$Q_n(x;\overset{c_I}{y_I}) =(-1)^{n+1}\Bigg\{ \sum_{\substack{\sigma=(n+1)\text{-cycle}\\ \sigma(1)=2}} \Bigg(- \frac{[\mathbf{\Psi}^{-1}(x_{\sigma^{-1}(1)})\mathbf{L}(x)\mathbf{\Psi}(x_{\sigma(2)})]_{c_{\sigma^{-1}(1)},c_{\sigma(2)}}}{\hbar}$$

$$\times \prod_{i=1}^{n-3} \mathbf{K}_{c_{\sigma^i(2)},c_{\sigma^{i+1}(2)}}(y_{\sigma^i(2)},y_{\sigma^{i+1}(2)})\Bigg)$$

$$+ \sum_{\substack{\sigma=(n+1)\text{-cycle}\\ \sigma(1)\neq 2,\ \sigma(2)\neq 1}} \Bigg(\prod_{j=1,2} \frac{(y_{\sigma^{-1}(j)}-y_{\sigma(j)})\mathbf{K}_{c_{\sigma^{-1}(j)},c_{\sigma(j)}}(y_{\sigma^{-1}(j)},y_{\sigma(j)})}{2(x-y_{\sigma^{-1}(j)})(x-y_{\sigma(j)})}$$

$$\times \prod_{\substack{i=1\\ \sigma^{i+1}(1)\neq 1,2}}^{n-2} \mathbf{K}_{c_{\sigma^i(1)},c_{\sigma^{i+1}(1)}}(x_{\sigma^i(1)},x_{\sigma^{i+1}(1)})\Bigg)\Bigg\}.$$

This expression is a rational function of x which can have poles only at x_i for $i\in I$, and at poles of $\mathbf{L}$. Therefore, we proved that $P_n(x;\overset{c_I}{y_I})$ is a rational function of x which can have poles only at those very points.

Proof of Lemma 2.1. We have the analog of (2.9) for $a=b$:

$$\widetilde{\mathcal{W}}_{n+1}(\overset{a}{x},\overset{a}{x},\overset{c_I}{y_I})g = (-1)^n\Bigg\{ -2\hbar^{-1}(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi})_{a,a}(x)$$

$$\times\Bigg(\sum_{\substack{\sigma=(n+1)\text{-cycles}\\ \sigma(1)=2}} \mathbf{K}_{a,c_{\sigma(2)}}(x,y_{\sigma(2)})\left[\prod_{i=1}^{n-1}\mathbf{K}_{c_{\sigma^i(2)},c_{\sigma^{i+1}(2)}}\Big(y_{\sigma^i(2)},y_{\sigma^{i+1}(2)}\Big)\right]$$

$$\times \mathbf{K}_{c_{\sigma^{n-1}(2)},a}(y_{\sigma^{n-1}(2)},x)\Bigg) + \sum_{\substack{1\leq j,k\leq n\\ j+k=n}} \sum_{\substack{\sigma=(n+1)\text{-cycles}\\ \sigma^{j+1}(1)=2}} \mathbf{K}_{a,c_{\sigma(1)}}(x,y_{\sigma(1)})$$

$$\times\left[\prod_{i=1}^{j-1}\mathbf{K}_{c_{\sigma^j(1)},c_{\sigma^{i+1}(1)}}\Big(y_{\sigma^i(1)},y_{\sigma^{i+1}(1)}\Big)\right]\mathbf{K}_{c_{\sigma^j(1)},a}(y_{\sigma^j(1)},x)\mathbf{K}_{a,c_{\sigma(2)}}(x,y_{\sigma(2)})$$

$$\times\left[\prod_{i=1}^{k-1}\mathbf{K}_{c_{\sigma^i(2)},c_{\sigma^{i+1}(2)}}(y_{\sigma^i(2)},y_{\sigma^{i+1}(2)})\right]\mathbf{K}_{c_{\sigma^k(2)},a}(y_{\sigma^k(2)},x)\Bigg\}.$$

We recognize in the first line $-2\mathcal{W}_1(\overset{a}{x})\mathcal{W}_n(\overset{a}{x},\overset{c_I}{y_I})$. Besides, the two last lines amount to a sum over two disjoint cycles of length $(j+1)$ and $(k+1)$, and we recognize each term correlators up to a sign factor. Namely:

$$\widetilde{\mathcal{W}}_{n+1}(\overset{a}{x},\overset{a}{x},\overset{a_I}{x_I}) = -2\mathcal{W}_1(\overset{a}{x})\mathcal{W}_n(\overset{a}{x},\overset{c_I}{y_I}) - \sum_{\varnothing\subset J\subset I}\mathcal{W}_{|J|+1}(\overset{a}{x},\overset{c_J}{y_J})\mathcal{W}_{n-|J|}(\overset{a}{x},\overset{c_{I\setminus J}}{y_{I\setminus J}}).$$

The first term completes the sum with the terms $J=\varnothing$ and $J=I$, hence the result.

Detailed Example. Let us redo the computation in the case $n = 1$ to illustrate the method of the proof. We have:

$$\hbar^2 P_1(x) = \sum_{1\leq a<b\leq d} -\big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{a,b}(x)\big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{b,a}(x) + \big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{a,a}\big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{b,b}(x).$$

Notice that the summand vanish if $a = b$. We can thus write:

$$\begin{aligned}\hbar^2 P_1(x) &= \sum_{a,b=1}^{d} -\big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{a,b}(x)\big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{b,a}(x) \\ &\quad + \big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{a,a}(x)\big(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}\big)_{b,b}(x) \\ &= \frac{1}{2}\big(-\operatorname{Tr}\mathbf{L}^2(x) + [\operatorname{Tr}\mathbf{L}(x)]^2\big).\end{aligned}$$

2.3. Spectral Curve

Definition 2.4. The *spectral curve* is the plane curve $\mathcal{S}$ of equation $\det(y - \mathbf{L}(x)) = 0$.

The eigenvalues of $\mathbf{L}(x)$ are algebraic functions.

Proposition 2.3. *The spectral curve can be expressed in terms of correlators:*

$$\det(y - \mathbf{L}(x)) = \sum_{k=0}^{d} y^{d-k} \sum_{1\leq a_1<\cdots<a_k\leq d} \overline{\mathcal{W}}_k(\overset{a_1}{x}, \ldots, \overset{a_k}{x}).$$

Proof. We first write the coefficients of a characteristic polynomial as a sum over minors:

$$\begin{aligned}\det(y - \mathbf{L}(x)) &= \det(y - \mathbf{\Psi}^{-1}(x)\mathbf{L}(x)\mathbf{\Psi}(x)) \\ &= \sum_{k=0}^{d} y^{d-k} \sum_{1\leq a_1<\cdots<a_k\leq d} \det_{1\leq i,j\leq k}[-\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi}]_{a_i,a_j}(x) \\ &= \sum_{k=0}^{d} y^{d-k}\hbar^k \sum_{1\leq a_1<\cdots<a_k\leq d} \det_{1\leq i,j\leq k} \widetilde{\mathbf{K}}_{a_i,a_j}(x,x),\end{aligned}$$

where we have defined $\widetilde{\mathbf{K}}_{a,b}(x,x) = -\hbar^{-1}(\mathbf{\Psi}^{-1}\mathbf{L}\mathbf{\Psi})_{a,b}(x)$. Notice that

$$\widetilde{\mathbf{K}}_{a,b}(x,x) = \mathbf{K}_{a,b}(x,x)$$

when $a \neq b$, whereas $\widetilde{\mathbf{K}}_{a,a}(x,x) = \mathcal{W}_1(\overset{a}{x})$. And, the specialization of the definition of non-connected correlators (2.2) to $x_i \equiv x$ for $i \in [\![1, d]\!]$ and $a_1 < \cdots < a_k$ yields:

$$\overline{\mathcal{W}}_k(\overset{a_1}{x}, \ldots, \overset{a_k}{x}) = \det_{1\leq i,j\leq k} \widetilde{\mathbf{K}}_{a_i,a_j}(x,x), \tag{2.10}$$

whence the announced formula. $\square$

We remark that the coefficient of y^{d-2} was already identified in Theorem 2.2.

2.4. Gauge Transformations

If $\mathbf{\Psi}$ is a solution of (1.1), and $\mathbf{G}$ is a matrix depending on x, $\widetilde{\mathbf{\Psi}} = \mathbf{G}\mathbf{\Psi}$ will also be solution of similar equation, with:

$$\widetilde{\mathbf{L}} = (\hbar\,\partial_x \mathbf{G})\mathbf{G}^{-1} + \mathbf{G}\mathbf{L}\mathbf{G}^{-1}.$$

Any two arbitrary $d \times d$ matrices $\mathbf{\Psi}(x)$ and $\widetilde{\mathbf{\Psi}}(x)$ can be related by a gauge transformation $\mathbf{G}(x) = \widetilde{\mathbf{\Psi}}(x)\mathbf{\Psi}(x)^{-1}$. Therefore, the concept of gauge transformations is only meaningful if we impose some restriction on the form of $\mathbf{G}(x)$. Here, the natural restriction to impose is that $\mathbf{G}$ is such that the new matrix $\widetilde{\mathbf{L}}$ has the same poles and same pole degrees than $\mathbf{L}$. Gauge transformations in general completely change the matrix kernel and the correlators. However, there are two special gauge transformations under which the correlators do not change. If $\mathbf{G}$ is independent of x:

$$\widetilde{\mathbf{L}} = \mathbf{G}\mathbf{L}\mathbf{G}^{-1}, \qquad \widetilde{\mathbf{P}} = \mathbf{G}\mathbf{P}\mathbf{G}^{-1}, \qquad \widetilde{\mathbf{K}} = \mathbf{K}, \qquad \widetilde{\mathcal{W}}_n = \mathcal{W}_n.$$

If $\mathbf{G}$ depends on x but is scalar $\mathbf{G} = G\mathbf{1}_d$:

$$\widetilde{\mathbf{L}} = \mathbf{L} + \hbar\,\partial_x \ln G, \qquad \widetilde{\mathbf{P}} = \mathbf{P}, \qquad \widetilde{\mathbf{K}}(x,y) = \frac{G(y)}{G(x)}\,\mathbf{K}(x,y), \qquad \widetilde{\mathcal{W}}_n = \mathcal{W}_n.$$

2.5. Insertion Operator

In this paragraph, we introduce a notion of "insertion operator". At the level of algebra, it enables passing from the n-point correlator to the $(n+1)$-point correlator, i.e. it "inserts" a new point. In practice, it is very useful to construct insertion operator with nice properties. For instance, in the application described in Sect. 5, proving a statement about $\mathcal{W}_n$ for all $n \geq 1$ can be reduced to checking it for $n = 1$ by means of a suitable insertion operator. Here, it also allows us to obtain higher-order loop equations from the linear and quadratic ones. In the context of integrable systems with a complete set of times $(t_j)_{j\geq 0}$, the formal differential operator $\sum_{j\geq 0} x^{-(j+1)}\,\partial/\partial t_j$ is a prototype of "insertion operator". This is also the case in the context of (may be non-integrable) matrix models, where the t_j corresponds to a perturbation of the matrix potential by a monomial of degree j, cf. Sect. D.1.3.

Before getting to the definition, we need to expose some notions of differential algebra. Let $(\mathbb{C}(x), \partial_x)$ be the differential ring generated by rational functions. We consider a Picard–Vessiot ring $\mathbb{B}$ of the differential system $\hbar\,\partial_x \mathbf{\Psi}(x) = \mathbf{L}(x)\mathbf{\Psi}(x)$ [27]. It is a simple extension of $(\mathbb{C}(x), \partial_x)$ by the matrix elements of $\mathbf{\Psi}(x)$ and $\big(\det \mathbf{\Psi}(x)\big)^{-1}$. Let $\mathbb{B}_n$ the n-variable analog of $\mathbb{B}$, i.e. the differential ring generated by rational functions in n variables $x_1, \ldots, x_n$ and by the matrix elements of $\mathbf{\Psi}(x_i)$ and $\big(\det \mathbf{\Psi}(x_i)\big)^{-1}$. We denote the projective limit $\mathbb{B}_\infty = \lim_{n\to\infty} \mathbb{B}_n$. By construction, the matrix elements of $\mathbf{P}(\overset{a}{x})$ or of $\mathbf{L}(x)$ are in $\mathbb{B}$, those of $\mathbf{K}(x_1, x_2)$ are in $\mathbb{B}_2$, and the n-point correlators $\mathcal{W}_n(\overset{a_1}{x}_1, \ldots, \overset{a_n}{x}_n)$ are in $\mathbb{B}_n$.

Definition 2.5. An *insertion operator* is a collection of derivations $(\delta_y^a)_{1\leq a\leq d}$ over $\mathbb{B}_\infty$, commuting with ∂_{x_i}, with the following properties:

- $\delta_y^a(\mathbb{B}_n) \subseteq \mathbb{B}_{n+1}$.
- $\delta_y^a(\mathbb{C}(x_i)) = 0$.
- there exist matrices $\mathbf{U}(\overset{a}{y})$ with entries in $\mathbb{B}$, so that:

$$\delta_y^a \mathbf{\Psi}(x) = \left(\frac{\mathbf{P}(\overset{a}{y})}{x-y} + \mathbf{U}(\overset{a}{y}) \right) \mathbf{\Psi}(x), \tag{2.11}$$

and such that $\mathbf{U}$ satisfies

$$\delta_x^a \mathbf{U}(\overset{b}{y}) - \delta_y^b \mathbf{U}(\overset{a}{x}) = [\mathbf{U}(\overset{a}{x}), \mathbf{U}(\overset{b}{y})]. \tag{2.12}$$

Since the matrix elements of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ generate $\mathbb{B}$ and $\delta_y a$ annihilates the constant matrix that gives the only algebraic relation $\mathbf{\Psi}\mathbf{\Psi}^{-1} = \mathbf{1}$, Eq. 2.11 uniquely defines the insertion operator by requiring that the Leibniz rule holds. The condition (2.12) on $\mathbf{U}$ ensures that $[\delta_x^a, \delta_y^b] = 0$. We included (2.12) in our definition because it is convenient not to worry about order of insertion operators, though it is not essential. For instance, in the application to the qth reduction of KP in Sect. 5.7.3, we construct a suitable insertion operator including the check of (2.12) in Proposition 5.6, but we do not really make use of this property. As a matter of fact, the key property we want insertion operators to satisfy is the last equation of the following:

Lemma 2.2. *If δ_y^a is an insertion operator, for any $n \geq 1$, any $a, b, a_1, \ldots, a_n \in [\![1, d]\!]$,*

$$\begin{aligned}
\delta_y^a \mathbf{K}(x_1, x_2) &= -\mathbf{K}(x_1, y)\mathbf{E}_a \mathbf{K}(y, x_2), \\
\delta_y^a \mathbf{P}(\overset{b}{x}) &= \Big[\frac{\mathbf{P}(\overset{a}{y})}{x-y} + \mathbf{U}(\overset{a}{y}), \mathbf{P}(\overset{b}{x}) \Big], \\
\delta_y^a \mathbf{L}(x) &= \Big[\frac{\mathbf{P}(\overset{a}{y})}{x-y} + \mathbf{U}(\overset{a}{y}), \mathbf{L}(x) \Big] - \frac{\mathbf{P}(\overset{a}{y})}{(x-y)^2}, \\
\delta_y^a \mathrm{Tr}\, \mathbf{L}(x) &= -\frac{1}{(x-y)^2}, \\
\delta_y^a \ln\det \mathbf{\Psi}(x) &= \frac{1}{x-y} + \mathrm{Tr}\, \mathbf{U}(\overset{a}{y}), \\
\delta_y^a \ln\left(\frac{\det \mathbf{\Psi}(x)}{\det \mathbf{\Psi}(z)} \right) &= \frac{1}{x-y} - \frac{1}{z-y}, \\
\delta_y^a \mathcal{W}_n(\overset{a_1}{x_1}, \ldots, \overset{a_n}{x_n}) &= \mathcal{W}_{n+1}(\overset{a}{y}, \overset{a_1}{x_1}, \ldots, \overset{a_n}{x_n}).
\end{aligned}$$

Proof. These results are proved by direct computations done in "Appendix A". They are a good illustration on how to manipulate insertion operators, and of the fact that giving (2.11) is enough to characterize them. □

We remark that the first and the last equations are independent of $\mathbf{U}$. We also remark that, because of the fifth relation, $\det \mathbf{\Psi}$ is not constant regarding the action of the insertion operator. Notice that in general, up to a scalar gauge transformation, one can always choose $\det \mathbf{\Psi}(x)$ to be a constant. What this means here is that the insertion operator δ_y^a does not commute with gauge transformations.

We are now in position to establish higher-order loop equations. Let us define the semi-connected correlators:

$$\mathcal{W}_{k;n}(\overset{a_1}{x_1},\overset{a_2}{x_2},\ldots,\overset{a_k}{x_k}\,;\,\overset{b_1}{y_1},\ldots,\overset{b_n}{y_n})$$
$$= \sum_{I\vdash[\![1,k]\!]}\ \sum_{J_1\dot\cup\cdots\dot\cup J_{\ell(\mu)}=[\![1,n]\!]}\ \prod_{j=1}^{\ell(I)} \mathcal{W}_{|I_j|+|J_j|}(\overset{a_{I_j}}{x_{I_j}},\overset{b_{J_j}}{y_{J_j}}).$$

Here, I is a partition of $[\![1,k]\!]$, i.e. a set of $\ell(I)$ non-empty, pairwise disjoint subsets $I_i \subseteq [\![1,k]\!]$ whose union is $[\![1,k]\!]$, whereas the subsets $J_i \subseteq [\![1,n]\!]$ could be empty.

Proposition 2.4 (Most general loop equations). *For every $k \leq d$ and every $\{\overset{b_1}{y_1},\ldots,\overset{b_n}{y_n}\}$,*

$$P_{k,n}(x;\overset{b_1}{y_1},\ldots,\overset{b_n}{y_n}) = \sum_{1\leq a_1<a_2<\cdots<a_k\leq d} \mathcal{W}_{k;n}(\overset{a_1}{x},\ldots,\overset{a_k}{x};\overset{b_1}{y_1},\ldots,\overset{b_n}{y_n})$$

is a rational function of x, with poles at $x = y_j$ for some j and at poles of $\mathbf{L}$.

Proof. The case $n = 0$ is Proposition 2.3. The cases $n \geq 1$ are obtained by recursively applying $\delta_{y_j}^{b_j}$, for any insertion operator δ. □

3. Asymptotics and Topological Expansion

Loop equations form an infinite system of equations, in general difficult to solve. In many applications, correlators have an asymptotic expansion (or are formal series) in powers of $\hbar$, and if this expansion is of "topological type" (Definition 3.3 below), loop equations can be solved recursively in powers of $\hbar$, by the topological recursion of [41]. This claim is justified in this section and turns the loop equation into an effective tool to study all-order WKB expansions.

We assume that $\mathbf{L}(x)$ has an asymptotic expansion in powers of $\hbar$, of the form:

$$\mathbf{L}(x) = \sum_{k\geq 0} \hbar^k\, \mathbf{L}^{[k]}(x),$$

which is uniform for x in some domain of the complex plane, or alternatively, $\mathbf{L}(x) \in \mathbb{C}[[\hbar]]$ is defined as a formal power series in $\hbar$. Let us denote

$$\mathbf{\Lambda}(x) = \mathrm{diag}(\lambda_1(x),\ldots,\lambda_d(x))$$

the diagonal matrix of eigenvalues of $\mathbf{L}(x)$ counted with multiplicities and ordered arbitrarily. $\mathbf{\Lambda}(x)$ also has an expansion in powers of $\hbar$:

$$\mathbf{\Lambda}(x) = \sum_{k\geq 0} \hbar^k\, \mathbf{\Lambda}^{[k]}(x).$$

Our first task, completed in Sects. 3.1–3.2, is to describe the singularities and monodromies of the coefficients of expansion of $\mathbf{\Psi}(x)$ when $\hbar \to 0$. We arrive at the classical and elementary result that they can be realized as the pushforward of meromorphic functions on the semiclassical spectral curve (Proposition 3.2). This fact is however crucial, since we will use later complex analysis on the closure of the semiclassical spectral curve—which is here a compact Riemann surface—to characterize those meromorphic functions.

3.1. The Semiclassical Spectral Curve

The semiclassical spectral curve is the locus of leading-order eigenvalues:

Definition 3.1. The semiclassical spectral curve is defined as:

$$\mathcal{S}^{[0]} = \overline{\left\{(x,y) \in \mathbb{C}^2 \mid \det(y\,\mathbf{1}_d - \mathbf{L}^{[0]}(x)) = 0\right\}}.$$

It can be seen as the immersion of a compact Riemann surface $\mathcal{S}^{[0]}$ into $\mathbb{C} \times \mathbb{C}$, through the maps $x : \mathcal{S}^{[0]} \to \mathbb{C}$ and $y : \mathcal{S}^{[0]} \to \mathbb{C}$. If x is of degree d (the degree in y of the algebraic equation defining $\mathcal{S}^{[0]}$, i.e. the size of the matrix $\mathbf{L}^{[0]}(x)$), then the preimage of $x_0 \in \mathbb{C}$ is denoted:

$$x^{-1}(\{x_0\}) = \{z^0(x_0), \ldots, z^{d-1}(x_0)\} \subseteq \mathcal{S}^{[0]}.$$

In other words, $\mathcal{S}^{[0]}$ is realized as a branch covering of $\mathbb{C}$ of degree d by the projection $x : \mathcal{S}^{[0]} \to \mathbb{C}$. The zeroes of $\mathrm{d}x$ in $\mathcal{S}^{[0]}$ are the *ramification points*, and their x-coordinate is the *branchpoints*. Branchpoints $\beta_i \in \mathbb{C}$ occur when $z^a(\beta_i) = z^b(\beta_i)$ for at least two distinct indices a and b, and we then denote $r_i = z^a(\beta_i) = z^b(\beta_i)$. Let us call $\mathbf{r}$ the set of ramification points.

$\lambda_a^{[0]}(x)$ are the eigenvalues of $\mathbf{L}^{[0]}(x)$, i.e. by definition they are the y coordinate of some $z^a(x) \in \mathcal{S}^{[0]}$:

$$\left\{y(z^a(x)) \quad a \in [\![1,d]\!]\right\} = \left\{\lambda_a^{[0]}(x) \quad a \in [\![1,d]\!]\right\}. \tag{3.1}$$

Double points $\alpha_i \in \mathbb{C}$ occur where two or more eigenvalues collide, i.e.

$$y(z^a(\alpha_i)) = \lambda_a^{[0]}(\alpha_i) = \lambda_b^{[0]}(\alpha_i) = y(z^b(\alpha_i))$$

for at least two distinct indices $a \neq b$, but $\mathrm{d}x(z^a(\alpha_i)) \neq 0$ and $\mathrm{d}x(z^b(\alpha_i)) \neq 0$—a fortiori, $z^a(\alpha_i)$ and $z^b(\alpha_i)$ must be distinct points in $\mathcal{S}^{[0]}$.

The space $H^1(\mathcal{S}^{[0]})$ of holomorphic 1-forms on $\mathcal{S}^{[0]}$ is a complex vector space of dimension $\mathfrak{g}$, where $\mathfrak{g}$ is the genus of $\mathcal{S}^{[0]}$. In particular, if $\mathfrak{g} = 0$, $H^1(\mathcal{S}^{[0]}) = \{0\}$ and a meromorphic form on $\mathcal{C}$ is completely determined by the singular behavior at its poles.

Definition 3.2. Let $\mathcal{B}(\mathcal{S}^{[0]})$ the set of *fundamental bidifferentials of the second kind*, i.e. $B(z_1,z_2)$ which are symmetric 2-form in $(\mathcal{S}^{[0]})^2$, with no residues, and a double pole at $z_1 = z_2$ with behavior in any local coordinate ξ:

$$B(z_1,z_2) \underset{z_1 \to z_2}{=} \frac{\mathrm{d}\xi(z_1)\,\mathrm{d}\xi(z_2)}{\big(\xi(z_1)-\xi(z_2)\big)^2} + O(1).$$

Since one can add to B any symmetric bilinear combination of holomorphic forms, $\mathcal{B}(\mathcal{S}^{[0]})$ is an affine space, whose underlying vector space is $\mathrm{Sym}^2[H^1(\mathcal{S}^{[0]})]$, so it has complex dimension $\mathfrak{g}(\mathfrak{g}+1)/2$.

3.2. Expansions in Powers of $\hbar$

We now assume that $\mathcal{S}^{[0]}$ is a regular plane curve, i.e. dx and dy do not have common zeroes. Therefore, $\mathbf{L}^{[0]}(x)$ has simple eigenvalues for any x which is not a branchpoint or double point, hence is diagonalizable. So $\mathbf{L}(x)$ must be diagonalizable with simple eigenvalues at least when $\hbar$ is small and x stays away from the branchpoints or double points. We can thus find a matrix of eigenvectors $\mathbf{V}(x)$:

$$\mathbf{L}(x) = \mathbf{V}(x)\mathbf{\Lambda}(x)\mathbf{V}^{-1}(x),$$

which admits an expansion in powers of $\hbar$:

$$\mathbf{V}(x) = \sum_{k\geq 0} \hbar^k\,\mathbf{V}^{[k]}(x). \tag{3.2}$$

Such a matrix is defined up to transformations $\mathbf{V}(x) \to \mathbf{V}(x)\mathbf{D}(x)\mathbf{\Sigma}$, where $\mathbf{D}(x)$ is a diagonal matrix and $\mathbf{\Sigma}$ a permutation matrix. We can use the first freedom to impose:

$$\forall a \in [\![1,d]\!], \quad \big(\mathbf{V}^{-1}(x)\,\partial_x\mathbf{V}(x)\big)_{a,a} = 0, \tag{3.3}$$

and we then say that $\mathbf{V}(x)$ is a normalized matrix of eigenvectors. Any two such matrices are related by a transformation $\mathbf{V}(x) \to \mathbf{V}(x)\mathbf{D}\mathbf{\Sigma}$, where $\mathbf{D}$ is a constant diagonal matrix and $\mathbf{\Sigma}$ a permutation matrix.

We would like to study solutions of (1.1) which have an expansion in powers of $\hbar$. For this purpose, we fix a base point o, an invertible matrix of constants $\mathbf{C}$, and introduce a matrix $\widehat{\mathbf{\Psi}}(x)$ such that:

$$\mathbf{\Psi}(x) = \mathbf{V}(x)\widehat{\mathbf{\Psi}}(x)\exp\left(\frac{1}{\hbar}\int_o^x \mathbf{\Lambda}(x')\mathrm{d}x'\right)\mathbf{C}. \tag{3.4}$$

$\mathbf{\Psi}(x)$ is a solution of (1.1) if and only if:

$$\hbar\,\partial_x\widehat{\mathbf{\Psi}}(x) = -\hbar\,\mathbf{T}(x)\widehat{\mathbf{\Psi}}(x) + [\mathbf{\Lambda}(x), \widehat{\mathbf{\Psi}}(x)], \tag{3.5}$$

where $\mathbf{T}(x) = \mathbf{V}(x)^{-1}\partial_x\mathbf{V}(x)$ also has an expansion in powers of $\hbar$ derived from (3.2):

$$\mathbf{T}(x) = \sum_{k\geq 0} \hbar^k\,\mathbf{T}^{[k]}(x).$$

We start by proving a classical result:

Proposition 3.1. *Equation 3.5 has a unique solution which is a formal power series in $\hbar$ of the form:*

$$\widehat{\mathbf{\Psi}}(x) = \mathbf{1}_d + \sum_{k\geq 1}\hbar^k\,\widehat{\mathbf{\Psi}}^{[k]}(x) \tag{3.6}$$

up to transformations $\widehat{\mathbf{\Psi}}^{[k]}(x) \to \widehat{\mathbf{\Psi}}^{[k]}(x) + \widehat{\mathbf{C}}^{[k]}$, where $\widehat{\mathbf{C}}^{[k]}$ is a diagonal matrix of constants. A priori, the entries of $\widehat{\mathbf{\Psi}}^{[k]}(x)$ are multivalued functions of x with monodromies around branchpoints, double points and poles at the poles of $(\mathbf{L}^{[j]}(x))_{j\geq 0}$.

Proof. Inserting the ansatz (3.6) in (3.5) and collecting the terms of order $\hbar^{k+1}$ yield, for any $a, b \in [\![1, d]\!]$:

$$\partial_x \widehat{\mathbf{\Psi}}^{[k]}_{a,b} = -\sum_{j=0}^{k} \left(\mathbf{T}^{[k-j]}\widehat{\mathbf{\Psi}}^{[j]}\right)_{a,b} + \left(\lambda_a^{[0]} - \lambda_b^{[0]}\right) \widehat{\mathbf{\Psi}}^{[k+1]}_{a,b} + \sum_{j=0}^{k} \left(\lambda_a^{[k-j]} - \lambda_b^{[k-j]}\right) \widehat{\mathbf{\Psi}}^{[j]}_{a,b}. \tag{3.7}$$

Since we assume that $\mathcal{S}^{[0]}$ is regular and x is away from a branchpoint or a double point, we have $\lambda_a^{[0]}(x) \neq \lambda_b^{[0]}(x)$ when $a \neq b$, which allows to write:

$$\widehat{\mathbf{\Psi}}^{[k+1]}_{a,b} = \frac{1}{\lambda_a^{[0]} - \lambda_b^{[0]}} \left(\partial_x \widehat{\mathbf{\Psi}}^{[k]}_{a,b} + \sum_{j=0}^{k} \left(\mathbf{T}^{[k-j]}\widehat{\mathbf{\Psi}}^{[j]}\right)_{a,b} - \left(\lambda_a^{[k-j]} - \lambda_b^{[k-j]}\right)\widehat{\mathbf{\Psi}}^{[j]}_{a,b}\right). \tag{3.8}$$

This equation determines the off-diagonal part of $\widehat{\mathbf{\Psi}}^{[k+1]}$ in terms of $\widehat{\mathbf{\Psi}}^{[j]}$ for $j \in [\![0, k]\!]$. For $a = b$ in (3.7), we rather find:

$$\partial_x \widehat{\mathbf{\Psi}}^{[k+1]}_{a,a} = \sum_{\substack{c=1 \\ c\neq a}} \mathbf{T}^{[0]}_{a,c} \widehat{\mathbf{\Psi}}^{[k+1]}_{c,a} + \sum_{j=0}^{k} (\mathbf{T}^{[k+1-j]}\widehat{\mathbf{\Psi}}^{[j]})_{a,a}. \tag{3.9}$$

We took into account the normalization[1] (3.3), so that the right-hand side involves only off-diagonal entries of $\widehat{\mathbf{\Psi}}^{[k+1]}$ or the entries of $\widehat{\mathbf{\Psi}}^{[j]}$ for $j \in [\![0, k]\!]$.

We proceed by recursion starting from the initial condition $\widehat{\mathbf{\Psi}}^{[0]} = \mathbf{1}_d$. Assuming that $\widehat{\mathbf{\Psi}}^{[j]}$ are completely known for $j \in [\![0, k]\!]$, we obtain the off-diagonal part of $\widehat{\mathbf{\Psi}}^{[k+1]}$ from (3.8), and solving the first-order differential equation (3.9) we then obtain the diagonal part of $\widehat{\mathbf{\Psi}}^{[k+1]}$ up to a diagonal matrix of integration constants $\widehat{\mathbf{C}}^{[k+1]}$. It is clear that the singularities of $\widehat{\mathbf{\Psi}}^{[k]}$ can only occur at singularities of $\lambda_a^{[j]}(x)$ and $\mathbf{T}^{[j]}(x)$, i.e. either at semiclassical branchpoints or poles of $(\mathbf{L}^{[j]}(x))_{j\geq 0}$, or at double points where $\lambda_a^{[0]} = \lambda_b^{[0]}$. $\square$

Proposition 3.2 (Analytic continuation). *The matrices $\mathbf{V}(x)$, $\mathbf{\Lambda}(x)$ and*

$$\widetilde{\mathbf{\Psi}}(x) = \mathbf{V}(x)\widehat{\mathbf{\Psi}}(x),$$

all have a power series expansion in $\hbar$, whose coefficients are such that their ath column vector is the evaluation of meromorphic function on $\mathcal{S}^{[0]}$ at $z^a(x)$. In particular, there exists a vector $\tilde{\psi}^{[k]}(z)$ such that:

$$\widetilde{\mathbf{\Psi}}_{i,a}(x) = \left(\mathbf{V}(x)\,\widehat{\mathbf{\Psi}}(x)\right)_{i,a} = \sum_{k\geq 0} \hbar^k\, \tilde{\psi}_i^{[k]}(z^a(x)). \tag{3.10}$$

Proof. For the diagonal matrix $\mathbf{\Lambda}$, we have already seen in (3.1) that $\lambda_a^{[0]}(x) = y(z^a(x))$. Solving $\det(\lambda_a(x)\,\mathbf{1}_d - \mathbf{L}(x)) = 0$ with $\mathbf{L}(x) = \sum_{k\geq 0} \hbar^k\, \mathbf{L}^{[k]}(x)$ and $\lambda_a(x) = \sum_{k\geq 0} \hbar^k\, \lambda_a^{[k]}(x)$, by recursion on k, shows easily that each $\lambda_a^{[k]}(x)$ is

[1] Notice that we only need (3.3) at leading order here.

a meromorphic function $\lambda^{[k]}(z^a(x))$ for all k. Similarly, Cramer's formula for computing the eigenvectors of $\mathbf{L}(x)$ shows that up to a normalization factor, the eigenvector corresponding to the a^{th} eigenvalue $\lambda_a(x)$ has also a power series expansion in $\hbar$ whose coefficients are meromorphic functions of $z^a(x)$ at each order. In other words, one can choose a matrix $\widehat{\mathbf{V}}(x)$ of eigenvectors of $\mathbf{L}(x)$ satisfying

$$\mathbf{L}(x) = \widehat{\mathbf{V}}(x)\mathbf{\Lambda}(x)\widehat{\mathbf{V}}^{-1}(x)$$

of the form

$$\widehat{\mathbf{V}}(x) = \sum_{k\geq 0} \hbar^k\, \widehat{\mathbf{V}}^{[k]}(x), \quad \widehat{\mathbf{V}}^{[k]}(x)_{i,a} = \hat{v}_i^{[k]}(z^a(x)).$$

Then, notice that any symmetric meromorphic function of $(z^1(x),\ldots,z^d(x))$ is a meromorphic function of x, and thus a meromorphic function of any $z^a(x)$. And, any symmetric meromorphic function of $(z^1(x),\ldots,z^d(x))_{\hat{a}}$ (i.e. all $z^j(x)$'s except $z^a(x)$) is a meromorphic function of x and of $z^a(x)$, and thus is a meromorphic function of $z^a(x)$. In particular, this implies that the determinant of $\widehat{\mathbf{V}}(x)$ is a power series of $\hbar$ whose coefficients are meromorphic function of $z^a(x)$, and the inverse matrix $\widehat{\mathbf{V}}^{-1}(x)$ takes the form:

$$\widehat{\mathbf{V}}^{-1}_{a,i}(x) = \sum_{k\geq 0} \hbar^k\, \hat{v}_i^{[k]}(z^a(x)).$$

This implies that

$$\left(\widehat{\mathbf{V}}^{-1}(x)\,\partial_x\widehat{\mathbf{V}}(x)\right)_{a,a} = \sum_{k\geq 0} \hbar^k\, \hat{t}^{[k]}(z^a(x)),$$

where each $\hat{t}^{[k]}(z)$ is a meromorphic function on the semiclassical spectral curve $\mathcal{S}^{[0]}$.

We chose to normalize our basis of eigenvectors $\mathbf{V}(x) = \widehat{\mathbf{V}}(x)\,\mathbf{D}(x)$ where $\mathbf{D}(x)$ is some diagonal matrix, so that (3.3) is satisfied, i.e. we have to choose $D(x)$ satisfying:

$$\mathbf{D}^{-1}_{a,a}(x)\,\partial_x\mathbf{D}_{a,a}(x) = -\left(\widehat{\mathbf{V}}^{-1}(x)\,\partial_x\widehat{\mathbf{V}}(x)\right)_{a,a} = -\sum_{k\geq 0} \hbar^k\, \hat{t}^{[k]}(z^a(x)).$$

This shows that $D_{a,a}(x)$ also has a power series expansion in $\hbar$ whose coefficients are meromorphic functions of $z^a(x)$. Finally, this shows that $\mathbf{V}(x)$ has the form:

$$\mathbf{V}_{i,a}(x) = \sum_{k\geq 0} \hbar^k\, v_i^{[k]}(z^a(x)),$$

where each $v_i^{[k]}(z)$ is a meromorphic function on the semiclassical spectral curve.

If we choose $\mathbf{C}$ to be diagonal, we see that:

$$\widetilde{\mathbf{\Psi}}(x) = \mathbf{V}(x)\,\widehat{\mathbf{\Psi}}(x) = \mathbf{\Psi}(x)\,\mathbf{C}^{-1}\,\exp\left(-\frac{1}{\hbar}\int_\alpha^x \mathbf{\Lambda}(x')\,\mathrm{d}x'\right)$$

obeys:

$$\hbar\,\partial_x \widetilde{\mathbf{\Psi}}(x) = \mathbf{L}(x)\,\widetilde{\mathbf{\Psi}}(x) - \widetilde{\mathbf{\Psi}}(x)\,\mathbf{\Lambda}(x).$$

The equation for the ath column of $\widetilde{\mathbf{\Psi}}(x)$ involves only $\mathbf{\Lambda}_{a,a}(x)$, and thus is order by order in $\hbar$ analytical in $z^a(x)$, and since we know that $\widetilde{\mathbf{\Psi}}(x)$ has only meromorphic singularities, we see again that the column vectors of $\widetilde{\mathbf{\Psi}}(x)$ have an $\hbar$ expansion such that the coefficients are meromorphic functions of $z^a(x)$. □

Corollary 3.1. *The coefficients $\tilde{\psi}_i^{[k]}(z)$ appearing in the expansion of $\widetilde{\mathbf{\Psi}}_{i,a}(x)$ are meromorphic functions of $z \in \mathcal{S}^{[0]}$ whose poles occur only at values of z such that $\exists a \neq b$ and $x \in \mathbb{C}$ with $z = z^a(x) = z^b(x)$, or at poles of $\mathbf{L}^{[l]}(x)$ for $l \leq k$. In other words, $\tilde{\psi}_i^{[k]}(z)$ can be singular only at ramification points, at preimages in $\mathcal{S}^{[0]}$ of double points, or at poles of $\mathbf{L}^{[l]}$ on the semiclassical spectral curve $\mathcal{S}^{[0]}$.*

Proof. $\widetilde{\mathbf{\Psi}}(x)$ was constructed so that it has at most meromorphic singularities at poles of $\mathbf{L}(x)$. Then, one can see in (3.8) that singularities can occur only when $\lambda_a^{[0]}(x) = \lambda_b^{[0]}(x)$ for some $a \neq b$, i.e. at branchpoints or double points. □

3.3. Expansion of the Correlators

In this section, we consider the projectors, the correlators, etc. (see Sect. 2.1) associated to the solution $\mathbf{\Psi}(x)$ deduced from Proposition 3.1 via (3.4).

Lemma 3.1. *Assume that the constant matrix $\mathbf{C}$ in (3.4) is diagonal. Then, the projectors have an expansion in powers of $\hbar$ of the form:*

$$\mathbf{P}(\overset{a}{x}) = \sum_{k\geq 0} \hbar^k\,\mathbf{P}^{[k]}(\overset{a}{x}),$$

and there exists a sequence of matrices $\mathbf{p}^{[k]}(z)$ of meromorphic functions in $z \in \mathcal{S}^{[0]}$, with poles at ramification points, at preimages in $\mathcal{S}^{[0]}$ of double points, and at poles of $(\mathbf{L}^{[j]}(x))_{j\geq 0}$, such that $\mathbf{p}^{[k]}(z^a(x)) = \mathbf{P}^{[k]}(\overset{a}{x})$.

Proof. Since we assume $\mathbf{C}$ to be diagonal, the exponentials—which might have prevented the existence of an expansion in powers of $\hbar$—disappear:

$$\begin{aligned}
\mathbf{P}(\overset{a}{x}) &= \mathbf{V}(x)\widehat{\mathbf{\Psi}}(x)\exp\left(\frac{1}{\hbar}\int_o^x \mathbf{\Lambda}(x')\mathrm{d}x'\right)\mathbf{C}\mathbf{E}_a\mathbf{C}^{-1} \\
&\quad \times \exp\left(-\frac{1}{\hbar}\int_0^x \mathbf{\Lambda}(x')\mathrm{d}x'\right)\widehat{\mathbf{\Psi}}^{-1}(x)\mathbf{V}^{-1}(x) \\
&= \mathbf{V}(x)\widehat{\mathbf{\Psi}}(x)\mathbf{E}_a\widehat{\mathbf{\Psi}}^{-1}(x)\mathbf{V}^{-1}(x) \\
&= \widetilde{\mathbf{\Psi}}(x)\,\mathbf{E}_a\widetilde{\mathbf{\Psi}}^{-1}(x).
\end{aligned}$$

From Proposition 3.2, $\widetilde{\mathbf{\Psi}}(x)$ has an expansion in $\hbar$, so $\mathbf{P}(\overset{a}{x})$ has an expansion in $\hbar$. Moreover, $\widetilde{\mathbf{\Psi}}(x)\,\mathbf{E}_a\widetilde{\mathbf{\Psi}}^{-1}(x)$ involves only the ath column of $\widetilde{\mathbf{\Psi}}(x)$ and the ath line of $\widetilde{\mathbf{\Psi}}^{-1}(x)$, i.e. the coefficients of the expansion are meromorphic functions of $z^a(x)$. From Corollary 3.1, those meromorphic functions can be singular only at ramification points, at preimages in $\mathcal{S}^{[0]}$ of double points, or at poles of $\mathbf{L}(x)$ in $\mathcal{S}^{[0]}$. $\square$

Notice that to leading order, $\widetilde{\mathbf{\Psi}}(x) = \mathbf{1}_d + O(\hbar)$, and

$$\mathbf{P}^{[0]}(\overset{a}{x}) = (\mathbf{V}^{[0]}(x))^{-1}\mathbf{E}_a\mathbf{V}^{[0]}(x) \tag{3.11}$$

is the projection on the ath eigenspace of $\mathbf{L}^{[0]}(x)$. From the expression of the correlators in terms of the projectors, we deduce:

Corollary 3.2. *For any $a \in [\![1, d]\!]$, $\mathcal{W}_1(\overset{a}{x})$ has an expansion in powers of $\hbar$, of the form:*

$$\mathcal{W}_1(\overset{a}{x}) = \sum_{k\geq -1} \hbar^k\, \mathcal{W}_1^{[k]}(\overset{a}{x}),$$

and there exist meromorphic functions $w_1^{[k]}(z)$ in $z \in \mathcal{S}^{[0]}$, with poles at the ramification points, or at preimages in $\mathcal{S}^{[0]}$ of double points, or at poles of $(\mathbf{L}^{[j]}(x))_{j\geq 0}$, so that $w_1^{[k]}(z^a(x)) = \mathcal{W}_1^{[k]}(\overset{a}{x})$.

For example we have:

$$\mathcal{W}_1^{[0]}(\overset{a}{x}) = -\lambda_a^{[0]}(x),$$

Corollary 3.3. *For any $n \geq 2$, any $a_1, \ldots, a_n \in [\![1, d]\!]$, the correlators have an expansion in powers of $\hbar$:*

$$\mathcal{W}_n(\overset{a_1}{x_1}, \ldots, \overset{a_n}{x_n}) = \sum_{k\geq 0} \hbar^k\, \mathcal{W}_n^{[k]}(\overset{a_1}{x_1}, \ldots, \overset{a_n}{x_n})$$

and there exist symmetric meromorphic functions $w_n^{[k]}(z_1, \ldots, z_n)$ in variables $z_i(\mathcal{S}^{[0]})$, with poles when z_i is at a ramification point or at a double pole or at a pole of $(\mathbf{L}^{[j]}(x))_{j\geq 0}$, and so that

$$w_n^{[k]}(z^{a_1}(x_1), \ldots, z^{a_n}(x_n)) = \mathcal{W}_n^{[k]}(\overset{a_1}{x_1}, \ldots, \overset{a_n}{x_n}).$$

On top of that, $w_2^{[0]}(z_1, z_2)$ has a double pole at $z_1 = z_2$ and behaves as:

$$w_2^{[0]}(z_1, z_2) \underset{z_1\to z_2}{=} \frac{x'(z_1)x'(z_2)}{\big(x(z_1) - x(z_2)\big)^2} + O(1).$$

3.4. Expansion in $\hbar$ with Poles Assumptions

Many interesting systems have the property that their leading asymptotic behavior at the poles of $\mathbf{L}(x)$ is governed by the $\hbar \to 0$ limit, i.e. in some sense that $\mathbf{L}^{[j]}(x)$ for $j > 0$ is somewhat "smaller" than $\mathbf{L}^{[0]}(x)$. When this holds, only $\mathcal{W}_1^{[0]}(\overset{a}{x})$ can have poles at the poles of $\mathbf{L}(x)$, all other $\mathcal{W}_n^{[g]}$ have no poles at the poles of $\mathbf{L}(x)$. Let us make it precise.

Assumption 3.1. Let us assume that $\mathbf{L}(x) = \sum_{j\geq 0} \hbar^j \, \mathbf{L}^{[j]}(x)$ has the property that for any $j > 0$ the poles of $\mathbf{L}^{[j]}(x)$ are a subset of the poles of $\mathbf{L}^{[0]}(x)$, and the expansion of its eigenvalues

$$\lambda_a(x) = \sum_{j\geq 0} \hbar^j \, \lambda_a^{[j]}(x)$$

is such that, for any $j > 0$, $\lambda_a^{[j]}(x) \to 0$ when x approaches a pole of $\mathbf{L}(x)$. Equivalently, this means that the characteristic polynomial of $\mathbf{L}(x)$ satisfies

$$Q(x,y) = \det\big(y\,\mathbf{1}_d - \mathbf{L}(x)\big) = \sum_{j\geq 0} \hbar^j \, Q^{[j]}(x,y),$$

where the coefficients, for $j > 0$, are such that:

$$D^{[0]}(x)\, Q^{[j]}(x,y) = \sum_{(m,n)\in \text{interior}(\mathcal{N})} \hat{Q}^{[j]}_{m,n-1}\, x^m\, y^{n-1},$$

where $D^{[0]}(x)$ is the common denominator of all coefficients of $Q^{[0]}(x,y)$ and $\mathcal{N}$ is the envelope of the Newton's polytope of $D^{[0]}(x)\,Q^{[0]}(x,y)$.

Corollary 3.4. *When assumption 3.1 is satisfied, only* $\mathcal{W}_1^{[0]}(\overset{a}{x})$ *can have poles at the poles of* $\mathbf{L}(x)$*, all other* $\mathcal{W}_n^{[k]}$ *are regular at the poles of* $\mathbf{L}(x)$.

Corollary 3.5. $\omega_2^{(0)}(z_1,z_2) = w_2^{[0]}(z_1,z_2)\mathrm{d}x(z_1)\mathrm{d}x(z_2)$ *defines an element of* $\mathcal{B}(\mathcal{S}^{[0]})$ (see Definition 3.2).

For instance, we have from Proposition 2.1 and (3.11):

$$\mathcal{W}_2^{[0]}(\overset{a_1}{x_1},\overset{a_2}{x_2}) = \frac{\Big[(\mathbf{V}^{[0]})^{-1}(x_1)\mathbf{V}^{[0]}(x_2\Big]_{a_1,a_2}\Big[(\mathbf{V}^{[0]})^{-1}(x_2)\mathbf{V}^{[0]}(x_1)\Big]_{a_2,a_1}}{(x_1-x_2)^2}.$$

3.5. Expansion of Topological Type and Topological Recursion

Definition 3.3 (TT property). We say that the correlators have an expansion of topological type (or have the TT property) when they have:

- the $\hbar \leftrightarrow -\hbar$ symmetry: $(\mathcal{W}_n)_{-\hbar} = (-1)^n(\mathcal{W}_n)_{\hbar}$.
- the $\hbar^{n-2}$ property: for any $n \geq 2$, $\mathcal{W}_n \in O(\hbar^{n-2})$. When these two properties are satisfied, the $\hbar$ expansion of the correlators looks like:

$$\forall n \geq 1, \quad \mathcal{W}_n = \sum_{g\geq 0} \hbar^{2g-2+n}\, \mathcal{W}_n^{(g)}. \tag{3.12}$$

- the pole property: when $(g,n) \neq (0,1),(0,2)$, the $\omega_n^{(g)}$ have poles only at the ramification points. In particular they must have no pole at the preimages in $\mathcal{S}^{[0]}$ of double points, or at the poles of $\mathbf{L}^{[k]}(x)$. And $\omega_2^{(0)}(z_1,z_2)$ has a double pole at $z_1 = z_2$, and no other pole.

In the Sect. 4, we shall study some sufficient conditions (related to integrable systems) to have the TT property, and in Sect. 5, we shall show that qth reductions of the KP hierarchy have the TT property. We believe that the TT property is closely related to integrability (see Question 4.1), but we do not have a proof of such a statement. Let us just mention that the $\hbar^{n-2}$ property is a highly non-trivial one. For example large random matrices, it is related to the "concentration" property [20].

When the TT property is satisfied, one can plug the $\hbar$ expansion (3.12) into the loop equations to obtain a set of equations satisfied by $\mathcal{W}_n^{(g)}$. The key point is that those equations can be solved recursively on $2g-2+n$. The prototype of such a result is known since [3–5]. The solution is given by the topological recursion developed in [43]. The topological recursion associates to a plane curve $(\mathcal{S}^{[0]}, x, y)$ (algebraic in our case) and $\omega_2^{(0)} \in \mathcal{B}(\mathcal{S}^{[0]})$, a sequence of symmetric meromorphic n-forms $\omega_n^{(g)}$ on $(\mathcal{S}^{[0]})^n$, defined by a recursion on $2g-2+n$ in terms of the geometry of the curve $\mathcal{S}^{[0]}$. It was first presented under the assumption that ramification points are simple [41] and extended to arbitrary ramification points in [23]. Then, it was shown [22] that the general formula of [23] is a limiting case of the formula of [41] for simple ramification points. For instance, the semiclassical spectral curve of r-KdV has one ramification point of order r. For readability, we present now the case of simple ramification points and refer to [22] for the case of arbitrary ramifications.

Theorem 3.1. *If the correlators have an expansion of topological type, and* $\mathrm{d}x$ *has only simple zeroes on the semiclassical spectral curve* $\mathcal{S}^{[0]}$: $\det(y\,\mathbf{1}_d - \mathbf{L}^{[0]}(x)) = 0$, *then the coefficients of* (3.12) *are given by:*

$$\mathcal{W}_n^{(g)}(\overset{a_1}{x_1}, \ldots, \overset{a_n}{x_n})\mathrm{d}x_1 \cdots \mathrm{d}x_n = \omega_n^{(g)}(z^{a_1}(x_1), \ldots, z^{a_n}(x_n))$$

and $\omega_n^{(g)}$ *satisfy:*

$$\begin{aligned}\omega_n^{(g)}(z_1, z_2, \ldots, z_n) = \sum_{r\in\mathbf{r}} \operatorname*{Res}_{z\to r} K_r(z_1, z) \Bigg[& \omega_{n+1}^{(g-1)}(z, \sigma_r(z), z_2, \ldots, z_n) \\ & + \sideset{}{'}\sum_{\substack{h+h'=g \\ I\dot{\cup} I'=[\![2,n]\!]}} \omega_{1+|I|}^{(h)}(z, z_I)\, \omega_{1+|I'|}^{(h')}(\sigma_r(z), z_{I'}) \Bigg] \\ & + H_n^{(g)}(z_1, \ldots, z_n)\end{aligned} \tag{3.13}$$

where $H_n^g(z_1, \ldots, z_n)$ *is some symmetric holomorphic* n*-form on* $(\mathcal{S}^{[0]})^n$, $\sum'$ *means that we exclude* $(h, I) = (0, \emptyset)$ *and* $(h', I') = (0, \emptyset)$, r *are the ramification points (i.e. the zeroes of* $\mathrm{d}x$*),* σ_r *is the local Galois involution near the ramification point* r*, i.e. the holomorphic map defined in the vicinity of* r*, such that* $x \circ \sigma_r = x$ *and* $\sigma_r \neq \mathrm{id}$. *And, the recursion kernel is:*

$$K_r(z_1, z) = \frac{\frac{1}{2}\int_{\sigma_r(z)}^{z} \omega_2^{(0)}(z_1, \cdot)}{\omega_1^{(0)}(z) - \omega_1^{(0)}(\sigma_r(z))} \tag{3.14}$$

where $\omega_1^{(0)} = -y\mathrm{d}x$ *on* $\mathcal{S}^{[0]}$.

Corollary 3.6. *If furthermore* $\mathcal{S}^{[0]}$ *has genus* 0, $H_n^{(g)} \equiv 0$ *(since there are no holomorphic* 1*-forms on* $\mathcal{S}^{[0]}$*) and* $\omega_n^{(g)}$ *are exactly given by the topological recursion of* [41] *applied to the initial data* $\omega_1^{(0)} = -y\mathrm{d}x$ *and* $\omega_2^{(0)}$ *(see Corollary* 3.5*).*

Proof. The proof is essentially done in [19,41]. To be self-contained, we redo it in "Appendix B". □

3.6. Symmetry $\hbar \leftrightarrow -\hbar$

Here, we give a sufficient condition for the existence of an $\hbar \leftrightarrow -\hbar$ symmetry. We do not know whether this criterion is also a necessary condition.

Proposition 3.3. *Assume there exists an invertible matrix* $\mathbf{\Gamma}$*, independent of* x*, such that:*

$$\mathbf{\Gamma}\mathbf{L}_\hbar^T(x)\mathbf{\Gamma}^{-1} = \mathbf{L}_{-\hbar}(x). \tag{3.15}$$

Then, if $\mathbf{\Psi}_+$ *is a solution of* (1.1), $\mathbf{\Psi}_- = \mathbf{\Gamma}(\mathbf{\Psi}_+^{-1})^T$ *is a solution of* (1.1) *with* $\hbar \to -\hbar$*. The projector associated to the two solutions are related by* $\mathbf{P}_+ = \mathbf{\Gamma}\mathbf{P}_-^T\mathbf{\Gamma}^{-1}$*, and the correlators by* $(\mathcal{W}_n)_+ = (-1)^n\,(\mathcal{W}_n)_-$ *for any* $n \geq 1$*.*

Proof. The relation between the projectors is an easy computation, and given Proposition 2.1 for the n-point correlators, we deduce $(\mathcal{W}_n)_+ = (-1)^n(\mathcal{W}_n)_-$ for any $n \geq 2$. For $n = 1$, we check it directly:

$$\begin{aligned}
(\mathcal{W}_1)_-(\overset{a}{x}) &= \hbar\,[\mathbf{\Psi}_-^{-1}(x)\mathbf{L}_{-\hbar}(x)\mathbf{\Psi}_-(x)]_{a,a} \\
&= \hbar\,\mathrm{Tr}\,\mathbf{\Psi}_-^{-1}(x)\mathbf{L}_{-\hbar}(x)\mathbf{\Psi}_-(x)\mathbf{E}_a = \mathrm{Tr}\,\mathbf{P}_-(\overset{a}{x})\mathbf{L}_{-\hbar}(x) \\
&= \hbar\,\mathrm{Tr}\,\mathbf{\Gamma}\mathbf{P}_+^T(\overset{a}{x})\mathbf{\Gamma}^{-1}\mathbf{L}_\hbar(x) = \mathrm{Tr}\,\mathbf{P}_+^T(\overset{a}{x})\mathbf{L}_\hbar^T(x) \\
&= \hbar\,\mathrm{Tr}\,\mathbf{P}(\overset{a}{x})\mathbf{L}_\hbar(x) = -(\mathcal{W}_1)_+(\overset{a}{x}).
\end{aligned}$$
□

4. Case of Isomonodromic Integrable Systems

We believe that integrable systems is the good setting to have the TT property satisfied. We give some arguments here, and then show in Sect. 5 that the special case of qth reduction of KP fits in our framework.

4.1. Behavior at the Poles and Isomonodromic Times

In this paragraph, we review classical results from the theory of linear differential systems. A $d \times d$ invertible matrix $\mathbf{\Psi}(x)$ which is the solution of $\hbar\,\partial_x\mathbf{\Psi}(x) = \mathbf{L}(x)\mathbf{\Psi}(x)$ that can have singularities only at poles of $\mathbf{L}(x)$. For any $p \in \mathcal{P}$, it can be put locally around $x = p$ in the form:[2]

[2] When $p = \infty$, the factors $(x - p)$ should be replaced by $1/x$.

$$\mathbf{\Psi}(x) = \widetilde{\mathbf{\Psi}}_p(x) \exp\Big(\mathbf{B}_p \ln(x-p) + \mathbf{A}_p(x)\Big)\mathbf{C}_p \tag{4.1}$$

$$\mathbf{A}_p(x) = \sum_{k=1}^{m_p} \frac{\mathbf{A}_{p;k}}{(x-p)^k} \tag{4.2}$$

$$\widetilde{\Psi}_p(x) \underset{x\to p}{\sim} \mathbf{1}_d, \tag{4.3}$$

where $\mathbf{A}_p(x)$ and $\mathbf{B}_p$ are Jordanized matrices. Such asymptotics can only be valid in an angular sector near $x = p$, and the constant matrix $\mathbf{C}_p$ depends on the sector. $\mathbf{B}_p$ describes the monodromy around p of the right-hand side of (4.1).

Imagine that $\mathbf{L}(x)$ depends smoothly on parameters $\vec{t} = (t_\alpha)_\alpha$, generically called "times". One can always define a matrix

$$\mathbf{M}_\alpha(x) = \hbar\,\partial_{t_\alpha}\mathbf{\Psi}(x)\,\mathbf{\Psi}(x)^{-1},$$

such that $\mathbf{\Psi}(x)$ satisfy on top of (1.1) the compatible systems:

$$\forall\alpha, \quad \hbar\,\partial_{t_\alpha}\mathbf{\Psi}(x) = \mathbf{M}_\alpha(x)\mathbf{\Psi}(x).$$

Requiring that $\mathbf{M}_\alpha(x)$ be rational is equivalent to requiring that the global monodromy data $\mathbf{B}_p$ and the Stokes matrices[3] do not depend on $\vec{t}$. If $\partial_{t_\alpha}\mathbf{B}_p \equiv 0$ for any $p \in \mathcal{P}$, we say that t_α is an *isomonodromic* time. Integrable systems in Lax form provide examples of such rational compatible differential systems. A second realization of this setting in the realm of formal series in $\vec{t}$ can be achieved by deformation of any given $\mathbf{L}(x)$ and solution $\mathbf{\Psi}(x)$ (independent of parameters) [6, Chapt. 5]. The latter might not be the restriction of an integrable system in Lax form (for $\mathbf{\Psi}(x,\vec{t})$ might not be defined as a function of $\vec{t}$). Our formalism applies equally well to the two cases.

One can try, for any given ODE (1.1), to embed it in a family of isomonodromic deformations depending on some times $\vec{t}$, and consider family of solutions $\mathbf{\Psi}_\hbar(x,\vec{t})$. The question about the TT property can then be reformulated as follows:

Question 4.1. Characterize the monodromy data of (1.1) for which the TT property is satisfied.

4.2. Isomonodromic Tau Function

In this section, we assume that $\mathbf{L}(x)$ depends on a family of isomonodromic times $\vec{t} = (t_\alpha)_\alpha$. If there is more than one time, we first need a remark. Let us define:

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[3] The Stokes matrices are the sequences of matrices $\mathbf{C}_p(\mathbf{C}'_p)^{-1}$ with $\mathbf{C}_p$ being $\mathbf{C}'_p$, the constant (in x) matrices appearing in (4.1) for two adjacent sectors.

$$\begin{aligned}\Upsilon_\alpha(\vec{t}) &= -\sum_{p\in\mathcal{P}} \operatorname*{Res}_{x\to p} \mathrm{d}x \,\mathrm{Tr}\left[\mathbf{\Psi}^{-1}(x)(\partial_x \mathbf{\Psi}(x))\, e^{-\mathbf{A}_p(x)}(\partial_{t_\alpha} e^{\mathbf{A}_p(x)})\right] \\ &= -\sum_{p\in\mathcal{P}} \operatorname*{Res}_{x\to p} \mathrm{d}x \,\mathrm{Tr}\left[\widetilde{\mathbf{\Psi}}_p^{-1}(x)(\partial_x \widetilde{\mathbf{\Psi}}_p(x))\, e^{-\mathbf{A}_p(x)}(\partial_{t_\alpha} e^{\mathbf{A}_p(x)})\right] \\ &= -\sum_{p\in\mathcal{P}} \operatorname*{Res}_{x\to p} \sum_{a=1}^{d} \left[\mathcal{W}_1(\overset{a}{x})\left(e^{-\mathbf{A}_p(x)}\partial_{t_\alpha} e^{\mathbf{A}_p(x)}\right)_{a,a}\right]. \end{aligned} \tag{4.4}$$

Lemma 4.1.

$$\forall \alpha,\beta,\quad \partial_{t_\beta}\Upsilon_\alpha(\vec{t}) = \partial_{t_\alpha}\Upsilon_\beta(\vec{t}). \tag{4.5}$$

Proof. The definition of Υ_α and this result is due to Jimbo, Miwa and Ueno for integrable systems in Lax form and diagonal $\mathbf{A}_{p,k}$ (see also [6]). It was generalized to non-diagonal $\mathbf{A}_{p,k}$ in [12]. The proof is essentially the same. □

Definition 4.1. We define the *isomonodromic Tau function* as a function $\mathcal{T}(\vec{t})$ (or as a power series in $\vec{t}$), such that:

$$\partial_{t_\alpha} \ln \mathcal{T}(\vec{t}) = \Upsilon_\alpha(\vec{t}). \tag{4.6}$$

It is defined up to a constant independent of $\vec{t}$.

Tau functions play an important role in the theory of integrable systems and its applications, and have been extensively studied; we refer to [6] and references therein.

4.3. Case of an Integrable System: Expansion of the Tau Function

If $\mathbf{L}$ depends on isomonodromic times $\vec{t}$, an isomonodromic Tau function $\mathcal{T}(\vec{t})$ has been defined in Definition (4.1). It is a consequence of Corollary 3.2 and the formula (4.4) for the isomonodromic Tau function that:

Corollary 4.1. *If $\mathbf{A}_p = \hbar^{-1}\,\mathbf{A}_p^{[0]}$ is diagonal for any pole p, we have an expansion of the form:*

$$\ln \mathcal{T}(\vec{t}) = \sum_{k\geq -2} \hbar^k\, F^{[k]}(\vec{t}), \tag{4.7}$$

where

$$\partial_{t_\alpha} F^{[k]}(\vec{t}) = -\sum_{p\in\mathcal{P}} \operatorname*{Res}_{x\to p} \sum_{a=1}^{d} \left[\mathrm{d}x\, \mathcal{W}_1^{[k+1]}(\overset{a}{x})\, \partial_{t_\alpha}(\mathbf{A}_p^{[0]}(x))_{a,a}\right]. \tag{4.8}$$

Corollary 4.2. *In particular, if the TT property holds, then only even powers of $\hbar$ appear:*

$$\ln \mathcal{T}(\vec{t}) = \sum_{g\geq 0} \hbar^{2g-2}\, F^{(g)}(\vec{t}), \tag{4.9}$$

where

$$\partial_{t_\alpha} F^{(g)}(\vec{t}) = -\sum_{p\in\mathcal{P}} \operatorname*{Res}_{z\to p} \left[\omega_1^{(g)}(z)\, f_\alpha(z)\right], \tag{4.10}$$

with

$$\frac{\mathrm{d}f_\alpha(z)}{\mathrm{d}x(z)} = \partial_{t_\alpha} y(z)\big|_{x(z)}. \tag{4.11}$$

Proof. Indeed, when there is an expansion of topological type, we have

$$\mathcal{W}_1^{[2g-1]}(\overset{a}{x})\,\mathrm{d}x = \omega_1^{(g)}(z^a(x)).$$

□

4.4. Compatibility of the Insertion Operator with Isomonodromic Deformations

The definition of Picard–Vessiot rings is easily generalized to a family of compatible differential systems

$$\hbar\partial_x \mathbf{\Psi}(x) = \mathbf{L}(x)\mathbf{\Psi}(x), \quad \hbar\partial_{t_\alpha}\mathbf{\Psi}(x) = \mathbf{M}_\alpha(x)\mathbf{\Psi}(x).$$

We amend Definition 2.5 of insertion operators:

Definition 4.2. We say that an insertion operator δ is *compatible* if it commutes with all ∂_{t_α}, i.e. if it satisfies:

$$\hbar\partial_{t_\alpha}\mathbf{U}(\overset{a}{y}) = \delta_y^a \mathbf{M}_\alpha(x) + [\mathbf{M}_\alpha(x), \mathbf{U}(\overset{a}{y})] + \left[\frac{\mathbf{M}_\alpha(x) - \mathbf{M}_\alpha(y)}{x-y}, \mathbf{P}(\overset{a}{y})\right]. \tag{4.12}$$

The existence of an insertion operator compatible with all times is not something obvious, but if it exists it is quite useful. For the qth reduction of KP, we construct in Sect. 5.7.3 a compatible differential operator, which enables to prove the $O(\hbar^{n-2})$ axiom of the TT property.

5. Application to Finite Reductions of KP

In this section, we show an important application of the former formalism, namely to the qth reductions of the KP hierarchy. They are related to the Drinfeld–Sokolov hierarchies [31] and contain as a more special case the (p,q) models exemplified in Sect. 6. They appear in one of the formulations of $2d$ quantum gravity [30] and conjecturally describe the algebraic critical points which can arise in hermitian multi-matrix models. In physics, the (p,q) models are expected to describe thermodynamic observables in the coupling of Liouville theory to the (p,q) minimal models of conformal field theory [28], the latter corresponding to the classification of finite representations of the conformal Virasoro symmetry of central charge $c = 1 - 6(p-q)^2/pq$ [29]. The qth reduction of KP is related to perturbations of this coupled theory by primary operators.

5.1. Pseudo-Differential Approach to the qth Reduction of KP

Let t be a 1-dimensional variable, and $\mathcal{C}^\infty$ denote an algebra of smooth functions of t. Let $\mathbb{D} = \mathcal{C}^\infty[\hbar\partial_t, \hbar^{-1}\partial_t^{-1}]$ be the graded algebra of pseudodifferential operators. Let $\mathbb{D}_+ = \mathcal{C}^\infty(\mathbb{R})[\hbar\,\partial_t]$ its subalgebra of differential operators, graded by the degree. We say that $D \in \mathbb{D}$ is monic of degree $r \geq 0$ if

$$D = \hbar^r \partial_t^r + \sum_{k=-\infty}^{r-1} a_k(t)(\hbar\,\partial_t)^k.$$

We then recall that there exists a unique pseudodifferential operator denoted $D^{1/r}$, which is monic of degree 1 and satisfies $(D^{1/r})^r = D$. We denote D_+, the projection of any $D \in \mathbb{D}$ to $\mathbb{D}_+$.

The *string equation* is a relationship

$$[P, Q] = \hbar \tag{5.1}$$

between differential operators P and Q. It can be written as the compatibility condition of two differential equations for a function $\psi(x,t)$:

$$x\psi(x,t) = Q\psi(x,t), \quad -\hbar\,\partial_x\psi(x,t) = P\psi(x,t). \tag{5.2}$$

We call (5.2) the *associated linear system*.

Let (p,q) be a couple of positive integers distinct from $(1,1)$. The (p,q) *model* is a hierarchy of 1-dimensional nonlinear differential equations for a sequence of functions $u(t)$, $u_k(t)$ for $k \in [\![1, p-3]\!]$, and $v_l(t)$ for $l \in [\![1, q-3]\!]$, ensuing by looking[4] for a solution of a string equation of the form:

$$P = \sum_{k=0}^{p} v_k(t)\,(\hbar\,\partial_t)^k, \quad v_p \equiv 1,\, v_{p-1} = 0,\, v_{p-2} = -pu, \tag{5.3}$$

$$Q = \sum_{l=0}^{q} u_l(t)\,(\hbar\,\partial_t)^l, \quad u_q \equiv 1,\, u_{q-1} = 0,\, u_{q-2} = -qu. \tag{5.4}$$

We thus have:

$$\begin{cases} P = (\hbar\,\partial_t)^p - pu(t)\,(\hbar\,\partial_t)^{p-2} + \sum_{l=0}^{p-3} v_l(t)\,(\hbar\,\partial_t)^l \\ Q = (\hbar\,\partial_t)^q - qu(t)\,(\hbar\,\partial_t)^{q-2} + \sum_{k=0}^{q-3} u_k(t)\,(\hbar\,\partial_t)^l \end{cases} \tag{5.5}$$

When P and Q assume the form (5.3), it is well known that:

Proposition 5.1. [28,31] *The most general solution of* (5.1) *is of the form:*

$$P = \sum_{l=0}^{p} t_l\,(Q^{l/q})_+, \quad Q = \sum_{k=0}^{q} \widetilde{t}_k\,(P^{k/p})_+.$$

for some constants t_l and $\widetilde{t}_k$ (with $t_p = 1$ and $\widetilde{t}_q = 1$).

---

[4] The choice $u_{q-1} = v_{p-1} = 0$ can always be achieved by a redefinition of the variable t. And then $u_{q-2}/q = v_{p-2}/p$ follows from the string equation, and we denote $u = -u_{q-2}/q = -v_{p-2}/p$.

For coprime (p,q), the (p,q) model is defined by the choice $P = (Q^{p/q})_+$. The string equation $[P,Q] = \hbar$ usually implies a non-linear equation for $u(t)$.

Example of PDEs for the $(p,q) = (3,2)$ model. Let us denote $\dot{u}(t) = \partial_t u(t)$. We have

$$P = (\hbar\partial_t)^3 - 3u\hbar\partial_t + v \quad Q = (\hbar\partial_t)^2 - 2u$$

and the string equation implies

$$v = -\frac{3}{2}\,\hbar\,\dot{u} + t_1$$

for some constant t_1, and the Painlevé I equation for $u(t)$:

$$-\frac{1}{2}\,\hbar^2\,\ddot{u} + 3u^2 = t.$$

5.2. Constructing the Lax Pair by "Folding"

In this paragraph, we show that the associated linear system is an integrable system in Lax form, i.e. it can be written:

$$\hbar\,\partial_x \mathbf{\Psi}(x,t) = \mathbf{L}(x,t)\mathbf{\Psi}(x,t), \quad \hbar\,\partial_t \mathbf{\Psi}(x,t) = \mathbf{M}(x,t)\mathbf{\Psi}(x,t), \tag{5.6}$$

for a matrix

$$\mathbf{\Psi}(x,t) = \begin{pmatrix} \psi_1(x,t) & \cdots & \psi_q(x,t) \\ (\hbar\,\partial_t)\psi_1(x,t) & \cdots & (\hbar\,\partial_t)\psi_q(x,t) \\ \vdots & & \vdots \\ (\hbar\,\partial_t)^{q-1}\psi_1(x,t) & \cdots & (\hbar\,\partial_t)^{q-1}\psi_q(x,t) \end{pmatrix}. \tag{5.7}$$

where the $\psi_j(x)$ are independent solutions of the associated linear system (5.2).

It is easy to achieve the second equation with the companion matrix:

$$\mathbf{M}(x,t) = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ x - u_0(t) & -u_1(t) & \cdots & -u_{q-2}(t) & -u_{q-1}(t) \end{pmatrix} \tag{5.8}$$

where we recall that $u_{q-2} = -qu$, and up to a redefinition of time t we can choose $u_{q-1} \equiv 0$. We now construct the matrix $\mathbf{L}(x,t)$ to realize the first equation. Naively, $\partial_x\partial_t^k\psi$ can be expressed by the action of a differential operator of order $(p+k)$ on ψ. But, if we want to write an equation like (5.6) with $\mathbf{L}(x,t)$ having coefficients which are functions of x—and not differential operators—only derivatives of order smaller than $(q-1)$ are allowed. To bypass this restriction, we can use the first member of (5.2) to express any qth order derivative of ψ in terms of derivatives of lower order. This can be systematized with the notion of folding operators. The folding procedure is a classical trick, which has been used earlier, e.g. in [51] and [58].

Definition 5.1. We define for any integer l the *folding operators*:

$$F_l(x,t) = \sum_{j\geq 0} F_{l,j}(x,t)\,(\hbar\,\partial_t)^j \in \mathbb{D}_+[x],$$

by the following recursion:

$$F_0(x,t) = 1, \quad F_{l+1}(x,t) = (\hbar\,\partial_t) F_l(x,t) + F_{l,q-1}(x,t)(x-Q).$$

They have the property that for every solution ψ_l of (5.2)

$$\forall\, i \in \mathbb{Z}_+,\ \ \forall\, l = 1,\ldots,q, \quad (\hbar\partial_t)^i\, \psi_l(x,t) = \sum_{j=0}^{q-1} F_{i,j}(x,t)\,(\hbar\partial_t)^j\, \psi_l(x,t)$$

in other words, they express any time derivative in terms of only up to order $q-1$ derivatives.

Notice that $F_l(x,t) = (\hbar\,\partial_t)^l$ for $l \in [\![0, q-1]\!]$, but:

$$F_q(x,t) = (\hbar\,\partial_t)^q + x - Q = x - qu(t)\,(\hbar\,\partial_t)^{q-2} - \sum_{k=0}^{q-3} u_k(t)\,(\hbar\,\partial_t)^k.$$

Lemma 5.1. *For any integer l, $F_{l,j}(x,t) \equiv 0$ whenever $j \geq q$. Besides, for every solution ψ of* (5.2)

$$-\,\hbar\,\partial_x\psi(x,t) = P\psi(x,t) = \Big(\sum_{l=0}^{p} v_l(t) F_l(x,t)\Big)\psi(x,t) \tag{5.9}$$

Proof. Since Q is monic of degree q, the last term in (5.9) prevents $F_l(x,t)$ to have terms of degree higher than $(q-1)$, as one can show by recursion. Then, recall that $(x-Q)\psi(x,t) = 0$, so these operators satisfy $(\hbar\,\partial_t)^l\psi(x,t) = F_l(x,t)\psi(x,t)$, hence (5.9). □

Definition 5.2. For any integer k, we define the operators:

$$L_k(x,t) = \sum_{j\geq 0} L_{k,j}(x,t)\,(\hbar\,\partial_t)^j \in \mathbb{D}_+[x]$$

by the following recursion:

$$L_0(x,t) = \ -\sum_{l=0}^{p} v_l(t) F_l(x,t)$$
$$L_{k+1}(x,t) = (\hbar\,\partial_t) L_k(x,t) + L_{k,q-1}(x,t)(x-Q).$$

We have similarly:

Lemma 5.2. *For any integer k, $L_{k,j}(x,t) \equiv 0$ whenever $j \geq q$.*

We are now in position to conclude:

Proposition 5.2. *The first equation of* (5.6) *is achieved with*

$$\mathbf{L}(x,t) = (L_{k,j}(x,t))_{0\leq k,j\leq q-1}.$$

In particular, the string equation is equivalent to the compatibility condition of this system:

$$[\mathbf{M}(x,t), \mathbf{L}(x,t)] = \hbar\,\partial_t \mathbf{L}(x,t) - \hbar\,\partial_x \mathbf{M}(x,t). \tag{5.10}$$

By a gauge transformation, one can choose $u_{q-1}(t) \equiv 0$, i.e. $\mathbf{M}(x,t)$ traceless and therefore $\det \mathbf{\Psi}(x,t)$ independent of t. If an initial condition $\mathbf{\Psi}(x,t_0)$ is invertible, $\mathbf{\Psi}(x,t)$ remains invertible for all t.

Example of folding for the $(3,2)$ model. We have:

$$P = (\hbar\partial_t)^3 - 3u\,\hbar\partial_t - \frac{3}{2}\,\hbar\,\dot{u} + t_1, \qquad Q = (\hbar\partial_t)^2 - 2u. \tag{5.11}$$

for which the string equation $[P,Q] = \hbar$ implies the Painlevé I equation for $u(t)$: $-\frac{1}{2}\,\hbar^2\ddot{u} + 3u^2 = t$. The first folding operators are

$$\begin{aligned} F_1 &= \hbar\partial_t \\ F_2 &= x + 2u \\ F_3 &= x\,\hbar\partial_t + 2u\hbar\partial_t + 2\hbar\dot{u} \\ F_4 &= x^2 + 4ux + 4u^2 + 4\hbar^2\dot{u}\,\partial + 2\hbar^2\ddot{u}. \end{aligned}$$

This gives

$$\begin{aligned} L_0 &= -F_3 + 3uF_1 + \Big(\frac{3}{2}\,\hbar\dot{u} - t_1\Big)F_0, \\ L_1 &= -F_4 + 3uF_2 + 3\hbar\dot{u}F_1 + \Big(\frac{3}{2}\,\hbar\dot{u} - t_1\Big)F_1 + \frac{3}{2}\hbar^2\ddot{u}F_0, \end{aligned}$$

and consequently

$$\mathbf{L}(x,t) = \begin{pmatrix} -\frac{1}{2}\hbar\dot{u} - t_1 & -x + u \\ -(x-u)(x+2u) - \frac{1}{2}\hbar^2\ddot{u} & \frac{1}{2}\hbar\dot{u} - t_1 \end{pmatrix}$$

and

$$\mathbf{M}(x,t) = \begin{pmatrix} 0 & 1 \\ x + 2u & 0 \end{pmatrix}.$$

5.3. Semiclassical Spectral Curve and Formal $\hbar$ Expansion

We consider formal solutions of the string equation, i.e. u_k and v_l which have a formal series expansion in $\hbar$. Let us denote:

$$u_k(t) = \sum_{m\geq 0} \hbar^m\, u_k^{[m]}(t), \quad v_l(t) = \sum_{m\geq 0} \hbar^m\, v_l^{[m]}(t). \tag{5.12}$$

Lemma 5.3. *$u_k^{[0]}(t)$ and $v_l^{[0]}(t)$ can be obtained by replacing $\hbar\,\partial_t$ by a variable $z \in \widehat{\mathbb{C}}$. Namely, one defines*

$$\begin{cases} X(z) := \sum_{k=0}^{q} u_k^{[0]}(t)\, z^k \\ Y(z) := \sum_{l=0}^{p} v_l^{[0]}(t)\, z^l; \end{cases} \tag{5.13}$$

(which are the $\hbar \to 0$ semiclassical limit of Q and P). The leading order in $\hbar$ of the string equation becomes a Poisson bracket:

$$\partial_z Y(z)\partial_t X(z) - \partial_z X(z)\partial_t Y(z) = 1, \tag{5.14}$$

which gives an algebraic constraint on $u_k^{[0]}$ and $v_l^{[0]}$.

Proof. The leading order of $[P, Q]$ is:

$$\hbar = [P, Q] = \sum_{k,l} \hbar\, l\, v_l^{[0]}\, \dot{u}_k^{[0]}\, \partial_t^{k+l-1} - \sum_{k,l} \hbar\, k\, u_k^{[0]}\, \dot{v}_l^{[0]}\, \partial_t^{k+l-1} + O(\hbar^2),$$

i.e. this means that

$$Y'(z)\dot{X}(z) - X'(z)\dot{Y}(z) = 1.$$

□

Lemma 5.4. *A solution of* (5.14) *is obtained as follows*

$$X(z) = (u^{[0]})^{q/2}\, f\big(z\, (u^{[0]})^{-1/2}\big), \quad Y(z) = (u^{[0]})^{p/2}\, g\big(z\, (u^{[0]})^{-1/2}\big), \tag{5.15}$$

where $u^{[0]} = (t/\rho)^{\frac{2}{p+q-1}}$, *and the functions* f *and* g *satisfy:*

$$qf(\zeta)g'(\zeta) - pg(\zeta)f'(\zeta) = (p+q-1)\rho, \tag{5.16}$$

and ρ *is chosen such that at large* ζ *the solution of* (5.16) *behaves as* $f(\zeta) = \zeta^q(1 - qu^{[0]}\zeta^{-2} + O(\zeta^{-3}))$ *and* $g(\zeta) = \zeta^p\big(1 - pu^{[0]}\zeta^{-2} + O(\zeta^{-3})\big)$. *We call it the* homogeneous solution.

Proof. The result is claimed in [28]. Let us prove it directly. If we assume the form (5.15), and write $\zeta = z\,(u^{[0]})^{-1/2}$, then we have

$$\begin{aligned}
\partial_z X &= (u^{[0]})^{(q-1)/2}\, f'(\zeta),\\
\partial_t X &= \frac{1}{2}\,\partial_t u^{[0]}\, \left(q\,(u^{[0]})^{(q-2)/2}\, f(\zeta) - (u^{[0]})^{(q-3)/2}\, f'(\zeta)\right),\\
\partial_z Y &= (u^{[0]})^{(p-1)/2}\, g'(\zeta),\\
\partial_t Y &= \frac{1}{2}\,\partial_t u^{[0]}\, \left(p\,(u^{[0]})^{(p-2)/2}\, g(\zeta) - (u^{[0]})^{(p-3)/2}\, g'(\zeta)\right).
\end{aligned}$$

It follows:

$$1 = \partial_t X\, \partial_z Y - \partial_t Y\, \partial_z X = \frac{1}{2}\,\partial_t u^{[0]}\, (u^{[0]})^{(p+q-3)/2}\, (qfg' - pgf'),$$

which is satisfied if $u^{[0]} = (t/\rho)^{\frac{2}{p+q-1}}$ and $qfg' - pgf' = (p+q-1)\rho$. □

Lemma 5.5. *If* $p+q \geq 4$, *this implies when* $\zeta \to \infty$ *that:*

$$\begin{aligned}
f(\zeta) &= g(\zeta)^{q/p} - \frac{\rho}{p}\,\zeta^{1-p}\left(1 + u^{[0]}\left(p - 2 + \frac{2}{p+q+1}\right)\zeta^{-2} + O(\zeta^{-3})\right),\\
g(\zeta) &= f(\zeta)^{p/q} - \frac{\rho}{q}\,\zeta^{1-q}\left(1 + u^{[0]}\left(q - 2 + \frac{2}{p+q+1}\right)\zeta^{-2} + O(\zeta^{-3})\right).
\end{aligned}$$

In particular:

$$f = (g^{q/p})_+, \quad g = (f^{p/q})_+.$$

Proof. Write $f = g^{q/p}\, h$, the equation then gives:

$$-\,p\,\frac{h'}{h} = \frac{(p+q-1)\,\rho}{fg} = \frac{(p+q-1)\rho}{\zeta^{p+q}}\,\big(1 + (p+q)u^{[0]}\zeta^{-2} + O(\zeta^{-3})\big),$$

and upon integration:

$$\ln h = \frac{\rho}{p}\,\zeta^{1-p-q}\left(1+\frac{(p+q)(p+q-1)\,u^{[0]}}{p+q+1}\,\zeta^{-2}+O(\zeta^{-3})\right).$$

Then, using $p+q \geq 4$ to ensure that $2(p+q-1) > p+q+1$, we can exponentiate:

$$h = 1+\frac{\rho}{p}\,\zeta^{1-p-q}\left(1+\frac{(p+q)(p+q-1)\,u^{[0]}}{p+q+1}\,\zeta^{-2}+O(\zeta^{-3})\right).$$

We then multiply by $g^{q/p} = \zeta^q\left(1-qu^{[0]}\zeta^{-2}+O(\zeta^{-3})\right)$ and get

$$f = g^{q/p}+\frac{\rho}{p}\,\zeta^{1-p}\left(1+(p-2+\frac{2}{p+q+1})\,u^{[0]}\,\zeta^{-2}+O(\zeta^{-3})\right).$$

We have the same proof for g. □

Special Solutions. In the (p,q) model, we have $P = (Q^{p/q})_+$ and similarly $Q = (P^{q/p})_+$. Therefore, at the semiclassical limit, we find $Y(z) = (X^{p/q}(z))_+$ and $X(z) = (Y^{q/p}(z))_+$. The relation (5.14) can be solved explicitly in the case $p = (2m+1)q \pm 1$ for some integer m [28]:

$$\begin{cases} f(\zeta) = \sum_{n=0}^{m} \frac{\Gamma(n+1)}{\Gamma(p/q+1)\Gamma(n-p/q+1)}\,T_{p-2nq}(\zeta) \\ g(\zeta) = T_q(\zeta) \end{cases}, \quad \rho = 2p.$$

where $T_l(2\cos\theta) = 2\cos(l\theta)$ are the Chebyshev polynomials of the first kind. In particular, for the so-called "unitary" models $p = q+1$, we find:

$$\begin{cases} f(\zeta) = T_{q+1}(\zeta) \\ g(\zeta) = T_q(\zeta) \end{cases}, \quad \rho = 2(q+1).$$

5.4. Semiclassical Spectral Curve

Proposition 5.3. *In the semiclassical limit $\hbar \to 0$, the eigenvalues of $\mathbf{M}(x,t)$ and $\mathbf{L}(x,t)$ are given by the functions $X(z)$ and $Y(z)$ defined in (5.13), by:*

$$z = \text{eigenvalue of } \mathbf{M}^{[0]}(x,t) \quad \Longleftrightarrow \quad x = X(z) = \sum_{k=0}^{q} u_k^{[0]}(t)\,z^k.$$

$$y = \text{eigenvalue of } \mathbf{L}^{[0]}(X(z),t) \quad \Longleftrightarrow \quad y = Y(z) = \sum_{l=0}^{p} v_l^{[0]}(t)\,z^k.$$

The leading-order spectral curve, i.e. the locus of eigenvalues of $\mathbf{L}^{[0]}(x(z),t)$ is a genus 0 algebraic plane curve.

Proof. Since $\mathbf{M}(x,t)$ is a companion matrix, its characteristic polynomial is

$$0 = \det\left(z\,\mathbf{1}_q - \mathbf{M}(x,t)\right) = x - \sum_{k=0}^{q} u_k(t)\,z^k;$$

therefore in the limit $\hbar \to 0$, the eigenvalues of $\mathbf{M}^{[0]}(x,t)$ are z such that $X(z) = x$:

$$\sum_{k=0}^{q} u_k^{[0]}(t)\, z^k = x = X(z),$$

where $X(z)$ is the function introduced in (5.13). It follows that in the limit $\hbar \to 0$, $\hbar\partial_t\psi(x,t) \sim z\,\psi(x,t)\,(1 + O(\hbar))$. The eigenvalues y of $\mathbf{L}(x,t)$, by definition, are such that

$$y\,\psi(x,t) = -\hbar\,\partial_x\psi(x,t) = \sum_{l=0}^{p} v_l(t)\,(\hbar\partial_t)^l\,\psi(x,t), \tag{5.17}$$

and thus in the $\hbar \to 0$ limit, the eigenvalues of $\mathbf{L}^{[0]}(x,t)$ are such that

$$y = Y(z) = \sum_{l=0}^{p} v_l^{[0]}(t)\, z^l.$$

The spectral curve $P(x,y) = \det(y\,\mathbf{1}_q - \mathbf{L}^{[0]}(x,t))$ is a polynomial of x and y, monic of degree q in y, which vanishes if and only if y is an eigenvalue of $\mathbf{L}^{[0]}(x)$, i.e. if and only if there exists some z such that $x = X(z)$ and $y = Y(z)$. Therefore, $P(x,y)$ is proportional to the resultant of the polynomials $X(z) - x$ and $Y(z) - y$:

$$(-1)^q\,\mathcal{P}(x,y) = \mathrm{Resultant}(X(z)-x, Y(z)-y)$$

$$= \det\begin{pmatrix}
1 & u_{q-1}^{[0]} & u_{q-2}^{[0]} & \dots & u_1^{[0]} & u_0^{[0]}-x & & & \\
 & 1 & u_{q-1}^{[0]} & u_{q-2}^{[0]} & \dots & u_1^{[0]} & u_0^{[0]}-x & & \\
 & & \ddots & & & & & \ddots & \\
 & & & 1 & u_{q-1}^{[0]} & u_{q-2}^{[0]} & \dots & u_1^{[0]} & u_0^{[0]}-x \\
1 & v_{p-1}^{[0]} & \dots & v_1^{[0]} & v_0^{[0]}-y & & & & \\
 & 1 & v_{p-1}^{[0]} & \dots & v_1^{[0]} & v_0^{[0]}-y & & & \\
 & & \ddots & & & & \ddots & & \\
 & & & \ddots & & & & \ddots & \\
 & & & & 1 & v_{p-1}^{[0]} & \dots & v_1^{[0]} & v_0^{[0]}-y
\end{pmatrix}$$

As mentioned above, it admits a parametric solution:

$$\mathcal{P}(X(z), Y(z)) = 0$$

with X and Y polynomials of z. This means that there is a holomorphic map $z \mapsto (X(z), Y(z))$ from the Riemann sphere $\widehat{\mathbb{C}}$ to the spectral curve (the locus of $\mathcal{P}(x,y) = 0$ in $\mathbb{C} \times \mathbb{C}$). In particular, this implies that the spectral curve is an algebraic plane curve of genus $\mathfrak{g} = 0$. $\square$

5.5. Asymptotic Expansion and TT Property

As in Sect. 3.2, we look for asymptotics of the form:

$$\mathbf{\Psi}(x,t) \sim \mathbf{V}(x,t)\,\widehat{\mathbf{\Psi}}(x,t)\,\mathrm{e}^{\frac{1}{\hbar}\,\mathbf{S}(x,t)},$$

where:

- $\mathbf{S}(x,t) = \mathrm{diag}(S(z_a))_{1\leq a\leq q}$ is such that $\partial_t S_a(z)|_{X(z)=x} = z^i$ are the eigenvalues of $\mathbf{M}^{[0]}(x,t)$, where $z = z_a$ is related to x by

$$x = X(z) = z^q - qu^{[0]}(t)\,z^{q-2} + \sum_{k=0}^{q-2} u_k^{[0]}(t)\,z^k.$$

  Thanks to (5.14), it also satisfies:

$$\partial_x S_a(z) = Y(z_a)$$

  where $Y(z_a)$ are the eigenvalues of $\mathbf{L}^{[0]}(x,t)$.

- $\mathbf{V}(x,t)$ is a matrix whose columns are eigenvectors of both $\mathbf{M}^{[0]}(x,t)$ and $\mathbf{L}^{[0]}(x,t)$, normalized such that $\mathbf{V}^{-1}\,\partial_x\mathbf{V}(x,t)$ has a vanishing diagonal. Since $\mathbf{M}^{[0]}(x,t)$ is a companion matrix, $\mathbf{V}(x,t)$ can be found rather explicitly, as a Vandermonde matrix, with columns normalized by a factor $1/\sqrt{X'(z_a)}$:

$$\mathbf{V}_{a,b}(x,t) = \frac{(z_b(x))^{a-1}}{\sqrt{X'(z_a)}} \quad \text{where } x = X(z_b) = \sum_{k=0}^{q} u_k^{[0]}(t)\,z_b^k.$$

  Its inverse is

$$(\mathbf{V}^{-1})_{a,b} = \frac{(X(z_a(x))z_a(x)^{-b})_+}{\sqrt{X'(z_a(x))}} = \frac{\sum_{k=b}^{q} u_k^{[0]}(t)\ z_a(x)^{k-b}}{\sqrt{X'(z_a(x))}}.$$

  It satisfies:

$$\begin{aligned}
&\text{if } a\neq b \quad (\mathbf{V}^{-1}\,\partial_x\mathbf{V})_{a,b} = \frac{\sqrt{X'(z_b)}}{\sqrt{X'(z_a)}}\,\frac{1}{z_a - z_b} = O(x^{-1/q}),\\
&\text{if } a = b \quad (\mathbf{V}^{-1}\,\partial_x\mathbf{V})_{a,a} = 0,\\
&\text{if } a\neq b \quad (\mathbf{V}^{-1}\,\partial_t\mathbf{V})_{a,b} = \frac{\partial_t X(z_b)}{\sqrt{X'(z_a)\,X'(z_b)}}\,\frac{1}{z_a - z_b} = O(x^{-2/q}),\\
&\text{if } a = b \quad (\mathbf{V}^{-1}\,\partial_t\mathbf{V})_{a,a} = -\,\frac{1}{2}\,\frac{\dot{X}'(z_a)}{X'(z_a)} = O(x^{-2/q}).
\end{aligned}$$

- The matrix $\widehat{\mathbf{\Psi}}(x,t) = \mathbf{1}_q + O(\hbar)$ has a formal asymptotic series as $\hbar \to 0$. From $\hbar\partial_t\mathbf{\Psi}\cdot\mathbf{\Psi}^{-1} = \mathbf{M} = \mathbf{M}^{[0]} - \mathbf{e}_q(\mathbf{u} - \mathbf{u}^{[0]})^T$, where $\mathbf{e}_q = (0,0,\ldots,0,1)$ and $\mathbf{u} = (u_0,\ldots,u_{q-1})$, we get the equation for $\widehat{\mathbf{\Psi}}$ involving the diagonal matrix $\mathbf{Z} = \mathrm{diag}(z_1,\ldots,z_q)$ of eigenvalues of $\mathbf{M}^{[0]}$:

$$[\mathbf{Z},\widehat{\mathbf{\Psi}}] = \mathbf{V}^{-1}\,\mathbf{e}_q\,(\mathbf{u} - \mathbf{u}^{[0]})^t\,\mathbf{V}\,\widehat{\mathbf{\Psi}} + \mathbf{V}^{-1}\,\hbar\partial_t\mathbf{V}\ \widehat{\mathbf{\Psi}} + \hbar\partial_t\widehat{\mathbf{\Psi}}, \tag{5.18}$$

i.e.

$$(z_a - z_b)\,\widehat{\mathbf{\Psi}}_{a,b} = \sum_{l=1}^{q} \frac{\sum_{k=0}^{q-2}(u_k - u_k^{[0]})\, z_l^k}{\sqrt{X'(z_a)\,X'(z_b)}}\;\widehat{\mathbf{\Psi}}_{l,b} + \hbar\sum_{l=1}^{q}(\mathbf{V}^{-1}\partial_t\mathbf{V})_{a,l}\,\widehat{\mathbf{\Psi}}_{l,b} + \hbar\partial_t\widehat{\mathbf{\Psi}}_{a,b}. \tag{5.19}$$

This equation uniquely determines $\widehat{\mathbf{\Psi}} = \mathbf{1}_q + O(\hbar)$ as its asymptotic expansion in powers of $\hbar$. In fact, it also uniquely determines $\widehat{\mathbf{\Psi}} = \mathbf{1}_q + O(x^{-1/q})$ as an asymptotic series at large x, in powers of $x^{1/q}$. From $\hbar\partial_x\mathbf{\Psi}\cdot\mathbf{\Psi}^{-1} = \mathbf{L}$ we also get an ODE for $\widehat{\mathbf{\Psi}}$:

$$\mathbf{V}^{-1}\mathbf{L}\mathbf{V}\,\widehat{\mathbf{\Psi}} - \widehat{\mathbf{\Psi}}\,\mathbf{\Lambda}^{[0]} = \hbar\mathbf{V}^{-1}\partial_x\mathbf{V}\;\widehat{\mathbf{\Psi}} + \hbar\partial_x\widehat{\mathbf{\Psi}}. \tag{5.20}$$

We observe that the semiclassical spectral curve has genus 0. Therefore, we will be able to apply Theorem 3.1 if we can show:

- the existence of a $\hbar \leftrightarrow -\hbar$ symmetry. This is a technical but simple check done in Sect. 5.6.
- that the n-point correlators $\mathcal{W}_n(x_1,\ldots,x_n)$ are $O(\hbar^{n-2})$ after a suitable gauge transformation. This is a non-trivial property of (p,q) models, that we establish in Sect. 5.7 by constructing an insertion operator δ_x^a which is compatible with ∂_t.
- the pole property, i.e. that $\omega_n^{(g)}$ have poles only at ramification points, established in Sect. 5.8.

The consequences of Theorem 3.1 for the (p,q) models are gathered in Sect. 5.10.

5.6. $\hbar \leftrightarrow -\hbar$ Symmetry

The goal of this subsection is that the (p,q) models admit conjugated solutions in the terminology of Sect. 3.6:

Proposition 5.4. *For any invertible solution $\mathbf{\Psi}(x,t)$ of (5.6) with coupling constant $\hbar$, there exists a solution $\mathbf{\Phi}(x,t)$ of (5.6) with coupling constant $-\hbar$, such that $\gamma(x,t) = \mathbf{\Phi}(x,t)\mathbf{\Psi}^T(x,t)$ is independent of x.*

This proposition is proved below, but to do so, we need some intermediate results and definitions. We first introduce a conjugation operator:

Definition 5.3. We define the conjugation $\dagger\;:\;\mathbb{D}\to\mathbb{D}$ such that, for any $f_0,\ldots,f_N\in\mathcal{C}^\infty$,

$$\left(\sum_{k=0}^{N} f_k\,(\hbar\partial_t)^k\right)^{\dagger} := \sum_{k=0}^{N}(-\hbar\partial_t)^k f_k = \sum_{k=0}^{N}(-1)^k\left\{\sum_{l=0}^{k}\binom{k}{l}\left[(\hbar\partial_t)^l f_k\right](\hbar\partial_t)^{k-l}\right\}$$

It is an antilinear operator that satisfies:

- for any $f\in\mathcal{C}^\infty\subseteq\mathbb{D}$, $f^\dagger = f$.
- $(\hbar\partial_t)^\dagger = -(\hbar\partial_t)^\dagger$.
- for any $D_1, D_2\in\mathbb{D}$, $(D_1D_2)^\dagger = D_2^\dagger D_1^\dagger$.

In particular, if $P, Q \in \mathbb{D}_+$ satisfy $[P, Q] = \hbar$, then $[P^\dagger, Q^\dagger] = -\hbar$. Moreover, if P and Q are differential operators of the form (5.5), so are $P^\dagger$ and $Q^\dagger$. To summarize, $\dagger$ puts in correspondence the models with coupling constant $\hbar$ and $-\hbar$. The linear system associated to $(P^\dagger, Q^\dagger)$ is:

$$x\phi(x,t) = Q^\dagger\phi(x,t), \quad \hbar\,\partial_x\phi(x,t) = P^\dagger\phi(x,t). \tag{5.21}$$

If $\phi_1(x,t),\ldots,\phi_q(x,t)$ denotes a family of solutions of (5.21), we can define a matrix:

$$\mathbf{\Phi}(x,t) = \begin{pmatrix} \phi_1(x,t) & \cdots & \phi_q(x,t) \\ (\hbar\,\partial_t)\phi_1(x,t) & \cdots & (\hbar\,\partial_t)\phi_q(x,t) \\ \vdots & & \vdots \\ (\hbar\,\partial_t)^{q-1}\phi_1(x,t) & \cdots & (\hbar\,\partial_t)^{q-1}\phi_q(x,t) \end{pmatrix},$$

As before, we can represent (5.21) in Lax form, and we denote $\mathbf{L}_{-\hbar}(x,t)$ and $\mathbf{M}_{-\hbar}(x,t)$ the corresponding Lax matrices:

$$-\hbar\,\partial_x\mathbf{\Phi}(x,t) = \mathbf{L}_{-\hbar}(x,t)\mathbf{\Phi}(x,t), \quad -\hbar\,\partial_t\mathbf{\Phi}(x,t) = \mathbf{M}_{-\hbar}(x,t)\mathbf{\Phi}(x,t).$$

The following result gives a correspondence between solutions of the associated linear systems of (P, Q) and $(P^\dagger, Q^\dagger)$.

Proposition 5.5. *Let $\psi_1,\ldots,\psi_q$ be a basis of solutions of* (5.2)*, $\Psi(x,t)$ as defined in* (5.7)*, and define:*

$$\begin{aligned} \Delta(x) &= \det\mathbf{\Psi}(x,t), \\ \Delta_{i_0-1,j_0}(x,t) &= \det\Big[(\hbar\,\partial_t)^{i-1}\psi_j(x,t)\Big]_{1\leq i,j\leq q}^{i\neq i_0,\ j\neq j_0}, \\ \widetilde{\phi}_j(x,t) &= \Delta_{q-1,j}(x,t). \end{aligned}$$

then $(\widetilde{\phi}_j(x,t))_{1\leq j\leq q}$ is a basis of solutions of (5.21)*.*

The proof of this proposition relies on a technical result:

Lemma 5.6. *Let $j \in [\![1,q]\!]$. With the convention $\Delta_{-1,j} \equiv 0$, we have for any $i \in [\![0,q-1]\!]$,*

$$\hbar\,\partial_t\Delta_{i,j}(x,t) = \Delta_{i-1,j}(x,t) + (-1)^{q-j}(u_i(t) - \delta_{i,0}x)\Delta_{q-1,j}(x,t), \tag{5.22}$$

and for any $k \in [\![1,q+1]\!]$,

$$\begin{aligned} \hbar\,\partial_t\Delta_{q-k,j}(x,t) = &\left(\sum_{l=1}^{k}(-1)^{l+1}(\hbar\,\partial_t)^{k-l}[u_{q-l+1}(t)\Delta_{q-1,j}(x,t)]\right) \\ &+\delta_{k,q+1}(-1)^q x\Delta_{q-1,j}(x,t). \end{aligned} \tag{5.23}$$

Proof. By multilinearity, we can differentiate the minors $\Delta_{i,j}$ line by line:

$$\hbar\,\partial_t \Delta_{i,j}$$
$$= \det \begin{pmatrix} \psi_m \\ \vdots \\ (\hbar\,\partial_t)^{i-2}\psi_m \\ (\hbar\,\partial_t)^{i}\psi_m \\ \hline (\hbar\,\partial_t)^{i+1}\psi_m \\ \vdots \\ (\hbar\,\partial_t)^{q-1}\psi_m \end{pmatrix}_{m\neq j} + \det \begin{pmatrix} \psi_m \\ \vdots \\ (\hbar\,\partial_t)^{i-1}\psi_m \\ \hline (\hbar\,\partial_t)^{i+1}\psi_m \\ \vdots \\ (\hbar\,\partial_t)^{q-2}\psi_m \\ (\hbar\,\partial_t)^{q}\psi_m \end{pmatrix}_{m\neq j} + \sum_{\substack{k=0 \\ k\neq i}}^{q-2} \det \begin{pmatrix} \psi_m \\ \vdots \\ (\hbar\,\partial_t)^{k-2}\psi_m \\ (\hbar\,\partial_t)^{k}\psi_m \\ (\hbar\,\partial_t)^{k}\psi_m \\ \vdots \\ \hline \vdots \\ (\hbar\,\partial_t)^{q-1}\psi_m \end{pmatrix}_{m\neq j}.$$

The non-zero contributions arise only from the terms where:

- the ith line is differentiated: we recognize the definition of $D_{i-1,j}(x,t)$.
- the $(q-1)$th line is differentiated. Since $\psi_1,\ldots,\psi_q$ are solutions of (5.2), we can replace $(\hbar\partial_t)^q\psi_m$ by a $x\psi_m - \sum_{l=0}^{q-2} u_l(t)\,(\hbar\,\partial_t)^k$. By subtraction of the other lines, we may keep in the latter only the term involving a derivative of order ith, which was absent from the minor. We thus recreate a minor $D_{q-1,j}(x,t)$, with a prefactor $(x\delta_{i,0} - u_i(t))$, and up to a sign $(-1)^{q-i}$ taking into account the ordering of the lines.

We therefore arrive to (5.22), and (5.23) follows by recursion. In particular, we obtain at the last step of the recursion $(k = q+1)$:

$$\begin{aligned} 0 = \Delta_{-1,j}(x,t) = &\Big(\sum_{l=1}^{q+1} (-1)^{l+1} (\hbar\,\partial_t)^{k-l} u_{q-l+1}(t) + (-1)^q x\Big)\Delta_{q-1,j}(x,t) \\ &= (-1)^q (x - Q^\dagger)\Delta_{q-1,j}(x,t). \end{aligned}$$

Accordingly, $\widetilde{\phi}_j(x,t) \equiv \Delta_{q-1,j}(x,t)$ provides a solution of (5.21) for any $j \in [\![1,q]\!]$. To show that $(\widetilde{\phi}_j)_j$ is a basis, we define the matrix $\mathbf{\Phi}(x,t) = [(\hbar\,\partial)^{i-1}\widetilde{\phi}_j]_{1\leq i,j\leq q}$ and compute its determinant. Thanks to (5.22), we may write:

$$\begin{aligned} \det \widetilde{\mathbf{\Phi}} = &\det \begin{pmatrix} \Delta_{q-1,m} \\ \hbar\,\partial_t \Delta_{q-1,m} \\ \vdots \\ (\hbar\,\partial_t)^{q-1}\Delta_{q-1,m} \end{pmatrix}_{1\leq m\leq j} \\ &= \det \begin{pmatrix} \Delta_{q-1,m} \\ \Delta_{q-2,m} + (u_{q-1}(t) - x\delta_{q,1})\Delta_{q-1,m} \\ \vdots \\ (\hbar\,\partial_t)^{q-1}\Delta_{q-1,m} \end{pmatrix}_{1\leq m\leq j}, \end{aligned}$$

and upon subtracting the first line in the second line, we can replace the second line by $[\Delta_{q-2,m}]_{1\leq m\leq q}$. We find recursively that the ith line can be replaced by $\Delta_{q-i,m}$, and thus:

$$\det \widetilde{\mathbf{\Phi}} = \det[\Delta_{q-k,j}]_{1\leq j,k\leq q} = (\det \Psi)^q \det \left[(-1)^{j-1}\frac{\Delta_{k-1,j}}{\det \Psi}\right]_{1\leq j,k\leq q}$$
$$= (\det \Psi)^{q-1}.$$

So, $(\widetilde{\phi}_j)_j$ is a basis of solutions of (5.21) if and only if $(\psi_j)_j$ is a basis of solutions of (5.2). $\square$

To obtain Proposition 5.4, we exploit the freedom to choose a normalization of $\phi_j(x,t)$ depending on x. As we shall see, an appropriate choice is:

$$\phi_j(x,t) = (-1)^j \frac{\widetilde{\phi}_j(x,t)}{\det \mathbf{\Psi}(x)} = (-1)^j \frac{\Delta_{q-1,j}(x,t)}{\Delta(x)} = (-1)^{q-1}\mathbf{\Psi}^{-1}_{j,q-1}(x,t),$$

and we define the matrix:

$$\mathbf{\Phi}(x,t) = \begin{pmatrix} \phi_1(x,t) & \cdots & \phi_q(x,t) \\ -\hbar\,\partial_t\phi_1(x,t) & \cdots & -\hbar\,\partial_t\phi_q(x,t) \\ \vdots & & \vdots \\ (-\hbar\,\partial_t)^q\phi_1(x,t) & \cdots & (-\hbar\partial_t)^q\phi_q(x,t) \end{pmatrix}.$$

It remains to show that:

$$\mathbf{C}_{i,j}(x,t) = \sum_{k=1}^{q}[(\hbar\,\partial_t)^{i-1}\phi_k(x,t)]\,[(\hbar\,\partial_t)^{j-1}\psi_k(x,t)], \quad i,j \in [\![1,q]\!] \tag{5.24}$$

does not depend on x. For this purpose, we first observe:

$$\forall j \in [\![1,q]\!], \quad \mathbf{C}_{1,j} = \sum_{k=1}^{q}(-1)^q\mathbf{\Psi}^{-1}_{k,q-1}\mathbf{\Psi}_{i-1,k} = (-1)^{q-1}\delta_{i,q}. \tag{5.25}$$

Besides, from the very structure of (5.24), we observe:

$$\forall i,j \in [\![1,q-1]\!], \quad \hbar\,\partial_t\mathbf{C}_{i,j} = \mathbf{C}_{i,j+1} - \mathbf{C}_{i+1,j},$$

and when $j = q$, we use the fact that ψ_j is solution to the system (5.2) to write:

$$\forall i \in [\![1,q-1]\!], \quad \hbar\,\partial_t\mathbf{C}_{i,q-1} = -\mathbf{C}_{i+1,q-1} - \sum_{l=0}^{q-2}(u_l(t) - \delta_{l,0}x)\mathbf{C}_{i,l+1}. \tag{5.26}$$

Considering (5.25) as an initial condition for (5.26), we obtain by recursion that $\mathbf{C}_{i,j} = 0$ whenever $i + j \leq q$. Hence, $\sum_{l=0}^{q-2}\delta_{l,0}\mathbf{C}_{i,l+1}$ always vanish. This implies that the recursion relation (5.26) does not depend on x. Since $\mathbf{C}_{i,j}$ is determined uniquely from (5.26) with the constant initial condition (5.25), we conclude that $\mathbf{C}$ does not depend on x, which completes the proof of Proposition 5.4.

5.7. The $\hbar^{n-2}$ Property

We are going to construct a suitable insertion operator allowing to prove the $\hbar^{n-2}$ property. Unfortunately, this construction is rather technical, and we have not found an easier route.

5.7.1. A Useful Decomposition. The very special form (5.8) of the matrix $\mathbf{M}(x,t)$ in (p,q) models allows a decomposition:

Lemma 5.7. $\mathbf{P}(\overset{a}{x}) = \mathbf{A}(\overset{a}{x}) + x\,\mathbf{B}(\overset{a}{x}) + \hbar\,\mathbf{C}(\overset{a}{x})$ *where* $\mathbf{A}$ *and* $\mathbf{B}$ *do not depend on* $\hbar$ *and have the properties:*

$$[\mathbf{A}(\overset{a}{x},t),\mathbf{A}(\overset{b}{y},t)] = 0 \tag{5.27}$$

$$[\mathbf{B}(\overset{a}{x},t),\mathbf{B}(\overset{b}{y},t)] = 0, \tag{5.28}$$

$$[\mathbf{A}(\overset{a}{x},t),\mathbf{B}(\overset{b}{y},t)] = [\mathbf{A}(\overset{b}{y},t),\mathbf{B}(\overset{a}{x},t)], \tag{5.29}$$

and $\mathbf{C}$ *depends on* $\hbar$, *is* $O(1)$, *and is expressible in terms of matrix elements of* $\mathbf{P}(x,t)$ *and their time derivatives.*

Proof. The projectors $\mathbf{P}$ satisfy the evolution equation:

$$\hbar\,\partial_t\mathbf{P}(\overset{a}{x},t) = [\mathbf{M}(x,t),\mathbf{P}(\overset{a}{x},t)]. \tag{5.30}$$

We have:

$$M_{l,m}(x,t) = \delta_{m,l+1} + \delta_{l,q}\Big(x\,\delta_{m,1} - u_{m-1}(t)\Big),$$

hence:

$$\big(\mathbf{M}(x,t)\,\mathbf{P}(\overset{a}{x},t)\big)_{l,n} = P_{l+1,n}(\overset{a}{x},t) + \delta_{l,q}\Big(x\,P_{1,n}(\overset{a}{x},t) - \sum_{m=1}^{q} u_{m-1}\,P_{m,n}(\overset{a}{x},t)\Big),$$

$$\big(\mathbf{P}(\overset{a}{x},t)\,\mathbf{M}(x,t)\big)_{l,n} = P_{l,n-1}(\overset{a}{x},t) + \big(x\,\delta_{n,1} - u_{n-1}(t)\big)P_{l,q}(\overset{a}{x},t).$$

Omitting to precise the variables, (5.30) implies the relations:

$$\begin{aligned}
1 \leq l < d \quad & \hbar\,\partial_t P_{l,1} = P_{l+1,1} - (x-u_0)P_{l,q},\\
1 \leq l < d,\ 1 < n \leq d \quad & \hbar\,\partial_t P_{q,n} = P_{l+1,n} - P_{l,n-1} + u_{n-1}\,P_{l,q},\\
1 \leq n \leq d \quad & \hbar\,\partial_t P_{q,n} = x\,P_{1,n} - \sum_{l=1}^{d} u_{l-1}\,P_{l,n} - P_{q,n-1}\\
& \qquad - (x\,\delta_{n,1} - u_{n-1})P_{q,q}.
\end{aligned}$$

These relations give an expression of the elements $P_{l,n}$ in terms of the elements $P_{k,q}$ of the last column and their time derivatives. If we introduce:

$$\Gamma_1 = \Gamma_q = 0, \quad \Gamma_k = P_{k,q} \text{ if } k \in [\![2,q-1]\!],$$

we find for elements above and on the diagonal:

$$1 \leq l \leq n \leq d, \quad P_{l,n} = \Gamma_{q+l-n} + \sum_{m=n}^{q-1} u_m\,\Gamma_{m+l-n} - \sum_{m=0}^{q-n-1} \hbar\,\partial_t P_{l+m,n+m+1},$$

and for elements below the diagonal:

$$1 \leq n < l \leq q, \quad P_{l,n} = x\,\Gamma_{l-n} - \sum_{m=0}^{n-1} u_m\,\Gamma_{m+l-n} + \sum_{m=0}^{n-1} \hbar\,\partial_t P_{l-m-1,n-m}.$$

Consequently, we may write:

$$\mathbf{P} = \mathbf{A} + x\,\mathbf{B} - \hbar\,\mathbf{C},$$

with:

$$\begin{aligned}
A_{l,n} &= \Gamma_{q+l-n} + \sum_{m=n}^{q-1} u_m\,\Gamma_{m+l-n}, 1 \leq l \leq n \leq q \\
A_{l,n} &= -\sum_{m=0}^{n-1} u_m\,\Gamma_{m+l-n}, \quad 1 \leq n < l \leq q \\
B_{l,n} &= \Gamma_{l-n}, 1 \leq l, n \leq d \\
C_{l,n} &= -\sum_{m=0}^{q-n-1} \partial_t P_{l+m,n+m+1}, \quad 1 \leq l \leq n \leq d \\
C_{l,n} &= \sum_{m=0}^{n-1} \partial_t P_{l-m-1,n-m} \quad 1 \leq n < l \leq d.
\end{aligned} \tag{5.31}$$

We now prove the commutation relations. We claim that, for any $\theta \in \mathbb{C}$ generic, the matrix

$$\mathbf{G}_\theta(\overset{a}{x}, t) = \mathbf{A}(\overset{a}{x}, t) + \theta\,\mathbf{B}(\overset{a}{x}, t)$$

has a basis of eigenvectors which is independent of x and a. This will imply:

$$[\mathbf{G}_\theta(\overset{a}{x}, t), \mathbf{G}_\theta(\overset{b}{y}, t)] = 0,$$

from which the relations (5.27)–(5.29) can be deduced by identification of the coefficients of θ. Let $(\zeta_i)_{1 \leq i \leq q}$ be the roots of:

$$X^q + \sum_{m=0}^{q-1} u_m\,X^m = \theta.$$

For generic θ, the roots are simple, so that the column vectors $\mathbf{v}_i(z) = (\zeta_i^j)_{0 \leq j \leq q-1}$ form a basis of $\mathbb{C}^q$. Let us set:

$$\lambda_i = (\mathbf{G}_\theta \mathbf{v}_i)_1 = \sum_{m=1}^{q} A_{1,m}\,\zeta_i^m.$$

Considering the second line:

$$(\mathbf{G}_\theta \mathbf{v}_i - \lambda_i\,\mathbf{v}_i)_2 = \theta\,B_{2,1} + \sum_{m=1}^{q} A_{2,m}\,\zeta_i^{m-1} - \sum_{m=1}^{q} A_{1,m}\,\zeta_i^m,$$

but since $B_{2,1} = \Gamma_1$, $A_{2,1} = -u_0\Gamma_1$ and $A_{1,d} = \Gamma_1$, using the polynomial equation (5.32) for ζ_i, it must vanish. If we proceed to the kth line, we have:

$$\begin{aligned}
&(\mathbf{G}_\theta \mathbf{v}_i - \lambda_i\, \mathbf{v}_i)_k \\
&\quad = \theta \sum_{m=1}^{k-1} B_{k,m}\, \zeta_i^{m-1} + \sum_{m=1}^{q} A_{k,m}\, \zeta_i^{m-1} - \sum_{m=q-k+2}^{q} A_{1,m}\, \zeta_i^{m+k-2} \\
&\quad = \sum_{m=1}^{k-1} (\theta\, B_{k,m} + A_{k,m}) \zeta_i^{m-1} + \sum_{m=k}^{q} (A_{k,m} - A_{1,m-k+1}) z^{m-1} \\
&\qquad - \sum_{m=q-k+2}^{q} A_{1,m}\, \zeta_i^{m+k-2}.
\end{aligned}$$

Using:

$$\begin{aligned}
1 \le m < k \le q \quad & \theta\, B_{k,m} + A_{k,m} = \theta\, \Gamma_{k-m} - \sum_{n=0}^{m-1} u_n\, \Gamma_{k-m+n}, \\
1 \le k \le m \le q \quad & A_{k,m} - A_{1,m-k+1} = -\sum_{n=1}^{k-1} u_{m+n-k}\, \Gamma_n, \\
1 \le m \le q \quad & A_{1,m} = \Gamma_{d-m+1} + \sum_{n=1}^{q-m} u_{m+n-1}\, \Gamma_n,
\end{aligned}$$

we may collect the terms relative to a given Γ_m and we obtain:

$$(\mathbf{G}_\theta \mathbf{v}_i - \lambda_i\, \mathbf{v}_i)_k = \left(\sum_{n=1}^{k-1} \Gamma_n\, \zeta_i^{k-n-1} \right) \left(\theta - \sum_{m=0}^{q-1} u_m\, \zeta_i^m - \zeta_i^q \right) = 0.$$

This concludes the proof. □

5.7.2. Main Argument of the Proof. Thanks to the decomposition of Lemma 5.7, we can prove:

Corollary 5.1. *If we choose* $\mathbf{U}(\overset{a}{y}) = \mathbf{B}(\overset{a}{y}, t) + \hbar\, \mathbf{V}(\overset{a}{y}, t)$ *to define an insertion operator, then*

$$\delta_y^a \mathbf{P}(\overset{b}{x}, t) \in O(\hbar),$$

and is expressible in terms of $\mathbf{V}(\overset{a}{y}, t)$*, matrix elements of* $\mathbf{P}(x, t)$ *and their time derivatives.*

Proof. From the second equation in Lemma 2.2, we have:

$$\begin{aligned}
\delta_y^a \mathbf{P}(\overset{a}{x}) = {} & \frac{1}{x-y} \left[\mathbf{A}(\overset{a}{y}, t) + x\, \mathbf{B}(\overset{a}{y}, t) + \hbar\, \mathbf{C}(\overset{a}{y}, t),\ \mathbf{A}(\overset{b}{x}, t) + x\, \mathbf{B}(\overset{b}{x}, t) + \hbar\, \mathbf{C}(\overset{b}{x}, t) \right] \\
& + \hbar\, [\mathbf{V}(\overset{a}{y}, t), \mathbf{P}(\overset{b}{x}, t)],
\end{aligned}$$

and using the commutation relations (5.27)–(5.29), we obtain:

$$\begin{aligned}\delta^a_y \mathbf{P}(\overset{a}{x},t) &= \frac{\hbar}{x-y}\big\{-[\mathbf{P}(\overset{b}{x},t),\mathbf{C}(\overset{a}{y},t)] + [\mathbf{P}(\overset{a}{y},t),\mathbf{C}(\overset{b}{x},t)]\big\} + \hbar\,[\mathbf{B}(\overset{a}{y},t),\mathbf{C}(\overset{b}{x},t)] \\ &\quad + \frac{\hbar^2}{(x-y)^2}\,[\mathbf{C}(\overset{b}{x},t),\mathbf{C}(\overset{a}{y},t)] + \hbar\,[\mathbf{V}(\overset{a}{y},t),\mathbf{P}(\overset{a}{x},t)].\end{aligned}$$

□

Corollary 5.2. *If δ^a_y is a compatible insertion operator such that $\mathbf{U}(\overset{a}{y},t) = \mathbf{B}(\overset{a}{y},t) + \hbar\,\mathbf{V}(\overset{a}{y})$ and $\mathbf{V}$ depends on $\hbar$, is of order 1 and is expressible in terms of matrix elements of $\mathbf{P}(\overset{a}{x})$ and their time derivatives, then:*

$$\delta^{a_1}_{y_1}\ldots\delta^{a_k}_{y_k}\mathbf{P}(\overset{a}{x}) \in O(\hbar^k),$$

and:

$$\mathcal{W}_n(\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n}) \in O(\hbar^{n-2}).$$

Proof. If δ^a_y commutes with ∂_t, we also have for any $k \geq 0$:

$$\delta^a_y \partial^k_t \mathbf{P}(\overset{b}{x}) \in O(\hbar). \tag{5.32}$$

Since δ^a_y itself is expressible in terms of elements of the matrices $\mathbf{P}$ and their time derivatives, we can apply repeatedly (5.32) to show that each application of the insertion operator to $\mathbf{P}(\overset{a}{x})$ increases at least by one the order in $\hbar$. Now, starting from the expression given in Proposition 2.1 for $\mathcal{W}_2$ and by successive applications of the insertion operator to compute $\mathcal{W}_n$ according to the last equation of Lemma 2.2, we obtain that $\mathcal{W}_n \in O(\hbar^{n-2})$. □

5.7.3. Existence of a Compatible Insertion Operator. It is possible to construct explicitly an insertion operator which commutes with ∂_t:

Proposition 5.6. *The choices:*

$$U(\overset{a}{x},t)_{k,m} = \sum_{l=0}^{k-m-1}\binom{m+l-1}{l}(\hbar\,\partial_t)^l P_{k-m-l,q}(\overset{a}{x},t) = B_{k,m}(\overset{a}{x},t) + O(\hbar),$$

$$\delta^a_y u_k(t) = P_{1,k}(\overset{a}{y},t) - \delta_{k,1}P_{q,q}(\overset{a}{y},t) + \sum_{m=k}^{q} u_m(t)\,U_{m+1,k}(\overset{a}{y},t),$$

where we used the convention $u_q(t) = -1$, define the unique insertion operator which commutes with ∂_t.

We remark in this case that, since we have a second differential system $\partial_t - \mathbf{M}$, we also need to specify how the insertion operator acts on the matrix elements[5] of $\mathbf{M}$, i.e. on $u_k(t)$. As a matter of fact, the second formula in Lemma 5.6 is actually a consequence of the first formula and the commutativity of the insertion operator with ∂_t.

[5] The action of v_l is already given since we know from Lemma 2.2 how δ^a_y act on $\mathbf{L}$.

Proof. The commutativity of δ_y^a and ∂_t is equivalent to:

$$\delta_y^a \partial_t \mathbf{\Psi}(x,t) = \partial_t \delta_y^a \mathbf{\Psi}(x,t),$$

that is:

$$\delta_y^a \mathbf{M}(x,t) = [\mathbf{U}(\overset{a}{y},t), \mathbf{M}(x,t)] + \hbar\, \partial_t \mathbf{U}(\overset{a}{y},t) + \left[\mathbf{P}(\overset{a}{y},t), \frac{\mathbf{M}(x,t)-\mathbf{M}(y,t)}{x-y}\right]. \tag{5.33}$$

With the expression (5.8) of $\mathbf{M}(x,t)$ for (p,q) models, we compute:

$$\frac{\mathbf{M}(x,t)-\mathbf{M}(y,t)}{x-y} = \mathbf{E}_{q,1}$$
$$\delta_y^a M_{k,m}(x,t) = -\,\delta_{k,q}\,\delta_y^a u_{m-1}(t).$$

The equation (5.33) gives a strong constraints upon the matrix $\mathbf{U}(\overset{a}{y},t)$. For instance, it cannot be zero since:

$$[\mathbf{E}_{q,1}, \mathbf{P}(\overset{a}{y},t)]_{k,m} = \delta_{k,q} \mathbf{P}_{1,m}(\overset{a}{y},t) - \delta_{m,1} \mathbf{P}_{k,q}(\overset{a}{y},t).$$

We compute:

$$\begin{aligned}[\mathbf{U}(\overset{a}{y},t), \mathbf{M}(x,t)]_{k,m} &= U_{k,m-1}(\overset{a}{y},t) + \Big(x\,\delta_{m,1} - u_{m-1}(t)\Big) U_{k,q}(\overset{a}{y},t) \\ &\quad - U_{k+1,m}(\overset{a}{y},t) + \delta_{m,q}\left(x\,U_{1,m}(\overset{a}{y},t) - \sum_{l=1}^{q} u_{l-1}(t)\, U_{l,m}(\overset{a}{y},t)\right).\end{aligned}$$

The condition (5.33) is an affine function of x. With the choice $U_{k,q} = U_{1,m} = 0$ for any $k,m \in [\![1,q]\!]$, the coefficient of x vanishes. The remaining constraint reads:

$$\begin{aligned}-\delta_{k,q}\delta_y^a u_{m-1}(t) &= U_{k,m-1}(\overset{a}{y},t) - U_{k+1,m}(\overset{a}{y},t) - \delta_{k,q} \sum_{l=1}^{q} u_{l-1}(t)\, U_{l,m}(\overset{a}{y},t) \\ &\quad + \hbar\, \partial_t U_{k,m}(\overset{a}{y},t) - \delta_{k,q} P_{1,m}(\overset{a}{y},t) + \delta_{m,1} P_{k,q}(\overset{a}{y},t).\end{aligned}$$

Omitting the dependence in y, a and t, we have for $k \neq q$:

$$U_{k+1,m} = U_{k,m-1} + \delta_{m,1} P_{k,q} - \hbar\, \partial_t U_{k,m}. \tag{5.34}$$

The solution at leading order in $\hbar$ is:

$$U_{k,m} = \begin{cases} P_{k-m,q} + O(\hbar) & m > k \\ O(\hbar) & m \leq k \end{cases},$$

which coincides with the definition of the matrix $\mathbf{B}$ in (5.31). Equation (5.34) can be solved recursively, and we find that its unique solution is given by the first equation of Proposition 5.6. To define completely an insertion operator, it remains to specify how it acts on the functions $u_k(t)$. The commutativity condition prescribes the second equation of Proposition 5.6. Finally, we have to check the last condition in our definition of an insertion operator:

Lemma 5.8. *For any $a,b \in [\![1,q]\!]$, we have $[\delta_x^a, \delta_y^b] = 0$. This is equivalent to:*

$$\delta_x^a \mathbf{U}(\overset{b}{y},t) - \delta_y^b \mathbf{U}(\overset{a}{x},t) + [\mathbf{U}(\overset{a}{x},t), \mathbf{U}(\overset{b}{y},t)] = 0. \tag{5.35}$$

This can be established by direct but long computations, that we give in "Appendix C". Remark that since $\mathbf{U}(\overset{a}{x},t)=\mathbf{B}(\overset{a}{x},t)+O(\hbar)$ and $\delta_x^a\mathbf{U}(\overset{b}{y})\in O(\hbar)$ owing to Lemma 5.2, the commutation relation (5.28) implies (5.35) at leading order. □

5.8. The Pole Property

We need to prove that ω_n^g has poles only at ramification points, in particular, no pole at ∞ or at double zeroes. For this purpose, we will use the observations of Sect. 5.5.

5.8.1. Double Points.

Lemma 5.9. *In the qth reduction of KP, for any n,g, ω_n^g are regular at preimages in $\mathcal{S}^{[0]}$ of double points.*

Proof. We remind that this property is not obvious because equations (3.8) and (3.9), which allow the computation of the WKB expansion of $\mathbf{\Psi}(x,t)=\mathbf{V}\widehat{\mathbf{\Psi}}\,\mathrm{e}^{\mathbf{S}/\hbar}\,\mathbf{C}$, may have a pole $1/(\lambda_a^{[0]}(x,t)-\lambda_b^{[0]}(x,t))$, i.e. at the double points. However, this analysis was performed for the differential equation with respect to x. But now, we have a second differential equation

$$\hbar\,\partial_t\mathbf{\Psi}(x,t)=\mathbf{M}(x,t)\mathbf{\Psi}(x,t), \tag{5.36}$$

from which we can perform a similar WKB analysis. One notices that solving (5.19) for $\widehat{\mathbf{\Psi}}(x,t)=\mathbf{1}_q+\sum_{k\geq 1}\hbar^k\,\widehat{\mathbf{\Psi}}^{[k]}(x,t)$ recursively, the only denominators are of the form $1/(z_a-z_b)$, and thus the only poles that are produced are when $x\to\alpha$ such that $z^a(\alpha)=z^b(\alpha)$ for $a\neq b$, i.e. when z goes to a ramification point. The conclusion is that poles at double points in x (and thus at preimages of double points in $z\in\mathcal{S}^{[0]}$) do not occur. □

5.8.2. Behavior at $z\to\infty$.

Lemma 5.10. *The qth reduction of KP satisfies Assumption 3.1.*

Proof. We now expand $\mathbf{\Psi}$ at large x as

$$\mathbf{\Psi}=\mathbf{V}\widehat{\mathbf{\Psi}}\,\mathrm{e}^{\mathbf{S}/\hbar}\,\mathbf{C},$$

where:

$$\partial_x S_i=\Lambda_i^{[0]}(x)=Y(z_i),\quad \mathbf{V}^{-1}\partial_x\mathbf{V}=O(x^{-1/q}),\quad \widehat{\mathbf{\Psi}}=\mathbf{1}_q+O(x^{-1/q}).$$

Moreover, as in Sect. 3.2, the equation $\hbar\,\partial_x\mathbf{\Psi}=\mathbf{L}\,\mathbf{\Psi}$ implies that there is also a large x expansion of the form:

$$\mathbf{\Psi}=\widetilde{\mathbf{V}}\widetilde{\mathbf{\Psi}}\,\mathrm{e}^{\widetilde{\mathbf{S}}/\hbar}\,\mathbf{C},$$

where $\partial_x\widetilde{\mathbf{S}}=\mathrm{Diag}(\Lambda_i(x))$, and $\widetilde{\mathbf{V}}^{-1}\partial_x\widetilde{\mathbf{V}}=O(x^{-1/q})$ and $\widetilde{\mathbf{\Psi}}=\mathbf{1}_q+O(x^{-1/q})$. This implies that:

$$\mathbf{\Lambda}=\mathbf{\Lambda}^{[0]}+O(x^{-1/q}),$$

and thus the pole property of Assumption 3.1 is satisfied. This implies that, for any $g,n\neq(1,0)$, the $\omega_n^{(g)}(z_1,\ldots,z_n)$ are regular when $z_i=\infty$. □

5.9. Tau Function

It is well known that for (p,q) model we have [28]:

Theorem 5.1.

$$\hbar^2\,\partial_t^2 \ln \mathcal{T}(t) = u(t). \tag{5.37}$$

5.10. Application of the Topological Recursion

Theorem 3.1, and in particular Corollary 3.6 (since our spectral curve has genus 0), implies that the correlators have the expansion:

$$W_n(\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n})\mathrm{d}x_1,\ldots \mathrm{d}x_n = \sum_{g\geq 0} \hbar^{2g-2+n}\,\omega_n^{(g)}(z^{a_1}(x_1),\ldots,z^{a_n}(x_n)),$$

where the $\omega_n^{(g)}(z_1,\ldots,z_n)$ are computed by the topological recursion. The initial data are:

$$\omega_1^{(0)} = -Y(z)\mathrm{d}X(z), \qquad \omega_2^{(0)}(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}.$$

To justify the second equation, we know from Corollary 3.5 that $\omega_2^{(0)} \in \mathcal{B}(\mathcal{S}^{[0]})$ and there is a unique such object on a genus 0 curve, which can be written as in the second equation in any uniformization variable z.

In particular, we can retrieve the expansion of the Tau function with Corollary 4.2.

$$\ln \mathcal{T} = \sum_{g\geq 0} \hbar^{2g-2}\,F^{(g)},$$

Since $Y'\dot{X} - X'\dot{Y} = 1$, we find that $\partial_t Y|_{X(z)} = -\mathrm{d}z/\mathrm{d}X$, hence:

$$\partial_t F^{(g)} = \operatorname*{Res}_{z\to\infty} z\,\omega_1^{(g)}(z). \tag{5.38}$$

Remember that $\mathcal{T}$ is defined up to a multiplicative constant, so the constant of integration to get $F^{(g)}$ from (5.38) is irrelevant here. A direct integration can be done explicitly for $F^{(0)}$ [32] and $F^{(1)}$ [40], but the formulae are complicated to state. In simple examples, it is more efficient to rely on (5.38).

Case of the Homogeneous Solution. For the homogeneous solution, we have

$$X(z) = (u^{[0]})^{q/2}\,f(\zeta),\quad Y(z) = (u^{[0]})^{p/2}\,g(\zeta),\quad \zeta = z\,(u^{[0]})^{-1/2}, \tag{5.39}$$

and where $u^{[0]}(t) = (t/\rho)^{\frac{2}{p+q-1}}$. By homogeneity of the topological recursion (see [41,43]) this implies:

$$\begin{aligned}\omega_n^{(g)}(z_1,\ldots,z_n) &= (u^{[0]})^{(2-2g-n)(p+q)/2}\;\check{\omega}_n^{(g)}(\zeta_1,\ldots,\zeta_n)\\ &= (t/\rho)^{(2-2g-n)(p+q)/(p+q-1)}\;\check{\omega}_n^{(g)}(\zeta_1,\ldots,\zeta_n).\end{aligned}$$

where $\check{\omega}_n^{(g)}$ is computed as if $u^{[0]}$ was equal to 1. In particular for $n=0$

$$\forall g\neq 1,\quad F^{(g)}(t) = t^{(2-2g)(p+q)/(p+q-1)}\;F^{(g)}(1).$$

For $F^{(1)}$, we have:

$$\begin{aligned}\partial_t F^{(1)} &= \operatorname*{Res}_{z\to\infty} z\,\omega_1^{(g)}(\zeta)\\ &= (u^{[0]})^{-(p+q-1)/2}\big\{\operatorname*{Res}_{\zeta\to\infty}\zeta\,\omega_1^{(1)}(\zeta)\big\} = \frac{\rho}{t}\big\{\operatorname*{Res}_{\zeta\to\infty}\zeta\,\omega_1^{(1)}(\zeta)\big\},\end{aligned}$$

therefore:

$$F^{(1)}(t) = c\,\ln t,\quad c = \rho\operatorname*{Res}_{\zeta\to\infty}\zeta\,\omega_1^{(1)}(\zeta).$$

where the arbitrary integration constant was set to 0 for $t = 1$.

For the homogeneous solution, we observe that the $\hbar \to 0$ expansion coincides with a $t \to \infty$ expansion:

$$\mathcal{T} = \exp\left(\sum_{g\geq 0}\hbar^{2g-2}F^{(g)}(t)\right) = t^c\exp\left(\sum_{g\geq 0}(\hbar\,t^{-(p+q)/(p+q-1)})^{2g-2}\,F_g(1)\right).$$

We see that $\hbar$ can be absorbed in a redefinition of the variable t. We also have:

$$u(t) = \hbar^2\,\partial_t^2\ln\mathcal{T} = t^{\frac{2}{p+q-1}}\sum_{g\geq 0}(\hbar\,t^{-(p+q)/(p+q-1)})^{2g}\,u^{\{g\}}(1),$$

where

$$u^{\{g\}}(1) = \frac{(p+q)(2-2g)\big((p+q)(2-2g)-1\big)}{(p+q-1)^2}\,F_g(1).$$

In particular, we see that

$$u^{\{0\}}(1) = \rho^{-2/(p+q-1)},\quad F^{(0)}(1) = \frac{1}{2}\,\frac{(p+q-1)^2}{(p+q)(p+q+1)}\,\rho^{-2/(p+q-1)}.$$

6. Examples

The qth reductions of KP, and in particular the (p,q) models describe universal behavior—provably or conjecturally—in statistical physics, random matrix theory, and integrable systems. For those reasons, many of them have received names referring to the problems where they appear. The $(1,2)$ model is known to appear when studying the double scaling limit of random matrices at a generic edge of the spectral density, and is related to the Airy process [59]. The $(3,2)$ model was shown, first in physics [30,58], then rigorously [51], to describe generating series of random maps with generic critical weights, and thus was called "pure gravity". The $(4,3)$ (resp. the $(6,5)$ model) is expected to describe the generating series of random maps carrying an Ising model (resp. 3-Potts model) with non-generic critical weights, and in fact, the theory we developed allows a proof of those conjectures [16].

All the (p,q) models are conjectured to describe the double-scaling limit in random matrices around an edge a where the spectral density behaves like $|x-a|^{p/q}$. This is also relevant for systems of vicious walkers via Dyson Brownian motion [34], and this is related to $2d$ quantum gravity for reasons dating back to [24]. This has been proven so far in a handful of cases (see e.g. [55] and references therein), but mainly for $q = 2$ cases—which correspond to the

Gelfand–Dikii hierarchies [46]. This conjecture is based on an ansatz [58] for the convergence of operators $\hat{P}$ and $\hat{Q}$—interpreted as differentiation and multiplication in the vector space generating by orthogonal polynomials—which has not been justified rigorously so far. Our methods do not provide a proof that double-scaling limits exist. However, once this existence is granted and it is characterized in terms of a Lax pair, it can actually prove that the semiclassical expansion of the limit laws is computed by the topological recursion. Moreover, if the semiclassical spectral curve of the Lax pair can be identified with a blow-up of the large N spectral curve of the matrix model when parameters become critical, it shows—combining the results of [41] and [20]—that the semiclassical expansion of the double-scaling limit does coincide with a limit of coefficients in an off-critical $1/N$ expansion when approaching criticality. This crossover is expected and we are able to justify it only relying on loop equations, i.e. by algebraic methods. We refer to [11,17] for applications relying on those ideas.

In the remaining of the text, we illustrate some (p, q) models, by describing the non-linear PDEs they generate, the spectral curves and the first few coefficients in the $\hbar \to 0$ expansion of the correlators and of the Tau function.

6.1. $(p, q) = (3, 2)$: Pure Gravity

Here, we chose $q = 2$ and $p = 3$

$$Q = (\hbar\partial_t)^2 - 2u, \qquad P = (\hbar\partial_t)^3 - 3u\,\hbar\partial_t - \frac{3}{2}\hbar\dot{u} + v.$$

The string equation $[P, Q] = \hbar$ implies that $\dot{v} = 0$ and the Painlevé I equation for $u(t)$:

$$-\frac{1}{2}\,\hbar^2\,\ddot{u} + 3u^2 = t, \quad v = t_1.$$

It has the $\hbar$ expansion:

$$u = \sqrt{\frac{t}{3}} - \frac{\hbar^2}{48}\,t^{-2} - \frac{49\,\hbar^4}{2^9 3^{3/2}}\,t^{-9/2} - \frac{5^2\,7^2\,\hbar^6}{2^{11} 3^2}\,t^{-7} + O(\hbar^8). \tag{6.1}$$

The Lax pair is given by

$$\mathbf{M}(x, t) = \begin{pmatrix} 0 & 1 \\ x + 2u & 0 \end{pmatrix},$$

and

$$\mathbf{L}(x, t) = \begin{pmatrix} \frac{1}{2}\hbar\,\dot{u}(t) - t_1 & x - u \\ (x - u)(x + 2u) + \frac{1}{2}\,\hbar^2\ddot{u} & -\frac{1}{2}\hbar\,\dot{u} - t_1 \end{pmatrix}.$$

The spectral curve is:

$$\det(y\,\mathbf{1}_2 - \mathbf{L}(x, t)) = (y + t_1)^2 - (x + 2u)(x - u)^2 - \frac{1}{2}\,\hbar^2\,\ddot{u}\,(x - u) - \frac{1}{4}\,\hbar^2\dot{u}^2.$$

To leading order in $\hbar$, the eigenvalues of $\mathbf{L}^{[0]}(x, t)$ are thus:

$$y = -t_1 \pm (x - u^{[0]})\sqrt{x + 2u^{[0]}}, \tag{6.2}$$

and are parametrized by:

$$\begin{cases} X(z) = z^2 - 2u^{[0]} \\ Y(z) = z^3 - 3u^{[0]}\, z - t_1 \end{cases} \quad \text{with} \quad u^{[0]} = \sqrt{\frac{t}{3}}.$$

Notice that with $\zeta = (u^{[0]})^{-1/2} z$, we recover the Chebyshev polynomials:

$$\begin{cases} X(z) = u^{[0]}\,(\zeta^2 - 2) = u^{[0]}\, T_2(\zeta) \\ Y(z) = (u^{[0]})^{3/2}\,(\zeta^3 - 3\zeta) - t_1 = (u^{[0]})^{3/2}\, T_3(\zeta) - t_1 \end{cases}.$$

Applying the topological recursion gives the coefficients of expansion of the correlators:

$$\begin{aligned}
\omega_1^{(0)}(z) &= -Y(z)\mathrm{d}X(z) = -2\,(z^4 - 3u^{[0]}\, z^2 - t_1 z)\,\mathrm{d}z, \\
\omega_2^{(0)}(z_1, z_2) &= \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{(z_1 - z_2)^2}, \\
\omega_3^{(0)}(z_1, z_2, z_3) &= \frac{-1}{6u^{[0]}}\ \frac{\mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3}{z_1^2\, z_2^2\, z_3^2}, \\
\omega_4^{(0)}(z_1, \dots, z_4) &= \frac{1}{36\,(u^{[0]})^3}\ \frac{\mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4}{z_1^2\, z_2^2\, z_3^2\, z_4^2} \left(1 + \sum_{i=1}^{4} \frac{3u^{[0]}}{z_i^2}\right),
\end{aligned}$$

$$\begin{aligned}
\omega_5^{(0)}(z_1, \dots, z_5) = {} & \frac{-1}{72\,(u^{[0]})^5} \left[\prod_{i=1}^{5} \frac{\mathrm{d}z_i}{z_i^2}\right] \\
& \times \left(1 + \sum_{i=1}^{5} \frac{3u^{[0]}}{z_i^2} + \sum_{i=1}^{5} \frac{5\,(u^{[0]})^2}{z_i^4} + \sum_{i<j} \frac{6(u^{[0]})^2}{z_i^2\, z_j^2}\right),
\end{aligned}$$

$$\begin{aligned}
\omega_1^{(1)}(z) &= -\ \frac{1}{144\,(u^{[0]})^2}\ \frac{\mathrm{d}z}{z^4}\ (z^2 + 3u^{[0]}), \\
\omega_2^{(1)}(z_1, z_2) &= \frac{1}{864\,(u^{[0]})^4}\ \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{z_1^2\, z_2^2} \Big(2 + 6u^{[0]}(z_1^{-2} + z_2^{-2}) + 9(u^{[0]})^2\, z_1^{-2}\, z_2^{-2} \\
&\quad + 15(u^{[0]})^2(z_1^{-4} + z_2^{-4})\Big), \\
\omega_1^{(2)}(z) &= -\ \frac{7}{2^{10}3^5\,(u^{[0]})^7}\ \frac{\mathrm{d}z}{z^{10}} \Big(4z^8 + 12u^{[0]}\, z^6 \\
&\quad + 36\,(u^{[0]})^2\, z^4 + 87\,(u^{[0]})^3\, z^2 + 135\,(u^{[0]})^4\Big), \\
\omega_1^{(3)}(z) &= -\ \frac{7}{2^{15}3^9\,(u^{[0]})^{12}}\ \frac{\mathrm{d}z}{z^{16}} \Big(1400z^{14} + 4200u^{[0]} z^{12} + 12600(u^{[0]})^2 z^{10} \\
&\quad + 34740(u^{[0]})^3 z^8 + 85860(u^{[0]})^4 z^6 + 181764(u^{[0]})^5 z^4 \\
&\quad + 297297(u^{[0]})^6 z^2 + 289575(u^{[0]})^7\Big).
\end{aligned}$$

The expansion of the Tau function $\ln\mathcal{T} = \sum_{g\geq 0} \hbar^{2g-2} F^{(g)}$ is obtained from:

$$\partial_t F^{(g)} = \operatorname*{Res}_{z\to\infty} z\,\omega_1^{(g)}(z) = 6u^{[0]}\dot{u}^{[0]} \operatorname*{Res}_{z\to\infty} \omega_1^{(g)}(z)$$

and the solution $u = u^{[0]} + \sum_{g\geq 1} \hbar^{2g}\, u^{\{g\}}$ from $u^{\{g\}} = \partial_t^2 F^{(g)}$. We emphasized that $1 = 6u^{[0]}\dot{u}^{[0]}$ to facilitate the integration. That gives:

$$\begin{aligned}
\partial_t F^{(1)} &= \frac{6u^{[0]}\dot{u}^{[0]}}{144\,(u^{[0]})^2} = \frac{\dot{u}^{[0]}}{24\,u^{[0]}},\\
\Rightarrow \quad F^{(1)} &= \frac{\ln u^{[0]}}{24} = \frac{1}{48}\ln(t/3)\\
\Rightarrow \quad u^{\{1\}} &= \frac{-1}{48\,t^2}\\
\partial_t F^{(2)} &= \frac{7\cdot 6u^{[0]}\dot{u}^{[0]}}{2^8 3^5 (u^{[0]})^7} = \frac{7\,\dot{u}^{[0]}}{2^7 3^4 (u^{[0]})^6},\\
\Rightarrow \quad F^{(2)} &= \frac{-7}{2^7 3^4 5\,(u^{[0]})^5} = \frac{-7}{2^7 3^{3/2} 5 t^{5/2}}\\
\Rightarrow \quad u^{\{2\}} &= \frac{-49}{2^9 3^{3/2} t^{9/2}}.\\
\partial_t F^{(3)} &= \frac{7\cdot 1400\cdot 6u^{[0]}\dot{u}^{[0]}}{2^{15} 3^9 (u^{[0]})^{12}} = \frac{5^2 7^2 \dot{u}^{[0]}}{2^{11} 3^8 (u^{[0]})^{11}},\\
\Rightarrow \quad F^{(3)} &= \frac{-5\cdot 7^2}{2^{12} 3^8 (u^{[0]})^{10}} = \frac{-5\cdot 7^2}{2^{12} 3^3 t^5}\\
\Rightarrow \quad u^{\{3\}} &= \frac{-5^2 7^2}{2^{11} 3^2 t^7}.
\end{aligned}$$

These results agree with the direct $\hbar$ expansion of the solution of the Painlevé I equation (6.1).

6.2. $(p, q) = (2, 3)$

Here, we consider pure gravity again, but exchange the role of P and Q, namely we choose $p = 2$ and $q = 3$. This gives the 3×3 Lax pair:

$$\mathbf{M}(x,t) = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ x + \frac{3}{2}\hbar\dot{u} - t_1 & 3u & 0 \end{pmatrix},$$

$$\mathbf{L}(x,t) = \begin{pmatrix} 2u & 0 & -1\\ t_1 - x + \frac{1}{2}\,\hbar\dot{u} & -u & 0\\ \frac{1}{2}\,\hbar^2\ddot{u} & t_1 - x - \frac{1}{2}\,\hbar\dot{u} & -u \end{pmatrix}.$$

The spectral curve is:

$$\det\left(y\,\mathbf{1}_3 - \mathbf{L}(x,t)\right) = y^3 - 2u^2 y - 2u^3 + (x-t_1)^2 + \frac{1}{2}\,\hbar^2\left(y\ddot{u} - \frac{1}{2}\,\dot{u}^2 + u\ddot{u}\right).$$

To leading order the spectral curve is thus:

$$y^3 - 2(u^{[0]})^2 y + (x-t_1)^2 - 2(u^{[0]})^3 = 0,$$

which admits the parametrization:

$$\begin{cases} X(z) = (u^{[0]})^{3/2}\, T_3(\zeta) = z^3 - 3u^{[0]}z \\ Y(z) = -u^{[0]}\, T_2(\zeta) = 2u^{[0]} - z^2 \end{cases} \qquad u^{[0]} = (t/3)^{1/2}.$$

The ramification points are at $\zeta = a_\pm = \pm 1$ and correspond to $X(a_\pm) = \mp 2\,(u^{[0]})^{3/2}$. The local Galois conjugate near $a = \pm 1$ is:

$$\sigma_a(\zeta) = \frac{-1}{2}\,\left(\zeta - a\sqrt{12 - 3\zeta^2}\right).$$

The topological recursion gives (we denote $\zeta = (u^{[0]})^{-1/2}z$) for the expansion of the correlators:

$$\begin{aligned}
\omega_1^{(0)}(z) &= \; -Y(z)\mathrm{d}X(z) = 3\,(u^{[0]})^{5/2}\,(\zeta^2 - 2)(\zeta^2 - 1)\,\mathrm{d}\zeta,\\
\omega_2^{(0)}(z_1, z_2) &= \frac{\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2}{(\zeta_1 - \zeta_2)^2},\\
\omega_3^{(0)}(z_1, z_2, z_3) &= \frac{-\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\,\mathrm{d}\zeta_3}{12\,(u^{[0]})^{5/2}}\left(\sum_{\varepsilon=\pm 1}\frac{1}{(\zeta_1+\varepsilon)^2(\zeta_2+\varepsilon)^2(\zeta_3+\varepsilon)^2}\right),\\
\omega_1^{(1)}(z) &= \frac{-\mathrm{d}\zeta}{288\,(u^{[0]})^{5/2}}\left(\sum_{\varepsilon=\pm 1}\frac{5 + 3\varepsilon\zeta + \zeta^2}{(\zeta+\varepsilon)^4}\right),\\
\omega_1^{(2)}(z) &= \frac{-\mathrm{d}\zeta}{2^{19}3^5\,(u^{[0]})^{15/2}}\sum_{\varepsilon=\pm 1}\frac{1}{(\zeta+\varepsilon)^{10}}\Big(7168\zeta^8 + 246834\zeta^6\\
&\quad + 1016572\zeta^4 + 1218226\zeta^2 + 369664\\
&\quad + \varepsilon\big(61957\zeta^7 + 602251\zeta^5 + 1271499\zeta^3 + 862277\zeta\big)\Big)
\end{aligned}$$

It is necessary to compute $\omega_2^{(1)}$ to obtain $\omega_1^{(2)}$, but we omitted its expression for conciseness. Then, for the expansion of the Tau function $\mathcal{T}$ and of u, we may use $6u^{[0]}\dot{u}^{[0]} = 1$ and get:

$$\begin{aligned}
\partial_t F^{(1)} &= \frac{6\,(u^{[0]})^{3/2}\dot{u}^{[0]}}{144\,(u^{[0]})^{5/2}} = \frac{\dot{u}^{[0]}}{24\,u^{[0]}},\\
\Rightarrow \quad F^{(1)} &= \frac{\ln u^{[0]}}{24} = \frac{1}{48}\,\ln(t/3),\\
\Rightarrow \quad u^{\{1\}} &= \frac{-1}{48\,t^2}.\\
\partial_t F^{(2)} &= \frac{6\,(u^{[0]})^{3/2}\dot{u}^{[0]}\,7168}{2^{18}3^5\,(u^{[0]})^{15/2}} = \frac{7\,\dot{u}^{[0]}}{2^7 3^4\,(u^{[0]})^6},\\
\Rightarrow \quad F^{(2)} &= \frac{-7}{2^7 3^4 5\,(u^{[0]})^5} = \frac{-7}{2^7 3^{3/2} 5\,t^{5/2}},\\
\Rightarrow \quad u^{\{2\}} &= \frac{-49}{2^9 3^{3/2}\,t^{9/2}}.
\end{aligned}$$

This again perfectly agrees with the direct $\hbar$ expansion of the solution of the Painlevé I equation (6.1), and this agrees with the (3, 2) model, as an illustration of the $(p, q) \to (q, p)$ duality.

6.3. $(p, q) = (4, 3)$: Ising Model

The model is defined by:

$$Q = (\hbar\partial_t)^3 - 3u\,\hbar\partial_t + u_0, \quad P = (\hbar\partial_t)^4 - 4u\,(\hbar\partial_t)^2 + v_1\,\hbar\partial_t + v_0.$$

where u, u_0, v_1, v_0 are functions of t. The string equation implies

$$u_0 = -\frac{3}{2}\,\hbar\,\dot{u} - 3w + t_1,$$

where w is a function of t, and:

$$v_1 = -4w - 4\,\hbar\,\dot{u}, \quad v_0 = 2u^2 - \frac{5}{3}\,\hbar^2\,\ddot{u} - 2\hbar\,\dot{w} + t_2,$$

where w satisfies

$$12uw - 2\hbar^2\,\ddot{w} = t_3,$$

and then $u(t)$ satisfies

$$\frac{1}{6}\,\hbar^4\,\ddddot{u} - 3\,\hbar^2 u\ddot{u} - \frac{3}{2}\,\hbar^2\dot{u}^2 + 4u^3 + 6w^2 = t,$$

where t_1, t_2, t_3 are integration constants. A particular choice is $t_1 = t_2 = t_3 = 0$ and $w = 0$, in which case we have

$$\frac{1}{6}\,\hbar^4\,\ddddot{u} - 3\hbar^2 u\ddot{u} - \frac{3}{2}\,\hbar^2\dot{u}^2 + 4u^3 = t.$$

The first few orders of expansion are:

$$u = \frac{1}{2}\,(2t)^{1/3} - \frac{1}{24}\,\frac{\hbar^2}{t^2} - \frac{1925}{1458}\,\frac{\hbar^4}{(2t)^{13/3}} - \frac{509575}{13122}\,\frac{\hbar^4}{(2t)^{20/3}} + O(\hbar^8), \quad (6.3)$$

Up to rescalings of u and t, this equation can be identified with the second member of the Painlevé I hierarchy studied, e.g. in [53]. In that context, t is considered as a "space variable". In Dubrovin universality conjecture [33], the solution (6.3) describes the shape—the variable of the generic solution of a hamiltonian perturbation hyperbolic PDE exactly at the catastrophe time.

From the relation $\hbar^2\partial_t^2 \ln Z = u$:

$$\ln Z = \frac{9}{224}\,\frac{(2t)^{7/3}}{\hbar^2} + \frac{1}{24}\,\ln t - \frac{55}{1296}\,\frac{\hbar^2}{(2t)^{7/3}} - \frac{29975}{81648}\,\frac{\hbar^4}{(2t)^{14/3}} + O(\hbar^6).$$

The Lax pair is:

$$\mathbf{M}(x,t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x + \frac{3}{2}\,\hbar\,\dot{u} + 3w - t_1 & 3u & 0 \end{pmatrix}$$

$$\mathbf{L}(x,t)$$

$$= \begin{pmatrix} 2u^2 + t_2 & x - t_1 - w & -u \\ (t_1 - x - 3w)u & -u^2 + t_2 & x - t_1 - w \\ (x - t_1)^2 + 2(x - t_1)w - 3w^2 & -2(t_1 - x + 3w)u & -u^2 + t_2 \end{pmatrix}$$

$$+ \hbar \begin{pmatrix} \dot{w} - \frac{1}{6}\,\hbar\ddot{u} & \frac{1}{2}\,\dot{u} & 0 \\ \frac{5}{2}\,u\dot{u} + \hbar^2\,\ddot{w} - \frac{1}{6}\,\hbar^3\,\dddot{u} & \frac{1}{3}\,\hbar\,\ddot{u} & -\frac{1}{2}\,\dot{u} \\ 9u\dot{w} + \hbar\big(\frac{7}{4}\,\dot{u}^2 + \frac{5}{2}\,u\ddot{u}\big) + \hbar\,\dddot{w} - \frac{1}{6}\,\hbar^2\,\ddddot{u} & -u\dot{u} + \hbar\ddot{w} - \frac{1}{6}\,\hbar^2\dddot{u} & -\dot{w} - \frac{1}{6}\,\hbar\,\ddot{u} \end{pmatrix}.$$

In the particular case where $t_1 = t_2 = t_3 = w = 0$, we have:

$$\mathbf{L}(x,t) = \begin{pmatrix} 2u^2 - \frac{1}{6}\,\hbar^2\ddot{u} & x + \frac{1}{2}\,\hbar\,\dot{u} & -u \\ -ux + \frac{5}{2}\,\hbar\,u\dot{u} - \frac{1}{6}\,\hbar^3\,\dddot{u} & -u^2 + \frac{1}{3}\,\hbar^2\,\ddot{u} & x - \frac{1}{2}\,\hbar\,\dot{u} \\ x^2 + \hbar^2\big(\frac{7}{4}\dot{u}^2 + \frac{5}{2}\,u\ddot{u}\big) - \frac{1}{6}\,\hbar^4\,\ddddot{u} & 2ux - \hbar\,u\dot{u} + \frac{1}{6}\,\hbar^3\,\dddot{u} & -u^2 - \frac{1}{6}\,\hbar^2\,\ddot{u} \end{pmatrix}.$$

The spectral curve is:

$$\det(y\,\mathbf{1}_3 - \mathbf{L}(x,t)) = y^3 - \Big(3u^4 - \frac{1}{6}\,\hbar\,\dot{u}u^3 - 3\hbar^2\,u^2\ddot{u} + \frac{1}{12}\,\hbar^2(\ddot{u}^2 + 2u\ddddot{u})\Big)y - x^4$$
$$+tx^2 + 2u^6 + \hbar^2(u^3\dot{u}^2 - 3u^4\ddot{u}) + \hbar^4\Big(-\frac{7}{16}\,\dot{u}^4 + \frac{1}{4}\,u\dot{u}\ddot{u} + \frac{3}{4}\,u^2\ddot{u}^2 - \frac{1}{2}\,u^2\dot{u}\dddot{u} + \frac{1}{6}\,u^3\ddddot{u}\Big)$$
$$+\hbar^6\Big(\frac{1}{108}\,\ddot{u}^3 + \frac{1}{36}(-\dot{u}\ddot{u}\dddot{u} + u\dddot{u}^2) + \frac{1}{24}\,\dot{u}^2\ddddot{u} - \frac{1}{18}\,u\ddot{u}\ddddot{u}\Big).$$

To leading order the spectral curve is thus:

$$y^3 - 3(u^{[0]})^4 y = x^4 - 4(u^{[0]})^3 x^2 + 2(u^{[0]})^6,$$

i.e. in terms of Chebyshev polynomials:

$$T_3\big(y/(u^{[0]})^2\big) = T_4\big(x/(u^{[0]})^{3/2}\big),$$

which admits the parametrization:

$$\begin{cases} X(z) = (u^{[0]})^{3/2}\,T_3(\zeta) = z^3 - 3u^{[0]}z \\ Y(z) = (u^{[0]})^2\,T_4(\zeta) = z^4 - 4u^{[0]}z^2 + 2(u^{[0]})^2 \end{cases} \qquad u^{[0]} = (t/4)^{1/3}.$$

The ramification points are at $\zeta = a_\pm = \pm 1$ and correspond to $X(a_\pm) = \mp 2$. The local Galois conjugate near $a = \pm 1$ is:

$$\sigma_a(\zeta) = \frac{-1}{2}\,\left(\zeta - a\sqrt{12 - 3\zeta^2}\right).$$

The topological recursion gives (we denote $\zeta = z/\sqrt{u^{[0]}}$) for the expansion of the correlators:

$$\omega_1^{(0)}(z) = -Y(z)\mathrm{d}X(z) = -3\,(u^{[0]})^{7/2}\,(\zeta^4 - 4\zeta^2 + 2)(\zeta^2 - 1)\,\mathrm{d}\zeta,$$
$$\omega_2^{(0)}(z_1, z_2) = \frac{\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2}{(\zeta_1 - \zeta_2)^2},$$
$$\omega_3^{(0)}(z_1, z_2, z_3) = \frac{-\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\,\mathrm{d}\zeta_3}{24\,(u^{[0]})^{7/2}}\left(\sum_{\varepsilon=\pm 1}\frac{1}{(\zeta_1 + \varepsilon)^2(\zeta_2 + \varepsilon)^2(\zeta_3 + \varepsilon)^2}\right),$$
$$\omega_1^{(1)}(z) = \frac{-\mathrm{d}\zeta}{576\,(u^{[0]})^{7/2}}\sum_{\varepsilon=\pm 1}\frac{7 + 7\varepsilon\zeta + 3\zeta^2}{(\zeta + \varepsilon)^4},$$
$$\omega_1^{(2)}(z) = \frac{-5\mathrm{d}\zeta}{2^{13}3^5\,(u^{[0]})^{21/2}}\,\frac{1}{(\zeta^2 - 1)^{10}}\Big(791 + 10831\zeta^2 + 5642\zeta^4 + 8010\zeta^6$$
$$-5060\zeta^8 + 6556\zeta^{10} - 4098\zeta^{12} + 1982\zeta^{14} - 539\zeta^{16} + 77\zeta^{18}\Big),$$
$$\omega_1^{(3)}(z) = \frac{-5\mathrm{d}\zeta}{2^{19}3^9\,(u^{[0]})^{35/2}}\,\frac{1}{(\zeta^2 - 1)^{16}}\Big(1534020 + 51852480\zeta^2$$
$$+139051115\zeta^4 + 126732801\zeta^6 + 14026336\zeta^8 + 136206860\zeta^{10}$$

$$-165273597\zeta^{12} + 227618305\zeta^{14} - 221591820\zeta^{16}$$
$$+175823400\zeta^{18} - 107773575\zeta^{20} + 51069755\zeta^{22}$$
$$-17959320\zeta^{24} + 4465420\zeta^{26} - 701415\zeta^{28} + 53955\zeta^{30}\Big).$$

The computation of $\omega_1^{(2)}$ (resp. $\omega_1^{(3)}$) required the knowledge of $\omega_2^{(1)}$ (resp. the knowledge of $\omega_4^{(0)}$, $\omega_3^{(1)}$ and $\omega_2^{(2)}$), but since their expression is lengthy we do not copy them here. We now come to the expansion of the Tau function $\mathcal{T}$, and the solution u. We can use $12(u^{[0]})^2\dot{u}^{[0]} = 1$ to perform the integration that gives:

$$\begin{aligned}
&\partial_t F^{(1)} = 12\,(u^{[0]})^{5/2}\dot{u}^{[0]}\,\frac{1}{2^5 3\,(u^{[0]})^{7/2}} = \frac{\dot{u}^{[0]}}{8\,u^{[0]}}\\
\Rightarrow\quad &F^{(1)} = \frac{\ln u_0}{8} = \frac{\ln(t/4)}{24}.\\
\Rightarrow\quad &u^{\{1\}} = -\frac{1}{24t^2},\\
&\partial_t F^{(2)} = 12\,(u^{[0]})^{5/2}\dot{u}^{[0]}\,\frac{5\cdot 7\cdot 11}{2^{13}3^5\,(u^{[0]})^{21/2}} = \frac{5\cdot 7\cdot 11}{2^{11}3^4\,(u^{[0]})^8}\\
\Rightarrow\quad &F^{(2)} = -\frac{5\cdot 11}{2^{11}3^4\,(u^{[0]})^7} = -\frac{55}{1296\,(2t)^{7/3}}.\\
\Rightarrow\quad &u^{\{2\}} = -\frac{1925}{1458\,(2t)^{13/3}},\\
&\partial_t F^{(3)} = 12\,(u^{[0]})^{5/2}\dot{u}^{[0]}\,\frac{5^2 11\cdot 109}{2^{19}3^7\,(u^{[0]})^{35/2}} = \frac{5^2 11\cdot 109\,\dot{u}^{[0]}}{2^{17}3^6\,(u^{[0]})^{15}}\\
\Rightarrow\quad &F^{(3)} = -\frac{5^2 11\cdot 109}{2^{18}3^6 7\,(u^{[0]})^{14}} = -\frac{29975}{81648\,(2t)^{14/3}}\\
\Rightarrow\quad &u^{\{3\}} = -\frac{509575}{13122\,(2t)^{20/3}}.
\end{aligned}$$

This matches (6.3).

Appendix A. Proof of Lemma 2.2

If δ_y^a is an insertion operator, we now prove the following formulae. For any $n \geq 1$, any $a, b, a_1, \ldots, a_n \in [\![1, d]\!]$,

$$\begin{aligned}
\delta_y^a \mathbf{K}(x_1, x_2) &= -\mathbf{K}(x_1, y)\mathbf{E}_a\mathbf{K}(y, x_2),\\
\delta_y^a \mathbf{P}(\overset{b}{x}) &= \Big[\frac{\mathbf{P}(\overset{a}{y})}{x-y} + \mathbf{U}(\overset{a}{y}), \mathbf{P}(\overset{b}{x})\Big],\\
\delta_y^a \mathbf{L}(x) &= \Big[\frac{\mathbf{P}(\overset{a}{y})}{x-y} + \mathbf{U}(\overset{a}{y}), \mathbf{L}(x)\Big] - \frac{\mathbf{P}(\overset{a}{y})}{(x-y)^2},\\
\delta_y^a \mathrm{Tr}\,\mathbf{L}(x) &= -\frac{1}{(x-y)^2},
\end{aligned}$$

$$\begin{aligned}
\delta_y^a \ln\det \mathbf{\Psi}(x) &= \frac{1}{x-y} + \mathrm{Tr}\,\mathbf{U}(\overset{a}{y}),\\
\delta_y^a \ln\left(\frac{\det \mathbf{\Psi}(x)}{\det \mathbf{\Psi}(z)}\right) &= \frac{1}{x-y} - \frac{1}{z-y},\\
\delta_y^a \mathcal{W}_n(\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n}) &= \mathcal{W}_{n+1}(\overset{a}{y},\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n}).
\end{aligned}$$

First we have by the Leibniz rule $\delta_y^b(\mathbf{\Psi}^{-1}(x)\mathbf{\Psi}(x)) = 0$, which leads to:

$$\delta_y^a \mathbf{\Psi}^{-1}(x) = -\mathbf{\Psi}^{-1}(x)(\delta_y^a\mathbf{\Psi}(x))\mathbf{\Psi}^{-1}(x) = \frac{\mathbf{\Psi}^{-1}(x)\mathbf{P}(\overset{a}{y})}{y-x} - \mathbf{\Psi}^{-1}(x)\mathbf{U}(\overset{a}{y}).$$

Then, we compute

$$\begin{aligned}
\delta_y^a \mathbf{K}(x_1,x_2) &= \frac{1}{x_1-x_2}\,\delta_y^b[\mathbf{\Psi}^{-1}(x_1)\mathbf{\Psi}(x_2)]\\
&= \frac{1}{x_1-x_2}\Big(\frac{\mathbf{\Psi}^{-1}(x_1)\mathbf{P}(\overset{a}{y})\mathbf{\Psi}(x_2)}{y-x_1} + \frac{\mathbf{\Psi}^{-1}(x_1)\mathbf{P}(\overset{a}{y})\mathbf{\Psi}(x_2)}{x_2-y}\\
&\quad + \mathbf{\Psi}^{-1}(x_1)\mathbf{U}(\overset{a}{y})\mathbf{\Psi}(x_2) - \mathbf{\Psi}^{-1}(x_1)\mathbf{U}(\overset{a}{y})\mathbf{\Psi}(x_2)\Big)\\
&= -\frac{\mathbf{\Psi}^{-1}(x_1)\mathbf{\Psi}(y)}{x_1-y}\mathbf{E}_a\,\frac{\mathbf{\Psi}^{-1}(y)\mathbf{\Psi}(x_2)}{y-x_2} = -\mathbf{K}(x_1,y)\mathbf{E}_a\mathbf{K}(y,x_2),
\end{aligned}$$

and notice that $\mathbf{U}$ disappears in this computation. Similarly,

$$\begin{aligned}
\delta_y^a \mathbf{P}(\overset{b}{x}) &= (\delta_y^a\mathbf{\Psi}(x))\mathbf{E}_b\mathbf{\Psi}^{-1}(x) + \mathbf{\Psi}(x)\mathbf{E}_b(\delta_y^a\mathbf{\Psi}^{-1}(x))\\
&= \frac{\mathbf{P}(\overset{a}{y})\mathbf{\Psi}(x)\mathbf{E}_b\mathbf{\Psi}^{-1}(x)}{x-y} - \frac{\mathbf{\Psi}(x)\mathbf{E}_b\mathbf{\Psi}^{-1}(x)\mathbf{P}(\overset{a}{y})}{x-y}\\
&= + \mathbf{U}(\overset{a}{y})\mathbf{\Psi}(x)\mathbf{E}_b\mathbf{\Psi}^{-1}(x) - \mathbf{\Psi}(x)\mathbf{E}_b\mathbf{\Psi}^{-1}(x)\mathbf{U}(\overset{a}{y})\\
&= \frac{[\mathbf{P}(\overset{a}{y}),\mathbf{P}(\overset{b}{x})]}{x-y} + [\mathbf{U}(\overset{a}{y}),\mathbf{P}(\overset{b}{x})].
\end{aligned}$$

Then we have

$$\begin{aligned}
\delta_y^a \mathbf{L}(x) &= \delta_y^a\left(\hbar\,\partial_x\mathbf{\Psi}(x)\ \mathbf{\Psi}^{-1}(x)\right)\\
&= \hbar\,\partial_x\left(\delta_y^a\mathbf{\Psi}(x)\right)\ \mathbf{\Psi}^{-1}(x) - \hbar\,\partial_x\,\mathbf{\Psi}(x)\ \delta_y^a\left(\mathbf{\Psi}^{-1}(x)\right)\\
&= \hbar\,\partial_x\left(\left(\frac{\mathbf{P}(\overset{a}{y})}{x-y} + \mathbf{U}(\overset{a}{y})\right)\mathbf{\Psi}(x)\right)\ \mathbf{\Psi}^{-1}(x) - \mathbf{L}(x)\ \delta_y^a\left(\mathbf{\Psi}^{-1}(x)\right)\\
&= -\frac{\hbar}{(x-y)^2}\,\mathbf{P}(\overset{a}{y}) + \left[\frac{\mathbf{P}(\overset{a}{y})}{x-y} + \mathbf{U}(\overset{a}{y}),\mathbf{L}(x)\right].
\end{aligned}$$

To compute the action of δ_y^a on the correlators, we consider $n=1$ separately:

$$\begin{aligned}
\delta_y^b \mathcal{W}_1(\overset{b}{x}) &= \delta_y^b \left(\lim_{z\to x} \mathbf{K}_{a,a}(x,z) - \frac{1}{x-z} \right) \\
&= \lim_{z\to x} \delta_y^b \mathbf{K}_{a,a}(x,z) \\
&= -\lim_{z\to x} \mathbf{K}_{a,b}(x,y)\, \mathbf{K}_{b,a}(y,z) \\
&= -\ \mathbf{K}_{a,b}(x,y)\, \mathbf{K}_{b,a}(y,x) = \mathcal{W}_2(x,y).
\end{aligned}$$

Then, for $n \geq 2$, we can use Definition 2.2:

$$\begin{aligned}
&\delta_y^a \mathcal{W}_n(x_1,\ldots,x_n) \\
&\quad = (-1)^{n+1} \sum_{\sigma = n\text{-cycle}} \delta_y^a \left[\prod_{i=1}^n \mathbf{K}_{a_i,a_{\sigma(i)}}(x_i,x_{\sigma(i)}) \right] \\
&\quad = (-1)^{n+2} \sum_{\sigma = n\text{-cycle}} \sum_{j=1}^n \mathbf{K}_{a_j,a}(x_j,y) \mathbf{K}_{a,a_{\sigma(j)}}(y,x_{\sigma(j)}) \prod_{i\neq j} \mathbf{K}_{a_i,a_{\sigma(i)}}(x_i,x_{\sigma(i)}) \\
&\quad = (-1)^{n+2} \sum_{\substack{\sigma=(n+1)\text{-cycle} \\ y=x_{n+1},\ a_{n+1}=a}} \prod_{i=1}^n \mathbf{K}_{a_i,a_{\sigma(i)}}(x_i,x_{\sigma(i)}) \\
&\quad = \mathcal{W}_{n+1}(\overset{a}{y},\overset{a_1}{x_1},\ldots,\overset{a_n}{x_n}).
\end{aligned}$$

□

Appendix B. Proof of Theorem 3.1

We assume that all ramification points are simple (see [22] for the case or higher ramifications), the embedding of the curve $\mathcal{S}^{[0]} \to \mathbb{C}^2$ by the functions (x,y) is regular, and that TT is satisfied. We shall prove the topological recursion using the linear (Proposition 2.1) and quadratic (Proposition 2.2) loop equations only. This is already done in [19,41], but we present here a self-contained proof. Contrarily to [19] which is more general, we take advantage here that the semiclassical spectral curve $\mathcal{S}^{[0]}$ is a compact Riemann surface of genus $\mathfrak{g}$, to identify more precisely the possible holomorphic term in (3.13).

From the TT hypothesis, we have that every $\omega_n^{(g)}$ with $(g,n) \neq (0,1)$ or $(0,2)$ has poles only at the ramification points. We have called $\mathbf{r} = \{r_1,\ldots,r_m\}$ the set of ramification points. Let $r \in \mathbf{r}$ be a ramification point, by definition and assumption there are exactly two indices $a \neq b$ such that $z^a(r) = z^b(r)$, and we define the local Galois involution σ_r in a vicinity of r, as the map $z^a(x) \mapsto z^b(x)$. Let $J = \{2,\ldots,n\}$ and $\mathbf{z}_J = (z_j)_{j\in J}$, and define:

$$\tilde{\mathcal{Q}}_n^{(g)}(z,z';\mathbf{z}_J) := \omega_{n+2}^{(g-1)}(z,z',\mathbf{z}_J) + \sideset{}{'}\sum_{h+h'=g,\, I\dot\cup I'=J} \omega_{1+|I|}^{(h)}(z,\mathbf{z}_I)\, \omega_{1+|I'|}^{(h')}(z',\mathbf{z}_{I'})$$

and

$$\mathcal{Q}_n^{(g)}(z,z';\mathbf{z}_J) := \omega_{n+2}^{(g-1)}(z,z',\mathbf{z}_J) + \sum_{h+h'=g,\, I\dot\cup I'=J}' \omega_{1+|I|}^{(h)}(z,\mathbf{z}_I)\,\omega_{1+|I'|}^{(h')}(z',\mathbf{z}_{I'}),$$

where $\sum'$ means that we exclude the cases $(h,I)=(0,\emptyset)$ and $(h,I)=(g,J)$, i.e.

$$\mathcal{Q}_n^{(g)}(z,z';\mathbf{z}_J) = \tilde{\mathcal{Q}}_n^{(g)}(z,z';\mathbf{z}_J) + \omega_1^{(0)}(z)\,\omega_{n+1}^{(g)}(z',\mathbf{z}_J) + \omega_{n+1}^{(g)}(z,\mathbf{z}_J)\,\omega_1^{(0)}(z').$$

Lemma B.1. *Near a ramification point r, we have:*

$$\sum_{a<b} \mathcal{Q}_n^{(g)}(z^a,z^b;\mathbf{z}_J) = \mathcal{Q}_n^{(g)}(z,\sigma_r(z);\mathbf{z}_J) + \text{analytical at } z\to r. \tag{B.1}$$

Proof. To simplify notations, we can always label 1 and 2 the sheets meeting at the ramification point r. I.e. if $z=z^1$, we have $\sigma_r(z)=z^2$. Let us decompose the sum over indices as:

$$\sum_{1\le a<b\le d} \mathcal{Q}_n^{(g)}(z^a,z^b;\mathbf{z}_J) = \mathcal{Q}_n^{(g)}(z^1,z^2;z_J) + \sum_{2<b\le d} \mathcal{Q}_n^{(g)}(z^1,z^b;\mathbf{z}_J)$$
$$+ \sum_{2<b\le d} \mathcal{Q}_n^{(g)}(z^2,z^b;\mathbf{z}_J) + \sum_{2<a<b\le d} \mathcal{Q}_n^{(g)}(z^a,z^b;\mathbf{z}_J).$$

The linear loop equation implies that:

$$\mathcal{Q}_n^{(g)}(z^1,z^b;\mathbf{z}_J) + \mathcal{Q}_n^{(g)}(z^2,z^b;\mathbf{z}_J) = -\sum_{2<a\le d} \mathcal{Q}_n^{(g)}(z^a,z^b;\mathbf{z}_J),$$

and thus:

$$\sum_{1\le a<b\le d} \mathcal{Q}_n^{(g)}(z^a,z^b;\mathbf{z}_J)$$
$$= \mathcal{Q}_n^{(g)}(z^1,z^2;\mathbf{z}_J) - \sum_{2<a,b\le d} \mathcal{Q}_n^{(g)}(z^a,z^b;\mathbf{z}_J) + \sum_{2<a<b\le d} \mathcal{Q}_n^{(g)}(z^a,z^b;\mathbf{z}_J).$$

The last two lines have no poles at the ramification point, hence the announced result. ☐

Remark B.1. Since the analytic term in r in (B.1) is a quadratic differential in z invariant under Galois involution, it must actually have a double zero at r.

Theorem B.1. *The $\omega_n^{(g)}$ satisfy the topological recursion:*

$$\omega_{n+1}^{(g)}(z_1,\mathbf{z}_J) = \frac{1}{2}\sum_{r\in\mathbf{r}} \operatorname*{Res}_{z\to r} \frac{\int_{\sigma_r(z)}^{z} \omega_2^{(0)}(z_1,\cdot)}{\omega_1^{(0)}(z)-\omega_1^{(0)}(\sigma_r(z))}\, \tilde{\mathcal{Q}}_n^{(g)}(z,\sigma_r(z);\mathbf{z}_J) + H_n^{(g)}(z_1,\mathbf{z}_J)$$

where $H_n^{(g)}$ is holomorphic in z_1.

Proof First, Lemma B.1 together with the quadratic loop equation implies that $\mathcal{Q}_n^{(g)}(z,\sigma_r(z);z_J)$ has no pole at the ramification point r. This means that:

$$\tilde{\mathcal{Q}}_n^{(g)}(z,\sigma_r(z);\mathbf{z}_J) = -\omega_1^{(0)}(z)\,\omega_{n+1}^{(g)}(\sigma_r(z),\mathbf{z}_J) - \omega_{n+1}^{(g)}(z,\mathbf{z}_J)\,\omega_1^{(0)}(\sigma_r(z)) + \cdots$$

where the dots are analytical at r. Moreover, using again the linear loop equation we have that

$$\omega_{n+1}^{(g)}(\sigma_r(z),\mathbf{z}_J) = -\omega_{n+1}^{(g)}(z,\mathbf{z}_J) + \cdots$$

and thus

$$\tilde{\mathcal{Q}}_n^{(g)}(z,\sigma_r(z);\mathbf{z}_J) = \left[\omega_1^{(0)}(z) - \omega_1^{(0)}(\sigma_r(z))\right]\omega_{n+1}^{(g)}(z,\mathbf{z}_J) + \cdots$$

According to the previous remark, the remainder has actually a double zero at $z = r$. We remind that $\omega_1^{(0)} = y\mathrm{d}x$, and since we assume that the embedding of $\mathcal{S}^{[0]}$ in $\mathbb{C}^2$ by (x,y) is regular, $\mathrm{d}y(r) \neq 0$. Combined with the assumption that x has simple ramification points, this implies that $\left[\omega_1^{(0)}(z) - \omega_1^{(0)}(\sigma_r(z))\right]$ has exactly a double zero at $z = r$. Therefore, we find:

$$\begin{aligned}
\mathcal{I}_n^{(g)} &= \frac{1}{2}\sum_{r\in\mathbf{r}} \operatorname*{Res}_{z\to r} \frac{\int_{\sigma_r(z)}^{z} \omega_2^{(0)}(z_1,\cdot)}{\omega_1^{(0)}(z) - \omega_1^{(0)}(\sigma_r(z))}\,\tilde{\mathcal{Q}}_n^{(g)}(z,\sigma_r(z);\mathbf{z}_J)\\
&= \frac{1}{2}\sum_{r\in\mathbf{r}} \operatorname*{Res}_{z\to r}\left(\int_{\sigma_r(z)}^{z}\omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(z,\mathbf{z}_J)\\
&= \frac{1}{2}\left\{\sum_{r\in\mathbf{r}} \operatorname*{Res}_{z\to r}\left(\int_{o}^{z}\omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(z,\mathbf{z}_J)\right.\\
&\qquad \left. - \operatorname*{Res}_{z\to r}\left(\int_{o}^{\sigma_r(z)}\omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(z,\mathbf{z}_J)\right\}
\end{aligned}$$

where o is an arbitrary base point on the spectral curve. We rename the integration variable in the second term $z \to \sigma_r(z)$, and get:

$$\begin{aligned}
\mathcal{I}_n^{(g)} &= \frac{1}{2}\sum_{r\in\mathbf{r}}\left\{\operatorname*{Res}_{z\to r}\left(\int_{o}^{z}\omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(z,\mathbf{z}_J)\right.\\
&\qquad \left. - \operatorname*{Res}_{z\to r}\left(\int_{o}^{z}\omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(\sigma_r(z),\mathbf{z}_J)\right\}
\end{aligned}$$

using again the linear loop equation, i.e. that $\omega_{n+1}^{(g)}(\sigma_r(z),\mathbf{z}_J) + \omega_{n+1}^{(g)}(z,\mathbf{z}_J)$ is analytical at r, we arrive to

$$\mathcal{I}_n^{(g)} = \sum_{r\in\mathbf{r}} \operatorname*{Res}_{z\to r}\left(\int_{o}^{z}\omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(z,\mathbf{z}_J).$$

Now, observe that $\omega_{n+1}^{(g)}(z,\mathbf{z}_J)$ has poles only at the ramification points, whereas $\omega_2^{(0)}(z,z_1)$ has a pole only at $z = z_1$ (a double pole). We may move the integration contours from surrounding the poles of $\omega_{n+1}^{(g)}(z,\mathbf{z}_J)$ to surrounding the poles of $\omega_2^{(0)}(z,z_1)$, i.e. using the Riemann bilinear identity:

$$\begin{aligned}\mathcal{I}_n^{(g)} = &- \operatorname*{Res}_{z\to z_1} \left(\int_o^z \omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(z,\mathbf{z}_J) \\ &+ \frac{1}{2\mathrm{i}\pi}\sum_{i=1}^{\mathfrak{g}}\left\{\Big(\oint_{\mathcal{A}_i}\omega_2^{(0)}(z_1,\cdot)\Big)\Big(\oint_{\mathcal{B}_i}\omega_{n+1}^{(0)}(\cdot,\mathbf{z}_J)\Big)\right. \\ &\left. -\left(\oint_{\mathcal{B}_i}\omega_2^{(0)}(z_1,\cdot)\right)\left(\oint_{\mathcal{A}_i}\omega_{n+1}^{(0)}(\cdot,\mathbf{z}_J)\right)\right\},\end{aligned}$$

where the cycles $\mathcal{A}_i, \mathcal{B}_j$ are chosen to form a basis of $2\mathfrak{g}$ non-contractible cycles on $\mathcal{S}^{[0]}$, with canonical intersections $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$. Observe that $\left(\int_o^z \omega_2^{(0)}(z_1,\cdot)\right)$ has a simple pole at $z_1 = z$ with residue 1, so the first term is:

$$-\operatorname*{Res}_{z\to z_1}\left(\int_o^z \omega_2^{(0)}(z_1,\cdot)\right)\omega_{n+1}^{(g)}(z,\mathbf{z}_J) = \omega_{n+1}^{(g)}(z_1,\mathbf{z}_J).$$

Since $\omega_2^{(0)} \in \mathcal{B}(\mathcal{S}^{[0]})$ (from Corollary 3.5), we also know that $\oint_{\mathcal{A}_i}\omega_2^{(0)}(z_1,\cdot)$ and $\oint_{\mathcal{B}_i}\omega_2^{(0)}(z_1,\cdot)$ are holomorphic forms of z_1, and thus we have obtained the decomposition:

$$\mathcal{I}_n^{(g)} = \omega_{n+1}^{(g)}(z_1,\mathbf{z}_J) + \text{holomorphic}\,(z_1).$$

This finishes the proof of Theorem 3.1.

Appendix C. Proof of Lemma 5.8

We want to prove:

$$\delta_x^a \mathbf{U}(\overset{b}{y}) - \delta_y^b \mathbf{U}(\overset{a}{x}) = [\mathbf{U}(\overset{a}{x}), \mathbf{U}(\overset{b}{y})] \tag{C.1}$$

for the choice:

$$U_{k,m}(\overset{a}{x}) = \sum_{l=0}^{k-1-m}\binom{m-1+l}{l}P_{k-m-l,q}^{(l)}(\overset{a}{x}) \tag{C.2}$$

where we adopted the notation $f^{(l)} = (\hbar\partial_t)^l f$. Notice that $\mathbf{U}$ is lower triangular, i.e. $U_{k,m} = 0$ when $k \leq m$ since the sum is empty. Subsequently, the right-hand side of (C.1) is equal to:

$$\begin{aligned}&[\mathbf{U}(\overset{a}{x}),\mathbf{U}(\overset{b}{y})]_{k,m} \\ &= \sum_{r=m+1}^{k-1}\sum_{l=0}^{k-1-r}\sum_{l'=0}^{r-1-m}\binom{r-1+l}{l}\binom{m-1+l'}{l'}P_{k-r-l,q}^{(l)}(\overset{a}{x})\,P_{r-m-l',q}^{(l')}(\overset{b}{y}) \\ &\quad -(\overset{a}{x}\leftrightarrow\overset{b}{y})\end{aligned} \tag{C.3}$$

Let us now compute the left-hand side of (C.1). Using the fact that δ^a_x commutes with ∂_t, we have:

$$\left\{\delta^a_x \mathbf{U}(\overset{b}{y}) - \delta^b_y \mathbf{U}(\overset{a}{x})\right\}_{k,m} = \sum_{l=0}^{k-1-m} \binom{m-1+l}{l} (\hbar\partial_t)^l \left\{\delta^a_x \mathbf{P}(\overset{b}{y}) - \delta^b_y \mathbf{P}(\overset{a}{x})\right\}_{k-m-l,q} \tag{C.4}$$

From Lemma 2.2 we remind:

$$\delta^b_y \mathbf{P}(\overset{a}{x}) = \frac{[\mathbf{P}(\overset{b}{y}), \mathbf{P}(\overset{a}{x})]}{x-y} + [\mathbf{U}(\overset{b}{y}), \mathbf{P}(\overset{a}{x})]$$

One can check that the proof of this equation given in "Appendix A" does not use the commutation relation (C.1) that we want to prove. We thus find:

$$\delta^a_x \mathbf{P}(\overset{b}{y}) - \delta^b_y \mathbf{P}(\overset{a}{x}) = [\mathbf{U}(\overset{a}{x}), \mathbf{P}(\overset{b}{y})] - \left(\overset{a}{x} \leftrightarrow \overset{b}{y}\right)$$

Then, we remark from the structure of $\mathbf{U}$ that the qth column of $\mathbf{P}(\overset{a}{x})\mathbf{U}(\overset{b}{y})$ is zero. So, we have:

$$\delta^a_x P_{k-m-l,q}(\overset{b}{y}) - \delta^b_y P_{k-m-l,q}(\overset{a}{x}) = \left\{\mathbf{U}(\overset{a}{x})\mathbf{P}(\overset{b}{y})\right\}_{k-m-l,q} - \left(\overset{a}{x} \leftrightarrow \overset{b}{y}\right) \tag{C.5}$$

and we can write:

$$\left\{\delta^a_x \mathbf{U}(\overset{b}{y}) - \delta^b_y \mathbf{U}(\overset{a}{x})\right\}_{k,m} = D_{k,m}(\overset{a}{x}, \overset{b}{y}) - \left(\overset{a}{x} \leftrightarrow \overset{b}{y}\right)$$

The expression for the D term is obtained by inserting (C.5) in (C.4), using the Leibniz rule to distribute the lth order derivative between the two factors $\mathbf{U}$ and $\mathbf{P}$, and replacing $\mathbf{U}$ by its value (C.2):

$$\begin{aligned} D_{k,m}(\overset{a}{x}, \overset{b}{y}) &= \sum_{l=0}^{k-m-1} \sum_{n=0}^{l} \sum_{r=k-m-l-1}^{q} \sum_{l'=0}^{k-m-l-r-1} \binom{l}{n} \binom{r+l'-1}{l'} \\ &\quad \times P^{(l'+n)}_{k-m-l-r-l',q}(\overset{a}{x})\, P^{(l-n)}_{r,q}(\overset{b}{y}). \end{aligned} \tag{C.6}$$

The structure is now similar to (C.3), except that we have a fourfold sum instead of a threefold sum. To make the comparison precise, let us redefine the indices of summation $(l, n, r, l') \mapsto (\underline{r}, \underline{l}, \underline{l'}, n)$ in (C.6) to mimic those of (C.3):

$$r = \underline{r} - m - \underline{l'}, \quad \underline{l'} = l - n, \quad \underline{l} = l' + n$$

Then, automatically the combination $k - \underline{r} - \underline{l}$ is equal to $k - m - l - r - l'$, and rewriting (C.6) in a form as close to (C.3) as possible:

$$D_{k,m} = \sum_{\underline{r},\underline{l},\underline{l'}} \binom{r-1+l}{l} \binom{m-1+l'}{l'} P^{(l)}_{k-r-l,q}(\overset{a}{x})\, P^{(l')}_{r-m-l',q}(\overset{b}{y}) \cdot S_{\underline{r},\underline{l},\underline{l'}} \tag{C.7}$$

The extra-summation over n was included in

$$S_{\underline{r},\underline{l},\underline{l}'} = S(\underline{l}+\underline{r}-1-m-\underline{l}', m-1+\underline{l}', \underline{l})$$
$$S(a,b,c) = \frac{(a+b-1+c)!\,c!}{b!(a-c)!(a+b+1)!}\sum_{n=0}^{c}\frac{(a-n)!(b+n)!}{n!(c-n)!} \tag{C.8}$$

To conclude, we must show that $S \equiv 1$, and describe carefully the range of summation in (C.7) following from the change of indices.

For the first task, we write a generating series:

$$\sum_{c=0}^{\infty}\frac{(a-n)!(b+n)!}{n!(c-n)!}\,z^c = b!\,(a-c)!\,T(b;z)\,T(a-c;z),$$

where $T(b;z) = \sum_{n\geq 0}\binom{b+n}{b}\,z^n$. We claim that $T(b;z) = (1-z)^{-(b+1)}$. Indeed, setting a new summation index $k = b+n$:

$$\sum_{b\geq 0} T(b)\,w^b = \sum_{k\geq 0} w^k\Big(\sum_{n=0}^{k}\binom{k}{n}\,(z/w)^n\Big) = \sum_{k\geq 0} w^k(1+w/z)^k$$
$$= \frac{1}{1-(w+z)}.$$

which entails the claim after expansion in $w \to 0$. Then:

$$\sum_{n=0}^{c}\frac{(a-n)!(b+n)!}{n!(c-n)!} = b!(a-c)!\,[z^c]\,\frac{1}{(1-z)^{b+a-c+2}}$$
$$= \frac{b!(a-c)!}{c!}\,\frac{(b+a+1)!}{(b+a-c+1)!}.$$

which proves that $S(a,b,c) \equiv 1$.

For the second task, we decompose as follows the change of indices $(l,n,r,l') \mapsto (\underline{r},\underline{l},\underline{l}',n)$, with the convention that the innermost summation indices appear at the bottom of the column.

$$\begin{cases} l & [0,k-1-m] \\ r & [1,k-1-m-l] \\ n & [0,l] \\ l' & [0,k-1-m-r-l] \end{cases} \xrightarrow[\underline{l}=l'+n]{r\leftrightarrow l} \begin{cases} r & [1,k-1-m] \\ l & [0,k-1-m-r] \\ n & [0,l] \\ \underline{l} & [n,k-1-m-r-l+n] \end{cases}$$

$$\xrightarrow{l\leftrightarrow n} \begin{cases} r & [1,k-1-m] \\ n & [0,k-1-m-r] \\ l & [n,k-1-m-r] \\ \underline{l} & [n,k-1-m-r-l+n] \end{cases} \xrightarrow[\underline{l}'=l-n]{n\leftrightarrow r} \begin{cases} n & [0,k-2-m] \\ r & [1,k-1-m-n] \\ \underline{l}' & [0,k-1-m-n-r] \\ \underline{l} & [n,k-1-m-r-\underline{l}'] \end{cases}$$

$$\xrightarrow{\underline{l}'\leftrightarrow r} \begin{cases} n & [0,k-2-m] \\ \underline{l}' & [0,k-2-m-n] \\ r & [1,k-1-m-n-\underline{l}'] \\ \underline{l} & [n,k-1-m-r-\underline{l}'] \end{cases} \xrightarrow[\underline{r}=r+m+\underline{l}']{} \begin{cases} n & [0,k-2-m] \\ \underline{l}' & [0,k-2-m-n] \\ \underline{r} & [1+m+\underline{l}',k-1-n] \\ \underline{l} & [n,k-1-\underline{r}] \end{cases}$$

$$\xrightarrow{\underline{l}'\leftrightarrow\underline{r}} \begin{cases} n & [0,k-2-m] \\ \underline{r} & [1+m,k-1-n] \\ \underline{l}' & [0,\underline{r}-m-1] \\ \underline{l} & [n,k-1-\underline{r}] \end{cases} \xrightarrow{n\leftrightarrow\underline{r}} \begin{cases} \underline{r} & [1+m,k-1] \\ \underline{l}' & [0,\underline{r}-m-1] \\ n & [0,k-1-\underline{r}] \\ \underline{l} & [n,k-1-\underline{r}] \end{cases}$$

Eventually, exchanging the sums over n and $\underline{l}$, we find the range of summations:

$$\begin{cases} \underline{r} \ [1+m, k-1] \\ \underline{l}' \ [0, \underline{r}-m-1] \\ \underline{l} \ [0, k-1-\underline{r}] \\ n \ [0, \underline{l}] \end{cases} \tag{C.9}$$

The summation over n indeed matches with (C.8) since we took $c = \underline{l}$, while the first threefold sum is exactly the same as in (C.3). Hence the claimed equality (C.1). $\square$

Appendix D. Matrix Models and Motivations

The theory we presented was initially motivated by particular cases at the intersection between random matrix models and integrable systems, such as the 1- and the 2-matrix model. We give a short summary of the correspondence between the notions introduced in this article and the one usually used to study those matrix models.

D.1. The 1-Matrix Model

Consider a hermitian matrix integral of the type

$$Z_N = \int_{H_N} dM \ \mathrm{e}^{-\hbar^{-1} \operatorname{Tr} V(M)},$$

where $\mathrm{d}M = \prod_{i=1}^N \mathrm{d}M_{i,i} \prod_{1 \le i < j \le N} \mathrm{dRe}\, M_{i,j} \cdot \mathrm{dIm}\, M_{i,j}$ is the usual invariant Lebesgue measure on the space of $N \times N$ hermitian matrices $\mathcal{H}_N$, and where $V(M) = t_0 + \sum_{k=1}^d t_k M^k/k$ is a polynomial, called the potential.

D.1.1. The Differential System. It is well known that the matrix integral Z_N is a Jimbo–Miwa isomonodromic Tau function [12,13]. After Heine formula [60], the expectation value of the characteristic polynomial is an orthogonal polynomial with respect to the measure $\mathrm{e}^{-\hbar^{-1} V(x)}\, \mathrm{d}x$:

$$p_N(x) = \mathbb{E}_N\big[\det(x\mathbf{1} - M)\big] = \frac{1}{Z_N} \int_{\mathcal{H}_N} \mathrm{d}M\, \mathrm{e}^{-\hbar^{-1} \operatorname{Tr} V(M)} \det(x\mathbf{1} - M),$$

i.e. it satisfies:

$$\int_{\mathbb{R}} p_N(x)\, p_L(x)\, \mathrm{e}^{-\hbar^{-1} V(x)}\, \mathrm{d}x = h_N \delta_{N,L}.$$

The orthogonality is a way of writing Hirota equation [54, Sect. 2.2]. The orthogonal polynomials satisfy a linear, second-order differential equation, which can be written in 2×2 matrix form:

$$\hbar\, \partial_x \mathbf{\Psi}_N(x) = \mathbf{L}_N(x) \mathbf{\Psi}_N(x), \qquad \mathbf{\Psi}_N(x) = \begin{pmatrix} \psi_N(x) & \phi_N(x) \\ \psi_{N-1}(x) & \phi_{N-1}(x) \end{pmatrix}, \tag{D.1}$$

where

$$\psi_N(x) = \frac{p_N(x)}{\sqrt{h_N}}\,\mathrm{e}^{-\frac{1}{2\hbar}\,V(x)}$$

is the normalized N-orthogonal polynomial up to an exponential prefactor, and

$$\phi_N(x) = \int_{\mathbb{R}} \frac{\psi_N(x')}{x - x'}\,\mathrm{e}^{\frac{1}{2\hbar}\,(V(x)-V(x'))}\,\mathrm{d}x'$$

is its Hilbert transform. It is characterized by the property that it is holomorphic in $\mathbb{C}\backslash\mathbb{R}$, and on the real line its lower and upper boundary values satisfy a Riemann Hilbert problem:

$$\phi_N(x+\mathrm{i}0) - \phi_N(x-\mathrm{i}0) = -2\mathrm{i}\pi\,\psi_N(x). \tag{D.2}$$

The matrix $\mathbf{L}_N(x)$ is traceless, and its entries are polynomial in x of maximal degree $(d-1)$: it can be found, e.g. in [13]. This is our starting point (1.1). There is also a compatible recursion relation in N, which is the Toda chain [47]. The times t_j generate the jth Toda flow and are isomonodromic parameters for the ODE (D.1).

D.1.2. Matrix Kernel and Correlators. The expectation values of ratios of characteristic polynomials obey Giambelli determinantal relations [45]:

$$\mathbb{E}_N\left[\frac{\prod_{i=1}^n \det(x_i\mathbf{1} - M)}{\prod_{i=1}^n \det(y_i\mathbf{1} - M)}\right] = \frac{\det_{1\leq i,j\leq n} K(y_i, x_j)}{\det_{1\leq i,j\leq n}\frac{1}{y_i - x_j}}, \tag{D.3}$$

where the scalar kernel K is defined as

$$K(y,x) = \frac{1}{y-x}\,\mathbb{E}_N\left[\frac{\det(x\mathbf{1}-M)}{\det(y\mathbf{1}-M)}\right] \propto \frac{\phi_N(y)\psi_{N-1}(x) - \phi_{N-1}(y)\psi_N(x)}{x-y}. \tag{D.4}$$

Since the matrix $\mathbf{L}_N(x)$ is traceless, $\det\boldsymbol{\Psi}_N(x)$ is constant, so up to a constant prefactor, the scalar kernel coincides with the $(1,1)$ entry of the matrix kernel $\mathbf{K}$ of our definition 2.1:

$$K(y,x) \;\propto\; \frac{1}{y-x}\,\left(\boldsymbol{\Psi}(y)^{-1}\,\boldsymbol{\Psi}(x)\right)_{1,1} \;\propto\; \mathbf{K}_{1,1}(y,x).$$

while the $(1,2)$ entry of the matrix kernel is proportional to the Christoffel–Darboux kernel of orthogonal polynomials:

$$\sum_{M=0}^{N-1}\psi_M(x)\psi_M(y) = \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y} \;\propto\; \mathbf{K}_{2,1}(x,y). \tag{D.5}$$

According to (D.2), it is obtained from the scalar kernel (D.4) by computing its discontinuity when y crossed the real axis. Computing likewise the discontinuity of (D.3) gives the famous formula of Dyson and Mehta [57] for the k-point density correlations of the eigenvalues as a $k\times k$ determinant of the Christoffel–Darboux kernel.

In the matrix model, the n-point correlation functions are defined as expectation values of traces of resolvents:

$$W_1(x) = \mathbb{E}_N\left[\operatorname{Tr}\frac{1}{x\mathbf{1} - M}\right], \tag{D.6}$$

and for $n \geq 1$

$$W_n(x_1,\ldots,x_n) = \mathbb{E}_N\left[\prod_{i=1}^{n}\operatorname{Tr}\left(\frac{1}{x_i\mathbf{1} - M}\right)\right]_c, \tag{D.7}$$

where the subscript c means that we take the cumulant. Since

$$\lim_{y\to x}\frac{\det(x\mathbf{1} - M)}{\det(y\mathbf{1} - M)} - \frac{1}{y-x} = \operatorname{Tr}(x\mathbf{1} - M)^{-1},$$

we have:

$$W_1(x) = \lim_{y\to x}\left(K(y,x) - \frac{1}{y-x}\right) = \mathcal{W}_1(\overset{1}{x}),$$

and similarly:

$$W_n(x_1,\ldots,x_n) = \mathcal{W}_n(\overset{1}{x}_1,\ldots,\overset{1}{x}_n)$$

in terms of the correlators in our Definition 2.2. By the remark of Equation (D.2), the entry $\mathcal{W}_n(\overset{a_1}{x}_1,\ldots,\overset{2}{x}_i,\ldots,\overset{a_n}{x}_n)$ can be expressed in terms of the discontinuity of $\mathcal{W}_n(\overset{a_1}{x}_1,\ldots,\overset{1}{x}_i,\ldots,\overset{a_n}{x}_n)$ when x_i crosses the real line.

The n-point correlation functions in the matrix model satisfy Schwinger–Dyson equations, which are obtained by integration by parts in the matrix integral—see, e.g. the review [37]. The first Schwinger–Dyson equation is:

$$W_2(x,x) + W_1(x)^2 - \frac{1}{\hbar}V'(x)\,W_1(x) = \mathbb{E}_N\left[\operatorname{Tr}\frac{V'(M) - V'(x)}{M - x}\right] \in \mathbb{C}[x].$$

And, for each $n \geq 2$, there exist similar equations involving $W_{n+1},\ldots,W_1$. Together with the linear loop equations of Proposition 2.1, they imply the quadratic loop equations of Proposition 2.2.

The correlators W_n and the matrix kernel $\mathbf{K}$ are related by the boson–fermion correspondence. We have seen that W_n satisfy determinantal formulae in terms of $\mathbf{K}$. Conversely, we notice that:

$$\frac{\det(x\mathbf{1} - M)}{\det(y\mathbf{1} - M)} = \mathrm{e}^{\operatorname{Tr}[\ln(x\mathbf{1}-M) - \ln(y\mathbf{1}-M)]} = \mathrm{e}^{\int_y^x \operatorname{Tr}(x'\mathbf{1}-M)^{-1}\,\mathrm{d}x'},$$

so we have an exponential formula for the matrix kernel in terms of the correlators:

$$K(y,x) = \frac{1}{y-x}\exp\Big(\sum_{n\geq 1}\frac{1}{n!}\int_y^x\cdots\int_y^x W_n(x'_1,\ldots,x'_n)\mathrm{d}x'_1\cdots\mathrm{d}x'_n\Big).$$

This relation is however formal since it is subjected to convergence of the series in right-hand side.

D.1.3. Insertion Operator. The formal differential operator:

$$\delta_x := -\frac{\hbar}{x}\frac{\partial}{\partial t_0} - \sum_{k=1}^{\infty}\frac{k\,\hbar}{x^{k+1}}\frac{\partial}{\partial t_k} \tag{D.8}$$

is such that:

$$\delta_x \mathrm{e}^{-\hbar^{-1}\mathrm{Tr}\,V(M)} = \mathrm{Tr}\,(x\mathbf{1} - M)^{-1}\,\mathrm{e}^{-\hbar^{-1}\mathrm{Tr}\,V(M)}.$$

Therefore,

$$\delta_x \ln Z_N = W_1(x), \qquad \delta_{x_{n+1}} W_n(x_1,\ldots,x_n) = W_{n+1}(x_1,\ldots,x_n,x_{n+1}).$$

So, in matrix models, (D.8) realizes the notion of insertion operator of Definition 2.5.

D.1.4. Asymptotic Expansions. The regime, where $N\hbar = t \neq 0$ is fixed while $N \to \infty$ and $\hbar \to 0$, is the most interesting for applications of random matrices and has been extensively studied. For potentials V such that the zeroes of orthogonal polynomials accumulate on a single segment (the so-called one-cut regime) and under some extra-technical assumptions, it was proved [2,15,20,35] that the asymptotic expansion takes the form:

$$\begin{aligned} \ln Z_N - \hbar^{-2}\ln(\hbar) &= \sum_{g\geq 0}\hbar^{2g-2}\,F^{(g)}, \\ W_n(x_1,\ldots,x_n) &= \sum_{g\geq 0}\hbar^{2g-2+n}\,W_n^{(g)}(x_1,\ldots,x_n), \end{aligned} \tag{D.9}$$

i.e. the correlators do have an expansion of topological type. The proof in [2,20] relies on the analysis of the Schwinger–Dyson equation of the matrix model, that is roughly speaking on the loop equations of Propositions 2.1 and 2.2. Actually, the main task of those works was to prove the existence of an expansion in $1/N$ and that $W_n \in O(N^{2-n})$. It is then a famous and early noticed fact—that can be attributed to [24] and [5]—that the Schwinger–Dyson equations of hermitian matrix models force the expansion of W_n to have parity $(-1)^n$, i.e. it is really an expansion in $1/N^2$. Taking (D.9) as a starting point, the solution of the loop equations was written for the first time in the form of Eq. (3.13) in [36] for this 1-hermitian matrix model. It was then realized that this universal form is relevant beyond the scope of the 1-matrix model, and even beyond matrix models.

Determinantal formulae are difficult to handle in the $\hbar \to 0$ limit, because many cancellations can occur in the alternating sum contained in the determinants. Informally put, the loop equation approach takes care of those cancellations and is better suited for the asymptotic analysis.

D.2. The 2-Matrix Model

The story is almost the same for the 2-matrix model, but the rank d of the differential equation can be larger than 2. Consider the 2-matrix integral:

$$Z_N = \iint_{H_N\times H_N} \mathrm{d}M\;\mathrm{d}\tilde{M}\,\mathrm{e}^{-\hbar^{-1}\,\mathrm{Tr}\,(V(M)+\tilde{V}(\tilde{M})-c\,M\tilde{M})},$$

where $V(M) = t_0 + \sum_{k=1}^{d} t_k M^k/k$ and $\tilde{V}(M) = \tilde{t}_0 + \sum_{k=1}^{\tilde{d}} \tilde{t}_k M^k/k$ are 2 polynomials, called the potentials. Z_N is again a Miwa–Jimbo isomonodromic Tau function [12]. The expectation values of the characteristic polynomials now form a family of bi-orthogonal polynomials:

$$p_N(x) = \mathbb{E}_N\big[\det(x\mathbf{1} - M)\big], \qquad \tilde{p}_N(y) = \mathbb{E}_N\big[\det(y\mathbf{1} - \tilde{M})\big],$$

with scalar product:

$$\int_{\mathbb{R}^2} p_N(x)\, \tilde{p}_L(y)\, \mathrm{e}^{-\hbar^{-1}\,(V(x)+\tilde{V}(y)-c\,xy)}\, \mathrm{d}x\, \mathrm{d}y = h_N \delta_{N,L}.$$

The p_N (resp. $\tilde{p}_M$) do satisfy a linear order $\tilde{d}$ (resp. d) isomonodromic ODE. For instance for $\psi_N(x)$, we have:

$$\hbar\, \partial_x \mathbf{\Psi}(x) = \mathbf{L}_N(x)\mathbf{\Psi}(x), \qquad \mathbf{\Psi}(x) = \begin{pmatrix} \psi_N(x) & \ldots \\ \vdots & \vdots \\ \psi_{N-\tilde{d}+1}(x) & \ldots \end{pmatrix}.$$

The entries in the first column of $\mathbf{\Psi}_N(x)$ are the normalized biorthogonal polynomials

$$\psi_N(x) = \frac{p_N(x)}{\sqrt{h_N}}\, \mathrm{e}^{-\hbar^{-1}\, V(x)},$$

and the other columns can be expressed in terms of Fourier and Hilbert transforms of the latter, cf. [9]. It is also shown that the t_j (resp. $\tilde{t}_j$) are isomonodromic times for the ODE (D.10) (resp. the analog ODE for $\tilde{\mathbf{\Psi}}_N(x)$).

Paragraph D.1.2 can be almost repeated here. In particular, it is a non-trivial result of [7,45] that the determinantal formulae of Sect. D.1.2 for expectation values of ratios of characteristic polynomials of any of the matrices (say M) hold as well in the 2-matrix model. The only difference is that the expression of the scalar kernel $K(y,x)$ in terms of ψ and ϕ is not as simple, but continues to coincide with the $(1,1)$ entry of the matrix kernel. Likewise, the density correlations are determinants of the kernel obtained from $K(y,x)$ by computing its discontinuity when y crosses the real axis. The result is a generalization of the Christoffel–Darboux formula, in particular (D.5) does not hold. The n-point correlators of the matrix M—defined in the matrix model by Equations (D.6)–(D.7) again—satisfy Schwinger–Dyson equations of degree $\tilde{d}$ written in [42]. Notice that, if $\tilde{d} = 2$, the measure for the matrix $\tilde{M}$ is Gaussian so it can be integrated out and we are left with a 1-matrix model, and we indeed find Schwinger–Dyson equations that are quadratic.

Establishing an all-order asymptotic expansion when $N \to \infty$ is notably more difficult in the 2-matrix model. A consequence of the general results of [48,56] is that, for $V(M)$ and $\tilde{V}(M)$ close enough to Gaussians and c small enough, the correlators (with respect to M, or with respect to $\tilde{M}$) do have an asymptotic expansion of topological type (D.9). In any case, if there is an expansion of topological type in a 2-matrix model, it was then shown in [42] that the $W_n^{(g)}$ are computed by the topological recursion of Theorem 3.1. The strategy of [42] actually consists in showing directly that if one inserts the topological expansion in the Schwinger–Dyson equation, after a long algebra, the $1/N$-expanded version of the loop equations of Propositions 2.1 and 2.2 is implied.

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