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The μ-ordinary Hasse invariant of unitary Shimura varieties

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Abstract. We construct a generalization of the Hasse invariant for any Shimura variety of PEL-type A over a prime of good reduction, whose non-vanishing locus is the open and dense μ-ordinary locus.

1. Introduction

Let p be a prime number and let sh be a special fiber modulo p of a Shimura variety of PEL-type at a neat level which is hyperspecial at p. The classical Hasse invariant H is, roughly speaking, an automorphic form mod p of weight $p-1$. The classical Hasse invariant satisfies the following four properties:

(Ha1) The non-vanishing locus of H is the ordinary locus of sh, namely the locus of points where the underlying abelian variety is ordinary.

(Ha2) The construction of H is compatible with varying the prime-to-p level.

(Ha3) A power of H extends to the minimal compactification of sh.

(Ha4) A power of H lifts to characteristic zero.

The Hasse invariant is the main tool to construct congruences modulo powers of p, both in the realms of automorphic forms and of Galois representations. However, when $\mathfrak{p}$ is a prime of the reflex field E of the Shimura variety for which the $\mathfrak{p}$-adic completion $E_{\mathfrak{p}}$ is strictly larger than $\mathbf{Q}_p$, the ordinary locus is empty and the Hasse invariant is identically zero.

To fix this, we construct a generalized Hasse invariant satisfying properties (Ha2)–(Ha4) and a "μ-ordinary" analogue of (Ha1) for any Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$ of PEL-type such that $\mathbf{G}$ is a group of unitary similitudes. The non-vanishing locus of our generalized Hasse invariant is the μ-ordinary locus, which, as Moonen has shown [13, Theorems 1.3.7, 3.2.7], is simultaneously the largest stratum of the Newton and of the Ekedahl–Oort stratifications. As an application, we use our new Hasse invariant to generalize the main result of [4], which

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concerns attaching Galois representations to automorphic representations whose archimedean component is a holomorphic limit of discrete series.

The main idea in this paper is to use the action of Frobenius $\mathbf{F}$ on the crystalline cohomology of abelian varieties. The use of this cohomology theory allows us to divide by p, i.e., to make sense of the operator "$\bigwedge^i \mathbf{F}/p^j$" for well-chosen positive integers i and j, see below. In the main body of the paper, we pursue the Newton point of view and apply the Newton–Hodge decomposition of Katz, a convenient tool in this context. In the first appendix, we illustrate how we can retrieve most of our results purely from the Ekedahl–Oort point of view. In the second appendix, we show how we can avoid the use of crystalline cohomology when the totally real field F^+ is equal to $\mathbf{Q}$ or, equivalently, that $\mathbf{G}(\mathbf{R})$ is isomorphic to the unitary group $\mathbf{GU}(a,b)$ for some $a, b \in \mathbf{N}_{>0}$.

We note that this article is the result of merging our two arXiv postings [5] and [6]. We also remark that, a little over one year after we posted [6] on arXiv, Koskivirta and Wedhorn posted a preprint in which they construct generalized Hasse invariants for Shimura varieties of Hodge type, see [9].

1.1. Main results. Throughout this paper, fix an isomorphism $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$.

Suppose $\mathcal{U} = (B, V, * <,>, \tilde{h})$ is a Kottwitz datum with associated Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$ such that the center of the simple $\mathbf{Q}$-algebra B is a totally imaginary quadratic field extension F of a totally real field F^+ ([4, Section 3.1]). Let d be the degree of F^+ over $\mathbf{Q}$. Suppose p is a prime of good reduction for $\mathcal{U}$ (see [4, Section 3.3]) and $\mathcal{K}^{(p)} \subset \mathbf{G}(\mathbf{A}_f^p)$ is a neat, open compact subgroup.

Let $E = E(\mathbf{G}, \mathbf{X})$ be the reflex field of $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$. Let $\mathrm{Sh} := \mathrm{Sh}_{\mathcal{K}^{(p)}}$ be the Kottwitz integral model of $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$ at level $\mathcal{K}^{(p)}$ over $\mathbf{Z}_{(p)} \otimes \mathcal{O}_E$. Let $\mathfrak{p}$ be a prime of E above p. Denote by $\mathrm{sh} := \mathrm{sh}_{\mathcal{K}^{(p)},\mathfrak{p}}$ the special fiber of $\mathrm{Sh}_{\mathcal{K}^{(p)}}$ at $\mathfrak{p}$. Let ω be the Hodge line bundle of sh as defined in Section 2.

Theorem 1.1. *There exists an explicit positive integer $m \in \mathbf{Z}_{\geq 1}$ and a section*

$$ {}^{\mu}H \in H^0(\mathrm{sh}, \omega^m) \tag{1.1} $$

satisfying the following four properties:

(μ-Ha1) *The non-vanishing locus of ${}^{\mu}H$ is the μ-ordinary locus of* sh*, as defined in* [15, 16].

(μ-Ha2) *The construction of ${}^{\mu}H$ is compatible with varying the level $\mathcal{K}^{(p)}$.*

(μ-Ha3) *The section ${}^{\mu}H$ extends to the minimal compactification.*

(μ-Ha4) *A power of ${}^{\mu}H$ lifts to characteristic zero.*

We call ${}^{\mu}H$ the μ-ordinary Hasse invariant.

Remark 1.2. The exponent m in Theorem 1.1 is explicitly defined in Definition 3.5, in terms of the action of Frobenius on the embeddings of F. In case p remains prime in F, the formula one finds there simplifies to $m = p^{2d} - 1$.

By ampleness of the Hodge line bundle ω on the minimal compactification (cf. [10, Theorem 7.2.4.1, no. 2]), we deduce the following corollary.

Corollary 1.3. *The μ-ordinary locus* $\mathrm{sh}^{min,\mu-\mathrm{ord}}$ *in the minimal compactification* $\mathrm{sh}^{\mathrm{min}}$ *is affine.*

1.2. Application to Galois representations. We also obtain an application to the construction of automorphic Galois representations which generalizes [4, Theorem 1.2.1]. To state the result we need some notation.

Suppose π is a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ with v-adic component π_v for every place v. Given a prime p, let $\mathcal{P}^{(p)}$ be the set of primes v different from p such that π_v is unramified and $\mathbf{G}$ is unramified at v. Let $\mathfrak{P}^{(p)}$ be the set of primes of F that are split over F^+ and lie over some $v \in \mathcal{P}^{(p)}$.

Assume $w \in \mathfrak{P}^{(p)}$. One has a decomposition $\mathbf{G}(\mathbf{Q}_v) \cong \mathrm{GL}(n, F_w) \times G_{v,\mathrm{rest}}$, for some group $G_{v,\mathrm{rest}}$, where n is given by $n^2 = \dim_F \mathrm{End}_B V$. Write $\pi_v \cong \pi_w \otimes \pi_{v,\mathrm{rest}}$, with π_w a representation of $\mathrm{GL}(n, F_w)$ and $\pi_{v,\mathrm{rest}}$ a representation of $G_{v,\mathrm{rest}}$.

Theorem 1.4. *Suppose π is a cuspidal automorphic representation of* $\mathbf{G}(\mathbf{A})$ *whose archimedean component π_∞ is an* $\mathbf{X}$*-holomorphic limit of discrete series representation of* $\mathbf{G}(\mathbf{R})$ *(see* [4, Section 2.3]*). Assume p is a prime of good reduction for $\mathcal{U}$. Then there exists a unique semisimple Galois representation*

$$R_{p,\iota}(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(n, \overline{\mathbf{Q}}_p) \tag{1.2}$$

satisfying the following two conditions:

(Gal1) *If $v \in \mathcal{P}^{(p)}$ and w is a prime of F dividing v, then $R_{p,\iota}(\pi)$ is unramified at w. In particular, $R_{p,\iota}(\pi)$ is unramified at all but finitely many places.*

(Gal2) *If $w \in \mathfrak{P}^{(p)}$, then there is an isomorphism of Weil–Deligne representations*

$$(R_{p,\iota}(\pi)|_{W_{F_w}})^{\mathrm{ss}} \cong \iota^{-1}\mathrm{rec}\big(\pi_w \otimes |\cdot|_w^{\frac{1-n}{2}}\big), \tag{1.3}$$

where W_{F_w} is the Weil group of F_w, the superscript ss *denotes semi-simplification and* rec *is the local Langlands correspondence, normalized as in* [7].

Remark 1.5. The argument given in [4, Section 6] carries over almost verbatim (see Section 5 for a minor correction) and shows that our main result Theorem 1.1 implies our application Theorem 1.4.

2. Preliminaries on F-crystals and the Hodge filtration

Let $E \subset E' \subset \mathbf{C}$, where E' is a finite extension of E such that B is split over E' and for every embedding $\tau : F \hookrightarrow \mathbf{C}$, one has $\tau(F) \subset E'$. Denote by $\mathfrak{p}$ a prime of E over p, and by $\mathfrak{p}'$ a prime of E' over $\mathfrak{p}$. Pick κ to be the smallest finite field containing the residue fields $\mathcal{O}_{E'}/\mathfrak{p}'$, for all $\mathfrak{p}'$ over $\mathfrak{p}$. Via $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$, there is a bijection $\tau \mapsto \iota^{-1} \circ \tau$ between the set of complex embeddings $\tau : F \hookrightarrow \mathbf{C}$ and the set of p-adic embeddings $\iota^{-1} \circ \tau : F \hookrightarrow \overline{\mathbf{Q}}_p$ and we denote either type of embedding simply by τ. After fixing an embedding $W(\kappa) \hookrightarrow \mathbf{C}$, there is further a bijection with the set of embeddings of $\mathcal{O}_F$ into $W(\kappa)$, and also with the set of homomorphisms to κ, noted $\mathrm{Hom}(\mathcal{O}_F, \kappa)$. The absolute Frobenius, noted σ, acts via composition on $\mathrm{Hom}(\mathcal{O}_F, \kappa)$.

Let $\mathcal{T}$ be the set of complex embeddings of F. Let r be the rank of B over F. From here onwards, fix the prime $\mathfrak{p}$ in $\mathcal{O}_E$. Let S be a smooth $\mathrm{Spec}(\mathcal{O}_E/\mathfrak{p})$-scheme and $\pi : A \to S$ a $\mathcal{U}^{(p)}$-enriched abelian scheme [4, Section 3.4]. Let

$$\omega = \bigwedge^{\mathrm{top}} \pi_* \Omega^1_{A/S}$$

be the Hodge bundle, i.e., the determinant of the pushforward of the sheaf of relative differentials on A. After extending scalars to κ, the Hodge bundle decomposes according to the embeddings $\tau \in \mathcal{T}$ and the standard idempotents in $M_r(\kappa)$:

$$\omega = \bigotimes_{\tau \in \mathcal{T}} \omega_\tau^{\otimes r}.$$

The Dieudonné crystal $H^1_{\mathrm{crys}}(A)$ also decomposes accordingly:

$$H^1_{\mathrm{crys}}(A) = \bigoplus_{\tau \in \mathcal{T}} H^1_{\mathrm{crys}}(A)_\tau^{\oplus r}.$$

Similarly for de Rham cohomology, one has

$$H^1_{\mathrm{dR}}(A) = \bigoplus_{\tau \in \mathcal{T}} H^1_{\mathrm{dR}}(A)_\tau^{\oplus r}.$$

Put

$$H^d_{\mathrm{crys}}(A)_{\tau_i} = \bigwedge^d H^1_{\mathrm{crys}}(A)_{\tau_i}, \quad H^d_{\mathrm{dR}}(A)_{\tau_i} = \bigwedge^d H^1_{\mathrm{dR}}(A)_{\tau_i}.$$

Let $\mathrm{Fil}^\bullet$ denote the Hodge filtration on the de Rham cohomology. Put

$$\mathrm{Fil}^1_\tau = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(A) \cap H^1_{\mathrm{dR}}(A)_\tau.$$

Then $(\mathrm{rank}\,\mathrm{Fil}^1_\tau, \mathrm{rank}\,\mathrm{Fil}^1_{\overline{\tau}})$ is the signature corresponding to the conjugate pair of embeddings $(\tau, \overline{\tau})$.

Given $\tau \in \mathcal{T}$, define $\mathfrak{o}_\tau$ to be the orbit of τ under the action of the absolute Frobenius σ. Let e_τ denote the cardinality of the orbit $\mathfrak{o}_\tau$. Write $\mathfrak{o}_\tau = \{\tau_1, \dots, \tau_{e_\tau}\}$ in such a way that $\mathrm{rank}\,\mathrm{Fil}^1_{\tau_1} \geq \dots \geq \mathrm{rank}\,\mathrm{Fil}^1_{\tau_{e_\tau}}$. The rank of $H^1_{\mathrm{dR}}(A)_\tau$ is independent of τ; we call it n. Define the multiplication type $\mathfrak{f} : \mathfrak{o}_\tau \to \{0, 1, \dots, n\}$ associated to $\mathfrak{o}_\tau$ by $\mathfrak{f}(\tau_i) = \mathrm{rank}\,\mathrm{Fil}^1_{\tau_i}$. To the pair $(n, \mathfrak{f})$ depending on $\mathfrak{o}_\tau$, Moonen [13, 1.2.5] associates a polygon $\mathrm{ord}_{\mathfrak{o}_\tau}(n, \mathfrak{f})$ that we call the μ-ordinary polygon associated to $\mathfrak{o}_\tau$. Recall that the slopes a_j, $1 \leq j \leq n$, of $\mathrm{ord}_{\mathfrak{o}_\tau}(n, \mathfrak{f})$ are defined by

$$a_j := \mathrm{card}(\{\tau' \in \mathfrak{o}_\tau \mid \mathfrak{f}(\tau') > n - j\}). \tag{2.1}$$

Now suppose $S = \mathrm{Spec}\, k$, where k is an algebraically closed field, so that A represents a geometric point of sh. Put $M = H^1_{\mathrm{crys}}(A)$ (resp. $M_\tau = H^1_{\mathrm{crys}}(A)_\tau$). Define the Hodge (resp. Newton) polygon of M_τ to be the Hodge (resp. Newton) polygon of $(M_\tau, \mathbf{F}^{e_\tau})$. Note that in general the Newton polygon of M_τ does not depend on τ but the Hodge polygon does.

Lemma 2.1. *Let $\mathcal{T} = \coprod \mathfrak{o}_\tau$ be the orbit decomposition of $\mathcal{T}$ according to the action of Frobenius. Let $M = H^1_{\mathrm{crys}}(A)$. Then the Newton polygon of $(M, \mathbf{F})$ is the Newton polygon* $\mathrm{NP}(\mathrm{sh}^{\mu-\mathrm{ord}})$ *of the μ-ordinary locus (i.e., A is μ-ordinary) if and only if for all $\tau \in \mathcal{T}$ the Newton polygon of M_τ is the μ-ordinary polygon* $\mathrm{ord}_{\mathfrak{o}_\tau}(n, \mathfrak{f})$.

Proof. See [16, 2.2.1]. □

Lemma 2.2. *Suppose the Newton polygon of M_τ is the μ-ordinary polygon* $\mathrm{ord}_{\mathfrak{o}_\tau}(n,\mathfrak{f})$. *Then the Hodge polygon of M_τ also coincides with* $\mathrm{ord}_{\mathfrak{o}_\tau}(n,\mathfrak{f})$.

In particular, under the assumption of being μ-ordinary, the Hodge polygon of M_τ depends only on the orbit $\mathfrak{o}_\tau$.

Proof. This follows from the proof of [13, 1.3.7]. □

Suppose A is μ-ordinary. Combining Lemmas 2.1 and 2.2, both the Newton polygon and the Hodge polygon of M_τ are equal to $\mathrm{ord}_{\mathfrak{o}_\tau}(n,\mathfrak{f})$. By the Hodge–Newton decomposition [8, 1.6.1], one can write

$$M_\tau = \bigoplus_{j=0}^{e_\tau} M_\tau^{[j]}, \tag{2.2}$$

where $M_\tau^{[j]}$ is an isoclinic subcrystal of slope j.

Theorem 2.3. *One has*

$$\mathrm{Fil}^1_{\tau_i} = \bigoplus_{j\geq i} \overline{M_{\tau_i}^{[j]}}. \tag{2.3}$$

Proof. By the explicit description of $\mathrm{ord}_{\mathfrak{o}_\tau}(n,\mathfrak{f})$, we see that the two sides of (2.3) have the same dimension. Hence it suffices to show the inclusion

$$\mathrm{Fil}^1_{\tau_i} \supset \bigoplus_{j\geq i} \overline{M_{\tau_i}^{[j]}}. \tag{2.4}$$

To this effect, our main tool will be Mazur's theorem which we recall now.

Theorem 2.4 (Mazur). *Let A be an abelian variety over an algebraically closed field k of characteristic p. Denote by $\overline{\square}$ the reduction modulo p of $\square$. Then for all $j,m \in \mathbf{Z}_{\geq 0}$, one has*

$$\mathrm{Fil}^j H^m_{\mathrm{dR}}(A) = \overline{\mathbf{F}^{-1}(p^j H^m_{\mathrm{crys}}(A))}, \tag{2.5}$$

where $\mathbf{F}$ is the canonical lifting of Frobenius on crystalline cohomology.

Proof. Let A be an abelian variety over an algebraically closed field of characteristic 0. Since the Hodge–de Rham spectral sequence of A degenerates at E_1 and since the crystalline cohomology of A torsion-free, the theorem is a special case of [1, 8.26]. □

Theorem 2.3 will now be proved as follows: Lemma 2.5–Corollary 2.9 are of a preparatory nature. The crux of the proof of Theorem 2.3 is contained in Lemmas 2.10 and 2.11.

Lemma 2.5. *Suppose, for $i \in \{1,2\}$, that $(M_i,\mathbf{F}_i)$ is a $W(k)$-module which is an ordinary $\mathbf{F}_i$-crystal, i.e., the Hodge and Newton polygons of $(M_i,\mathbf{F}_i)$ coincide. Let*

$$\varphi : (M_1,\mathbf{F}_1) \to (M_2,\mathbf{F}_2)$$

be an isogeny, so in particular the Newton polygon of $(M_1,\mathbf{F}_1)$ is the same as that of $(M_2,\mathbf{F}_2)$.

Let $0 \leq \lambda_1 < \cdots < \lambda_s$ *be the slopes of* $(M_i, \mathbf{F}_i)$ *with multiplicities* $m_1, \ldots, m_s$. *Let*

$$M_i = \bigoplus_{j=1}^{s} M_{i,j} \tag{2.6}$$

be the Newton–Hodge decomposition [8, 1.6.1] *applied to* $(M_i, \mathbf{F}_i)$ *so that* $(M_{i,j}, \mathbf{F}_i)$ *is an isoclinic subcrystal of rank* m_j *and slope* λ_j. *Then* $\varphi(M_{1,j}) \subset M_{2,j}$

Proof. Since φ is an isogeny and $M_{1,j}$ is an $\mathbf{F}_1$-subcrystal of M_1, the image $\varphi(M_{1,j})$ is an $\mathbf{F}_2$-subcrystal of M_2. Since the Newton polygon is invariant under isogeny, $(\varphi(M_{1,j}), \mathbf{F}_2)$ is isoclinic of slope λ_j with multiplicity m_j. Let M' denote the $\mathbf{F}_2$ subcrystal of M_2 generated by $\varphi(M_{1,j})$ and $M_{2,j}$. Then M' is isoclinic of slope λ_j, so the rank of M' is m_j. Since $M_2/M_{2,j}$ is free, we conclude that $M' = M_{2,j}$. Therefore $\varphi(M_{1,j}) \subset M_{2,j}$. □

Remark 2.6. *Lemma* 2.5 *also follows more generally from the fact that homomorphisms of* F*-crystals respect the slope decomposition, see* [2, Property e), p. 81].

Lemma 2.7. *Let* $M = H^1_{\text{crys}}(A)$, *where* A *is an abelian variety. Suppose* $k \in \mathbf{Z}_{\geq 2}$, $x \in M$ *and* $\mathbf{F}(x) \in p^k M$. *Then* $x \in p^{k-1} M$.

Proof. Since $\mathbf{F}(x) \in p^k M$, Mazur's theorem entails that $\bar{x} \in \mathrm{Fil}^k \bar{M}$. But $k \geq 2$, so $\mathrm{Fil}^k \bar{M} = \{0\}$. Hence $\bar{x} = 0$, so $x \in pM$. If $k = 2$, we are done. So assume $k > 2$ and write $x = py$, for some $y \in M$. Then $\mathbf{F}(x) = \mathbf{F}(py) = p\mathbf{F}(y)$ and $\mathbf{F}(x) \in p^k M$ implies $\mathbf{F}(y) \in p^{k-1} M$. By induction on k, one has $y \in p^{k-2} M$, whence $x \in p^{k-1} M$. □

Corollary 2.8. *Let* $M = H^1_{\text{crys}}(A)$. *Suppose* $j, k \in \mathbf{Z}_{\geq 2}$, $x \in M$ *and* $\mathbf{F}^j(x) \in p^k M$. *Then* $\mathbf{F}^{j-1}(x) \in p^{k-1} M$.

Proof. Write $\mathbf{F}^j(x) = \mathbf{F}(\mathbf{F}^{j-1}(x))$ and apply Lemma 2.7. □

Corollary 2.9. *Let* $M = H^1_{\text{crys}}(A)$. *Suppose* $k \in \mathbf{Z}_{\geq 1}$, $x \in M$ *and* $\mathbf{F}^k(x) \in p^k M$. *Then* $\bar{x} \in \mathrm{Fil}^1 \bar{M}$.

Proof. Applying Corollary 2.8 repeatedly $k-1$ times gives $\mathbf{F}(x) \in pM$. Then the conclusion follows from Mazur's theorem. □

Lemma 2.10. *The following inclusion holds:*

$$\mathrm{Fil}^1_{\tau_i} \supset \overline{M^{[e_\tau]}_{\tau_i}}. \tag{2.7}$$

Proof. Since $M^{[e_\tau]}_{\tau_i}$ is isoclinic of slope e_τ, we have

$$\mathbf{F}^{e_\tau}(M^{[e_\tau]}_{\tau_i}) \subset p^{e_\tau} M^{[e_\tau]}_{\tau_i}.$$

Suppose $x \in M^{[e_\tau]}_{\tau_i}$. Then

$$\mathbf{F}^{e_\tau}(x) \in p^{e_\tau} M^{[e_\tau]}_{\tau_i},$$

so the conclusion follows from Corollary 2.9. □

Lemma 2.11. *Let $\nu \in \{0, 1, \dots, e_\tau - 1\}$. Then for all $i \leq e_\tau - \nu$, one has*

$$\mathrm{Fil}^1_{\tau_i} \supset M^{[e_\tau - \nu]}_{\tau_i} \tag{2.8}$$

Proof. The proof is by induction on ν. The case $\nu = 0$ is Lemma 2.10. Suppose (2.8) holds up to $\nu - 1$. Then we have

$$\mathrm{Fil}^1_{\tau_{e_\tau - \beta}} = \bigoplus_{j \geq e_\tau - \beta} \overline{M^{[j]}_{\tau_{e_\tau - \beta}}}$$

for all $\beta \leq \nu - 1$.

Consider the diagram

$$M_{\tau_i} \xrightarrow{\mathbf{F}} M_{\sigma\tau_i} \xrightarrow{\mathbf{F}} \cdots \xrightarrow{\mathbf{F}} M_{\sigma^{e_\tau - 1}\tau_i} \xrightarrow{\mathbf{F}} M_{\tau_i}.$$

Let $t_1 \geq t_2 \geq \cdots \geq t_\nu$ such that for all α, $1 \leq \alpha \leq \nu$, one has $\sigma^{t_\alpha}\tau_i = \tau_{j_\alpha}$ and $j_\alpha > e_\tau - \nu$. Let $x \in M^{e_\tau - \nu}_{\tau_i}$. Then $\mathbf{F}^{e_\tau}(x) \in p^{e_\tau - \nu} M_{\tau_i}$. By Corollary 2.9, we can subtract $e_\tau - t - 1$ from the exponents on both sides, thus obtaining

$$\mathbf{F}^{t_1 + 1}(x) \in p^{t_1 + 1 - \nu} M_{\sigma^{t_1 + 1}\tau_i}.$$

Writing

$$\mathbf{F}\left(\frac{\mathbf{F}^{t_1}}{p^{t_1 - \nu}}(x)\right) \in p M_{\sigma^{t_1 + 1}\tau_i},$$

we see by Mazur's theorem that

$$\frac{\mathbf{F}^{t_1}}{p^{t_1 - \nu}}(\overline{x}) \in \mathrm{Fil}^1_{\sigma^{t_1}\tau_i} = \mathrm{Fil}^1_{\tau_{j_1}}.$$

Since $j_1 > e_\tau - \nu$, by the induction hypothesis and equality of dimensions, we have

$$\mathrm{Fil}^1_{\tau_{j_1}} = \bigoplus_{j \geq j_1} \overline{M^{[j]}_{\tau_{j_1}}}.$$

On the other hand, by assumption, $x \in M^{[e_\tau - \nu]}_{\tau_i}$. Since $\mathbf{F}^{t_1}/p^{t_1 - \nu}$ is an isogeny, Lemma 2.5 implies that

$$\frac{\mathbf{F}^{t_1}}{p^{t_1 - \nu}}(x) \in M^{e_\tau - \nu}_{\tau_{j_1}}.$$

Hence

$$\frac{\mathbf{F}^{t_1}}{p^{t_1 - \nu}}(\bar{x}) \in \overline{M^{e_\tau - \nu}_{\tau_{j_1}}} \cap \bigoplus_{j \geq j_1} \overline{M^{[j]}_{\tau_{j_1}}} = \{0\}.$$

Therefore $\mathbf{F}^{t_1}(x) \in p^{t_1 - \nu + 1} M_{\tau_{j_1}}$.

Repeating the same argument with t_2 we obtain $\mathbf{F}^{t_2}(x) \in p^{t_2 - \nu + 2} M_{\tau_{j_2}}$. Continuing in this way we finally arrive at $\mathbf{F}^{t_\nu}(x) \in p^{t_\nu} M_{\tau_{j_\nu}}$ and one last application of Corollary 2.9 yields $\bar{x} \in \mathrm{Fil}^1_{\tau_i}$. □

Lemma 2.11 completes the proof of Theorem 2.3. □

3. The generalized Hasse invariants

Based on the results and notation of the previous section, we are in position to define the desired generalized Hasse invariants.

Let $\mathcal{A} \to \mathrm{sh}$ be a representative of the universal isogeny class. The absolute Frobenius morphism

$$\mathbf{F} : \mathcal{A} \to \mathcal{A}$$

induces a σ-linear map

$$\mathbf{F} : H^1_{\mathrm{crys}}(\mathcal{A}) \to H^1_{\mathrm{crys}}(\mathcal{A}). \tag{3.1}$$

As we have seen in Section 2, this map permutes non-trivially the factors indexed by the embeddings τ. This permutation can be decomposed into cycles according to the orbits $\mathfrak{o}_\tau$. Consider such an orbit $\mathfrak{o}_\tau = \{\tau_1, \dots, \tau_{e_\tau}\}$. Let $\mathrm{Gr}^0_{\tau_i} = H^1_{\mathrm{dR}}(\mathcal{A})_{\tau_i}/\mathrm{Fil}^1_{\tau_i}$. Set $d_i = \dim \mathrm{Gr}^0_{\tau_i}$ and $c_i = (i-1)d_i - (d_1 + \cdots + d_{i-1})$.

Lemma 3.1. *The map*

$$\bigwedge^{d_i} \mathbf{F}^{e_\tau} : H^{d_i}_{\mathrm{crys}}(\mathcal{A})_{\tau_i} \to H^{d_i}_{\mathrm{crys}}(\mathcal{A})_{\tau_i} \tag{3.2}$$

is divisible by p^{c_i}.

Proof. Since the μ-ordinary locus $\mathrm{sh}^{\mu\text{-ord}}$ is open and dense [16, Theorem 1.6.2], it suffices to prove the divisibility for every μ-ordinary geometric point A. (We thank David Geraghty for pointing out to us that this follows from [3], specifically remarks in [3, Sections 1.1–1.2 and Section 2.3.4], using the fact that our Shimura variety sh is smooth over a field.) By Lemma 2.2 we know that the Hodge polygon of M_{τ_i} is $\mathrm{ord}_{\mathfrak{o}_\tau}(n, \mathfrak{f})$. Since the smallest slope of the Hodge polygon of $\bigwedge^{d_i} M_{\tau_i}$ is the sum of the d_i smallest slopes of the Hodge polygon of M_{τ_i}, the smallest slope of the Hodge polygon of $\bigwedge^{d_i} M_{\tau_i}$ is

$$\sum_{j=1}^{i-1} j(d_{j+1} - d_j) = c_i, \tag{3.3}$$

so the lemma follows from [8, 1.2.1]. □

Lemma 3.2. *The restriction of the map*

$$\frac{\bigwedge^{d_i} \mathbf{F}^{e_\tau}}{p^{c_i}} : H^{d_i}_{\mathrm{dR}}(\mathcal{A})_{\tau_i} \to H^{d_i}_{\mathrm{dR}}(\mathcal{A})_{\tau_i} \tag{3.4}$$

to $\mathrm{Fil}^1 H^{d_i}_{\mathrm{dR}}(\mathcal{A})_{\tau_i}$ *is zero.*

Proof. Again, because the μ-ordinary locus is open and dense [16, Theorem 1.6.2], it suffices to prove the vanishing for every μ-ordinary geometric point A. Let

$$W_{\tau_i} = \bigoplus_{j<i} M^{[j]}_{\tau_i}.$$

By Theorem 2.3, we have a decomposition

$$\overline{M_{\tau_i}} = \mathrm{Fil}^1_{\tau_i} \oplus \overline{W_{\tau_i}}.$$

Thus

$$\mathrm{Fil}^1 H^{d_i}_{\mathrm{dR}}(A)_{\tau_i} = \bigoplus_{s=1}^{d_i} \Big(\bigwedge^s \mathrm{Fil}^1_{\tau_i} \otimes \bigwedge^{d_i - s} \overline{W_{\tau_i}} \Big).$$

Therefore Lemma 3.2 is equivalent to showing that the restriction of $(\bigwedge^{d_i} \mathbf{F}^{e_\tau})/p^{c_i})$ to $\bigwedge^s \mathrm{Fil}^1_{\tau_i} \otimes \bigwedge^{d_i-s} \overline{W_{\tau_i}}$ is zero for all $s \geq 1$. So fix s and let $x \in \bigwedge^s \mathrm{Fil}^1_{\tau_i} \otimes \bigwedge^{d_i-s} \overline{W_{\tau_i}}$.

Let

$$M' = \bigwedge^s \Big(\bigoplus_{j \geq i} M^{[j]}_{\tau_i} \Big) \otimes \bigwedge^{d_i - s} \Big(\bigoplus_{j < i} M^{[j]}_{\tau_i} \Big).$$

By Theorem 2.3, there exists a lift $\tilde{x}$ of x to $H^{d_i}_{\mathrm{crys}}(A)_{\tau_i}$ which lies in M'. The smallest slope $\lambda_{\min}$ of the crystal $(M', (\bigwedge^{d_i} \mathbf{F}^{e_\tau}))$ is, by definition, the sum of the s smallest slopes of $\bigoplus_{j \geq i} M^{[j]}_{\tau_i}$ plus the sum of the $d_i - s$ smallest slopes of $\bigoplus_{j<i} M^{[j]}_{\tau_i}$. Since $s \geq 1$, $\lambda_{\min}$ is strictly bigger than the sum of the d_i smallest slopes of M_{τ_i}, and the latter is precisely c_i by definition. Thus, by [8, 1.2.1],

$$\Big(\bigwedge^{d_i} \mathbf{F}^{e_\tau} \Big)(\tilde{x}) \in p^{\lambda_{\min}} M_{\tau_i}$$

and therefore $(\bigwedge^{d_i} \mathbf{F}^{e_\tau} / p^{c_i})(\tilde{x}) \in p M_{\tau_i}$. □

By Lemma 3.2, we get an induced map

$$\frac{\bigwedge^{d_i} F^{e_\tau}}{p^{c_i}} : \mathrm{Gr}^0 H^{d_i}_{\mathrm{dR}}(\mathcal{A})_{\tau_i} \to \mathrm{Gr}^0 H^{d_i}_{\mathrm{dR}}(\mathcal{A})_{\tau_i}.$$

Since $\mathrm{Gr}^0 H^{d_i}_{\mathrm{dR}}(\mathcal{A})_{\tau_i} \cong \omega^{\vee}_{\tau_i}$, we obtain a section

$${}^{\tau_i}H \in H^0(\mathrm{sh}, \omega_{\tau_i}^{p^{e_\tau} - 1}).$$

Definition 3.3. The section ${}^{\tau_i}H$ is called the τ_i-Hasse invariant of sh.

As in [4, Theorem 4.2.1], the τ_i-Hasse invariant is compatible with isogenies in the sense that if $\varphi : \mathcal{A} \to \mathcal{B}$ is an isogeny preserving the $\mathcal{U}^{(p)}$-structure, then $\varphi^*({}^{\tau_i}H(\mathcal{B})) = {}^{\tau_i}H(\mathcal{A})$. Therefore the τ_i-Hasse invariant is well-defined.

Remark 3.4. *Compatibility with* [5]*: If $F^+ = \mathbf{Q}$, then ${}^{\tau_1}H$ is equal to the μ-Hasse invariant of* [5] *(see Appendix* B*).*

We are now in a position to define the μ-ordinary Hasse invariant in complete generality for unitary Shimura varieties.

Definition 3.5. Let $m = \mathrm{lcm}_{\tau \in \mathcal{T}}\{p^{e_\tau} - 1\}$, and let $m_\tau = m/(p^{e_\tau} - 1)$. We define the μ-ordinary Hasse invariant ${}^{\mu}H$ as the product

$$ {}^{\mu}H = \prod_{\tau \in \mathcal{T}} ({}^{\tau}H)^{m_\tau} \in H^0(\mathrm{sh}, \omega^m). \tag{3.5}$$

4. The non-vanishing loci of the Hasse invariants

We will describe the non-vanishing locus of the τ-Hasse invariant one embedding τ at a time. In the end, the non-vanishing locus of the μ-ordinary Hasse invariant will easily be read off as the μ-ordinary locus.

Theorem 4.1. *Let A be a geometric point of the special fiber* $\mathrm{sh}_{K^{(p)},\mathfrak{p}}$. *Then*

$$^{\tau_i}H(A) \neq 0$$

if and only if the Newton polygon of M_{τ_i} meets $\mathrm{ord}_{\mathfrak{o}_{\tau_i}}(n, \mathfrak{f})$ *at (d_i, c_i) in the notation of Section* 3.

Proof. By Rapoport–Richartz's version of Mazur's inequality (see [13, Lemma 1.3.4]), the Newton polygon of M_{τ_i} sits on or above the ordinary polygon $\mathrm{ord}_{\mathfrak{o}_{\tau_i}}(n, \mathfrak{f})$. Let (d_i, g_i) be the unique point on the Newton polygon of M_{τ_i} whose first coordinate is d_i. Since the point (d_i, c_i) lies on the polygon $\mathrm{ord}_{\mathfrak{o}_{\tau_i}}(n, \mathfrak{f})$, the point (d_i, g_i) is on or above (d_i, c_i), meaning that $g_i \geq c_i$. The rational number g_i is the sum of the first d_i slopes of the Newton polygon of M_{τ_i}, hence g_i is the smallest slope of the Newton polygon of $\bigwedge^{d_i} M_{\tau_i}$. Therefore the smallest Newton slope of

$$\left(\bigwedge^{d_i} M_{\tau_i}, \frac{\bigwedge^{d_i} \mathbf{F}^{e_\tau}}{p^{c_i}}\right)$$

is $g_i - c_i$. By [8, 1.3.3] the action of $\bigwedge^{d_i} \mathbf{F}^{e_\tau}/p^{c_i}$ on $\overline{\bigwedge^{d_i} M_{\tau_i}}$ is nilpotent if and only if this smallest slope is positive, i.e., if and only if $g_i > c_i$.

Since $\mathrm{Gr}^0 H^{d_i}_{\mathrm{dR}}(A)_{\tau_i}$ is a line, $\bigwedge^{d_i} \mathbf{F}^{e_\tau}/p^{c_i}$ acts on it by a scalar, namely $^{\tau_i}H(A)$. By Lemma 3.2, the action of $\bigwedge^{d_i} \mathbf{F}^{e_\tau}/p^{c_i}$ on $\overline{\bigwedge^{d_i} M_{\tau_i}}$ is nilpotent if and only if $^{\tau_i}H(A) = 0$. □

Corollary 4.2. *Let A be a geometric point of the special fiber* $\mathrm{sh}_{K^{(p)},\mathfrak{p}}$. *Then*

$$^{\mu}H(A) \neq 0$$

if and only if A is μ-ordinary.

Proof. By definition, $^{\mu}H(A) \neq 0$ if and only if $^{\tau}H(A) \neq 0$ for all $\tau \in \mathcal{T}$. By Theorem 4.1, for every orbit $\mathfrak{o}_\tau$ we have that $^{\tau'}H(A) \neq 0$ for all $\tau' \in \mathfrak{o}_\tau$ if and only if the Newton polygon of M_τ meets $\mathrm{ord}_{\mathfrak{o}_\tau}(n, \mathfrak{f})$ at every breakpoint of $\mathrm{ord}_{\mathfrak{o}_\tau}(n, \mathfrak{f})$, so $^{\tau'}H(A) \neq 0$ for all $\tau' \in \mathfrak{o}_\tau$ if and only if the Newton polygon of M_τ equals $\mathrm{ord}_{\mathfrak{o}_\tau}(n, \mathfrak{f})$. An application of Lemma 2.1 completes the proof. □

Proof of Theorem 1.1. Corollary 4.2 establishes (μ-Ha1). Properties (μ-Ha2)–(μ-Ha4) are proved in exactly the same way as in [4, Lemma 4.4.1, Theorem 6.2.1] □

5. Correction to [4]

Thanks to Jay Pottharst for pointing out the need to make the following minor modifications in [4, Section 6.2]: In the second and third sentences of the proof of [4, Theorem 6.2.1], the phrase "is non-zero" (resp. "is also non-zero") should be replaced with the phrase "is a non-zero divisor" (resp. "is also a non-zero divisor"). Moreover, in the third sentence, the word "separable" should be replaced with the word "finite". Finally, in the fifth sentence, the phrase "Since the product of two sections that are each non-zero modulo λ" should be replaced with the phrase "Since the product of a section which is non-zero modulo λ with section which is a non-zero divisor modulo λ". It should have also been pointed out in loc. cit. that the integral models defined there using normalization have the same number of connected components as the Kottwitz integral model, because this is so for the generic fibers.

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A. The point of view of Ekedahl–Oort

We keep the notation introduced in the main text. In particular, recall that $\mathfrak{o}_\tau$ denotes the orbit of embeddings of τ under Frobenius and e_τ denotes the cardinality of this orbit.

We begin by recalling Moonen's definition of "standard ordinary objects" [13, 1.2.3]. Given an orbit $\mathfrak{o}_\tau$ and its type $(n, \mathfrak{f})$, we have a Dieudonné module $M^{\mathrm{ord}_{\mathfrak{o}_\tau}}(n, \mathfrak{f})$ defined as follows: As $W(k)$-module, let $M^{\mathrm{ord}_{\mathfrak{o}_\tau}}(n, \mathfrak{f})$ be the free module generated by the basis consisting of symbols $\epsilon_{\tau_i, j}$ such that $\tau_i \in \mathfrak{o}_\tau$ and $1 \leq j \leq n$. On this basis, Frobenius acts by

$$\text{(A.1)} \qquad F(\epsilon_{\tau_i, j}) = \begin{cases} \epsilon_{\sigma\tau_i, j} & \text{if } \mathfrak{f}(\tau_i) \leq n - j, \\ p\epsilon_{\sigma\tau_i, j} & \text{if } \mathfrak{f}(\tau_i) > n - j, \end{cases}$$

and Verschiebung is given by

$$\text{(A.2)} \qquad V(\epsilon_{\sigma\tau_i, j}) = \begin{cases} p\epsilon_{\tau_i, j} & \text{if } \mathfrak{f}(\tau_i) \leq n - j, \\ \epsilon_{\tau_i, j} & \text{if } \mathfrak{f}(\tau_i) > n - j. \end{cases}$$

Put

$$M_{\tau_i}^{\mathrm{ord}_{\mathfrak{o}_{\tau_i}}}(n, \mathfrak{f}) = \mathrm{span}(\{\epsilon_{\tau_i, j} \mid 1 \leq j \leq n\}).$$

Note that the module $M_{\tau_i}^{\mathrm{ord}_{\mathfrak{o}_{\tau_i}}}(n, \mathfrak{f})$ is stable under $F^{e_{\tau_i}}$.

The key role played by the modules $M^{\mathrm{ord}_{\mathfrak{o}_\tau}}(n, \mathfrak{f})$ and $M_{\tau_i}^{\mathrm{ord}_{\mathfrak{o}_{\tau_i}}}(n, \mathfrak{f})$ stems from the following result of Moonen:

Theorem A.1 (Moonen [13, Theorem 1.3.7]). *Let A be a geometric point of the special fiber* sh. *Then A is μ-ordinary if and only if the Dieudonné module of A is isomorphic to*

$$\text{(A.3)} \qquad \bigoplus_{\textit{orbits } \mathfrak{o}_\tau} M^{\mathrm{ord}_{\mathfrak{o}_\tau}}(n, \mathfrak{f})^{\oplus r}.$$

Henceforth assume A is a μ-ordinary geometric point of the special fiber $\mathrm{sh}_{K^{(p)}, \mathfrak{p}}$. We identify the Dieudonné module (A.3) with $H^1_{\mathrm{crys}}(A)$ in such a way that the Frobenii F and $\mathbf{F}$ correspond to one another. Then the submodule $M_{\tau_i}^{\mathrm{ord}_{\mathfrak{o}_{\tau_i}}}(n, \mathfrak{f})$ corresponds to $H^1_{\mathrm{crys}}(A)_{\tau_i}$.

In the basis

$$\mathcal{B}_{d_i} = \{\epsilon_{\tau_i, j} \mid 1 \leq j \leq n\},$$

the matrix of $\mathbf{F}^{e_{\tau_i}}$ acting on $M_{\tau_i}^{\mathrm{ord}_{\mathfrak{o}_{\tau_i}}}(n, \mathfrak{f})$ is the diagonal matrix $\mathrm{diag}(p^{a_1}, \dots, p^{a_n})$, where the a_j are the slopes of $\mathrm{ord}_{\mathfrak{o}_{\tau_i}}(n, \mathfrak{f})$, whose definition was recalled in (2.1). Therefore the matrix of $\bigwedge^{d_i} \mathbf{F}^{|\mathfrak{o}_{\tau_i}|}$ in the basis

$$\mathcal{B}_{d_i} = \{\epsilon_{\tau_i, j_1} \wedge \cdots \wedge \epsilon_{\tau_i, j_{d_i}} \mid 1 \leq j_1 < \cdots < j_{d_i} \leq n\}$$

is the diagonal matrix with entry

$$p^{a_{j_1} + \cdots + a_{j_{d_i}}}$$

corresponding to the basis vector

$$\epsilon_{\tau_i, j_1} \wedge \cdots \wedge \epsilon_{\tau_i, j_{d_i}}.$$

Since c_i is the sum of the d_i smallest slopes of $M_{\tau_i}^{\mathrm{ord}_{\mathfrak{o}_{\tau_i}}}(n, \mathfrak{f})$, we see that $\bigwedge^{d_i} \mathbf{F}^{e_{\tau_i}}$ is divisible by p^{c_i}, thus reproving Lemma 3.1.

Applying Mazur's theorem (Theorem 2.4) to (A.1), we see that

$$\mathrm{Fil}^1_{\tau_i} = \overline{\mathrm{span}(\{\epsilon_{\tau_i, j} \mid \mathfrak{f}(\tau_i) > n - j\})}.$$

Hence

$$\mathrm{Fil}^1 H^{d_i}_{\mathrm{dR}}(A)_{\tau_i} = \overline{\mathrm{span}(\mathcal{B}_{d_i} - \{\epsilon_{\tau_i, 1} \wedge \cdots \wedge \epsilon_{\tau_i, d_i}\})} \tag{A.4}$$

It follows from the description of the matrix of $\bigwedge^{d_i} \mathbf{F}^{|\mathfrak{o}_{\tau_i}|}$ in the basis $\mathcal{B}_{d_i}$ that p^{c_i+1} divides $\bigwedge^{d_i} \mathbf{F}^{e_{\tau_i}}(\mathrm{span}(\mathcal{B}_{d_i} - \{\epsilon_{\tau_i, 1} \wedge \cdots \wedge \epsilon_{\tau_i, d_i}\}))$. Combining this with (A.4) reproves Lemma 3.2. We also get that $\bigwedge^{d_i} \mathbf{F}^{e_{\tau_i}} / p^{c_i}$ is non-zero on $\mathrm{Gr}^0 H^{d_i}_{\mathrm{dR}}(A)_{\tau_i}$, from which we recover the "if" part of Corollary 4.2.

B. An elementary construction for the case $F^+ = \mathbf{Q}$

Suppose that $F^+ = \mathbf{Q}$, and therefore $\mathbf{G}(\mathbf{R}) = \mathbf{GU}(a, b)$ for some positive integers a, b. Assume henceforth, without loss of generality, that $a \leq b$. The assumption of Section 1.1 that p is a prime of good reduction for $\mathcal{U}$ implies that p is unramified in E. If $a = b$, then $E = \mathbf{Q}$, so p is necessarily split in E and the classical ordinary locus is open dense. Hence we assume from now on that $a < b$ and that p is inert in E. It follows that the Hodge bundle Ω decomposes over E as

$$\Omega = \Omega_a^{\oplus r} \oplus \Omega_b^{\oplus r}, \tag{B.1}$$

where Ω_a (resp. Ω_b) has rank a (resp. b) and r is the rank of B over F. Let ω_a (resp. ω_b) be the determinant of Ω_a (resp. Ω_b).

Let $\mathcal{A}$ be an abelian scheme representing the universal isogeny class above sh. The Verschiebung $\mathrm{Ver} : \mathcal{A}^{(p)} \to A$ induces a map

$$\mathrm{Ver}^* : \Omega \to \Omega^{(p)}. \tag{B.2}$$

Since $\mathfrak{p}$ is inert, the restrictions of Ver^* to Ω_a (resp. $\Omega_{\mathcal{K}(p), b}$) have the form

$$\mathrm{Ver}^*_{|\Omega_a} : \Omega_a \to \Omega_b^{(p)} \quad \text{and} \quad \mathrm{Ver}^*_{|\Omega_b} : \Omega_b \to \Omega_a^{(p)}. \tag{B.3}$$

Therefore, if $(\mathrm{Ver}^*)^2$ denotes the composite of Ver^* with itself, then we have

$$(\mathrm{Ver}^*)^2_{|\Omega_a} : \Omega_a \to \Omega_a^{(p^2)}. \tag{B.4}$$

Let

$$ {}^{\mu}h(\mathcal{A}) : \omega_a \to \omega_a^{p^2} \tag{B.5}$$

be the top exterior power of that map, where we have used that $\omega_a^{(p^2)} = \omega_a^{p^2}$ since ω_a is a line bundle. The map ${}^{\mu}h(\mathcal{A})$ induces a global section

$$ {}^{\mu}H(\mathcal{A}) \in H^0(\mathrm{sh}, \omega_a^{p^2-1}). \tag{B.6}$$

If $\mathcal{B}$ is another representative of the universal isogeny class above sh and $\varphi : \mathcal{A} \to \mathcal{B}$ is an isogeny compatible with the endomorphism actions of $\mathcal{A}$, $\mathcal{B}$, then as in [4, Section 4.2], the compatibility of Verschiebung with isogenies ([4, Lemma 4.2.3]) implies that

$$\varphi^*({}^{\mu}H(\mathcal{B})) = {}^{\mu}H(\mathcal{A}).$$

Hence we may omit reference to the representatives $\mathcal{A}$ or $\mathcal{B}$ and we have a section

$${}^{\mu}H \in H^0(\mathrm{sh}, \omega_a^{p^2-1}).$$

Remark B.1. Note that applying the above construction is entirely done modulo p. Applying it to ω_b gives nothing but the zero section. With hindsight, this shows the necessity to lift our setup to characteristic zero to divide by higher powers of p.

Lemma B.2. *The Newton polygon $\mathcal{N}^{\mathrm{ord}}$ of the underlying isogeny class of abelian schemes of a μ-ordinary geometric point of* sh *has the following slopes:*

$$\left.\begin{matrix} 0 \\ 1/2 \\ 1 \end{matrix}\right\} \quad \textit{with multiplicity} \quad \left\{\begin{matrix} 2ar \\ 2(b-a)r \\ 2ar. \end{matrix}\right.$$

Proof. The case $r = 1$ follows from [16, 2.3.2]. The case of general r follows subsequently from [13, 1.3.1 and 3.2.9]. □

Proposition B.3. *The μ-ordinary locus is the maximal p-rank stratum of* sh.

Proof. The key point is that, by [15, Proposition 2.4 (iv) and Theorem 4.2], the Newton polygon $\mathcal{N}^{\mathrm{ord}}$ described in Lemma B.2 is the lowest among the Newton polygons of the underlying isogeny classes of abelian schemes corresponding to geometric points of sh. Let A be an abelian scheme with Newton polygon $\mathcal{N}(A)$. Then $\mathcal{N}(A)$ is symmetric and the p-rank of A is the multiplicity of 0 (= the multiplicity of 1) as a slope of $\mathcal{N}(A)$. But if the multiplicity of 0 in $\mathcal{N}(A)$ is at least the multiplicity of 0 in $\mathcal{N}^{\mathrm{ord}}$ and $\mathcal{N}(A)$ lies on or above $\mathcal{N}^{\mathrm{ord}}$, then by Lemma B.2 we must have $\mathcal{N}(A) = \mathcal{N}^{\mathrm{ord}}$. □

Corollary B.4. *The maximal p-rank stratum of* sh *has p-rank $2ar$.*

Proof. This follows directly from Lemma B.2 and the proof of Proposition B.3. □

Lemma B.5. *Suppose A is an abelian scheme which is a representative of the underlying isogeny class of a geometric point of* sh. *Then ${}^{\mu}H(A) \neq 0$ if and only if the p-rank of A is equal to $2ar$.*

Proof. One has

$$H^1(A, \mathcal{O}_A) \cong H^0(A, \Omega^1_A)$$

and under this isomorphism the action of Frobenius on $H^1(A, \mathcal{O}_A)$ corresponds to that of Verschiebung on $H^0(A, \Omega^1_A)$. Hence [14, Section 15] implies that the p-rank of A equals the semisimple rank of $(\mathrm{Ver}^*)^j : \Omega \to \Omega^{(p^j)}$ for all $j \in \mathbf{N}$. Since $\dim A = (a+b)r$, keeping in mind (B.1) and using [14, last corollary in Section 14], $(\mathrm{Ver}^*)^j$ is semisimple for $j \geq a+b$. Therefore the p-rank of A equals the rank of $(\mathrm{Ver}^*)^j$ for $j \geq a+b$. We take the $(a+b)$th power of the section ${}^{\mu}H(A)$, see (B.5). It is clear that ${}^{\mu}H(A) \neq 0$ if and only if ${}^{\mu}H(A)^n \neq 0$ for any $n \in \mathbf{N}_{>0}$, in particular for $n = a+b$.

Since $a \leq b$, both $\mathrm{Ver}^*_{|\Omega_a}$ and $\mathrm{Ver}^*_{|\Omega_b}$ have rank at most a. So also

$$\mathrm{rank}(\mathrm{Ver}^*)^j_{|\Omega_{\mathcal{K}(p),a}} \leq a \quad \text{and} \quad \mathrm{rank}(\mathrm{Ver}^*)^j_{|\Omega_{\mathcal{K}(p),b}} \leq a.$$

By (B.1), $(\mathrm{Ver}^*)^j$ has rank at most $2ar$.

The p-rank of A equals $2ar$ if and only if the rank of $(\mathrm{Ver}^*)^j$ is $2ar$ for $j \geq a+b$. In turn, the rank of $(\mathrm{Ver}^*)^j$ is $2ar$ if and only if both $\mathrm{Ver}^*_{|\Omega_a}$ and $\mathrm{Ver}^*_{|\Omega_b}$ have rank a. Since Ω_a and $\Omega_a^{(p^{2(a+b)})}$ are rank a vector bundles, the determinant of a map between them is nonzero if and only if it has rank a. □

The main properties of the Hasse invariant follow by standard arguments. For the liftability to characteristic zero, we may cite [12, Proposition 7.14] (or [11] in the compact case), to argue that there exists $k \in \mathbf{N}$ such that $\omega_a^{k(p^2-1)}$ itself extends to an ample line bundle on the minimal compactification $\mathrm{sh}^{\min}$.

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