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Perturbing Misiurewicz Parameters in the Exponential Family

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Abstract: In one-dimensional real and complex dynamics, a map whose post-singular (or post-critical) set is bounded and uniformly repelling is often called a Misiurewicz map. In results hitherto, perturbing a Misiurewicz map is likely to give a non-hyperbolic map, as per Jakobson's Theorem for unimodal interval maps. This is despite genericity of hyperbolic parameters (at least in the interval setting). We show the contrary holds in the complex exponential family $z \mapsto \lambda \exp(z)$: Misiurewicz maps are Lebesgue density points for hyperbolic parameters. As a by-product, we also show that Lyapunov exponents almost never exist for exponential Misiurewicz maps. The lower Lyapunov exponent is $-\infty$ almost everywhere. The upper Lyapunov exponent is non-negative and depends on the choice of metric.

1. Introduction

Jakobson's Theorem [15] from 1981 is one of the more celebrated and striking results in dynamical systems. In the real quadratic (or logistic) family $f_a : x \mapsto ax(1-x)$, Jakobson showed that there is a positive measure set of parameters a close to the Chebyshev parameter $a = 4$ for which the map has an absolutely continuous, f_a-invariant probability measure μ_a. One can contrast this with the result [13,19], due to Graczyk and Świątek and to Lyubich, which states that the set of hyperbolic parameters is open and dense, to emphasise the intricacy of quadratic dynamics. Rees in [26] generalised Jakobson's result to rational maps of the Riemann sphere. Benedicks and Carleson extended these results to the Hénon family in [5]. In these settings, one starts with a map with a repelling post-critical set, and sufficiently small perturbations are likely to give non-hyperbolic parameters. In this paper we present a counter-example to this paradigm in the complex exponential family.

In the exponential family $f_\lambda : z \mapsto \lambda e^z$, a parameter λ is called a *Misiurewicz* parameter if $\overline{\{f_\lambda^n(0) : n \geq 0\}} \subset \mathbb{C}$ is a bounded, hyperbolic repelling set. The simplest example is for $\lambda = 2\pi i$. For Misiurewicz parameters, the Julia set is the entire complex plane (or, regarding f as a meromorphic map, the Julia set is the entire Riemann sphere). In particular, there are dense orbits.

A parameter λ is called *hyperbolic* if f_λ has an attracting periodic orbit. For hyperbolic λ, almost every orbit is in the basin of attraction of the attracting periodic orbit. Any λ with $|\lambda| < 1/e$ is hyperbolic.

Main Theorem. *In the complex exponential family, Misiurewicz parameters are Lebesgue density points for the set of hyperbolic parameters.*

By this we mean, if λ_0 is a Misiurewicz parameter, H is the set of hyperbolic parameters and m denotes Lebesgue measure, then

$$\lim_{r\to 0^+} \frac{m(B(\lambda_0, r) \cap H)}{m(B(\lambda_0, r))} = 1.$$

For Misiurewicz parameters in the exponential family, there is a conservative, σ-finite, ergodic, absolutely-continuous invariant measure. It even has a real-analytic density off the post-singular set [8]. However, it was shown in [10,17] that no absolutely continuous invariant probability measure can exist. To prove the main theorem, strong estimates on the dynamics of Misiurewicz maps are required. The same estimates, with only a slight extension, permit one to show that for Misiurewicz maps, the Lyapunov exponent of a point exists almost nowhere. We use Df to denote the derivative of f with respect to the Euclidean metric.

Theorem 1. *Let f be a Misiurewicz map from the exponential family. For Lebesgue almost every $z \in \mathbb{C}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log |Df^n(z)| = -\infty,$$

while

$$\limsup_{n\to\infty} \frac{1}{n} \log |Df^n(z)| = +\infty. \tag{1}$$

However, the plane is not compact and there is a choice of Riemannian metric. For any metric ρ, let $D_\rho g$ denote the derivative of g with respect to ρ. In particular, for the spherical metric σ,

$$D_\sigma g(z) := \frac{1 + |z|^2}{1 + |g(z)|^2} Dg(z).$$

Theorem 2. *Let f be a Misiurewicz map from the exponential family. For Lebesgue almost every $z \in \mathbb{C}$ and every Riemannian metric ρ,*

$$\liminf_{n\to\infty} \frac{1}{n} \log |D_\rho f^n(z)| = -\infty \tag{2}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log |D_\rho f^n(z)| \geq 0,$$

while for the spherical metric σ,

$$\limsup_{n\to\infty} \frac{1}{n} \log |D_\sigma f^n(z)| = 0.$$

One could replace log in Eq. (1) by any finite composition of logarithms and the result would still hold, though we do not quite show this (and similarly for (2), remembering to take absolute values); the number 4 in Lemma 25 was chosen rather arbitrarily.

For a class of maps of the unit interval with negative Schwarzian derivative, Keller [16] showed that if $\limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)| > 0$ for almost every x, then there exists an absolutely continuous invariant probability measure. Theorem 1 implies that the same does not hold generally in the exponential family, at least for the Euclidean metric. It would be interesting to know whether the following conjectures are equivalent.

Conjecture 1. *Let $f : z \mapsto \lambda e^z$. For the spherical metric σ and Lebesgue almost every $z \in \mathbb{C}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log |D_\sigma f^n(z)| \le 0.$$

Conjecture 2. *No map from the exponential family admits an absolutely continuous invariant probability measure.*

Misiurewicz parameters (and maps) have a long and involved history in the field of one-dimensional dynamics. Introduced by Misiurewicz in [23] for smooth maps of the interval, they became the first examples where some non-trivial condition on the behaviour of critical orbits guaranteed the existence of absolutely continuous invariant probability measures. This result was superseded by many more in interval dynamics, see [7] for one of the latest and strongest. The concept of Misiurewicz parameter exists in other contexts too, see [3,6,10,14] for example. The articles [4,15,26,30] all find positive measure sets of non-hyperbolic parameters (indeed one admitting absolutely continuous invariant probability measures) in a neighbourhood of Misiurewicz parameters. On the other hand, Misiurewicz parameters have zero Lebesgue measure, in general [1,3,29].

In [30], Thunberg finds positive measure sets of non-hyperbolic parameters in unimodal families of interval maps with critical points of type $\exp(-|x|^{-\alpha})$, provided $\alpha < 1/8$. We showed in [9] that if $\alpha \ge 1$, no absolutely continuous invariant probability measure with positive entropy can exist, as was shown for Misiurewicz parameters in the same setting in [6].

Structural instability of Misiurewicz parameters in the exponential family was shown in [12,20,31]. For the (non-Misiurewicz) map $z \mapsto e^z$, the orbit of 0 is a (wild) metric attractor attracting almost every orbit [18,25], although generic orbits are dense. This map is a density point for hyperbolic maps in the exponential family [32]. For those interested in the structure of parameter space of the exponential family (as opposed to metric properties), we refer to [27].

It has been suggested by Hubbard that hyperbolic parameters should have full measure in the exponential family, see [24] (where it is shown that non-hyperbolic parameters have full Hausdorff dimension). This would be a stronger conjecture than density of hyperbolic parameters, and this paper and [32] could be viewed as first small steps in that direction.

One could ask about the complex quadratic family $f_c : z \mapsto z^2 + c$ with Julia set $\mathcal{J}_c$. Rivera-Letelier [28] showed that if $c \in \mathcal{J}_c$ and c is non-recurrent, then c is a density point for hyperbolic parameters (ones for which c is in the basin of a periodic attractor, finite or at infinity). Aspenberg in [2] extended this result to more general rational maps for which the Julia set is not the whole sphere. In both of these cases, basins of periodic attractors are open and dense in the sphere. It is natural in these cases to expect that, with expansion along the post-critical orbits, a small perturbation is likely to send the critical orbits into the attracting basins. What is strange in the exponential setting is that Misiurewicz parameters are density points for hyperbolic parameters even though the Julia set at the Misiurewicz parameter is the whole space.

For a map f_λ from the exponential family, $f_\lambda(z) = Df_\lambda(z)$ and $|f_\lambda(z)| = |\lambda| e^{\Re(z)}$, so f_λ is $2\pi i$-periodic, f_λ maps vertical lines to circles, horizontal lines to rays emanating from 0, and rectangles of height 2π onto annuli centred at 0. Points far to the left get mapped extremely close to 0, and points far to the right get mapped extremely far from 0.

In Appendix D of [22], Milnor shows that our choice of realisation of the exponential family is, in some sense, as good as any other: any entire map with asymptotic values at ∞ and at some finite point, and without critical points, is conjugate to a map from the exponential family. Alternative reasonable choices are $g_\kappa : z \mapsto e^z + \kappa$ and $g_\kappa : z \mapsto e^{\kappa z}$.

2. Global Definitions

Throughout the paper, let $f = f_{\lambda_0} : z \mapsto \lambda_0 \exp(z)$, for some Misiurewicz parameter $\lambda_0 \in \mathbb{C}$; in particular the post-singular set

$$P(f) := \overline{\{f^n(0) : n \geq 0\}}$$

is a bounded hyperbolic repelling set, so there are $n_0, \alpha > 0$ such that $|Df^{n_0}(z)| > \exp(2\alpha)$ for all $z \in P(f)$. By continuity, we can fix $\varepsilon_0, \delta \in (0, \frac{1}{2})$ such that for all $\lambda \in B(\lambda_0, \varepsilon_0)$ and all $z \in B(P(f), 3\delta)$,

$$|Df_\lambda^{n_0}(z)| > \exp(\alpha).$$

Set $V := B(P(f), \delta)$.

We shall denote by $\Delta > 1$ the modulus giving a Koebe *distortion bound* of 2, that is, the minimal number such that for any univalent map g on $B(0, \Delta)$, the distortion of g on $B(0, 1)$ is bounded by 2:

$$\sup_{y,z \in B(0,1)} \left| \frac{Dg(y)}{Dg(z)} \right| \leq 2.$$

We shall repeatedly use the following fact.

Lemma 3. *For any simply-connected open set U with* $\operatorname{dist}(U, P(f)) > \Delta \operatorname{diam}(U)$*, if $f^n(z) \in U$ then a neighbourhood of z is mapped biholomorphically onto U with distortion bounded by* 2.

Proof. Since $P(f) \cap B(f^n(z), \Delta \operatorname{diam}(U)) = \emptyset$, there is a neighbourhood of z mapped biholomorphically onto $B(f^n(z), \Delta \operatorname{diam}(U))$. By definition of Δ, a neighbourhood of z gets mapped with distortion bounded by 2 onto $B(f^n(z), \operatorname{diam}(U)) \supset U$, as required. □

The notation $A(y; a_1, a_2)$ is used for the annulus centred on $y \in \mathbb{C}$ with inner and outer radii of lengths a_1, a_2.

Denote by $\mathcal{R}(x)$ the right half-plane

$$\mathcal{R}(x) := \{z \in \mathbb{C} : \Re(z) \geq x\}$$

and denote by $\mathcal{L}(x)$ the left half-plane $\mathbb{C}\backslash\mathcal{R}(x)$. Denote by $\mathcal{Q}$ the collection of squares of the form

$$\{z : 2k\pi \leq \Re(z) < (2k+2)\pi; 2j\pi \leq \Im(z) < (2j+2)\pi\},$$

for $j, k \in \mathbb{Z}$. Each square has diameter $2\sqrt{2}\pi < 9$.

Further definitions occur throughout the paper. These include constants $\delta_0 \in (0, \delta)$ and $N_1, M > 0$ at the start of Sect. 5; following Lemma 34, constants M_0, r_0 and holomorphic motion h with $h(0, \lambda) = a_K(\lambda - \lambda_0)^K$ to first order, with $a_K \in \mathbb{C}\backslash\{0\}$ and K a positive integer; just prior to Proposition 35, $\xi_n : \lambda \mapsto f_\lambda^n(0)$.

3. Structure of the Proof

The following simple lemma guides the proof of the Main Theorem.

Lemma 4. *For each $C > 0$, for all x large enough, the following holds. Let $\lambda \in \mathbb{C}$ satisfy $|\lambda| < x$, let $n \geq 1$ and suppose that the following holds.*

(i) *f_λ^n maps a neighbourhood of 0 biholomorphically onto $B(f_\lambda^n(0), 1)$;*
(ii) *$B(f_\lambda^n(0), 1) \subset \mathcal{L}(-e^{x+\sqrt{x}} + 3\pi)$;*
(iii) *$|Df_\lambda^n(0)| < \exp(Ce^x)|\Re(f_\lambda^n(0))|^4$.*

Then f_λ has a hyperbolic attracting periodic orbit.

Proof. Set $v = f_\lambda^n(0)$ for the sake of readability. If x is large, so is $-\Re(v)$, by (ii). Thus

$$\exp(\Re(v)/2)|\Re(v)|^4 < 1. \tag{3}$$

Since $\Re(v) < -e^{x+\sqrt{x}} + 3\pi$,

$$ex \exp(\Re(v)/2)2\Delta \exp(Ce^x) < 1. \tag{4}$$

Using a (Koebe) distortion bound of 2, f_λ^n maps $B_\lambda := B(0, 1/2\Delta|Df_\lambda^n(0)|)$ into $B(v, 1/\Delta)$. Thus $f_\lambda^{n+1}(B_\lambda)$ is contained in $B(0, r)$, with $r = e|\lambda| \exp(\Re(v))$. Thanks to (iii) and the bound $|\lambda| < x$,

$$r2\Delta|Df_\lambda^n(0)| < ex \exp(\Re(v))2\Delta \exp(Ce^x)\Re(v)^4 < 1,$$

the latter inequality obtained combining (3) and (4). Hence $\overline{B(0, r)} \subset B_\lambda$.

Since $f_\lambda^{n+1}(B_\lambda) \subset B(0, r)$ and $\overline{B(0, r)} \subset B_\lambda$, f_λ has a hyperbolic attracting periodic orbit. □

With expansion along the post-singular orbit of f, one can often transfer estimates for large sets of points in phase space for f into estimates on the post-singular orbit of f_λ for large sets of parameters λ. It is natural, therefore, to try to find, for f, a large set of points which enter the left half-plane with estimates related to those of the lemma. A substantial portion of the paper comprises of this effort.

Note that as λ approaches λ_0, there will be a huge build-up of derivative initially as $f_\lambda^j(0)$ spends a long time near $P(f)$. To counteract this, one needs to land, eventually, far off to the left, before getting mapped extremely close to 0 to cancel out the derivative build-up.

A general result [12] implies that for exponential Misiurewicz maps (amongst others), every forward-invariant compact set is hyperbolic repelling (but with no rate estimate). This in turn implies that the measure of the set of points remaining in any bounded set for n iterates is exponentially small in n. However, to deal with smaller parameter perturbations (for λ closer to λ_0), we need to find large sets of points going ever further to the left. Thus we need to know how the exponential rate depends on the size of the bounded set. In Proposition 11, we obtain the relevant hyperbolicity estimates. For an exponential Misiurewicz map the derivative grows exponentially fast, except when it is slowed by the occasional passage close to zero.

Lemma 14 is useful and curiously does not hold for quadratic maps, say. The lemma implies a distortion bound, which together with the derivative growth estimates, allows one to relate large and small scales and hence, via a porosity-type argument, to estimate how long it takes for a large proportion of points to make a first entry into a right half-plane $\mathcal{R}(x)$. Boot-strapping, we show that for most of these points, the first entry actually lands in $\mathcal{R}(x+2\sqrt{x})$. In Sect. 6 we study the dynamics of points in a far-right half-plane, showing that most points go further and further to the right before eventually landing far out to the left.

These results are gathered together in Proposition 26, which says that if you start from a reasonable, reasonably large set close to $P(f)$, then most points in that set first enter a far-left half-plane in a bounded amount of time and with a derivative bound. Points may land far to the left in that half-plane, so the derivative bound depends not only on the half-plane but also on the real part of the landing location.

With the necessary ingredients in place, we pause to prove Theorems 1 and 2 in Sect. 8. Only the estimate for the spherical derivative is a little complicated.

Returning to the proof of the main theorem, we begin our parameter-based estimates. We show that points from Proposition 26 do not move too, too fast as the parameter moves. The continuation of a point is defined to have a similar orbit and the same first entry to the left half-plane. The estimates depend on the parameters considered being in a tiny ball, the time being bounded and the orbit being a certain distance away from 0.

Going backwards from a large neighbourhood of a point in $P(f)$ to a tiny neighbourhood of 0 (or of λ_0) is more delicate, though at this stage the arguments are well-understood. The estimates are also a little less cumbersome going forwards than backwards. For completeness, and because the desired estimates are not simple to extract from [3] (itself based on [1]), we include proofs of the estimates. Sectors of small annuli centred on λ_0 in parameter space get mapped biholomorphically and with bounded distortion onto reasonably large sets near $P(f)$ by the map $\lambda \mapsto f_\lambda^n(0)$, for some n depending on the annulus, see Lemma 37.

With the various (parametric) derivative estimates, it is not too hard then, in Proposition 39, to match up most parameters with orbits and, using Lemma 4, to show that for these parameters the maps are hyperbolic.

A note of comparison, Wang and Zhang [32] showed that 1 is a density point for hyperbolic parameters. For $g : z \mapsto e^z$, most points near 1 follow the orbit of 1 out towards infinity until escaping a small neighbourhood of the orbit, then take more steps towards $+\infty$, then get mapped extremely close to $-\infty$ and then super-close to 0 with derivative close to 0. One only needs to study the dynamics close to infinity and along the orbit of 1, so the arguments are relatively elegant and straightforward. Moreover, one can calculate by hand that the derivative of $\lambda \mapsto f_\lambda^n(0)$ is positive (and increasing in n) at 1, so in particular it is non-zero. If this were known to be the case for Misiurewicz parameters, one would have $K = 1$ in Eq. (35) and one could deal with balls instead of annuli, a minor simplification. One could attempt the parameter exclusion method as per [32] in the current setting, and it would work as long as one remains close to $P(f)$, though something along the lines of Proposition 35 would of course still need to be shown. However, continuing on beyond the comfort of a neighbourhood of $P(f)$, where one has injectivity and distortion control, would likely lead to many sleepless nights.

4. Non-uniform Hyperbolicity

In this section we gather some estimates on the growth of the derivative along individual orbits and their neighbourhoods. In the following section we will use these estimates to compare small and large scales and derive some measure estimates. Recall that $f = f_{\lambda_0}$ and λ_0 is a Misiurewicz parameter.

Lemma 5. *The Julia set is* $\overline{\mathbb{C}}$ *and* $|\lambda_0| \geq 1/e$.

Proof. The first statement follows immediately from Theorems 3–5 of [11], since the post-singular set is uniformly repelling. Were $|\lambda_0| < 1/e$, then $\overline{f(B(0,1))} \subset B(0,1)$ and f would have an attracting fixed point. □

Lemma 6. *For each* z *in* $\mathbb{C} \setminus P(f)$*, there are arbitrarily small neighbourhoods* U_z *on which the first return map* ϕ *to* U_z *is expanding (that is,* $|D\phi| > \gamma_z > 1$*).*

Proof. This is part (iii) of [8, Lemma 11], knowing that the Julia set is $\mathbb{C}$. □

Lemma 7. *Given any* $\theta > 0$*, there is a* $\beta \in (0,1)$ *such that, for any* $z \in \mathbb{C}$ *and* $k \geq 0$*, if* $\operatorname{dist}(f^k(z), P(f)) \geq \theta$ *then* $|Df^k(z)| > \beta$.

Proof. By Lemma 3, some neighbourhood W of z is mapped biholomorphically onto $B(f^k(z), \theta/\Delta)$ with distortion bounded by 2. But f is not univalent on any ball of radius π, so W cannot strictly contain a ball of radius π. Combining these two facts, the derivative of f^k on W cannot be too small. □

Lemma 8. *There is an* $M > 3$ *such that, for all* $z \in \mathbb{C}$ *and* $k \geq 1$*, if* $|f^k(z)| \geq M$ *then* $|Df^k(z)| > 3$.

Proof. Let $\theta > 0$ and let β be given by Lemma 7, Take $M > 3/\beta$ sufficiently large that $f(B(P(f), \theta)) \subset B(0, M)$. If $f^k(z) \geq M$ then $\operatorname{dist}(f^{k-1}(z), P(f)) \geq \theta$, so $|Df^{k-1}(z)| > \beta$. But $|Df^k(z)| = |f^k(z)||Df^{k-1}(z)| \geq M\beta > 3$. □

Lemma 9. *Given* $M_1 > 1$ *there is an* $M_2 > 1$ *such that, for all* $z \in \mathbb{C}$ *and* $k \geq 2$*, if* $|f^k(z)| \geq M_2$ *then* $|Df^k(z)| > M_1|f^k(z)|$.

Proof. Take M_2 large enough that $|f^{k-1}(z)|$ must be larger than M_1 and $|f^{k-2}(z)|$ must be larger than M, where M comes from Lemma 8. □

Lemma 10. *There is some* $\beta_1 > 0$ *such that, for each* $z \in \mathbb{C}$ *and* $k \geq 1$*,*

$$|Df^k(z)| \geq \beta_1 \inf_{1 \leq j \leq k} |f^j(z)|. \tag{5}$$

Proof. Let $n \leq k$ be maximal such that $|Df^n(z)| \geq 1$. If $n \geq 1$ then

$$|Df^n(z)| = \prod_{1 \leq j \leq n} |f^j(z)| \geq \inf_{1 \leq j \leq n} |f^j(z)|.$$

Note that if $n = k$, (5) holds with $\beta_1 = 1$. Assume now that $n < k$. Let M be given by Lemma 8, so $|f^j(z)| < M$ for $j = n + 1, \dots k$. By the chain rule and choice of n, $|Df^k(z)| \geq |Df^{k-n}(f^n(z))|$. It now suffices to prove that

$$|Df^{k-n}(f^n(z))| \geq \beta_1 \inf_{n+1 \leq j \leq k} |f^j(z)|.$$

Rewriting, it suffices to prove the lemma under the assumption $|f^j(z)| < M$ for $j = 1, \dots, k$.

Recall that V is globally defined in Sect. 2 as a small neighbourhood of the postsingular set. We can divide the orbit into three pieces: a first stretch which ends outside V, a final stretch spent entirely inside V, and in between a single iterate. So, let $n \leq k$ be maximal such that $f^n(z) \notin V$, if it exists, otherwise set $n = 0$. Let $\beta \in (0, 1)$ be given by Lemma 7. Then

$$|Df^n(z)| \geq \beta \geq \beta |f^n(z)|/M,$$

by assumption. If $n = k$, we are done (provided $\beta_1 \leq \beta/M$). So assume $n < k$. Now $f^{n+1}(z), \dots, f^k(z) \in V$, by definition of n. Since $|Df^{n_0}| > 1$ on V, it follows that $|Df^{k-n-1}(f^{n+1}(z))|$ is bounded below by the constant

$$\beta_2 := \min_{1 \leq j < n_0} \inf_{y \in V} |Df^j(y)| > 0.$$

Then $|Df^k(z)| \geq \beta\beta_2 |Df(f^n(z))|$. Taking $\beta_1 := \min(\beta/M, \beta\beta_2\beta_3)$ works, where $\beta_3 = |\lambda_0| e^{-M}$. □

In the following proposition, we use exponential growth when one remains in a neighbourhood of $P(f)$, exponential growth when one remains in a bounded region disjoint from that neighbourhood, plus absolute growth if an iterate lands outside a large bounded region, to give some sort of non-uniform hyperbolicity statement for Misiurewicz maps.

Proposition 11. *There are $N, N_1 > 0$ such that for each z there is a $j \leq N + N|\log|f(z)||$ with $|Df^j(z)| > 3$ and $|Df^i(z)|, |f^i(z)| \leq N_1 + |f(z)|$ for $i = 1, \dots, j$.*

Proof. We can assume $|f(z)| \leq 3$, otherwise one can simply take $j = 1$. Set $p := 1 + n_0 \lceil (2 - \log|f(z)|)/\alpha \rceil$, and note that p is bounded by an affine function of $|\log|f(z)||$. Let $k \geq 1$ be minimal such that $f^k(z) \notin V$. If $k \geq p$,

$$|Df^p(z)| \geq |f(z)| \exp(p\alpha/n_0) \geq e^2$$

and, setting $j = p$, we are done, for appropriately chosen N.

Otherwise, $1 \leq k < p$. Since $f^k(z) \notin V$, Lemma 7 provides a constant $\beta > 0$ (for $\theta = \delta$, say) for which $|Df^k(z)| > \beta$. Moreover, $f^k(z) \in Z := (B(0, 3) \cup f(V)) \setminus V$. Therefore it suffices to show that there is an N such that, for each $y \in Z$, there is a $j \leq N$ with $|Df^j(y)| > 3/\beta$.

By Lemma 9, we can choose N_1 large enough that $Z \subset B(0, N_1)$ (trivially) and that, for any $z \in \mathbb{C}$, if $|f^n(z)| \geq N_1$ then $|Df^n(z)| > 3/\beta$. Thus we restrict our attention to those y which do not leave $B(0, N_1)$ for the first N iterates, for some large N to be defined. We can cover the compact set $W := \overline{B(0, N_1)} \setminus V$ by a finite collection of balls $\{W_l\}_{l=1}^{L}$ on which the first return map is expanding, by Lemma 6, so there is a $\gamma > 1$ and each return map $\phi_l : W_l \to W_l$ has derivative greater than γ.

Let $q, r \in \mathbb{N}$ satisfy $\beta\gamma^q > 3/\beta$ and $\beta e^{r\alpha}|\lambda_0|e^{-N_1} > 3/\beta$. Set $N := qLrn_0$.

Consider the successive passages of y into W, at times $k_0, k_1, \ldots$, say. By time k_{qL}, if such exists, there must be some W_l which is passed through at least q times. Then $|Df^{k_{qL}}(y)| > \beta\gamma^q > 3/\beta$ and if $k_{qL} \leq N$ we are done.

Otherwise, at some point the orbit must spend a long period, at least rn_0 long, in $B(0, N_1)\setminus W \subset V$. That is, there is some $a \geq 0$ such that $f^l(y) \in V$ for $l = a+1, \ldots, a+rn_0 < N$ and such that $a = 0$ or $f^a(y) \in W$. Since $f^a(y) \in B(0, N_1)$, $|f^{a+1}(y)| \geq |\lambda_0|e^{-N_1}$. But by definition of V,

$$|Df^{rn_0}(f^{a+1}(y))| \geq \exp(r\alpha).$$

The choice of r entails $\beta|Df^{rn_0}(f^{a+1}(y))||f^{a+1}(y)| > 3/\beta$, so $|Df^{a+1+rn_0}(y)| > 3/\beta$. Noting that $a + 1 + rn_0 \leq N$, we conclude the proof. □

Recall that $\Delta > 1$ is the constant giving a Koebe distortion bound of 2. The following two lemmas are stated for maps in a neighbourhood of f which are uniformly expanding on $B(P(f), 3\delta)$, see Sect. 2.

Lemma 12. *Given $\varepsilon > 0$ there is a $\delta_0 \in (0, \delta)$ such that the following holds. Let $\lambda \in B(\lambda_0, \varepsilon_0)$. If $f_\lambda^k(z) \in V$ for all $0 \leq k \leq p$ then there is a neighbourhood W_p of z, contained in $B(z, \varepsilon/|Df_\lambda^p(z)|)$, mapped biholomorphically by f_λ^p onto $B(f_\lambda^p(z), \Delta\delta_0)$.*

Proof. First we consider a bounded number of iterates. The distortion of f_λ on V is uniformly bounded (independently of $\lambda \in B(\lambda_0, \varepsilon_0)$). Therefore, given $\varepsilon > 0$ there is a $\delta_0 \in (0, \varepsilon/2\Delta)$ such that, if $y, f_\lambda(y), \ldots, f_\lambda^j(y) \in V$ and $0 \leq j \leq n_0 - 1$, there is a neighbourhood of y contained in $B(y, 2\delta)$ which is mapped biholomorphically by f_λ^j onto $B(f_\lambda^j(y), \delta_0\Delta^2)$.

Meanwhile, $|Df_\lambda^{n_0}| > 1$ on $B(V, 2\delta)$. It follows that, writing $p = an_0 + j$ with $a, j \in \mathbb{N}$ and $0 \leq j \leq n_0 - 1$, a neighbourhood of z is mapped biholomorphically by $f_\lambda^{an_0}$ onto $B(f_\lambda^{an_0}(z), 2\delta)$. Combined with the previous paragraph, we deduce that a neighbourhood of z is mapped by f_λ^p biholomorphically onto $B(f_\lambda^p(z), \delta_0\Delta^2)$. Shrinking the target, a neighbourhood W_p of z is mapped by f_λ^p biholomorphically with distortion bounded by 2 onto $B(f_\lambda^p(z), \delta_0\Delta)$. Because of the distortion bound,

$$W_p \subset B(z, 2\delta_0\Delta/|Df_\lambda^p(z)) \subset B(z, \varepsilon/|Df_\lambda^p(z)|),$$

as required. □

Lemma 13. *There exists $\delta_0 > 0$ such that if $\lambda \in B(\lambda_0, \varepsilon_0)$, if $f_\lambda^j(z) \in V$ for $j = 1, \ldots, k$ and if $|Df_\lambda^k(z)| > 1$, then there is a neighbourhood U of z mapped biholomorphically by f_λ^k onto $B(f_\lambda^k(z), \Delta\delta_0)$ with $U \subset B(z, \delta)$.*

Proof. By hypothesis, $|Df_\lambda^k(z)| = |f_\lambda(z)||Df_\lambda^{k-1}(f_\lambda(z))| > 1$, so letting $\varepsilon < \delta/e$ and taking δ_0 from the preceding lemma, there is a neighbourhood W of $f_\lambda(z)$ contained in $B(f_\lambda(z), \varepsilon|f_\lambda(z)|)$ mapped biholomorphically onto $B(f_\lambda^k(z), \Delta\delta_0)$. Since f_λ is an exponential map, $f_\lambda(B(z,\delta)) \supset B(f_\lambda(z), |f_\lambda(z)|(1-e^{-\delta}))$. Since $0 < \delta < 1$, $1 - e^{-\delta} > \delta/e$. By choice of ε, we deduce that $B(f_\lambda(z), \varepsilon|f_\lambda(z)|) \subset f_\lambda(B(z,\delta))$, so $W \subset f_\lambda(B(z,\delta))$. Therefore the relevant pullback U of W (that is, with $z \in U$) is contained in $B(z,\delta)$. □

The following lemma requires that the postsingular set is contained in V, so it only holds for $f = f_{\lambda_0}$.

Lemma 14. *Let $\delta_0 > 0$ be given by Lemma 13. Let $z \in \mathbb{C}$ and suppose $|Df^k(z)| > |Df^j(z)|$ for all $j = 0, \ldots k-1$. Then there is a neighbourhood of z mapped biholomorphically by f^k onto $B(f^k(z), \Delta\delta_0)$.*

Proof. If $f^j(z) \in V$ for $j = 1, \ldots, k$, Lemma 13 produces the required neighbourhood. Otherwise, there is a maximal $j \le k$ for which $f^j(z) \notin V$. By Lemma 13 again, there is a neighbourhood U of $f^j(z)$ mapped by f^{k-j} biholomorphically onto $B(f^k(z), \Delta\delta_0)$, and $U \subset B(f^j(z), \delta)$. But $f^j(z) \notin V$, so $U \cap P(f) = \emptyset$. Therefore there is a neighbourhood of z mapped by f^j biholomorphically onto U. □

5. First Entry to a Right-Half Plane

Proposition 22 is the principal result of this section. It states that a large proportion of points in a neighbourhood of $P(f)$ get mapped, in not too long time, far out to the right and with derivative which is not too large. The idea behind the proof is porosity: at every small scale, a certain proportion gets mapped far out. We use the expansivity estimates from the previous section to transfer estimates from the large scale to the small scale. We upgrade the proposition in Lemma 23 both topologically, obtaining a well-behaved partition, and distance-wise, showing most points land a little further to the right than claimed by the proposition. In the following two sections we will examine the dynamics far out to the right and obtain estimates for first entry maps to a (far) left half-plane.

Let $\delta_0 > 0$ be the minimum of the δ_0 given by Lemma 14 and by Lemma 12 (with $\varepsilon < 1/2$, say). Let N_1 be given by Proposition 11. By Lemma 9, there is an $M > 100$ such that, if $|f^n(z)| > M$ then $|Df^n(z)| > |f^n(z)|/\delta_0$. We can suppose moreover that

$$M > |\lambda_0| e^{N_1 + \mathrm{diam}(P(f)) + 10\Delta}.$$

This choice of M is for future use [which the reader may choose to remember as a *sufficiently large constant*], for example to obtain (7) in the proof of Lemma 20.

Recall $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{L}$ are globally defined in Sect. 2.

Lemma 15. *There is a finite collection of sets $U_1, \ldots, U_p$ with corresponding numbers $n_k \ge 0$, $k = 1, \ldots, p$, such that f^{n_k} maps U_k biholomorphically onto an element of $\mathcal{Q}$ contained in $\mathcal{R}(M)$, and such that for each y with $|y| \le 2M$, $B(y, \delta_0)$ contains some U_k.*

Proof. By transitivity of f, there is finite set Z such that $\mathrm{dist}(y, Z) < \delta_0/2$ for all y with $|y| \le 2M$, and such that for each $z \in Z$, there is an n such that $f^n(z) \in \mathcal{R}(M + 2\pi)$. For such z, n, let $Q \in \mathcal{Q}$ be the square containing $f^n(z)$, so $Q \subset \mathcal{R}(M)$. By choice of M and by Lemma 3, there is a neighbourhood U of z which gets mapped by f^n biholomorphically onto Q with distortion bounded by 2. Since $|Df^n(z)| > |f^n(z)|/\delta_0 > M/\delta_0$, we deduce that the diameter of U is bounded by $2\sqrt{2}\pi 2\delta_0/M < \delta_0/2$. Thus if $|y - z| < \delta_0/2$, $U \subset B(y, \delta_0)$. The result follows. □

The following lemma deals with points in $\mathcal{L}(M)$. We shall deal with points to the right subsequently.

Lemma 16. *There is a countable collection of sets $\{U_i\}_{i\in\mathbb{Z}}$ and a constant $C > 1$ such that the following holds. Each U_i is mapped by some f^n, $n \geq 0$, onto a square $Q \in \mathcal{Q}$ with $Q \subset \mathcal{R}(M)$ with derivative bounded by C and distortion bounded by 2. If $\Re(y) \leq M$, then $B(y, \delta_0)$ contains as a subset an element of $\{U_i\}_{i\geq 0}$.*

Proof. Let U_k, n_k for $k = 1, \dots, p$ be given by Lemma 15. Taking translates by multiples of $2\pi i$ of the sets $U_1, \dots U_p$ deals with the points $y \in \mathbb{C}$ with $-M \leq \Re(y) \leq M$.

If $\Re(y) < -M$, by Proposition 11 there is a least $j \geq 1$ with $3 < |Df^j(y)|$ and for this j, $|f^j(y)|, |Df^j(y)| < N_1 + 1 < M/2$. By Lemma 14, there is a neighbourhood of y mapped biholomorphically by f^j onto $B(f^j(y), \Delta\delta_0)$ with a corresponding subneighbourhood W mapped by f^j onto $B(f^j(y), \delta_0)$ with distortion bounded by 2 (by choice of Δ). On W we deduce $1 < 3/2 < |Df^j| < M$. The lower bound implies $W \subset B(y, \delta_0)$. Now $B(f^j(y), \delta_0)$ contains some U_k, with $1 \leq k \leq p$. Thus there is some $U_y \subset W$ mapped by f^j onto U_k. The derivative $|Df^{j+n_k}|$ on U_y is bounded by $2N_1 \sup_{U_k} |Df^{n_k}|$. Thus one can take $C := M \max_{1\leq k\leq p} \sup_{U_k} |Df^{n_k}|$.

Countability of the collection of U_y obtained follows from countability of $\mathcal{Q}$ (and its preimages). The distortion bound comes from Lemma 3. □

Lemma 17. *Let Z denote the cone of positive linear combinations of $1+i$ and $1-i$. Let $y \geq M$. Let $Q \in \mathcal{Q}$ satisfy $Q \subset \mathcal{R}(y)\setminus\mathcal{R}(y+7)$. Then there is a subset of Q mapped biholomorphically onto a square $Q' \in \mathcal{Q}$ satisfying $Q' \subset Z \cap \mathcal{R}(|\lambda_0|e^y/2)\setminus\mathcal{R}(|\lambda_0|e^y e^7)$.*

Proof. One quarter of any square of $\mathcal{Q}$ gets mapped injectively into Z. We have $f(Q) \cap Z \subset \mathcal{R}(|\lambda_0|e^y/\sqrt{2})$, and $f(Q) \cap \mathcal{R}(|\lambda_0|e^{y+7}) = \emptyset$. Only a small proportion of squares from $\mathcal{Q}$ in $f(Q) \cap Z$ intersect $f(\partial Q)$, so we can pull back one of the other squares to get the required subset. □

Lemma 18. *Suppose $Q \in \mathcal{Q}$ satisfies $Q \subset \mathcal{R}(M)$. Let $x > M$. For some $z \in Q$ and some $k \geq 0$, the ball $B(z, 1/x^3) \subset Q$ is mapped by f^k univalently into $\mathcal{R}(x)$.*

Proof. Suppose $Q \subset \mathcal{R}(y)\setminus\mathcal{R}(y+7)$. We can assume $y < x$, otherwise the statement holds trivially, with $k = 0$. By repeatedly applying Lemma 17, we can construct an increasing sequence of numbers $y = y_0 < y_1 < y_2 < \cdots$ and a decreasing sequence of sets $Q = V_0 \supset V_1 \supset \cdots$ such that the following holds. For each $k \geq 0$,

- $f^k(V_k) \in \mathcal{Q}$;
- $f^k(V_k) \subset \mathcal{R}(y_k)\setminus\mathcal{R}(y_k + 7)$;
- $|\lambda_0|e^{y_k/2} < y_{k+1} < e^7|\lambda_0|e^{y_k}$;
- $|f^k(z)| \leq \sqrt{2}(y_k + 7)$ for $z \in V_k$ (noting $f^k(z)$ is in the cone Z);
- the distortion of f^k on V_k is bounded by 2 (by Lemma 3).

Since $\sqrt{y_{j+1}} > \sqrt{|\lambda_0|}e^{y_j/4} > 4\sqrt{2}(y_j + 7) > y_j$, we deduce that

$$\prod_{j=1}^{k} \sqrt{2}(y_j + 7) < ((y_k + 7)/2) \prod_{j=1}^{k-1} \sqrt{y_{j+1}} \leq (y_k + 7)y_k/2 < y_k^2.$$

Thus on V_k the derivative bound

$$|Df^k| = \prod_{j=1}^{k} |f^j(z)| \leq \prod_{j=1}^{k} \sqrt{2}(y_j + 7) < y_k^2$$

applies.

Let $k \geq 1$ be minimal such that $y_k \geq x$. If $f^k(V_{k-1}) \subset \mathcal{L}(2ex)$ (equivalently, if $f^k(V_{k-1}) \cap \mathcal{R}(2ex) = \emptyset$) then $y_k \leq 2ex$ and $|Df^k|$ on V_k is bounded by $(2ex)^2$. Therefore V_k easily contains a ball of radius $1/x^3$. Otherwise, $f^k(V_{k-1})$ is a geometric annulus centred on zero and intersecting $\mathcal{R}(2ex)$, and the square $f^{k-1}(V_{k-1})$ contains a ball of radius $1/16$ mapped by f into $\mathcal{R}(x)$, as is easy to check. The derivative of f^{k-1} on V_{k-1} is bounded by $y_{k-1}^2 < x^2$, so pulling back the ball we get a set containing a ball of radius $1/x^3$ once again, as required. □

Lemma 19. *There is a constant $\gamma > 0$ such that if $x > M$ the following holds.*

If $Q \in \mathcal{Q}$, there is a ball of radius γ/x^3 inside Q which gets mapped univalently by f^n, for some $n \geq 0$, into $\mathcal{R}(x)$ with distortion bounded by 2.

If $\Re(y) < M$, then there is a ball of radius γ/x^3 inside $B(y, \delta_0)$ which gets mapped univalently by f^n, for some $n \geq 0$, into $\mathcal{R}(x)$ with distortion bounded by 2.

Proof. This follows from Lemmas 16 and 18. □

The preceding lemma says that a certain proportion of everything at the large scale gets mapped far out to the right. The next lemma deduces the same, but at small scales.

Lemma 20. *There are constants $\kappa > 0$, $M_0 \geq M$ such that the following holds. Given $r \in (0, 1)$, $x \geq M_0$ and $z \in \mathbb{C}$, there is a finite collection of pairwise-disjoint balls $B_i \subset B(z, r)$, each of radius $> e^{-2x}r$, and numbers $n_i \geq 0$ such that*

- $m(\bigcup_i B_i)/m(B(z, r)) > \kappa/x^6$;
- *f^{n_i} maps B_i univalently into $\mathcal{R}(x)$;*
- $|Df^{n_i}_{|B_i}| < e^{3x}/r$.

Proof. Note first that if f^k maps a ball B into $\mathcal{R}(x)$, then f^k is univalent on B, as $P(f) \cap \mathcal{R}(x) = \emptyset$.

Let n be minimal such that $|Df^n(z)| > 20/r$. If there is some minimal $k < n$ with $f^k(z) \in \mathcal{R}(x)$, we can just pull back $B(f^k(z), 1)$ to get a set containing $B(z, r/40)$, using the derivative estimate and a distortion bound of 2. Some large sector of $B(z, r/40)$ gets mapped by f^k to $\mathcal{R}(\Re(f^k(z)))$ and the lemma follows easily.

Otherwise, $f^{n-1}(z) \notin \mathcal{R}(x)$, implying

$$|Df^n(z)| \leq |\lambda_0| e^x 20/r, \tag{6}$$

a bound we use later in the proof.

If $|f^n(z)| < M$, then f^n maps some neighbourhood W of z univalently onto $B(f^n(z), \delta_0)$ with distortion bounded by 2, by Lemma 14. With γ given by Lemma 19, for some $j \geq 0$ there is a ball of radius γ/x^3 in $B(f^n(z), \delta_0)$ which gets mapped by f^j with distortion bounded by 2 into $\mathcal{R}(x)$. As $|Df^n| < 2|f^n(z)|20/r < 40M/r$ on W, pulling back this ball gives a subset of W containing a ball of radius $(\gamma/x^3)r/40M$, as required.

Now we treat the case $|f^n(z)| \geq M$. Let $r' \leq r$ be maximal such that $f^{n-1}(B(z, r')) \subset B(f^{n-1}(z), 1)$. Set $W := B(z, r')$. As a neighbourhood of z gets mapped biholomorphically onto by f^{n-1} onto $B(f^{n-1}(z), 1)$ and f is univalent on each ball of radius 1, f^n is biholomorphic on W. Since

$$|f^{n-1}(z)| \geq \Re(f^{n-1}(z)) > \mathrm{diam}(P(f)) + 10\Delta \tag{7}$$

by choice of M, Lemma 3 implies that the distortion of f^{n-1} on W is bounded by 2. Thus $|Df^{n-1}| < 40/r$ on W, so $W \supset B(z, r/40)$. The distortion of f on any ball of radius 1 is e^2, so the distortion of f^n on W is bounded by $2e^2$.

The advantage of choosing W in this way is due to the distortion bound: if we can show $f^n(W)$ contains at least one square $Q \in \mathcal{Q}$, then the squares

$$\{Q \in \mathcal{Q} : Q \subset f^n(W)\}$$

fill some definite proportion of $f^n(W)$. We now have two further subcases.

Suppose first that $r' = r$, so $W = B(z, r)$. There is a $Q \in \mathcal{Q}$ containing $f^n(z)$, so (by Lemma 3, as usual) a neighbourhood W_z of z gets mapped biholomorphically onto Q by f^n with distortion bounded by 2. Since $|Df^n(z)| > 20/r$, we deduce that $\text{diam}(W_z) < r\,\text{diam}(Q)/10$, hence $W_z \subset B(z, r) = W$. In particular, $f^n(W)$ contains at least one square from $\mathcal{Q}$.

If we assume, on the other hand, that $r' < r$, then $f^{n-1}(W) \supset B(f^{n-1}(z), 1/2)$, by bounded distortion, and $f^n(W)$ is huge, in particular it contains at least one square $Q \in \mathcal{Q}$.

We have shown that in both subcases (so whenever $|f^n(z)| \geq M$), the squares $\{Q \in \mathcal{Q} : Q \subset f^n(W)\}$ fill some definite proportion of $f^n(W)$. Consequently, there is some independent constant $\gamma' > 0$ and a collection of pairwise-disjoint subsets $W_i \subset W$, each mapped by f^n onto an element Q_i of $\mathcal{Q}$ with $m(\bigcup_i W_i)/m(W) > \gamma'$, say. One can apply Lemma 19 on each Q_i to obtain a ball $B(y, \gamma/x^3) \subset Q_i$ say and some $l \geq 0$ such that f^l maps the ball univalently into $\mathcal{R}(x)$. Let $Z_i := B(y, \gamma/\Delta x^3)$ and let $V_i = W_i \cap f^{-n}(Z_i)$. By the Koebe principle, if $j, k \geq 0$ and $j + k \leq n + l$, the distortion of f^j is bounded by 2 on $f^k(V_i)$.

The distortion bound implies V_i contains a ball B_i of radius $\text{diam}(V_i)/4$, so $m(B_i)/m(V_i) > 1/16$. The bound (6) gives a bound on $|Df^n_{|B_i}|$ of $|\lambda_0|e^x 40/r$, which implies B_i has radius $\geq (\gamma/\Delta x^3)r/40|\lambda_0|e^x > e^{-2x}r$, provided x is large enough. This is the required estimate on the radii.

Continuing on, let $k \leq n + l$ be minimal such that $f^k(B_i) \subset \mathcal{R}(x)$. Thus there is a point in $f^{k-1}(B_i)$ not in $\mathcal{R}(x)$, so, by bounded distortion, $|Df| < 2|\lambda_0|e^x$ on $f^{k-1}(B_i)$. Univalence on $f^{k-1}(B_i)$ implies this set does not contain a ball of radius π, so the distortion bound of 2 for f^{k-1} on B_i and the estimate for the radius of B_i combine to imply

$$|Df^{k-1}_{|B_i}| < 80\pi|\lambda_0|e^x x^3 \Delta/r\gamma.$$

Thus $|Df^k_{|B_i}| < 160\pi|\lambda_0|^2 e^{2x}\Delta x^3/r\gamma < e^{3x}/r$, if x is large enough.

We note to finish that $m(Q_i) = 4\pi^2$ while $m(f^n(V_i)) = \pi\gamma^2/\Delta^2 x^6$, so

$$m(V_i)/m(W_i) > \gamma^2/\Delta^2 x^6 16\pi$$

for each i. Combining this with the uniform estimates for $m(B_i)/m(V_i)$, $m(\bigcup_i W_i)/m(W)$ and $m(W)/m(B(z, r)$, we conclude $m(\bigcup_i B_i)/m(B(z, r)) > \kappa/x^6$ for some $\kappa > 0$ independent of x. This completes the proof of the case $|f^n(z)| \geq M$. □

We call a square D *dyadic* if $2\pi 2^k D$ is an element of $\mathcal{Q}$ for some integer $k \geq 1$; 2^{-k} is then called the *scale* of D. Since each ball contains a square of comparable size, and vice versa, the previous lemma also holds for dyadic squares, with perhaps a slightly smaller scale (which we estimate crudely).

Lemma 21. *There are constants $\kappa > 0$, $M_0 \geq M$ such that the following holds. Let $k \geq 3$. Let $x \geq M_0$ and let D be a dyadic square of scale 2^{-k}. Then there is a finite collection of pairwise-disjoint dyadic squares $D_i \subset D$, each of scale $>e^{-3x}2^{-k}$, such that*

- $m(\bigcup_i D_i)/m(D) > \kappa/x^6$;
- *for each D_i there is an $n_i \geq 0$ with $f^{n_i}(D_i) \subset \mathcal{R}(x)$;*
- *f^{n_i} is univalent on $B(z, \Delta\,\mathrm{diam}(D_i))$ for all $z \in D_i$;*
- $|Df^{n_i}_{|D_i}| < e^{3x}2^k$.

If at all scales, a certain proportion gets mapped far out to the right, then almost every point does. The next lemma gives bounds on the time needed for a *large* proportion of points to get mapped far out to the right, together with a bound on the corresponding derivatives.

Proposition 22. *Let S be a bounded set. There is a constant M_0 such that the following holds. Let $x > M_0$. Let S_* denote the set of points z such that the first entry to $\mathcal{R}(x)$ happens at time $n(z)$ with*

- $|Df^{n(z)}(z)| < e^{x^9}/2$;
- $n(z) \leq e^{2x}$.

Then $m(S \backslash S_*) \leq 1/x$.

Proof. Let κ, M_0 come from Lemma 21. We can cover S with a finite number of dyadic squares of scale 2^{-3}, each contained in $B(S, 1)$, and with total area a, say. If $M' > M_0$ is sufficiently large, $x > M'$ and $p = x^7$, then

$$(1 - \kappa/x^6)^p a < e^{-\kappa x/2} a < 1/x.$$

At least a proportion κ/x^6 of each of these dyadic squares is covered by dyadic squares of scale $\geq 2^{-3}e^{-3x}$ given by Lemma 21. The remainder, less than $(1 - \kappa/x^6)$, can be covered by other dyadic squares of scale $\geq 2^{-3}e^{-3x}$ and we can apply Lemma 21 to each of these squares. Proceeding inductively, after p such applications, we end up with a collection $\mathcal{D}$ of dyadic squares such that

$$m\left(S \backslash \bigcup_{D \in \mathcal{D}} D\right) \leq (1 - \kappa/x^6)^p a < 1/x$$

and such that each $D \in \mathcal{D}$ satisfies

- the scale of D is $\geq 2^{-3}(e^{-3x})^p$;
- there is an $n_D \geq 0$, with $f^{n_D}(D) \subset \mathcal{R}(x)$;
- f^{n_D} is univalent on $B(z, \Delta\,\mathrm{diam}(D))$ for all $z \in D$;
- $|Df^{n_D}_{|D}| < (e^{3x})^{p+1}$.

We wish to show that S_* contains $\bigcup_{D \in \mathcal{D}} D$. For a point $y \in D \in \mathcal{D}$, n_D is not necessarily the first entry time $n(y)$ to $\mathcal{R}(x)$, but for all $j < n_D$, Lemma 8 implies $3|Df^j(y)| < |Df^{n_D}(y)|$, so $|Df^{n(y)}(y)| < (e^{3x})^{p+1} < e^{x^9}/2$.

It remains to show that n_D is not too large. It can be assumed that n_D is minimal such that $f^{n_D}(D) \subset \mathcal{R}(x)$. Now f^j on $B(z, \mathrm{diam}(D))$ is univalent with distortion bounded by 2 for all $j \leq n_D$, by choice of Δ, so $f^j(B(z, \mathrm{diam}(D)))$ cannot contain a ball of radius π for any $j < n_D$ and thus has diameter bounded by 4π. In particular, it does not intersect $\mathcal{R}(x + 4\pi)$. Thus for $1 \le j \le n_D$, $f^j(D) \subset B(0, |\lambda_0|e^{x+4\pi})$.

By Proposition 11, inside the region $B(0, |\lambda_0|e^{x+4\pi})$ the derivative multiplies by at least 3 at least every $C_0 e^x$ steps for some $C_0 > 0$. Therefore

$$3^{n_D/C_0e^x} < |Df^{n_D}_{|D}|,$$

so taking logs and using the estimate for the derivative,

$$\begin{aligned} n_D/C_0e^x &< 3x(p+1), \\ n_D &< C_0(p+1)(3x)e^x < e^{2x}, \end{aligned}$$

provided x is large enough, $x > M''$ say. We reset $M_0 := \max(M', M'')$. □

Next we show that the first entry usually happens a bit further to the right, and we recover some Markov property (*equal or disjoint*) which keeps the subsequent arguments from getting too messy.

Lemma 23. *Given $C > 0$, there exists M_0 such that, if $A \subset B(P(f), 1)$ is a simply-connected open set with ∂A of length at most C, then for all $x > M_0$ the following holds.*

There exists a set $A_* \subset A \backslash B(\partial A, x^{-1/4})$ and a partition $\mathcal{W}$ of A_* into elements W with associated numbers n_W, such that

- *$m(A \backslash A_*) < 1/2 \log x$;*
- *$|Df^{n_W}| < e^{x^9}$ on W;*
- *$n_W \le e^{2x}$;*
- *n_W is the first entry time to $\mathcal{R}(x + \log \frac{3}{2})$;*
- *f^{n_W} maps W biholomorphically onto a square from $\mathcal{Q}$;*
- *$f^{n_W}(W) \subset \mathcal{R}(x + 2\sqrt{x})$.*

Proof. In the proof, the sets W obtained will be mapped biholomorphically by corresponding f^{n_W} onto unions of squares of $\mathcal{Q}$ rather than onto single squares. This is of no import, as there will be a subpartition of each W whose elements each get mapped by f^{n_W} onto an element of $\mathcal{Q}$.

For large x, a standard estimate for the area of a tubular neighbourhood gives

$$m(B(\partial A, 2x^{-1/4})) \le 4Cx^{-1/4} + 4\pi x^{-1/2} < 8Cx^{-1/4}.$$

Therefore, setting $A_x := A \backslash B(\partial A, 2x^{-1/4})$, we have $m(A_x) > m(A) - 8Cx^{-1/4}$.

Let S_* be given by Proposition 22 for $S = B(P(f), 1)$ (and x sufficiently large). Set $A' := S_* \cap A_x$, so

$$m(A \backslash A') < 1/x + 8Cx^{-1/4}. \tag{8}$$

Let $z \in A'$ and let $n_0 = n_0(z)$ be the associated number $n(z)$ given by Proposition 22. Then n_0 is the first entry time of z to $\mathcal{R}(x)$, while $n_0 \le e^{2x}$ and

$$|Df^{n_0}(z)| < e^{x^9}/2. \tag{9}$$

Suppose first, in case one, that $\Re(f^{n_0}(z)) < 2\pi\lfloor(x + x^{3/4})/2\pi\rfloor$. Let T denote the partial strip

$$\{w : x \le \Re(w) < 2\pi\lfloor(x + x^{3/4})/2\pi\rfloor; 2j\pi \le \Im(w) < (2j+2)\pi\}$$

containing $f^{n_0}(z)$, for the relevant integer j. By Lemma 9 (with $M_1 = 2$ say), the connected set W_* containing z mapped univalently by f^{n_0} onto T has diameter less than $x^{3/4}/x = x^{-1/4}$, while $z \in A_x$, so $W_* \subset A \backslash B(\partial A, x^{-1/4})$. Let $T_+ := T \cap \mathcal{R}(2\pi \lfloor (x + 3\sqrt{x})/2\pi \rfloor)$, and set $W_z := W_* \cap f^{-n_0}(T_+)$. Note W_z does not necessarily contain z. Then $m(T \backslash T_+)/m(T) < 4\sqrt{x}/x^{3/4}$, so

$$m(W_z)/m(W_*) \geq 1 - 16x^{-1/4},$$

using a distortion bound of 2 from Lemma 3.

If, in case two, $\Re(f^{n_0}(z)) \geq 2\pi \lfloor (x + x^{3/4})/2\pi \rfloor$, let T denote the partial strip

$$\{w : 2\pi k \leq \Re(f^{n_0}(w)) < 2\pi(k+1); 2j\pi \leq \Im(w) < (2j+2)\pi\}$$

containing $f^{n_0}(z)$, for the relevant integers k, j. As before, by Lemma 9 (with $M_1 = 2\pi\sqrt{2}$ say), the connected set $W_z = W_*$ containing z mapped univalently by f^{n_0} onto T has diameter less than $1/x < x^{-1/4}$, and f^{n_0} on W_z has distortion bounded by 2. Again we deduce $W_z \subset A \backslash B(\partial A, x^{-1/4})$.

In both cases, for $j < n_0$, $f^j(W_z)$ has diameter bounded by $2x^{3/4}/x < \log \frac{3}{2}$, so $f^j(W_z) \cap \mathcal{R}(x + \log \frac{3}{2}) = \emptyset$. Meanwhile, $f^{n_0}(W_z) \subset \mathcal{R}(x + 2\sqrt{x})$. In particular, on W_z, n_0 is the first entry time to $\mathcal{R}(x + \log \frac{3}{2})$.

We claim that for $z_1, z_2 \in A'$, the sets $W_1 = W_{z_1}$, $W_2 = W_{z_2}$ are either equal or disjoint. Let $n_1 = n_0(z_1)$, $n_2 = n_0(z_2)$. The partial strips $f^{n_1}(W_1)$, $f^{n_2}(W_2)$ are either equal or disjoint. If $n_1 = n_2$ it follows that W_1, W_2 are either equal or disjoint. So suppose $n_1 < n_2$ and $W_1 \cap W_2 \neq \emptyset$. But $f^{n_1}(W_1) \subset \mathcal{R}(x + \sqrt{x})$, so $f^{n_1}(W_2) \cap \mathcal{R}(x + \sqrt{x}) \neq \emptyset$, contradicting $f^j(W_2) \cap \mathcal{R}(x + \log \frac{3}{2}) = \emptyset$ for $j < n_2$, from the previous paragraph. We conclude that the claim holds.

We thus obtain a (necessarily finite) pairwise-disjoint collection $\mathcal{W}$ of (such) subsets $W \subset A \backslash B(\partial A, x^{-1/4})$ with

$$m\left(\bigcup_{W \in \mathcal{W}} W\right) = \sum_{\mathcal{W}} m(W) \geq (1 - 16x^{-1/4})m(A'). \tag{10}$$

Set $A_* := \bigcup_{W \in \mathcal{W}} W$. Together with (8), (10) implies

$$\begin{aligned} m(A \backslash A_*) &\leq m(A) - m(A') + 16x^{-1/4}m(A') \\ &< 1/x + 8Cx^{-1/4} + 16x^{-1/4}m(B(P(f), 1)) \\ &< 1/2 \log x. \end{aligned}$$

If $W = W_z$ for some $z \in A'$, set $n_W := n_0(z)$, so $n_0 < e^{2x}$. The distortion bound of 2 combined with (9) gives the required derivative estimate $|Df^{n_W}| < e^{x^9}$ on W. □

6. Far-Right Dynamics

The dynamics far to the right is relatively easy to understand (and long-known, see for example [18,21,25]). Far-right squares from $\mathcal{Q}$ get mapped to enormous annuli, with approximately half getting mapped farther to the left, and half getting mapped farther to the right. That which gets mapped to the right, subsequently half of it gets mapped even farther to the left, half even farther to the right, and so on. Thus most points far to the right get mapped reasonably quickly very far to the left. A mathematical formulation is given by the following two lemmas.

Lemma 24. *Suppose n, S are such that f^n maps S biholomorphically onto some $Q \in \mathcal{Q}$. Provided the real rumber y satisfying $\Re(Q) = [y, y + 2\pi)$ is large enough, there is a finite partition of S into subsets $S_*, S_L, S_1, S_2, \dots, S_p$ such that the following holds:*

- $m(S_*) < m(S)/2y$;
- $f^{n+1}(S_L) \subset \mathcal{L}(-e^{y-\sqrt{y/2}})$;
- $m(S)/9 < m(S_1 \cup \dots \cup S_p) < \frac{7}{8}m(S)$;
- *each S_l, $1 \le l \le p$ is mapped by f^{n+1} biholomorphically onto an element of $\mathcal{Q}$ contained in $\mathcal{R}(e^{y-\sqrt{y/2}})$;*
- $|\Re(f^{n+1}(z))|^2 > |f^{n+1}(z)|$ *for all $z \in S \backslash S_*$.*

Proof. The proof will use that $A := f(Q)$ is a gigantic annulus, so most of it (by area) is a long way from the imaginary axis. By Lemma 3, taking y large, the distortion of f^n on S is bounded by 2. Note that on Q, the distortion of f is bounded by $e^{2\pi}$, so on S, the distortion of f^{n+1} is bounded by $2e^{2\pi}$.

For $r = |\lambda_0|e^y$, the annulus A has inner radius r and outer radius $re^{2\pi}$. Its area is $\pi r^2(e^{4\pi} - 1)$. Let X be the subset of A consisting of points close to the imaginary axis and close to $f(\partial Q)$ defined by

$$X := \{z \in A : |\Re(z)| \le |\lambda_0|^{-1} r e^{-\sqrt{y/2}} + 2\pi\} \cup B(f(\partial Q), 2\pi).$$

Then $m(X)$ is bounded by $2|\lambda_0|e^{2\pi} r^2 e^{-\sqrt{y/2}}$. Thus $m(X)/m(A) < e^{-\sqrt{y/3}}$, say, for large y. From this and the distortion bound we deduce that $m(S \cap f^{-n-1}(X)) < m(S)/2y$, provided y is large enough.

Set $S_L := f^{-n-1}(A \cap \mathcal{L}(0) \backslash X)$. Then $f^{n+1}(S_L) \subset \mathcal{L}(-e^{y-\sqrt{y/2}})$.

Let Y be the union of squares from $\mathcal{Q}$ containing points of $A \cap \mathcal{R}(0) \backslash X$. From the definition of X, $Y \subset A \backslash f(\partial Q)$ and $Y \subset \mathcal{R}(e^{y-\sqrt{y/2}})$. As

$$\frac{4}{9} m(Q) < m(f^{-1}(Y) \cap Q) < m(Q)/2,$$

using a distortion bound of 2 we deduce $m(S)/9 < m(f^{-n-1}(Y) \cap S) < \frac{7}{8}m(S)$ (one could improve this estimate to approximately $\frac{1}{2}m(S)$, but it is unnecessary). One can clearly partition the pullback of Y into the required sets $S_1, \dots, S_p$.

Set $S_* := S \backslash (S_L \cup S_1 \cup \dots \cup S_p)$. Since $f^{n+1}(S_*) \subset X$, we have from above that $m(S_*) < m(S)/2y$.

For $z \in S \backslash S_*$, we have

$$e^{y-\sqrt{y/2}} \le |\Re(f^{n+1}(z))| \le |f^{n+1}(z)| \le |\lambda_0| e^y e^{2\pi} < e^{3y/2} \le |\Re(f^{n+1}(z))|^2.$$

□

The square root terms in the following lemma are not exactly elegant, but they are used in the proof of Proposition 26.

Lemma 25. *Let $E : y \to e^y$. Let $Q \in \mathcal{Q}$ and suppose $Q \subset \mathcal{R}(x + 2\sqrt{x})$. If $x > 0$ is sufficiently large, there is a set $Q_0 \subset Q$ such that $m(Q_0)/m(Q) > 1/x$ and for all $z \in Q_0$, there is an integer $k = k(z)$ such that the following holds:*

- $1 \le k \le x$;
- $f^k(z) \in \mathcal{L}(-e^{x+\sqrt{x}}) \cap \mathcal{L}(-E^k(x))$;

- $|Df^k(z)| < |f^k(z)|^2 < |\Re(f^k(z))|^4$;
- $m(\{z \in Q_0 : k(z) \geq 4\}) > m(Q)/1000$.

Moreover, for $1 \leq j < k$, $f^j(z) \in \mathcal{R}(E^j(x))$ *and* $|Df^j(z)| < |f^j(z)|^2$.

Proof. Note that if $y \geq x + 2\sqrt{x}$, then

$$y - \sqrt{y/2} \geq x + 2\sqrt{x} - \sqrt{(x + 2\sqrt{x})/2} > x + \sqrt{x}.$$

Moreover $e^{y-\sqrt{y/2}} > e^{x+\sqrt{x}} > e^x + 2\sqrt{e^x}$. Inductively applying Lemma 24, we obtain sets $Q = Y^0 \supset Y^1 \supset \cdots$ and a collection of pairwise-disjoint sets $S_L^0, S_L^1, \ldots, S_*^0, S_*^1, \ldots$ for which

- for $0 \leq j \leq l$, $f^j(Y^l) \subset \mathcal{R}\left(E^j(x) + 2\sqrt{E^j(x)}\right) \subset \mathcal{R}(E^j(x))$;
- Y^l can be partitioned into sets mapped biholomorphically by f^l onto squares from $\mathcal{Q}$ (which together with the previous point allows one to proceed inductively);
- $Y^l = S_L^l \cup S_*^l \cup Y^{l+1}$;
- $m(S_*^l) < m(Q)\left(\frac{1}{2E^l(x)}\right)$;
- $m(Q)/9^l < m(Y_l) < m(Q)(\frac{7}{8})^l$;
- for $z \in S_L^l$, $f^{l+1}(z) \in \mathcal{L}(-e^{x+\sqrt{x}}) \cap \mathcal{L}(-E^{l+1}(x))$;
- for $z \in S_L^l$ and $1 \leq j \leq l+1$, $|f^j(z)| < |\Re(f^j(z))|^2$.

Thus $Y^l = Q\backslash(S_L^0 \cup \cdots \cup S_L^{l-1} \cup S_*^0 \cup \cdots \cup S_*^{l-1})$. Set $Q_0 := S_L^0 \cup \cdots \cup S_L^{\lfloor x\rfloor -1}$, so $Q_0 = Q\backslash\left(Y^{\lfloor x\rfloor} \cup S_*^0 \cup \cdots \cup S_*^{\lfloor x\rfloor}\right)$. From the two measure estimates,

$$m(Q_0)/m(Q) > \left(1 - \left(\frac{7}{8}\right)^{\lfloor x\rfloor} - \sum_{l\geq 0} \frac{1}{2E^l(x)}\right) > 1/x.$$

For $z \in S_L^l$, we set $k(z) := l + 1$. If $z \in Y^3\backslash Q_*$ then $k(z) \geq 4$, and $m(Y^3\backslash Q_*)/m(Q) \geq 9^{-3} - 1/x > 1/1000$.

It only remains to check the derivative. We have, for $z \in S_L^l$ and $1 \leq j \leq l$,

$$|f^j(z)|^2 < |\Re(f^j(z))|^2 < |\lambda_0|e^{\Re(f^j(z))} = |f^{j+1}(z)|$$

so, for $0 \leq j \leq l$,

$$\begin{aligned}|Df^{j+1}(z)| &= \prod_{a=1}^{j+1} |f^a(z)| \\ &\leq |f^{j+1}(z)|^{1+\frac{1}{2}+\frac{1}{4}+\cdots+2^{-j}} \\ &\leq |f^{j+1}(z)|^2 \leq |\Re(f^{j+1}(z))|^4,\end{aligned}$$

as required. □

7. First Entry to the Left Half-Plane

A key claim in the following proposition is that for many points, the first entry to $\mathcal{L}(-|\lambda_0|e^x)$ actually lands in $\mathcal{L}(-e^{x+\sqrt{x}})$. This added distance will be needed, see Lemma 4.

Proposition 26. *Given $C > 0$, there exists M_0 such that, if $A \subset B(P(f), 1)$ is a simply-connected open set with ∂A of length at most C, then for all $x > M_0$ the following holds. There is a set A_0 of points $z \in A \backslash B(\partial A, x^{-1/4})$ such that the first entry to $\mathcal{L}(-2|\lambda_0|e^x)$ happens at time $n(z)$ with*

(i) $f^{n(z)}(z) \in \mathcal{L}(-e^{x+\sqrt{x}})$

(ii) $e^x < |Df^{n(z)}(z)| < e^{x^9}|\Re(f^{n(z)}(z))|^4$;

(iii) $n(z) \leq e^{3x}$;

(iv) *there exists $n_0(z) < n(z)$ for which $|Df^l(z)| < e^{x^9}$ for $l \leq n_0$ and for which, for $l = n_0(z)+1, \ldots, n(z)$,*

$$|Df^l(z)| < e^{x^9}|f^l(z)|^2;$$

(v) $\inf_{j+k \leq n(z)} |Df^j(f^k(z))| > 2\exp(-2|\lambda_0|e^x)$;

and with $m(A\backslash A_0) \leq 1/\log x$.

Proof. Let A_*, with its attendant partition $\mathcal{W}$, be given by Lemma 23. Let $W \in \mathcal{W}$ and let n_W be given by Lemma 23. Let $Q = f^{n_W}(W) \in \mathcal{Q}$, and note $Q \subset \mathcal{R}(x+2\sqrt{x})$. Let $Q_0(W) = Q_0$ be given by Lemma 25. Set

$$A_0 := \bigcup_{W \in \mathcal{W}} W \cap f^{-n_W}(Q_0(W)).$$

Then (i)–(iv) are immediately obtained combining the estimates of Lemma 23 and Lemma 25, with $n_0(z) = n_W$ for $z \in W$.

It remains to justify (v) and the measure estimate. Now $n_0(z)$ is the first entry time to $\mathcal{R}(x + \log\frac{3}{2})$, so for $1 \leq j \leq n(z)$,

$$|f^j(z)| \geq |\lambda_0|\exp(-\frac{3}{2}|\lambda_0|e^x) > 2\exp(-2|\lambda_0|e^x)/\beta_1,$$

where β_1 comes from Lemma 10, and (5) implies (v). For the measure estimate, note $m(Q_0)/m(Q) > 1 - 1/x$ so, with a distortion bound of 2 for f^{n_W} on W,

$$m(A_0)/m(A_*) > 1 - 4/x.$$

Meanwhile, $m(A\backslash A_*) < 1/2\log x$ and $m(A_*) < m(B(P(f), 1))$ so

$$m(A\backslash A_0) < 1/2\log x + m(A_*)4/x < 1/\log x,$$

as required. □

8. Lyapunov Exponents Almost Never Exist

In this section we prove Theorems 1 and 2. We shall use the fact that Lebesgue measure is conservative and ergodic, see [10], to go from statements about positive-measure subsets to statements about full-measure subsets.

Lemma 27. *For almost every z and any Riemannian metric ρ,*

$$\limsup_{n\to\infty} \frac{1}{n} \log |D_\rho f^n(z)| \geq 0.$$

Proof. By Lemma 8 say, there is an M such that the first return map ϕ to $B(M+1,1)$ has $|D\phi| > 3$. Since Lebesgue measure is conservative and ergodic, almost every z enters $B(M+1,1)$ infinitely often. Thus for almost every z, there is a sequence n_k with $f^{n_k}(z) \in B(M+1,1)$ and $|Df^{n_k}(z)| \to +\infty$. Since $B(M+1,1)$ is bounded, $|Df_\rho^{n_k}(z)| \to +\infty$. □

Lemma 28. *For almost every z and any Riemannian metric ρ,*

$$\liminf_{n\to\infty} \frac{1}{n} \log |D_\rho f^n(z)| = -\infty. \tag{11}$$

For almost every z and the Euclidean metric,

$$\limsup_{n\to\infty} \frac{1}{n} \log |Df^n(z)| = +\infty. \tag{12}$$

Proof. Let $x > 0$ be large. Let $A = B(0,1)$, say, and let A_* and its attendant partition $\mathcal{W}$ be given by Lemma 23. Then $m(A_*) > \pi/2$ say. Let $W \in \mathcal{W}$ and let $n_W \leq e^{2x}$ be given by Lemma 23. Then $|Df^{n_W}| < e^{x^9}$ on W, and $Q_W := f^{n_W}(W) \in \mathcal{Q}$ and $Q_W \subset \mathcal{R}(x + 2\sqrt{x})$.

By Lemma 25 there is a subset $S_W \subset Q_W$ with $m(S_W) \geq m(Q_W)/1000$ for which the following holds. Let $z \in W \cap f^{-n_W}(S_W)$ and set $w := f^{n_W}(z)$. There is a $k = k(z)$ with $4 \leq k \leq x$,

- $f^k(w) \in \mathcal{L}(-E^4(x))$, where $E : y \mapsto e^y$;
- for $1 \leq j \leq k$, $|Df^j(w)| < |f^j(w)|^2 < |\Re(f^j(w)|^4$.

Then (Lemma 9 (with $M_1 > 1$ and $x > M_2$) gives the first inequality)

$$|Df^{n_W+k}(z)| > |f^{n_W+k}(z)| > E^4(x).$$

Meanwhile, $n_W + k \leq e^{2x} + x < 2e^{2x}$. Thus

$$\frac{1}{n_W + k} \log |Df^{n_W+k}(z)| > E^3(x)/2e^{2x} \gg x.$$

Going one step further will give us a tiny derivative.

$$\begin{aligned}
|Df^{n_W+k+1}(z)| &\leq e^{x^9} |\Re(f^k(w)|^4 |\lambda_0| \exp(\Re(f^k(w))) \\
&\leq e^{x^9} \exp(\Re(f^k(w))/2) \\
&\leq \exp(-E^4(x)/2 + x^9) \\
&\leq \exp(-E^4(x)/3).
\end{aligned}$$

Again, $n_W + k + 1 < 2e^{2x}$, from which we deduce

$$\frac{1}{n_W + k + 1} \log |Df^{n_W+k+1}(z)| \ll -x.$$

Let $X_x = \bigcup_{W\in\mathcal{W}}(W \cap f^{-n_W}(S_W))$. Using a distortion bound of 2, we obtain from the construction that

$$m(X_x) > m(A_*) \min_W \frac{m(S_W)}{4m(Q_W)} > \pi/8000$$

and that for each $z \in X_x$, there is an n with

$$\frac{1}{n} \log |Df^n(z)| > x,$$
$$\frac{1}{n+1} \log |Df^{n+1}(z)| < -x.$$

Necessarily, $f^{n+1}(z) \in B(0, 1)$, so for some $C > 0$ depending only on ρ,

$$\frac{1}{n+1} \log |Df_\rho^{n+1}(z)| < -Cx.$$

Taking a sequence of x_j tending to $+\infty$, we obtain sets X_{x_j} each with measure at least $\pi/8000$ and contained in the bounded set $B(0, 1)$. Thus there is a set X_∞ of positive measure for which each $z \in X_\infty$ is in infinitely many of the X_{x_j}. Thus (11), (12) hold for all $z \in X_\infty$, which implies (11), (12) hold for all $z \in \bigcup_{n\geq 0} f^{-n}(X_\infty)$. Using ergodicity and conservativity of Lebesgue measure [10], $\bigcup_{n\geq 0} f^{-n}(X_\infty)$ has full measure, completing the proof. □

Showing that the upper Lyapunov exponent is 0 almost everywhere for the spherical metric is more subtle. We need the following lemma.

Let $H : t \mapsto \exp(t^{1/10})$. For t large enough, $H(t) > t$ and $H^2(t) > e^t$.

Lemma 29. *Let $R > 0$ be sufficiently large and let $Q \in \mathcal{Q}$ be a subset of $\mathcal{L}(-R)$ satisfying $|z| < 2|\Re(z)|^2$ for all $z \in Q$. Let $Z \subset \mathbb{C}$ and $n_Z \geq 0$ be such that f^{n_Z} maps Z biholomorphically onto Q. There is a subset $Z_0 \subset Z$ and for each $z \in Z_0$ a number $n(z) \geq 1$ such that the following holds.*

- *For $j = 1, \ldots, n(z)$,*

$$\frac{1}{j} \log |D_\sigma f^j(f^{n_Z}(z))| < 1/\log R;$$

- *$m(Z\backslash Z_0)/m(Z) < 1/\log\log R$;*
- *if $z \in Z_0$, $f^{n_Z+n(z)}(z) \in \mathcal{L}(-H(R))$;*
- *$|f^{n_Z+n(z)}(z)| < 2|\Re(f^{n_Z+n(z)}(z))|^2$;*
- *there is a finite partition of Z_0 into sets U_i with associated numbers n_i, such that $n(z) = n_i$ for $z \in U_i$, and such that $f^{n_Z+n_i}$ maps U_i biholomorphically onto an element of $\mathcal{Q}$.*

Proof. Let $y \geq R$ satisfy $\Re(Q) = [-y-2\pi, -y)$. Let $B_y = B(0, |\lambda_0|e^{-y})$, so $f(Q) \subset B_y$. Let δ_0 be given by Lemma 12, so $0 < \delta_0 < \delta$. Let n_Q be the maximal positive integer such that $f^j(B_y) \subset B(f^j(0), \delta_0)$ for $j = 0, 1, \ldots, n_Q$. According to Lemma 12 then, a neighbourhood of 0 is mapped biholomorphically onto $B(f^{n_Q}(0), \Delta\delta_0)$. Thus the distortion of f^{n_Q} on B_y is bounded by 2. Since $\operatorname{diam}(B_y)/2 = |\lambda_0|e^{-y} \geq |Df|$ on Q and since $\delta_0 < \delta < 1/2$, it follows that

$$|Df^{1+j}| < 1 \tag{13}$$

on Q for $j = 0, \ldots, n_Q$. Meanwhile, since $\delta_0 < \delta$, for $j = 0, \ldots, n_Q$ we have $f^j(B_y) \subset V \subset B(0, M)$, so the derivative at each step is bounded by M.

For $j < y/2 \log M$,

$$|Df^j| < e^{j \log M} < e^{y/2} \tag{14}$$

on B_y. Thus for $z \in Q$, for $j < y/2 \log M$,

$$\begin{aligned} |D_\sigma f^{1+j}(z)| &< \left(1 + |z^2|\right) |\lambda_0| e^{-y} e^{j \log M} \\ &< \left(1 + |\Re(z)|^4\right) |\lambda_0| e^{-y+y/2} < (y+3\pi)^4 e^{-y/2} < e^{-y/3}, \end{aligned} \tag{15}$$

say. In particular, for $z \in Q$ and $j = 1, \ldots, \lfloor y/2 \log M \rfloor$,

$$\frac{1}{j} \log |D_\sigma f^j(z)| < 0. \tag{16}$$

This is our first estimate on the spherical derivative along the initial orbits of points in Q. From (13) we obtain, for $z \in Q$ and $j = 1 + \lfloor y/2 \log M \rfloor, \ldots, 1 + n_Q$,

$$\frac{1}{j} \log |D_\sigma f^j(z)| < \frac{1}{j} \log(1 + |z|^2) < \frac{2 \log M}{y} \log(y + 3\pi)^4 < y^{-1/2} \leq R^{-1/2} \tag{17}$$

say. Combining (16) and (17) gives

$$\frac{1}{j} \log |D_\sigma f^j(z)| < R^{-1/2} \tag{18}$$

for all $z \in Q$ and $j = 1, \ldots, 1 + n_Q$.

Now we have to study what happens at times greater than n_Q. By choice of n_Q, we deduce $\operatorname{diam}(f^{n_Q}(B_y)) > \delta_0/M$. Combined with (15), it follows that $n_Q \geq y/2 \log M$. It follows from the distortion bound that there is some $\nu_0 > 0$, independent of R, Q, for which $m(f^{n_Q+1}(Q)) > \nu_0$. Furthermore, $f^{n_Q+1}(\partial Q)$ has length bounded by $10\pi\delta_0 < 5\pi < 20$.

Let $x := y^{1/10}$. We claim that if W_j, n_j for $j = 1, 2$ are such that n_j is the first entry time of points in W_j to $\mathcal{L}(-2|\lambda_0|e^x)$, such that $f^{n_j}(W_j) \subset \mathcal{L}(-e^{x+\sqrt{x}} + 2\pi)$ and such that f^{n_j} maps W_j biholomorphically onto an element of $\mathcal{Q}$, then W_1 and W_2 are pairwise disjoint. If $n_1 = n_2$, this is obvious since $\mathcal{Q}$ is a partition. If $n_1 < n_2$, then $\operatorname{diam}(f^{n_1}(W_2)) < e^{-x}$ by Lemma 9, so $f^{n_1}(W_1) \cap f^{n_1}(W_2) = \emptyset$, by the first entry property, proving the claim.

Set $A := f^{n_Q+1}(Q \backslash \partial Q)$, so A is a simply-connected open set and, from above, $\partial A < 20$. $C = 20$ and let $A_0 \subset A$ be given by Proposition 26, and for $z \in A_0$, let $k_0(z), k(z)$ be the numbers $n_0(z), n(z) \leq e^{3x}$ given by Proposition 26. Then $k(z)$ is the first entry time of $z \in A_0$ to $\mathcal{L}(-e^x)$ and $f^{k(z)}(z) \in \mathcal{L}(-e^{x+\sqrt{x}})$. Let W_z be the neighbourhood of z mapped biholomorphically by $f^{k(z)}$ onto the element of $\mathcal{Q}$ containing $f^{k(z)}(z)$, so $f^{k(z)}(W_z) \subset \mathcal{L}(-e^{x+\sqrt{x}})$. Since $\operatorname{dist}(A_0, \partial A) \geq x^{-1/4}$, $W_z \subset A$.

By the claim, we obtain a cover of A_0 by a finite collection $\mathcal{W}$ of pairwise-disjoint sets W of the form W_z, $z \in A_0$. Extend the definition of k_0, k to $z' \in W_z$ by $k_0(z') = k_0(z)$, $k(z') = k(z)$. Set $k_W = k(z)$ for $z \in W$. On each W the distortion of f^j is bounded by 2 for $j = 1, \dots, k_W$ (as $P(f) \cap B(f^{k_W}(W), \Delta \operatorname{diam}(f^{k_W}(W))) = \emptyset$). Let us denote

$$A' := \bigcup_{W \in \mathcal{W}} W.$$

The measure estimate of Proposition 26 implies

$$m(A') > m(f^{n_Q+1}(Q)) - 1/\log x > m(f^{n_Q+1}(Q))(1 - 1/\nu_0 \log x). \tag{19}$$

Let

$$Z_0 := Z \cap f^{-n_Q-1-n_Z}(A').$$

The required partition of Z_0 is

$$\{Z \cap f^{-n_Q-1-n_Z}(W) : W \in \mathcal{W}\}.$$

With the distortion of f on Q bounded by $e^{2\pi}$, and distortion bounds of 2 for f^{n_Z} on Z and for f^{n_Q} on $f(Q)$, we derive from (19) that

$$m(Z_0)/m(Z) > 1 - (4e^{2\pi})^2/\nu_0 \log x > 1 - 1/\log\log x^{10} \geq 1 - 1/\log\log R.$$

Let $z \in Z_0$ and let $w := f^{n_Z}(z) \in Q$. Since $w \in Q$, in (18) we estimated $\frac{1}{j}\log|D_\sigma f^j(w)|$ for $j = 1, \dots, n_Q + 1$, while $f^{n_Q+1}(w) \in B(0, M)$. Now we consider higher iterates. For $j = n_Q + 2, \dots, 1 + n_Q + k_0(f^{1+n_Q}(w))$, we have the estimate $|Df^j(w)| < 2e^{x^9}$ coming from Proposition 26, whence

$$\begin{aligned}\frac{1}{j}\log|D_\sigma f^j(w)| &< \frac{1}{n_Q}\log((1+M^2)2e^{x^9}) < \frac{4\log M}{y}(y^{9/10} + \log(1+M^2))\\ &< 5(\log M)y^{-1/10}\\ &< 1/\log R.\end{aligned} \tag{20}$$

For $j = 2 + n_Q + k_0(f^{1+n_Q}(w)), \dots, 1 + n_Q + k(f^{1+n_Q}(w))$, we have the estimate $|Df^j(w)| < 2e^{x^9}|f^j(w)|^2$ again coming from Proposition 26, whence

$$\begin{aligned}\frac{1}{j}\log|D_\sigma f^j(w)| &< \frac{1}{n_Q}\log\left(\frac{1+M^2}{1+|f^j(w)|^2}2e^{x^9}|f^j(w)|^2\right)\\ &< \frac{1}{n_Q}\log((1+M^2)2e^{x^9})\\ &< 1/\log R,\end{aligned} \tag{21}$$

as before.

Set $n(z) := n_Z + 1 + n_Q + k(f^{n_Z+1+n_Q}((z)))$. Combining (18), (20) and (21) gives the required estimates on the spherical derivatives.

Once more from Proposition 26, for $z \in Z_0$,

$$|f^{n(z)}(z)| < 2|\Re(f^{n(z)}(z))|^2,$$

and, since $e^x = e^{y^{1/10}} \geq H(R)$,

$$f^{n(z)}(z) \in \mathcal{L}(-e^x) \subset \mathcal{L}(-H(R)),$$

as required. □

Lemma 30. *For almost every z and the spherical metric σ,*

$$\limsup_{n\to\infty} \frac{1}{n} \log |D_\sigma f^n(z)| = 0.$$

Proof. As before, by conservativity and ergodicity, we only need to show the result for a positive-measure set. Let $R \gg 0$ and let $S \in \mathcal{Q}$ with $S \subset \mathcal{L}(-R)$. Let $E : t \mapsto e^t$. Repeatedly applying Lemma 29, in the limit we obtain a set S_∞ for which

$$\begin{aligned} m(S_\infty)/m(S) &\geq \prod_{j=0}^{\infty} \left(1 - \frac{1}{\log\log H^j(R)}\right) \\ &\geq \prod_{j=0}^{\infty} \left(1 - \frac{1}{\log\log H^{2j}(R)}\right)\left(1 - \frac{1}{\log\log H^{2j+1}(R)}\right) \\ &> \prod_{j=0}^{\infty} \left(1 - \frac{1}{\log\log E^j(R)}\right)^2 \\ &> 0, \end{aligned}$$

and for which, for each $z \in S_\infty$, there is a strictly increasing sequence n_j, $j = 0, 1, \ldots$ such that

$$\frac{1}{k} \log |D_\sigma f^k(f^{n_j}(z))| < \frac{1}{\log H^j(R)}$$

for $k = 1, \ldots, n_{j+1} - n_j$. Consequently, for each z in the positive-measure set S_∞,

$$\limsup_{n\to\infty} \frac{1}{n} \log |D_\sigma f^n(z)| \leq 0.$$

□

Theorems 1 and 2 follow immediately from Lemmas 27, 28 and 30. □

9. Basic Parametric Estimates

We denote by log the principal branch of logarithm; it sends a neighbourhood of 1 in $\mathbb{C}$ to a neighbourhood of 0. In this section we commence our study of maps with parameters λ in a neighbourhood of λ_0.

Let $z, \lambda_1, \lambda_2 \in \mathbb{C}$ and suppose $|\log(\lambda_1/\lambda_2)|$ is small. Let $g_i : z \mapsto \lambda_i e^z$ for $i = 1, 2$. write $z_j := g_1^j(z)$ for $j \geq 0$. Suppose we have constructed $y_{k+1}, \ldots, y_n$ for some $0 \leq k < n$ and that $1 - y_j/z_j$ is small for $j = k+1, \ldots, n$. We can formally set

$$\alpha_j = \alpha_j(\lambda_1, \lambda_2, z) := \log(\lambda_1/\lambda_2) + \log(y_j/z_j) - (y_j - z_j)/z_j. \tag{22}$$

While $|1 - y_j/z_j| < \frac{1}{2}$, (22) gives

$$|\alpha_j| < |\log(\lambda_1/\lambda_2)| + |(y_j - z_j)/z_j|^2. \tag{23}$$

Set

$$y_k := z_k + (y_{k+1} - z_{k+1})/z_{k+1} + \alpha_{k+1}, \tag{24}$$

so $g_2(y_k) = y_{k+1}$. It follows that

$$y_k - z_k = \frac{y_n - z_n}{Dg_1^{n-k}(z_k)} + \sum_{j=k+1}^{n} \frac{\alpha_j}{Dg_1^{j-k-1}(z_k)}. \tag{25}$$

We shall use the above in Lemmas 31 and 33. The following proof just uses that λ_1, λ_2 are super-close and n is not too big, while to prove Lemma 33, we use expansion to get summability in (25).

Lemma 31. *Let $x > 10$ and $c_0 \geq 1/e$. Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ with $\beta := |\log(\lambda_1/\lambda_2)| < \exp(-9c_0e^x)$, and let $g_i : z \mapsto \lambda_i e^z$ for $i = 1, 2$. Let $n \leq e^{3x}$ and let $z = z_0 \in \mathbb{C}$. Suppose that*

$$\inf_{j+k \leq n} |Dg_1^k(g_1^j(z))| > \exp(-2c_0e^x).$$

Then there is a $y_0 = y(z, \lambda_1, \lambda_2, n)$ with $g_2^n(y_0) = g_1^n(z)$ and, for all $j \leq n$,

$$|g_2^j(y_0) - g_1^j(z_0)| \leq \beta \exp(3c_0e^x) < \exp(-c_0e^x) \tag{26}$$

Moreover, for all $j + k \leq n$,

$$|\log Dg_2^k(g_2^j(y_0))/Dg_1^k(g_1^j(z_0))| < \exp(-e^x). \tag{27}$$

Proof. The second inequality in (26) follows from the definition of β.

We commence by proving existence of y_0 satisfying (26) by induction on n. Write $z_j = g^j(z)$ for $j = 0, \ldots, n$. So assume, for $j = 1, \ldots, n$, that there exists $y_j = y(z_j, \lambda_1, \lambda_2, n-j)$ satisfying $|y_j - z_j| \leq \beta \exp(3c_0e^x)$ and, for $j = 1, \ldots, n-1$, $g_2(y_j) = y_{j+1}$. Existence of $y_n = z_n = y(z_n, \lambda_1, \lambda_2, 0)$ is trivial.

Define y_0 as per (24), so $g_2(y_0) = y_1$. From (25) and the hypotheses on n and the derivatives, one deduces for $k \geq 0$ that

$$|y_k - z_k| \leq e^{3x} \exp(2c_0e^x) \max_{j>k} |\alpha_j|.$$

For $k \geq 1$, $|z_k| > \exp(-2c_0 e^x)$ (by the derivative estimate), so

$$|y_k - z_k|/|z_k| \leq e^{3x} \exp(4c_0 e^x) \max_{j>k} |\alpha_j|. \tag{28}$$

By (28) and (23), for $k \geq 1$,

$$|\alpha_k| < \beta + 3e^{6x} \exp(8c_0 e^{x+1}) \max_{j>k} |\alpha_j|^2 < \beta + \beta^{-1} \max_{j>k} |\alpha_j|^2/4.$$

Now $|\alpha_n| = \beta$, so by induction it follows that $|\alpha_j| \leq 2\beta$ for $j = 1, \ldots, n$. Hence $|y_0 - z_0| \leq 2\beta e^{3x} \exp(2c_0 e^x) \leq \beta \exp(3c_0 e^x)$. Thus y_0 satisfies (26), completing the inductive argument.

To show (27), recall $|\alpha_l| \leq 2\beta$ and (28) and note that

$$\begin{aligned}\left|\log \frac{Dg_2^k(g_2^j(y_0))}{Dg_1^k(g_1^j(z_0))}\right| &= \left|\sum_{l=j+1}^{j+k} \log y_l/z_l\right| \leq \sum_{l=j+1}^{j+k} 2\left|\frac{y_l - z_l}{z_l}\right| \\ &\leq e^{3x} 4\beta e^{3x} \exp(4c_0 e^x) \\ &< \exp(-e^x).\end{aligned}$$

□

Given a function $R : \mathbb{C}^2 \to \mathbb{C}$, for $j = 1, 2$ we let $D_j R(z_1, z_2)$ denote the partial derivative of R with respect to the j^{th} variable, evaluated at the point (z_1, z_2).

Lemma 32. *Let $x > 10$. Let $B := \{\lambda \in \mathbb{C} : |\log(\lambda/\lambda_0)| < \exp(-10|\lambda_0|e^x)\}$. Suppose U is a simply-connected open set. Let $n \leq e^{3x}$. Suppose for all $z \in U$ that*

$$\inf_{j+k \leq n} |Df^j(f^k(z))| > 2\exp(-2|\lambda_0|e^x). \tag{29}$$

Then there is a holomorphic map $R : U \times B \to \mathbb{C}$ such that

$$f_\lambda^n(R(z, \lambda)) = f^n(z) \tag{30}$$

with

- *for $j = 0, \ldots, n$,*
$$|f^j(z) - f_\lambda^j(R(z, \lambda))| < e^{-x}; \tag{31}$$
- $|D_1 R(z, \lambda)| < \exp(-e^x)$;
- $|D_2 R(z, \lambda)| < \exp(4|\lambda_0|e^x)$.

Proof. Note that if $\lambda_1, \lambda_2 \in B$ then $|\log(\lambda_1/\lambda_2)| < 2\exp(-10|\lambda_0|e^x) < \exp(-9|\lambda_0|e^x)$. With $c_0 = |\lambda_0|$, for each $z \in U$, $\lambda \in B$, Lemma 31 spits out a point $R(z, \lambda) := y(z, \lambda_0, \lambda, n)$ with $|f^j(z) - f_\lambda^j(R(z, \lambda))| < \exp(-|\lambda_0|e^x) < e^{-x}$ for $j = 0, \ldots, n$. We can immediately write $R(z, \lambda) = \phi_\lambda \circ f^n(z)$ where ϕ_λ is the appropriate inverse branch of f_λ^n, but it takes some work to show what appropriate is, and in particular that the branches vary continuously and so are well-defined.

By (26),

$$|f_\lambda^j(R(z, \lambda)) - f^j(z)| < \exp(-|\lambda_0|e^x) < 1/2 \tag{32}$$

for $j = 0, \ldots, n$. Since f_λ is univalent on each ball of radius π, $R(z, \lambda)$ is the unique point z' with $f_\lambda^n(z') = f^n(z)$ for which $|f_\lambda^j(z') - f^j(z)| < 1$ for all $j = 0, \ldots, n$. Now (27) and (29) imply

$$\inf_{j+k \le n} |Df_\lambda^j(f_\lambda^k(R(z, \lambda))| > \exp(-2|\lambda_0|e^x),$$

so, for $\lambda' \in B$, we can apply Lemma 31 again to obtain points $y(R(z, \lambda), \lambda, \lambda', n)$. Again, for $j = 0, \ldots, n$,

$$|f_\lambda^j(R(z, \lambda)) - f_{\lambda'}^j(y(R(z, \lambda), \lambda, \lambda', n))| < \exp(-|\lambda_0|e^x) < 1/2$$

so with (32), the triangle inequality and uniqueness, one obtains

$$y(R(z, \lambda), \lambda, \lambda', n) = R(z, \lambda').$$

The estimate (26) then implies that

$$|R(z, \lambda) - R(z, \lambda')| \le |\log(\lambda/\lambda')| \exp(3|\lambda_0|e^x)$$

so $R(z, \cdot)$ is continuous, with Lipschitz bound $\exp(4|\lambda_0|e^x)$, say. Therefore the 'appropriate' inverse branches ϕ_λ vary holomorphically, and $R(z, \cdot)$ is holomorphic with $|D_2 R(z, \lambda)| \le \exp(4|\lambda_0|e^x)$.

Differentiating (30) gives $D_1 R(z, \lambda) = Df^n(z)/Df_\lambda^n(R(z, \lambda))$, so (27) implies

$$|\log D_1 R(z, \lambda)| < \exp(-e^x),$$

and holomorphicity of R, as required. □

The following lemma concerning existence of the holomorphic motion h is well-known. We include the elementary proof for completeness, and because it gives the Lipschitz-type constant M_0 without invoking λ-lemmas.

Lemma 33. *There exists $r_0, M_0 > 0$ and a function $h : P(f) \times B(\lambda_0, r_0)$ for which the following hold. For each $z \in P(f)$ and for $\lambda \in B(\lambda_0, r_0)$, $\lambda \mapsto h(z, \lambda)$ is holomorphic, while $z \mapsto h(z, \lambda)$ is injective, and $|h(z, \lambda) - z| \le M_0|\lambda - \lambda_0|$. For such z, λ and all $n \ge 0$,*

$$f_\lambda^n(h(z, \lambda)) = h(f^n(z), \lambda). \tag{33}$$

Proof. Note that if (33) holds with $n = 1$ then it holds for all $n \ge 0$.

Since $P(f)$ is a compact, forward-invariant, hyperbolic repelling set, there is a constant $M_1 > 1$ such that $\sum_{j \ge 1} |Df^j(z)|^{-1} < M_1$ for all $z \in P(f)$, and there is an $\eta \in (0, 1)$ such that $B(0, \eta) \cap P(f) = \{0\}$. Choose $r_0 > 0$ such that, for all $\lambda \in B(\lambda_0, r_0)$,

$$r_\lambda := \max(|\log(\lambda/\lambda_0)|, |\lambda - \lambda_0|) < \eta^2/4M_1^2.$$

As an intermediate step, we shall inductively construct functions h_n which shall converge to h. Let $h_0 : (z, \lambda) \mapsto z$ and suppose for $j = 1, \ldots, n-1$ we have functions $h_j : P(f) \times B(\lambda_0, r_0) \to \mathbb{C}$ such that, for all $(z, \lambda) \in P(f) \times B(\lambda_0, r_0)$,

- $h_{j-1}(f(z), \lambda) = f_\lambda(h_j(z, \lambda))$;
- $|h_j(z, \lambda) - z| \le 2M_1 r_\lambda$.

Then for each such pair (z, λ) we have the sequences $z = z_0, z_1 = f(z), \ldots, z_n = f^n(z)$ and $y_1 = h_{n-1}(z_1, \lambda), \ldots, y_n = z_n$ and the corresponding sequence of $\alpha_j = \alpha_j(\lambda_0, \lambda, z)$ as defined in (22). Then define y_0 by (24), whence $f_\lambda(y_0) = y_1$. For $j \geq 1$, by supposition, $|y_j - z_j| \leq 2M_1 r_\lambda$, while $z_j \in P(f)\backslash\{0\}$ so $|z_j| \geq \eta$. In particular, $|(y_j - z_j)/z_j| \leq 2M_1 r_\lambda/\eta$. Inserting this estimate into (23), we obtain

$$|\alpha_j| \leq |\log(\lambda/\lambda_0)| + 4M_1^2 r_\lambda^2/\eta^2 \leq 2r_\lambda.$$

By (25) and the definition of M_1, we deduce that $|y_0 - z_0| \leq 2M_1 r_\lambda$. Define $h_n(z, \lambda) := y_0$. Then

$$h_{n-1}(f(z), \lambda) = f_\lambda(h_n(z, \lambda)) \quad \text{and} \quad |h_n(z, \lambda) - z| \leq 2M_1 r_\lambda. \tag{34}$$

To conclude the inductive construction of h_n, note that a h_1 clearly exists satisfying the required properties. Thus (34) holds for each n.

Consequently $|h_{n-1}(f(z), \lambda) - f(z)| \leq 2M_1 r_\lambda < \eta$, while $|f(z)| \geq \eta$, so $h_{n-1}(f(z), \lambda) \neq 0$ and

$$\lambda \mapsto h_n(z, \lambda) = f_\lambda^{-1}(h_{n-1}(f(z), \lambda))$$

is well-defined and holomorphic, upon choosing the appropriate branch of f_λ^{-1}.

Since the $h_n(z, \cdot)$ are uniformly bounded, we can extract a convergent subsequence with holomorphic limit $h(z, \cdot)$ with the same Lipschitz bound $|h(z, \lambda) - z| \leq 2M_1 r_\lambda$. One can take $M_0 := 2M_1$. The map h satisfies (33) for $n = 1$ and thus for all n. We claim that, for given λ, $h(z, \lambda)$ is the unique point z_λ such that $|f_\lambda^n(z_\lambda) - f^n(z)| < \delta$ for all $n \geq 0$. Now $f_\lambda^{n_0}$ is uniformly expanding on $B(f^n(z), 3\delta)$ for each n. Therefore there is only one point, z', for which $f_\lambda^n(z') \in B(f_\lambda^n(z_\lambda), 2\delta)$ for all $n \geq 0$ and $z' = z_\lambda$, proving the claim. Therefore the map h is unique and $z \mapsto h(z, \lambda)$ is injective. □

10. Parameter Space to Phase Space Near $P(f)$

The following lemma is another form of the standard Koebe distortion lemma.

Lemma 34. *Given $\varepsilon' > 0$ there is a $\delta' > 0$ such that if g is any univalent function on the unit disc, one can write*

$$Dg(z) = Dg(0)[1 + \theta(z)],$$

where θ is a holomorphic function on $B(0, \delta')$ with $|\theta| < \varepsilon'$.

Proof. The distortion of g is bounded by 2 on $B(0, 1/\Delta)$, so $|Dg(z)| \leq 2|Dg(0)|$ on that ball. By Cauchy's integral formula, $|D^2 g| \leq 4\Delta|Dg(0)|$ on $B(0, 1/2\Delta)$. Integrating gives $|Dg(z) - Dg(0)| \leq 4|z|\Delta|Dg(0)|$, on $B(0, 1/2\Delta)$. Taking $\delta' = \varepsilon'/4\Delta$, the result follows. □

The ideas in this section are not especially new, though the exposition and the formulation of results are. The reader may wish to compare this section with [3, Sections 3, 4] and [1, Section 3]. The useful result is Lemma 37; it follows easily from the following proposition.

Recall the definitions of Sect. 2, where $n_0, \alpha, \varepsilon_0, \delta$ and V were fixed. Let h, M_0, r_0 be given by Lemma 33. Now $h(0, \cdot)$ is a holomorphic function of λ. *A priori* it could be identically zero, however Misiurewicz maps are not structurally stable [20,31], so $h \not\equiv 0$, see [3, Lemma 2.1]. Therefore, there exist an integer $K \geq 1$ and a non-zero constant a_K such that $h(0, \lambda) = a_K(\lambda - \lambda_0)^K$ + higher order terms. Thus given $\varepsilon_1 \in (0, 1)$, there is an $r(\varepsilon_1) > 0$ for which we can write

$$h(0, \lambda) = a_K(\lambda - \lambda_0)^K [1 + \theta_0(\lambda)], \tag{35}$$

where θ_0 is holomorphic on $B(\lambda_0, r(\varepsilon_1))$ with norm bounded by ε_1. In particular, for $\lambda \in B(\lambda_0, r(\varepsilon_1))$,

$$\left| a_K(\lambda - \lambda_0)^K \right| /2 \leq |h(0, \lambda)| \leq 2 \left| a_K(\lambda - \lambda_0)^K \right|. \tag{36}$$

For $n \geq 0$, let us denote by ξ_n the holomorphic map defined by

$$\xi_n(\lambda) = f_\lambda^n(0).$$

Proposition 35. *Given $\varepsilon > 0$, there exist constants $\delta_1, r_3, C_0, C_1 > 0$ such that, for all $r \in (0, r_3)$, the following holds. Let $n = n(r, \delta_1)$ be maximal such that*

$$f_\lambda^j(B(0, 2|h(0, \lambda)|)) \subset B(f^j(0), \delta_1) \subset V \tag{37}$$

for $j = 0, \ldots, n$ and all $\lambda \in B(\lambda_0, 2r)$.

Then $\xi_n(B(\lambda_0, 2r)) \subset B(f^n(0), \delta_1)$,

$$D\xi_n(\lambda) = -Df^n(0) K a_K (\lambda - \lambda_0)^{K-1} [1 + \theta_5(\lambda)],$$

where θ_5 is a holomorphic function on the annulus $A(\lambda_0; r/4, r)$ with $|\theta_5| < \varepsilon$, and

$$1/C_0 r < |D\xi_n(\lambda)| < C_0/r.$$

Moreover, $|Df_\lambda^n(0)| \leq C_1/r^K$ for all $\lambda \in B(\lambda_0, r)$.

Proof. Taking $\delta_1 < \delta$, $B(f^j(0), \delta_1) \subset V$. From (37), the statement $\xi_n(B(\lambda_0, 2r)) \subset B(f^n(0), \delta_1)$ is trivial. We shall expend much effort to compare $Df_\lambda^n(z)$, $Df_\lambda^n(0)$ and $Df^n(0)$.

Assume $\varepsilon \in (0, 1)$ and set $\varepsilon_1 = \varepsilon/16$. Let δ' be given by Lemma 34 for $\varepsilon' = \varepsilon_1$ and let $\delta_1 \in (0, \min(\delta_0\delta', \delta)/2)$ satisfy

$$\delta_1 e^{Mn_0+\alpha} \sum_{k\geq 0} e^{-k\alpha/n_0} < \varepsilon_1/8. \tag{38}$$

Let r_1 be the number $r(\varepsilon_1) > 0$ for which (35) holds. Let r satisfy

$$0 < r < \min(\varepsilon_0, r_0, r_1, \delta_1/M_0)/2 \tag{39}$$

and let $n = n(r, \delta_1)$ be given by (37).

By (37) and choice of δ_1,

$$f_\lambda^n(B(0, 2|h(0, \lambda)|)) \subset B(f_\lambda^n(0), 2\delta_1) \subset B(f_\lambda^n(0), \delta_0). \tag{40}$$

By Lemma 12, noting $\Delta > 1$, a neighbourhood of 0 is mapped biholomorphically by f_λ^n onto $B(f_\lambda^n(0), \delta_0)$. By (40), this neighbourhood necessarily contains $B(0, 2|h_\lambda(0, \lambda)|)$, and $\delta_0/2\delta_1 \geq \delta'$ plus choice of δ' then implies

$$Df_\lambda^n(z) = Df_\lambda^n(0)[1 + \gamma(z, \lambda)], \tag{41}$$

where $\gamma(\cdot, \lambda)$ is a holomorphic function on $B(0, 2|h(0, \lambda)|)$ bounded by ε_1, and this for each $\lambda \in B(\lambda_0, 2r)$.

Integrating along a ray from 0 to z, we obtain

$$f_\lambda^n(z) = f_\lambda^n(0) + zDf_\lambda^n(0)[1 + \gamma_1(z, \lambda)], \tag{42}$$

where $\gamma_1(z, \lambda) := \frac{1}{z}\int_0^z \gamma(w, \lambda)$, with $|\gamma_1(z, \lambda)| \le \varepsilon_1$.

Applying (42), with $z = h(0, \lambda)$, gives

$$f_\lambda^n(0) - f_\lambda^n(h(0, \lambda)) = -Df_\lambda^n(0)h(0, \lambda)\,[1 + \theta(\lambda)]\,, \tag{43}$$

where θ is the holomorphic function $\lambda \mapsto \theta(\lambda) := \gamma_1(h(0, \lambda), \lambda)$ with norm bounded by ε_1.

Now we wish to compare Df_λ^n with Df^n at 0. First we show n is not too large.

By (36), there is a $\lambda_1 \in B(\lambda_0, 2r)$ for which $|h(0, \lambda_1)| > |a_K|r^K$. Since $|Df^{n_0}| > \exp(\alpha)$ on V, if $kn_0 \le n$ then

$$B(f^{kn_0}(0), 2\delta_1) \supset f^{kn_0}(B(0, 2|h(0, \lambda_1)|)) \supset B(f^{kn_0}(0), e^{k\alpha}a_K r^K).$$

Thus $\alpha n/n_0 \le \log(2\delta_1 r^{-K}/|a_K|)$. In particular, there exists a $c_0 > 0$ for which

$$n = n(r, \delta_1) < -c_0 \log r.$$

This implies that $rn(r, \delta_1) \to 0$ as $r \to 0$.

Recall $|\lambda_0| \ge \frac{1}{e}$ and $V \subset B(0, M-2)$, so $|Df_\lambda| \ge e^{-M}$ on V for all $\lambda \in B(\lambda_0, 1/2e)$. By the same Koebe distortion bound that gave (41), and the estimates

$$|Df_\lambda^k(f_\lambda^j(0))| \ge e^{-n_0 M}\exp(\lfloor k/n_0\rfloor\alpha)$$

for $k + j = n$, we deduce that the images of $B(0, 2|h(0, \lambda)|)$ under f^j are exponentially small in $n - j$:

$$\operatorname{diam}(f^j(B(0, 2|h(0, \lambda)|))) \le 2\delta_1 e^{n_0 M+\alpha} e^{(j-n)\alpha/n_0}. \tag{44}$$

Meanwhile, by definition of h, for all $j \ge 0$,

$$h(f^j(0), \lambda) = f_\lambda^j(h(0, \lambda)) \in f_\lambda^j(B(0, 2|h(0, \lambda)|)),$$

while $|h(z, \lambda) - z| \le M_0|\lambda - \lambda_0|$. Hence

$$\operatorname{dist}(f^j(0), f_\lambda^j(B(0, 2|h(0, \lambda)|))) \le M_0|\lambda - \lambda_0|. \tag{45}$$

For $j \le n$, combining (45) and (44) gives

$$|f^j(0) - f_\lambda^j(0)| \le M_0|\lambda - \lambda_0| + 2\delta_1 e^{n_0 M+\alpha} e^{(j-n)\alpha/n_0}.$$

As an exponential map, $Df(y)/Df(y') = e^{y-y'}$. By (38), there is a uniform bound

$$\sum_{j=0}^{n-1} |\log|Df(f^j(0))/Df(f_\lambda^j(0))|| \le \sum_{j=0}^{n-1} |f^j(0) - f_\lambda^j(0)| < 2M_0 rn + \varepsilon_1/4, \tag{46}$$

while $Df/Df_\lambda = \lambda_0/\lambda$. Thus for $k \le n$,

$$\begin{aligned} |\log|Df^k(0)/Df_\lambda^k(0)|| &< 2M_0rn + \varepsilon_1/4 + |n\log(\lambda_0/\lambda)| \\ &< 2M_0rn + \varepsilon_1/4 + 2n|\lambda-\lambda_0|/|\lambda_0|. \end{aligned} \tag{47}$$

But from above, $rn \to 0$. Thus if r is sufficiently small,

$$|\log|Df^n(0)/Df_\lambda^n(0)|| < \varepsilon_1/2$$

so

$$Df_\lambda^n(0) = Df^n(0)[1+\theta_1(\lambda)] \tag{48}$$

with θ_1 a holomorphic function on $B(\lambda_0, 2r)$ with norm bounded by ε_1. With (43), we obtain

$$f_\lambda^n(0) - f_\lambda^n(h(0,\lambda)) = -Df^n(0)h(0,\lambda)\,[1+\theta_2(\lambda)]\,, \tag{49}$$

where $\theta_2 = (1+\theta)(1+\theta_1)$ is a holomorphic function with norm bounded by $3\varepsilon_1$.

Using (35), we can substitute in for h to obtain

$$f_\lambda^n(0) - f_\lambda^n(h(0,\lambda)) = -Df^n(0)a_K(\lambda-\lambda_0)^K\,[1+\theta_3(\lambda)]\,, \tag{50}$$

where $\theta_3 := (1+\theta_2)(1+\theta_0)$ is holomorphic with norm bounded by $5\varepsilon_1$ on $B(\lambda_0, 2r)$. By Cauchy's integral formula, $|D\theta_3(\lambda)| < 10\varepsilon_1/r$ on $B(\lambda_0, r)$, whence $|\lambda-\lambda_0||D\theta_3(\lambda)| < 10\varepsilon_1$. Thus, on $B(\lambda_0, r)$, the derivative of (50) can be written

$$-Df^n(0)Ka_K(\lambda-\lambda_0)^{K-1}\,[1+\theta_4(\lambda)]\,, \tag{51}$$

where $1+\theta_4(\lambda) := (1+\theta_3(\lambda)) + (\lambda-\lambda_0)D\theta_3(\lambda)/K$, so $|\theta_4(\lambda)| < 15\varepsilon_1$.

Now we have all the distortion-like estimates we need, let us estimate the size of the derivative. By maximality of n, there exists $\lambda_1 \in B(\lambda_0, 2r)$ for which $f_{\lambda_1}^{n+1}(B(0, 2|h(0,\lambda_1)|)) \not\subset B(f^{n+1}(0), \delta_1)$, which, combined with (45) implies

$$\operatorname{diam}(f_{\lambda_1}^{n+1}(B(0, 2|h(0,\lambda_1)|))) \ge \delta_1 - M_0|\lambda_1-\lambda_0| > \delta_1 - M_0r > \delta_1/2. \tag{52}$$

The derivative is bounded by M on V, so (52) implies

$$\operatorname{diam}(f_{\lambda_1}^{n}(B(0, 2|h(0,\lambda_1)|))) \ge \delta_1/2M. \tag{53}$$

Therefore, for some $z \in B(0, 2|h(0,\lambda_1)|)$,

$$|Df_{\lambda_1}^n(z)| \ge \frac{\delta_1}{4M|h(0,\lambda_1)|}. \tag{54}$$

The bounds (48) and (41) give good distortion control, combining to give $Df_\lambda^n(z) = Df^n(0)[(1+\theta(\lambda))(1+\gamma(z,\lambda))]$, so (54) implies

$$|Df^n(0)||h(0,\lambda_1)| > \delta_1/8M,$$

in turn implying, via (36),

$$|Df^n(0)||a_K|(2r)^K > \delta_1/16M. \tag{55}$$

If $|\lambda-\lambda_0| \ge r/4$ then

$$\frac{|\lambda-\lambda_0|^{K-1}}{r^K} \ge \frac{1}{4^{K-1}r}. \tag{56}$$

From (55) and (56), we deduce that, on the annulus $A(\lambda_0; r/4, 2r)$,

$$|Df^n(0)|K|a_K||\lambda - \lambda_0|^{K-1}|1 + \theta_4(\lambda)| > K\delta_1/2^{8K}Mr. \tag{57}$$

Now $\xi_n(\lambda) = f_\lambda^n(0)$, so adding and subtracting the same term,

$$\xi_n(\lambda) = f_\lambda^n(0) - f_\lambda^n(h(0, \lambda)) + h(f^n(0), \lambda),$$

and (50) gives, on $B(\lambda_0, 2r)$,

$$\xi_n(\lambda) = -Df^n(0)a_K(\lambda - \lambda_0)^K [1 + \theta_3(\lambda)] + h(f^n(0), \lambda). \tag{58}$$

Let D_2h denote the partial derivative of h with respect to the second variable. Taking the derivative on both sides of (58), and using (51),

$$D\xi_n(\lambda) = Df^n(0)Ka_K(\lambda - \lambda_0)^{K-1} [1 + \theta_4(\lambda)] + D_2h(f^n(0), \lambda). \tag{59}$$

Now $|h(z, \lambda) - z| \le M_0|\lambda - \lambda_0|$ for $z \in P(f)$ and $\lambda \in B(\lambda_0, r_0)$, so by Cauchy's integral formula, $|D_2h(z, \lambda)| \le 2M_0$ on $B(\lambda_0, r)$. Therefore, if r is small enough the bound (57) together with (59) entails that

$$D\xi_n(\lambda) = -Df^n(0)Ka_K(\lambda - \lambda_0)^{K-1} [1 + \theta_5(\lambda)], \tag{60}$$

where θ_5 is a holomorphic function on $A(\lambda_0; r/4, r)$ with norm bounded by $16\varepsilon_1$. Setting $C_0 := 2^{9K}M/K\delta_1$, taking absolute values of (60) and using (55), we obtain

$$|D\xi_n(\lambda)| > 1/C_0r.$$

It remains to provide the upper bound for $|Df_\lambda^n(0)|$. This follows simply from (55) and (48). □

Lemma 36. *Let g be a holomorphic map defined on an open convex set U. Suppose $\Re(Dg(z)) > 0$ for all $z \in U$. Then g is injective.*

Proof. Integrating Dg along a line from z_1 to z_2 in U, one cannot obtain 0. □

Given an annulus $A(y; a_1, a_2)$ and $k \ge 2$, the k rays leaving y with angles $2j\pi/k$ for $j \le k$ divide $A(y; a_1, a_2)$ into k (open) congruent pieces which we will call *k-sectors* of $A(y; a_1, a_2)$.

Lemma 37. *Given $\varepsilon' > 0$, there exists $r_3, \gamma \in (0, 1)$ and $\nu_0, C, C_0, C_1 > 0$ such that for all $r \in (0, r_3)$, the following holds. There exists $n \ge 1$ such that ξ_n maps each $4K$-sector W of $A(\lambda_0; \gamma r, r)$ injectively onto a simply-connected, open set $\xi_n(W)$ with $m(\xi_n(W)) > \nu_0$ and the length of $\partial\xi_n(W)$ bounded by C. For $j \le n$, $\xi_j(W) \subset V$.*

For $\lambda, \lambda' \in W$,

$$1/C_0r \le |D\xi_n(\lambda)|$$

and

$$\left|\frac{D\xi_n(\lambda)}{D\xi_n(\lambda')}\right| < 1 + \varepsilon'.$$

Moreover, $|Df_\lambda^n(0)| \le C_1/r^K$ for all $\lambda \in B(\lambda_0, r)$.

Proof. Let $\gamma < 1$ satisfy $\gamma^K > 1 - \varepsilon'/3$. With $\varepsilon = \varepsilon'/3$, let $\delta_1, r_3, C_0, C_1, \theta_5$ be given by Proposition 35, let $r \in (0, r_3)$ and let n be defined as per Proposition 35. Then $\xi_j(B(\lambda_0, r)) \subset V$ for $j \leq n$.

Let $\gamma \in (\frac{1}{2}, 1)$ and let W be a $4K$-sector of $A(\lambda_0; \gamma r, r)$. Let $\widehat{W}$ denote the convex hull of W, so $\widehat{W}$ is contained in a $4K$-sector W' of $A(\lambda_0; r/4, r)$. Now

$$\{(\lambda - \lambda_0)^{K-1} : \lambda \in W'\}$$

lies (strictly) in a quadrant of the plane. Since $|\theta_5| < |\varepsilon| < 1/\sqrt{2}$ on $A(\lambda_0; r/4, r)$,

$$\{1 + \theta_5(\lambda) : \lambda \in W'\}$$

is also a subset of a quadrant. Thus

$$D\xi_n(\lambda) = -Df^n(0) K a_K (\lambda - \lambda_0)^{K-1} [1 + \theta_5(\lambda)] \tag{61}$$

lies in a fixed half-plane for all $\lambda \in \widehat{W}$. By Lemma 36, ξ_n is injective on $\widehat{W}$ and thus is injective on W.

The derivative estimate $|D\xi_n(\lambda)| > 1/C_0 r$ on W implies the image has measure at least ν_0, for some $\nu_0 > 0$ depending on γ but not on r. Injectivity and bounded distortion give an upper bound on $r|D\xi_n|$, since the measure of V is bounded. The length of ∂W is bounded by a constant times r, so the upper bound on $r|D\xi_n|$ implies that the length of $\partial \xi_n(W)$ is bounded by a constant $C > 0$.

The distortion estimate follows from (61), as choice of γ and the bound $|\theta_5| < \varepsilon'/3$ give

$$\left| \frac{D\xi_n(\lambda)}{D\xi_n(\lambda')} \right| < 1/(1 - \varepsilon'/3)^2 < 1 + \varepsilon'.$$

The derivative estimate of $|Df_\lambda^n(0)| \leq C_1/r^K$ comes directly from Proposition 35. □

11. Parameter Dependence at the Large Scale

Lemma 32 allows us to show that some sets which get mapped eventually onto a square far out to the left do not move very fast as the parameter λ varies, so if λ does not vary much, the intersection remains large. Later on we will show that for relatively large sets of parameters, the orbit of 0 under f_λ lands in one of these intersections.

Lemma 38. *Let $C, \nu_0 > 0$. There is an $M_2 > 0$ such that for $x > M_2$, the following holds. Suppose $A \subset B(P(f), 1)$ is a simply-connected open set satisfying $m(A) > \nu_0$ and with ∂A having length at most C. Let $B := \{\lambda : |\log(\lambda_0/\lambda)| < \exp(-10|\lambda_0|e^x)\}$. There is a collection $\{U_l\}_{l=1}^L$ of pairwise-disjoint subsets of A and numbers n_l, together with a map $R : \bigcup_l U_l \times B \to A \backslash B(\partial A, e^{-x})$ such that*

- *$m(\bigcup_l U_l)/m(A) \geq 1 - 1/\log\log x$;*
- *$R(z, \lambda_0) = z$;*
- *on each $U_l \times B$, R is holomorphic, $|\log D_1 R| < \exp(-e^x)$ and $|D_2 R| < \exp(4|\lambda_0|e^x)$;*
- *for $z \in R(U_l, \lambda)$,*

$$|Df_\lambda^{n_l}(z)| < 3e^{x^9} |\Re(f^{n_l}(z))|^4;$$

- *for each* λ*, the sets* $R(U_l, \lambda)$ *for* $l = 1, \dots, L$ *are pairwise-disjoint;*
- *for* $z \in R(U_l, \lambda)$*, a neighbourhood* V_z *of* z *with diameter bounded by* e^{-x} *gets mapped biholomorphically by* $f_\lambda^{n_l}$ *onto*

$$B(f_\lambda^{n_l}(z), 1) \subset \mathcal{L}(-e^{x+\sqrt{x}} + 3\pi).$$

Proof. Given $C, \nu_0 > 0$, let $x \gg 0$ be large enough to apply Proposition 26. Let $A_0 \subset A \setminus B(\partial A, x^{-1/4})$ and $n(z)$ for $z \in A_0$ be given by Proposition 26, so $m(A \setminus A_0) \leq 1/\log x$ and $n(z) \leq e^{3x}$.

For $z \in A_0$, let Q_z be the element of $\mathcal{Q}$ containing $f^{n(z)}(z)$. Let U_z be the neighbourhood of z mapped biholomorphically by $f^{n(z)}$ onto Q_z. Clearly $Q_z \subset \mathcal{L}(-e^{x+\sqrt{x}} + 2\pi)$, and for $j < n(z)$, the diameter of $f^j(U_z)$ is bounded by e^{-x} (see Lemma 9 to treat $j \leq n(z) - 2$, while $|Df| \geq e^{x+\sqrt{x}}$ on $f^{n(z)-1}(U_z)$). Since $n(z)$ is also the first entry time of z to $\mathcal{L}(-2|\lambda_0|e^x)$, $f^j(U_z) \subset \mathcal{R}(-2|\lambda_0|e^x - 1)$ for $j < n(z)$. It follows that if $z' \in A_0$ and $U_z \cap U_{z'} \neq \emptyset$ then $n(z) = n(z')$ and $U_z = U_{z'}$. Thus the neighbourhoods U_z, for $z \in A_0$, form a finite (since $n(z)$ is bounded), pairwise-disjoint collection which we can write as $\{U_l\}_{l=1}^L$, setting $n_l := n(z)$ for some $z \in U_l \cap A_0$. The collection is a cover of A_0 and thus has measure at least $m(A) - 1/\log x$. Since $m(A) > \nu_0$, for large x we obtain the required measure estimate.

We can write $Q_l = f^{n_l}(U_l) \in \mathcal{Q}$. Let $\widehat{U}_l \supset U_l$ denote the set containing U_l mapped biholomorphically by f^{n_l} onto $B(Q_l, 1) \subset \mathcal{L}(-e^{x+\sqrt{x}} + 3\pi)$. Applying Lemma 3, the distortion of f^{n_l} is bounded by 2 on each $\widehat{U}_l$, and since $A_0 \cap \widehat{U}_l \neq \emptyset$, the estimates of Proposition 26 imply that for $z \in \widehat{U}_l$,

$$|Df^{n_l}(z)| < 2e^{x^9} \sup_{y \in \widehat{U}_l} |\Re(f^{n_l}(y))|^4 < 2e^{x^9} |\Re(f^{n_l}(z)) + 2\pi|^4 \tag{62}$$

and

$$\inf_{j+k \leq n_l} |Df^j(f^k(z))| > 2\exp(-2|\lambda_0|e^x).$$

We can therefore apply Lemma 32 to obtain a holomorphic map $R_l : \widehat{U}_l \times B \to \mathbb{C}$, where $R_l(z, \lambda_0) = z$ and, for $(z, \lambda) \in \widehat{U}_l \times B$,

$$f_\lambda^{-n_l} \circ R_l(z, \lambda) = f^{n_l}(z).$$

By Lemma 32, $|\log D_1 R_l| < \exp(-e^x)$ and $|D_2 R_l(z, \lambda)| < \exp(4|\lambda_0|e^x)$. The former implies

$$|Df_\lambda^{n_l}(R(z, \lambda))|/|Df^{n_l}(z)| \approx 1,$$

for all $\lambda \in B$, which combined with (62) produces the bound

$$|Df_\lambda^{n_l}(y)| < 2e^{x^9} |\Re(f^{n_l}(z)) + 2\pi|^4 < 3e^{x^9} |\Re(f^{n_l}(z))|^4$$

for $y \in R_l(\widehat{U}_l, \lambda)$. As $f_\lambda^{n_l}$ maps $R_l(\widehat{U}_l, \lambda)$ biholomorphically onto $B(Q_l, 1)$, for each $z \in R_l(U_l, \lambda)$ there is a neighbourhood V_z mapped biholomorphically by $f_\lambda^{n_l}$ onto $B(f_\lambda^{n_l}(z), 1) \subset B(Q_l, 1)$. By Lemma 9, say, the diameter of V_z is bounded by e^{-x}.

From before, $f^j(U_l) \subset \mathcal{R}(-2|\lambda_0|e^x - 1)$ and the diameter of $f^j(U_l)$ is bounded by e^{-x} for $j < n_l$. From (31), $\text{dist}(f_\lambda^j(z), f^j(U_l)) < e^{-x}$ for all $z \in R_l(U_l, \lambda)$. Thus n_l is the first entry time for each point of $R(U_l, \lambda)$ (under iteration by f_λ) to $\mathcal{L}(-2|\lambda_0|e^x - 2)$. Thus if $R(U_l, \lambda) \cap R(U_{l'}, \lambda) \neq \emptyset$, $n_l = n_{l'}$, so $Q_l = Q_{l'}$ (as Q_l and $Q_{l'}$ either coincide or are disjoint), so $R(U_l, \lambda) = R(U_{l'}, \lambda)$. In particular, the sets $R_l(U_l, \lambda)$, $1 \leq l \leq L$, are pairwise-disjoint. Define R as the map whose restriction to each U_l is R_l.

It remains to show that $R(U_l, B) \subset A \backslash B(\partial A, e^{-x})$. From above, $\text{dist}(z, U_l) < e^{-x}$ for every $z \in R(U_l, \lambda)$ and each $\lambda \in B$, and U_l has diameter less than e^{-x}. Therefore

$$\sup_{z', z \in U_l} \sup_{\lambda \in B} |R(z', \lambda) - z| < 2e^{-x}.$$

Since there exists $z \in U_l \cap A_0$, so $z \in A \backslash B(\partial A, x^{-1/4})$, and $x^{-1/4} > 3e^{-x}$, we deduce that $R(U_l, B) \subset A \backslash B(\partial A, e^{-x})$, as required. □

12. Proof of Main Theorem

The main theorem follows from the following proposition. The number K is, we recall, the local degree of $h(0, \cdot)$ at λ_0, while $\xi_n(\lambda) = f_\lambda^n(0)$. We denote by H the set of hyperbolic parameters.

We shall use the estimates for passing from parameter to phase space near $P(f)$ of Lemma 37, and the estimates of Lemma 38 to go from near $P(f)$ to far out to the left. Their combination allows us to apply Lemma 4 to find large sets of hyperbolic parameters.

Proposition 39. *Given $\varepsilon > 0$, there exists $\gamma, r_4 > 0$ such that for every $r \in (0, r_4)$ and every $4K$-sector W of $A(\lambda_0; \gamma r, r)$,*

$$\frac{m(H \cap W)}{m(W)} > 1 - \varepsilon.$$

Proof. Let $r_3, \gamma, \nu_0, C, C_0, C_1$ be given by Lemma 37, for $\varepsilon' = \varepsilon/4$. For these C, ν_0, let M_2 be given by Lemma 38. Let $C_2 > 0$ be large enough that

$$C_1 \exp(11K|\lambda_0|e^x) 3e^{x^9} < \exp(C_2 e^x)$$

for all $x > M_2$. Let $M_3 > M_2$ be large enough that

- $1/\log\log M_3 < \varepsilon/3$;
- $M_3 > C_0, C_2$;
- Lemma 4 holds for the constant C_2 for all $x > M_3$;
- $r_4 := \exp(-11|\lambda_0|e^{M_3}) < r_3$.

Let $r \in (0, r_4)$, so we can fix $x > M_3$ satisfying $r = \exp(-11|\lambda_0|e^x)$. Let n be given by Lemma 37, let W be as per the statement and set $A := \xi_n(W)$. From Lemma 37, ξ_n is injective with distortion bounded by $1 + \varepsilon/4$ and A is a simply-connected open set with length of ∂A bounded by C. Moreover $1/C_0 r < |D\xi_n|$ on W. The distortion bound implies

$$|D\xi_n| < \sqrt{\frac{m(A)}{m(W)}}(1 + \varepsilon/4). \tag{63}$$

Meanwhile, $B(\lambda_0, r) \subset B := \{\lambda : |\log(\lambda_0/\lambda)| < \exp(-10|\lambda_0|e^x)\}$, so we can apply Lemma 38, obtaining $R : \bigcup_{l=1}^{L} U_l \times B \to A \backslash B(\partial A, e^{-x})$ together with the numbers $\{n_l\}_{l=1}^{L}$ and the estimates $|\log D_1 R| < \exp(-e^x)|$ and $|D_2 R| < \exp(4|\lambda_0|e^x)$.

Fix l for now, and let $z \in U_l$. Let

$$Y_z := \{R(z, \lambda) : \lambda \in B\} \subset A \setminus B(\partial A, e^{-x}).$$

As $\exp(11|\lambda_0|e^x)/C_0 = 1/C_0 r < |D\xi_n|$,

$$|D_2 R| < \exp(e^{-x})|D\xi_n|. \tag{64}$$

Hence the map $y \mapsto R(z, \xi_n^{-1}(y))$ is a strict contraction on Y_z and it has a unique fixed point $y_z \in \overline{Y_z} \subset A$. Let $\Lambda(z) := \xi_n^{-1}(y_z)$, so $\xi_n(\Lambda(z)) = R(z, \Lambda(z))$.

Now $D\xi_n - D_2 R \neq 0$, so we can apply the implicit function theorem to deduce that $z \mapsto \Lambda(z)$ is holomorphic on each U_l. Suppose $\Lambda(z) = \Lambda(z_1)$. From Lemma 38, for each λ, the sets $R(U_l, \lambda)$ are pairwise-disjoint, so z and z_1 must be in the same U_l. But on each $U_l \times \{\lambda\}$, R is a homeomorphism, so $z = z_1$. Thus $\Lambda(z)$ is injective on $U := \bigcup_{l=1}^{L} U_l$. The map Λ gives the link between parameter space and phase space.

Taking derivative of $\xi_n(\Lambda(z)) = R(z, \Lambda(z))$ with respect to z,

$$D\xi_n(\Lambda(z))D\Lambda(z) = D_1 R(z, \Lambda(z)) + D_2 R(z, \Lambda(z))D\Lambda(z),$$

so

$$D\Lambda(z) = \frac{D_1 R(z, \Lambda(z))}{-D_2 R(z, \Lambda(z)) + D\xi_n(\Lambda(z))}. \tag{65}$$

Together with (64) and the estimate for $|\log D_1 R|$, (65) implies $|D\Lambda(z)| > (1 - e^{-x})/|D\xi_n(\Lambda(z))|$, say. Using (63) and integrating $|D\Lambda|^2$ over U,

$$m(\Lambda(U)) > m(U)\frac{(1 - e^{-x})^2}{(1 + \varepsilon/4)^2}\frac{m(W)}{m(A)}.$$

From Lemma 38 and choice of M_3, $m(U)/m(A) \geq 1 - 1/\log\log x > 1 - \varepsilon/3$. Thus

$$m(\Lambda(U))/m(W) \geq \frac{(1 - e^{-x})^2}{(1 + \varepsilon/4)^2}(1 - \varepsilon/3) > 1 - \varepsilon. \tag{66}$$

We have shown that $\Lambda(U)$ is a relatively large set. Next we show that it consists of hyperbolic parameters.

Let $\lambda \in \Lambda(U_l)$ say and set $z := R(\Lambda^{-1}(\lambda), \lambda) = f_\lambda^n(0)$. Let V_z be given by Lemmma 38, so V_z of z with diameter bounded by e^{-x} gets mapped biholomorphically onto $B(f_\lambda^{n_l}(z), 1)$. For $j \leq n$, we know $f_\lambda^j(0) \in V$, so by Lemma 12, a neighbourhood of 0 gets mapped biholomorphically onto $B(f_\lambda^n(0), \Delta\delta_0) \supset B(z, e^{-x})$. Therefore a neighbourhood of 0 gets mapped biholomorphically by $f_\lambda^{n+n_l}$ onto

$$B(f_\lambda^{n+n_l}(0), 1) \subset \mathcal{L}(-e^{x+\sqrt{x}} + 3\pi).$$

From Lemma 38, we have

$$|Df_\lambda^{n_l}(z)| < 3e^{x^9}|\Re(f^{n_l}(z))|^4,$$

while Lemma 37 states that $|Df_\lambda^n(0)| < C_1/r^K$. Recalling $r = \exp(-11|\lambda_0|e^x)$ and the choice of C_2, we obtain

$$|Df_\lambda^{n+n_l}(0)| < C_1 \exp(11K|\lambda_0|e^x)3e^{x^9}|\Re(f_\lambda^{n+n_l}(0))|^4 < \exp(C_2 e^x)|\Re(f_\lambda^{n+n_l}(0))|^4.$$

Applying Lemma 4, λ is a hyperbolic parameter. This holds for each $\lambda \in \Lambda(U)$, so $\Lambda(U) \subset H$. Thus (66) gives

$$\frac{m(H \cap W)}{m(W)} \geq \frac{m(\Lambda(U))}{m(W)} \geq 1 - \varepsilon,$$

as required. □

The statement of the main theorem follows immediately from Proposition 39, so its proof is now complete.

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