Research Article

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Energy and area minimizers in metric spaces

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Abstract: We show that in the setting of proper metric spaces one obtains a solution of the classical 2-dimensional Plateau problem by minimizing the energy, as in the classical case, once a definition of area has been chosen appropriately. We prove the quasi-convexity of this new definition of area. Under the assumption of a quadratic isoperimetric inequality we establish regularity results for energy minimizers and improve Hölder exponents of some area-minimizing discs.

Keywords: Plateau's problem, energy minimizers, Sobolev maps, quasi-convex areas, Hölder regularity, metric spaces

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1 Introduction

1.1 Motivation

The classical Plateau problem concerns the existence and properties of a disc of smallest area bounded by a given Jordan curve. In a Riemannian manifold X, a solution of the Plateau problem is obtained by a disc of minimal energy, where one minimizes over the set $\Lambda(\Gamma, X)$ of all maps u in the Sobolev space $W^{1,2}(D, X)$, whose boundary $\mathrm{tr}(u) : S^1 \to X$ is a reparametrization of the given Jordan curve Γ. This approach has the useful feature that the area minimizer obtained in this way is automatically conformally parametrized.

Recently, the authors of the present article solved the classical Plateau problem in the setting of arbitrary proper metric spaces in [17]. In particular, we proved existence of area minimizing discs with prescribed boundary in any proper metric space and with respect to any quasi-convex definition of area (in the sense of convex geometry). It should be noted that the classical approach (described above) to the Plateau problem cannot work literally in the generality of metric spaces. This is due to the fact that there are many natural but different definitions of area and of energy. Moreover, different definitions of area may give rise to different minimizers as was shown in [17]. Finally, the presence of normed spaces destroys any hope of obtaining a conformal area minimizer and the inevitable lack of conformality is the source of difficulties when trying to compare or identify minimizers of different energies and areas.

One of the principal aims of the present article is to show that the classical approach (of minimization of the area via the simpler minimization of the energy) does in fact work for some definitions of energy and area. As a byproduct we obtain new definitions of area which are quasi-convex (topologically semi-elliptic in the language of [12]), which might be of some independent interest in convex geometry.

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1.2 Energy and area minimizers

For a metric space X, the Sobolev space $W^{1,2}(D, X)$ consists of all measurable, essentially separably valued maps $u : D \to X$ which admit some function $g \in L^2(D)$ with the following property (cf. [21], see also [10]): For any 1-Lipschitz function $f : X \to \mathbb{R}$ the composition $f \circ u$ lies in the classical Sobolev space $W^{1,2}(D)$, and the norm of the gradient of $f \circ u$ is bounded from above by g at almost every point of D. In $L^2(D)$ there exists a unique minimal function g as above, called the *generalized gradient* of u. This generalized gradient g_u coincides with the *minimal weak upper gradient* of a representative of u in the sense of [10]. The square of the L^2-norm of this *generalized gradient* g_u is the Reshetnyak energy of u, which we denote by $E_+^2(u)$. A different but equivalent definition of the Sobolev space $W^{1,2}(D, X)$ is due to Korevaar–Schoen [16] and comes along with another definition of energy $E^2(u)$ generalizing the classical Dirichlet energy.

If X is a Riemannian manifold, then $g_u(z)$ is just the point-wise sup-norm of the weak differential $Du(z)$ for almost all $z \in D$. The Dirichlet–Korevaar–Schoen energy $E^2(u)$ is obtained in this case by integrating over D the sum of squares of eigenvalues of $Du(z)$. It is the heart of the classical approach to Plateau's problem by Douglas and Rado, extended by Morrey to Riemannian manifolds, that any minimizer of the Dirichlet energy E^2 in $\Lambda(\Gamma, X)$ is conformal and minimizes the area in $\Lambda(\Gamma, X)$.

Turning to general proper metric spaces X, we recall from [17] that for any Jordan curve Γ in X one can find minimizers of E^2 and E_+^2 in the set $\Lambda(\Gamma, X)$, whenever $\Lambda(\Gamma, X)$ is not empty. The first special case of our main result Theorem 4.3 identifies any minimizer of the Reshetnyak energy E_+^2 in $\Lambda(\Gamma, X)$ as a minimizer of the inscribed Riemannian area μ^i investigated by Ivanov in [12], see also Sections 2.3–2.4 below.

Theorem 1.1. *Let Γ be any Jordan curve in a proper metric space X. Then every map $u \in \Lambda(\Gamma, X)$ which minimizes the Reshetnyak energy E_+^2 in $\Lambda(\Gamma, X)$ also minimizes the μ^i-area in $\Lambda(\Gamma, X)$.*

Any minimizer u of the Reshetnyak energy as in Theorem 1.1 is $\sqrt{2}$-quasiconformal. This means, roughly speaking, that u maps infinitesimal balls to ellipses of aspect ratio at most $\sqrt{2}$, see [17] and Section 3.2 below. We emphasize that our notion of quasiconformal map is different from the notion of quasiconformal homeomorphism studied in the field of quasiconformal mappings. For any map $v \in W^{1,2}(D, X)$ there is an energy-area inequality $E_+^2(v) \geq \mathrm{Area}_{\mu^i}(v)$; and for any u as in Theorem 1.1 equality holds.

We find a similar phenomenon in the case of the more classical Korevaar–Schoen energy E^2, which generalizes the Dirichlet energy from the Riemannian to the metric setting. However, the corresponding *Dirichlet definition of area μ^D* seems to be new, see Section 3.3.

Theorem 1.2. *There exists a quasi-convex definition of area μ^D such that the following holds true. For any Jordan curve Γ in a proper metric space X, and for any map $u \in \Lambda(\Gamma, X)$ with minimal Korevaar–Schoen energy $E^2(u)$ in $\Lambda(\Gamma, X)$, this map u minimizes the μ^D-area in $\Lambda(\Gamma, X)$.*

Recall that quasi-convexity of the definition of area is a very important feature in the present context, since it is equivalent to the lower semi-continuity of the corresponding area functional in all Sobolev spaces [17], Theorem 5.4, and therefore, closely related to the question of the existence of area minimizers.

In order to describe the definition of area μ^D, we just need to fix the values of the μ^D-areas of one subset in every normed plane V. Considering the subset to be the ellipse arising as the image $L(D)$ of a linear map $L : \mathbb{R}^2 \to V$ (see Section 2 below), this value $\mathrm{Area}_{\mu^D}(L)$ equals

$$\mathrm{Area}_{\mu^D}(L) = \frac{1}{2} \inf\{E^2(L \circ g) \mid g \in \mathrm{SL}_2\}. \tag{1.1}$$

For any Sobolev map $v \in W^{1,2}(D, X)$ the energy-area inequality $E^2(v) \geq 2 \cdot \mathrm{Area}_{\mu^D}(v)$ holds true, with equality for any minimizer u as in Theorem 1.2. The minimizers in Theorem 1.2 are Q-quasiconformal with the non-optimal constant $Q = 2\sqrt{2} + \sqrt{6}$ ([17] and Section 3.3 below). An answer to the following question would shed light on the structure of energy minimizers from Theorem 1.2, cf. [8, p. 723] for the "dual" question.

Problem 1.3. For which $g \in \mathrm{SL}_2$ is the infimum in (1.1) attained? Is it possible to describe the measure μ^D appearing in Theorem 1.2 in a geometric way? What is the optimal quasiconformality constant of the minimizers of the Korevaar–Schoen energy?

All definitions of area of Sobolev maps agree with the parametrized Hausdorff area if X is a Riemannian manifold, a space with one-sided curvature bound or, more generally, any space with the property (ET) from [17, Section 11]. In this case, Theorem 1.2 directly generalizes the classical result of Douglas–Rado–Morrey.

Our results apply to all other *quasi-convex definitions of energy*, see Theorem 4.2. We refer to Section 2 for the exact definitions and mention as a particular example linear combinations $a \cdot E^2 + b \cdot E_+^2 + c \cdot \mathrm{Area}_\mu$, where $a, b, c \geq 0$ with $a^2 + b^2 > 0$ and where μ is some quasi-convex definition of area. For any such energy E there exists a quasi-convex definition of area μ^E such that a minimizer of E automatically provides a quasi-conformal minimizer of μ^E as in Theorem 1.1 and Theorem 1.2. The definition of area μ^E is given similarly to (1.1).

Remark 1.1. We would like to mention a related method of obtaining an area-minimizer for any quasi-convex definition of area μ. In the Riemannian case this idea can be found in [11], cf. [7, Section 4.10]: Consider the energy $E_\epsilon = \epsilon E_+^2 + (1-\epsilon)\,\mathrm{Area}_\mu$. Then a minimizer u_ϵ of E_ϵ in $\Lambda(\Gamma, X)$ can be found in the same way as the minimizer of E_+^2. This minimizer is automatically $\sqrt{2}$-quasiconformal and minimizes the area functional $\epsilon\,\mathrm{Area}_{\mu^i} + (1-\epsilon)\,\mathrm{Area}_\mu$ in $\Lambda(\Gamma, X)$. Due to the quasiconformality these minimizers have uniformly bounded energy. Therefore one can go to the limit (fixing three points in the boundary circle) and obtain a minimizer of Area_μ.

This remark also shows that the set of quasi-convex areas obtained via the minimization of energies as in (1.1) is a dense convex subset in the set of all quasi-convex definitions of area. It seems to be a natural question which definitions of area correspond in this way to some energies. In particular, if it is the case for the most famous Hausdorff, Holmes–Thompson and Benson definitions of area.

Remark 1.2. From Theorem 4.2 and Theorem 1.1 one can deduce the quasi-convexity of the inscribed Riemannian area μ^i. However, a much stronger convexity property of this area has been shown in [12].

1.3 Regularity of energy minimizers

In the presence of quadratic isoperimetric inequalities the regularity results for area minimizers obtained in [17] imply regularity of energy minimizers, once we have identified energy minimizers as area minimizers in Theorem 4.3. Recall that a complete metric space X is said to admit a (C, l_0)-isoperimetric inequality with respect to a definition of area μ if for every Lipschitz curve $c : S^1 \to X$ of length $l \leq l_0$ there exists some $u \in W^{1,2}(D, X)$ with

$$\mathrm{Area}_\mu(u) \leq C \cdot l^2$$

and such that the trace $\mathrm{tr}(u)$ coincides with c. We refer to [17] for a discussion of this property satisfied by many interesting classes of metric spaces. If μ is replaced by another definition of area μ', then in the definition above only the constant C will be changed at most by a factor of 2. If the assumption is satisfied for some triple (C, l_0, μ), we say that X satisfies a *uniformly local quadratic isoperimetric inequality*.

As far as qualitative statements are concerned the constants and the choice of the area do not play any role. As a consequence of Theorem 1.2 and the regularity results for area minimizers in [17] we easily deduce continuity up to the boundary and local Hölder continuity in the interior for all energy minimizers in $\Lambda(\Gamma, X)$ for any quasi-convex definition of energy. We refer to Theorem 4.4 for the precise statement.

1.4 Improved regularity of μ-minimal discs

We can use Theorem 1.1 to slightly improve the regularity results for solutions of the Plateau problem obtained in [17]. Assume again that Γ is a Jordan curve in a proper metric space X and let μ be a definition of area. We introduce the following

Definition 1.1. We say that a map $u \in \Lambda(\Gamma, X)$ is μ-minimal if it minimizes the μ-area in $\Lambda(\Gamma, X)$, and if it has minimal Reshetnyak energy E_+^2 among all such minimizers of the μ-area.

Due to Theorem 1.1, for the inscribed Riemannian definition of area $\mu = \mu^i$, a μ-minimal disc is just a minimizer of the Reshetnyak energy E_+^2 in $\Lambda(\Gamma, X)$. It follows from [17] that, for any quasi-convex μ, one finds some μ-minimal disc in any non-empty $\Lambda(\Gamma, X)$. Moreover, any such μ-minimal map is $\sqrt{2}$-quasiconformal. Assume further that X satisfies the (C, l_0, μ)-isoperimetric inequality. In [17], we used the quasiconformality to deduce that any such map has a locally α-Hölder continuous representative with $\alpha = \frac{1}{8\pi C}$. However, μ-minimal maps satisfy a stronger infinitesimal condition than $\sqrt{2}$-quasiconformality, and this can be used to improve α by a factor of $2 \cdot q(\mu) \in [1, 2]$ depending on the definition of area μ. The number $q(\mu)$ equals 1 for the maximal definition of area $\mu = \mu^i$. For other definitions of area μ, the number $q(\mu)$ is smaller than 1 and measures the maximal possible deviation of μ from μ^i, see (2.1). For instance, $q(\mu^b) = \frac{\pi}{4}$ for the Hausdorff area μ^b. Thus the following result improves the above Hölder exponent by 2 in the case of the inscribed Riemannian definition of area $\mu = \mu^i$ and by $\frac{\pi}{2}$ in the case of the Hausdorff area $\mu = \mu^b$:

Theorem 1.4. *Let Γ be a Jordan curve in a proper metric space X. Assume that X satisfies the (C, l_0, μ)-isoperimetric inequality and let u be a μ-minimal disc in $\Lambda(\Gamma, X)$. Then u has a locally α-Hölder continuous representative with $\alpha = q(\mu) \cdot \frac{1}{4\pi C}$.*

For $\mu = \mu^i$ we get the optimal Hölder exponent $\alpha = \frac{1}{4\pi C}$ as examples of cones over small circles show (see [19] and [17, Example 8.3]).

1.5 Some additional comments

The basic ingredient in the proof of Theorem 1.1 and its generalization Theorem 4.3 is the localized version of the classical conformality of energy minimizers. This was already used in [17]. This idea shows that almost all (approximate metric) derivatives of any minimizer u of the energy E in $\Lambda(\Gamma, X)$ have to minimize the energy in their corresponding SL_2-orbits, as in (1.1).

The proof of the quasi-convexity of μ^D, generalized by Theorem 4.2, is achieved by applying an idea from [13]. We obtain a special parametrization of arbitrary Finsler discs by minimizing the energy under additional topological constraints. This idea might be of independent interest as it provides canonical parametrizations of any sufficiently regular surface.

2 Preliminaries

2.1 Notation

By $(\mathbb{R}^2, s)$ we denote the plane equipped with a seminorm s. If s is not specified, $\mathbb{R}^2$ is always considered with its canonical Euclidean norm, always denoted by s_0. By D we denote the open unit disc in the Euclidean plane $\mathbb{R}^2$ and by S^1 its boundary, the unit circle. Integration on open subsets of $\mathbb{R}^2$ is always performed with respect to the Lebesgue measure, unless otherwise stated. By d we denote distances in metric spaces. Metric spaces appearing in this note will be assumed complete. A metric space is called proper if its closed bounded subsets are compact.

2.2 Seminorms and convex bodies

By $\mathfrak{S}_2$ we denote the proper metric space of seminorms on $\mathbb{R}^2$ with the distance given by

$$d_{\mathfrak{S}_2}(s, s') = \max_{v \in S^1}\{|s(v) - s'(v)|\}.$$

A seminorm $s \in \mathfrak{S}_2$ is *Q-quasiconformal* if for all $v, w \in S^1$ the inequality $s(v) \leq Q \cdot s(w)$ holds true. A convex body C in $\mathbb{R}^2$ is a compact convex subset with non-empty interior. Convex, centrally symmetric bodies are in one-to-one correspondence with unit balls of norms on $\mathbb{R}^2$. Any convex body C contains a unique ellipse of largest area, called the *John ellipse* of C. The convex body is said to be in John's position if its John ellipse is a Euclidean ball. We call a seminorm $s \in \mathfrak{S}_2$ isotropic if it is the 0 seminorm, or if s is a norm and its unit ball B is in John's position. In the last case the John ellipse of B is a multiple $t \cdot \bar{D}$ of the closed unit disc. By John's theorem (cf. [2]) B is contained in $\sqrt{2} \cdot t \cdot \bar{D}$. Therefore, every isotropic seminorm is $\sqrt{2}$-quasiconformal.

2.3 Definitions of area

While there is an essentially unique natural way to measure areas of Riemannian surfaces, there are many different ways to measure areas of Finsler surfaces, some of them more appropriate for different questions. We refer the reader to [2, 5, 12] and the literature therein for more information.

A definition of area μ assigns a multiple μ_V of the Lebesgue measure on any 2-dimensional normed space V, such that natural assumptions are fulfilled. In particular, it assigns the number $\mathbf{J}^\mu(s)$, *the μ-Jacobian* or *μ-area-distortion*, to any seminorm s on $\mathbb{R}^2$ in the following way. By definition, $\mathbf{J}^\mu(s) = 0$ if the seminorm is not a norm. If s is a norm, then $\mathbf{J}^\mu(s)$ equals the $\mu_{(\mathbb{R}^2,s)}$-area of the Euclidean unit square $I^2 \subset \mathbb{R}^2$. Indeed, the choice of the definition of area is equivalent to a choice of the *Jacobian* in the following sense.

Definition 2.1. A (2-dimensional definition of) Jacobian is a map

$$\mathbf{J} : \mathfrak{S}_2 \to [0, \infty)$$

with the following properties:
(1) Monotonicity: $\mathbf{J}(s) \geq \mathbf{J}(s')$ whenever $s \geq s'$.
(2) Homogeneity: $\mathbf{J}(\lambda \cdot s) = \lambda^2 \cdot \mathbf{J}(s)$ for all $\lambda \in [0, \infty)$.
(3) SL_2-invariance $\mathbf{J}(s \circ T) = \mathbf{J}(s)$ for any $T \in SL_2$.
(4) Normalization: $\mathbf{J}(s_0) = 1$.

Properties (2) and (3) can be joined to the usual transformation rule for the area: $\mathbf{J}(s \circ T) = |\det(T)| \cdot \mathbf{J}(s)$. It follows that $\mathbf{J}(s) = 0$ if and only if the seminorm s is not a norm. Moreover, properties (1)–(3) imply that $\mathbf{J}$ is continuous. This is due to the following crucial fact: If norms s_i converge to a norm s in $\mathfrak{S}_2$, then, for any $\epsilon > 0$ and all large i, the inequalities $(1 - \epsilon) \cdot s_i \leq s \leq (1 + \epsilon) \cdot s_i$ hold true.

A definition of area μ gives rise to a Jacobian $\mathbf{J}^\mu$ described above. On the other hand, any Jacobian $\mathbf{J} : \mathfrak{S}_2 \to [0, \infty)$ provides a unique definition of area $\mu^{\mathbf{J}}$ in the following way. On any $(\mathbb{R}^2, s)$ the definition of area $\mu^{\mathbf{J}}$ assigns the $\mathbf{J}(s)$-multiple of the Lebesgue area of $\mathbb{R}^2$. For any normed plane V, we choose a linear isometry to some $(\mathbb{R}^2, s)$ and pull back the corresponding measure from $(\mathbb{R}^2, s)$ to V. By construction, the assignments $\mu \to \mathbf{J}^\mu$ and $\mathbf{J} \to \mu^{\mathbf{J}}$ are inverses of each other.

Remark 2.1. We refer to [5] for another similar geometric interpretation of a definition of area.

There are many non-equivalent definitions of area/Jacobian. Any two of them differ at most by a factor of 2, due to John's theorem, [2]. The most prominent examples are the Busemann (or Hausdorff) definition μ^b, the Holmes–Thompson definition μ^{ht}, the Benson (or Gromov mass*) definition m^* and the inscribed Riemannian (or Ivanov) definition μ^i. We refer to [2] for a thorough discussion of these examples and of the whole subject; and to [5, 6, 12] for recent developments. Here, we just mention the Jacobians of these four examples (cf. [5]). In the subsequent examples, B will always denote the unit ball of the normed plane $(\mathbb{R}^2, s)$.
(1) The Jacobian $\mathbf{J}^b$ corresponding to the Hausdorff (Busemann) area μ^b equals $\mathbf{J}^b(s) = \frac{\pi}{|B|}$, where $|B|$ is the Lebesgue area of B.
(2) The Jacobian $\mathbf{J}^{ht}$ corresponding to the Holmes–Thompson area μ^{ht} equals $\mathbf{J}^{ht}(s) = \frac{|B^*|}{\pi}$, where $|B^*|$ is the Lebesgue area of the unit ball B^* of the dual norm s^* of s.
(3) The Jacobian $\mathbf{J}^*$ corresponding to Benson (Gromov mass*) definition of area m^* equals $\mathbf{J}^*(s) = \frac{4}{|P|}$, where $|P|$ is the Lebesgue area of a parallelogram P of smallest area which contains B.
(4) The Jacobian $\mathbf{J}^i$ corresponding to the inscribed Riemannian definition of area μ^i equals $\mathbf{J}^i(s) = \frac{\pi}{|L|}$, where $|L|$ is Lebesgue area of the John ellipse of B.

2.4 Comparison of the definitions of area

Below we denote by $|C|$ the Lebesgue area of a subset $C \subset \mathbb{R}^2$. Let s be a norm on $\mathbb{R}^2$, let B be its unit ball and let $L \subset B$ denote the John ellipse of B. If s_L denotes the norm whose unit ball is L, then $s \leq s_L$ and s_L is Euclidean. Thus, for any definition of area μ with Jacobian $\mathbf{J}^\mu$ we have $\mathbf{J}^\mu(s_L) = \frac{\pi}{|L|}$ and $\mathbf{J}^\mu(s) \leq \mathbf{J}^\mu(s_L)$.

For the inscribed Riemannian area μ^i and its Jacobian $\mathbf{J}^i$ we have equality $\mathbf{J}^i(s) = \mathbf{J}^i(s_L)$ in the above inequality. Hence, for any other definition of area μ we must have $\mathbf{J}^i \geq \mathbf{J}^\mu$. In particular, the inscribed Riemannian area is the largest definition of area. On the other hand, by John's theorem, $\mathbf{J}^i \leq 2\mathbf{J}^\mu$.

We set

$$q(\mu) := \inf \frac{\mathbf{J}^\mu(s)}{\mathbf{J}^i(s)}, \tag{2.1}$$

where s runs over all norms on $\mathbb{R}^2$. As we have just observed, $q(\mu^i) = 1$ and $\frac{1}{2} \leq q(\mu) < 1$ for any other definition of area μ.

Lemma 2.1. *For the Hausdorff area μ^b we have $q(\mu^b) = \frac{\pi}{4}$.*

Proof. Let B be the unit ball of the norm s on $\mathbb{R}^2$. In order to compare $\mathbf{J}^i(s)$ and $\mathbf{J}^b(s)$ we just need to evaluate μ^i and μ^b on B. For the Busemann definition of area we have $\mu^b(B) = \pi$. On the other hand, $\mu^i(B) = \pi \cdot \frac{|B|}{|L|}$, where L is the John ellipse of B. The *volume ratio* $\frac{|B|}{|L|}$ is maximal when B is a square, see [4, Theorem 6.2], in which case it is equal to $\frac{4}{\pi}$. □

Since we will not need further statements about the function q, we just summarize here some properties without proofs. For any definition of area μ, there exists a norm s with $\mathbf{J}^\mu(s) = q(\mu) \cdot \mathbf{J}^i(s)$. Moreover, using John's theorem one can show that this norm s can be chosen to have a square or a hexagon as its unit ball. One can show that $q(\mu^{ht}) = \frac{2}{\pi}$, where again on the supremum norm s_∞ the difference between $\mathbf{J}^i$ and $\mathbf{J}^{ht}$ is maximal. Finally, for Gromov's definition of area m^* one can show that $q(m^*) = \frac{\sqrt{3}}{2}$. Here the maximal deviation of $\mathbf{J}^i$ from $\mathbf{J}^*$ is achieved for the norm whose unit ball is a regular hexagon.

2.5 Definitions of energy

An assignment of a definition of area or Jacobian is essentially equivalent to the assignment of an area functional on all Lipschitz and Sobolev maps defined on domains in $\mathbb{R}^2$, see below. Similarly, the choice of an energy functional is essentially equivalent to the following choice of a *definition of energy*:

Definition 2.2. A (2-dimensional conformally invariant) *definition of energy* is a continuous map

$$\mathcal{I} : \mathfrak{S}_2 \to [0, \infty)$$

which has the following properties:
(1) Monotonicity: $\mathcal{I}(s) \geq \mathcal{I}(s')$ whenever $s \geq s'$.
(2) Homogeneity: $\mathcal{I}(\lambda \cdot s) = \lambda^2 \cdot \mathcal{I}(s)$ for all $\lambda \in [0, \infty)$.
(3) SO_2-invariance: $\mathcal{I}(s \circ T) = \mathcal{I}(s)$ for any $T \in SO_2$.
(4) Properness: The set $\mathcal{I}^{-1}([0, 1])$ is compact in $\mathfrak{S}_2$.

Due to properness and homogeneity, we have $\mathcal{I}(s) = 0$ only for $s = 0$. The properness of $\mathcal{I}$ implies that a definition of energy is *never* SL_2-invariant, in contrast to a definition of area. The set of all definitions of energy is a convex cone. Moreover, for any Jacobian $\mathbf{J}$, any definition of energy $\mathcal{I}$ and any $\epsilon > 0$ the map $I_\epsilon := \mathbf{J} + \epsilon \cdot \mathcal{I}$ is a definition of energy. Thus the closure (in the topology of locally uniform convergence) of the set of definitions of energy contains all definitions of Jacobians.

The following two definitions of energy are most prominent: the Korevaar–Schoen-Dirichlet energy I^2 given by

$$I^2(s) = \frac{1}{\pi} \int_{S^1} s(v)^2 \, dv$$

and the Reshetnyak energy

$$I_+^2(s) = \sup\{s(v)^2 \mid v \in S^1\}.$$

Due to properness and homogeneity any two definitions of energy are comparable: for any definition of energy $\mathcal{I}$ there is a constant $k_{\mathcal{I}} \geq 1$ such that

$$\frac{1}{k_{\mathcal{I}}} \cdot \mathcal{I} \leq I_+^2 \leq k_{\mathcal{I}} \cdot \mathcal{I}. \tag{2.2}$$

2.6 Energy and area of Sobolev maps

We assume some experience with Sobolev maps and refer to [17] and the literature therein. In this note we consider only Sobolev maps defined on bounded open domains $\Omega \subset \mathbb{R}^2$. Let Ω be such a domain and let $u \in W^{1,2}(\Omega, X)$ be a Sobolev map with values in X. Then u has an approximate metric derivative at almost every point $z \in \Omega$ ([14],[17]), which is a seminorm on $\mathbb{R}^2$ denoted by $\operatorname{ap\,md} u_z$. When $\operatorname{ap\,md} u_z$ exists, it is the unique seminorm s for which the following approximate limit is 0:

$$\operatorname{ap}\lim_{y\to z} \frac{d(u(z), u(y)) - s(y-z)}{|y-z|} = 0.$$

We refer the reader to [14, 17] and mention here only that in the case of locally Lipschitz maps u, the approximate metric derivative is just the metric derivative defined by Kirchheim ([15], cf. also [3, 12]). If the target space X is a Finsler manifold, then the approximate metric derivative at almost all points z is equal to $|D_z u|$, where $D_z u$ is the usual (weak) derivative and $|\cdot|$ is the given norm on the tangent space $T_{u(z)}X$. A map $u \in W^{1,2}(\Omega, X)$ is called *Q-quasiconformal* if the seminorms $\operatorname{ap\,md} u_z \in \mathfrak{S}_2$ are Q-quasiconformal for almost all $z \in \Omega$.

For a definition of energy $\mathcal{I}$, the $\mathcal{I}$-energy of a map $u \in W^{1,2}(\Omega, X)$ is given by

$$E_{\mathcal{I}}(u) := \int_\Omega \mathcal{I}(\operatorname{ap\,md} u_z)\, dz.$$

This value is well-defined and finite for any $u \in W^{1,2}(\Omega, X)$, due to (2.2). If $\mathcal{I}$ is the Korevaar–Schoen definition of energy I^2, respectively the Reshetnyak definition of energy I_+^2, then $E_{\mathcal{I}}(u)$ is the Korevaar–Schoen respectively the Reshetnyak energy of u described in [16, 21] and in the introduction. We will denote $E_{\mathcal{I}}$ in these cases as before by E^2 and E_+^2, respectively.

Similarly, given a definition of area μ and the corresponding Jacobian $\mathbf{J}^\mu$, one obtains the μ-area of u by integrating $\mathbf{J}^\mu(\operatorname{ap\,md} u_z)$ over Ω. We will denote it by

$$\operatorname{Area}_\mu(u) := \int_\Omega \mathbf{J}^\mu(\operatorname{ap\,md} u_z)\, dz.$$

Pointwise comparison of μ with the inscribed Riemannian definition of area μ^i discussed in Section 2.4 gives us for any Sobolev map u:

$$q(\mu)^{-1} \cdot \operatorname{Area}_\mu(u) \geq \operatorname{Area}_{\mu^i}(u) \geq \operatorname{Area}_\mu(u). \tag{2.3}$$

2.7 Quasi-convexity

A definition of energy $\mathcal{I} : \mathfrak{S}_2 \to [0, \infty)$ is called *quasi-convex* if linear 2-dimensional subspaces of normed vector spaces have minimal $\mathcal{I}$-energy. More precisely, if for every finite-dimensional normed space Y and every linear map $L : \mathbb{R}^2 \to Y$ we have

$$E_{\mathcal{I}}(L|_D) \leq E_{\mathcal{I}}(\psi) \tag{2.4}$$

for every smooth immersion $\psi : \bar{D} \to Y$ with $\psi|_{\partial D} = L|_{\partial D}$.

Similarly, one defines the quasi-convexity of a definition of area with corresponding functional

$$\mathbf{J} : \mathfrak{S}_2 \to [0, \infty),$$

see [17, Section 5]. As has been shown in [17], in extension of the classical results (cf. [1]), a definition of energy is quasi-convex if and only if the map $u \mapsto E_{\mathcal{I}}(u)$ is semi-continuous on any Sobolev space $W^{1,2}(\Omega, X)$ (with respect to L^2-convergence). Similarly, the quasi-convexity of a definition of area μ is equivalent to the semi-continuity property of the μ-area on all Sobolev spaces $W^{1,2}(\Omega, X)$.

Recall that the Reshetnyak and Korevaar–Schoen definitions of energy are quasi-convex [16, 17, 21]. The four definitions of area mentioned in Section 2.3 are quasi-convex as well [2, 6, 12, 17].

We dwell a bit discussing the properties of a definition of area μ which is not quasi-convex (cf. [20]). Let $L : \mathbb{R}^2 \to Y$ be a linear map to a finite-dimensional normed vector space and let $\psi : \bar{D} \to Y$ be a smooth map which coincides with L on S^1 and satisfies

$$\mathrm{Area}_\mu(\psi) < \mathrm{Area}_\mu(L|_D).$$

By enlarging Y if needed and by using a general position argument, we can assume that ψ is a diffeomorphism onto its image. Now we can obtain a special sequence of maps $\psi_m : \bar{D} \to Y$ converging to $L : \bar{D} \to Y$ and violating the semi-continuity property in the following way. The map ψ_m differs from L on at least $\delta \cdot m^2$ disjoint balls of radius m^{-1}, where $\delta > 0$ is a sufficiently small, fixed constant. The difference between ψ_m and L on any of these balls is given by the corresponding translate of ψ, rescaled by the factor m^{-1}.

Then there is a number $K > 0$, such that any of the maps ψ_m is bi-Lipschitz with the same bi-Lipschitz constant K. The maps ψ_m converge uniformly to the linear map L. Finally, for $\epsilon = \mathrm{Area}_\mu(L|_D) - \mathrm{Area}_\mu(\psi)$, we deduce $\mathrm{Area}_\mu(\psi_m) \leq \mathrm{Area}_\mu(L|_D) - \delta \cdot \epsilon$ for all m. In particular, $\mathrm{Area}_\mu(L|_D) > \lim_{m\to\infty}(\mathrm{Area}\, \mu(\psi_m))$.

3 Area definition corresponding to an energy

3.1 General construction

Let now $\mathcal{I}$ be any definition of energy. Consider the function $\hat{J} : \mathfrak{S}_2 \to [0, \infty)$:

$$\hat{J}(s) := \inf\{\mathcal{I}(s \circ T) \mid T \in \mathrm{SL}_2\}$$

given by the infimum of $\mathcal{I}$ on the SL_2-orbit of s. Due to the properness of $\mathcal{I}$, the infimum in the above equation is indeed a minimum, unless the seminorm is not a norm. On the other hand, if s is not a norm then the SL_2-orbit of s contains the 0 seminorm in its closure, and we get $\hat{J}(s) = 0$. By construction, the function $\hat{J} : \mathfrak{S}_2 \to [0, \infty)$ is SL_2-invariant. Since $\mathcal{I}$ is monotone and homogeneous, so is $\hat{J}$. Finally, $\hat{J}(s_0)$ is different from 0. Thus, setting the constant $\lambda_{\mathcal{I}}$ to be $\frac{1}{\hat{J}(s_0)}$, we see that

$$\mathbf{J}^{\mathcal{I}}(s) := \lambda_{\mathcal{I}} \cdot \hat{J}(s)$$

is a definition of a Jacobian in the sense of the previous section. The definition of area which corresponds to the Jacobian $\mathbf{J}^{\mathcal{I}}$ will be denoted by $\mu^{\mathcal{I}}$. By construction,

$$\mathbf{J}^{\mathcal{I}}(s) \leq \lambda_{\mathcal{I}} \cdot \mathcal{I}(s) \tag{3.1}$$

with equality if and only if $\mathcal{I}$ assumes the minimum on the SL_2-orbit of s at the seminorm s.

Definition 3.1. We will call a seminorm s minimal for the definition of energy $\mathcal{I}$, or just $\mathcal{I}$-minimal, if $\mathcal{I}(s) \leq \mathcal{I}(s \circ T)$ for all $T \in \mathrm{SL}_2$.

Thus a seminorm s is $\mathcal{I}$-minimal if and only if we have equality in inequality (3.1). By homogeneity and continuity, the set of all $\mathcal{I}$-minimal seminorms is a closed cone. Any $\mathcal{I}$-minimal seminorm is either a norm or the trivial seminorm $s = 0$. We therefore deduce by a limiting argument:

Lemma 3.1. *There is a number $Q_{\mathcal{I}} > 0$ such that any $\mathcal{I}$-minimal seminorm s is $Q_{\mathcal{I}}$-quasiconformal.*

3.2 The Reshetnyak energy and the inscribed Riemannian area

We are going to discuss the application of the above construction to the main examples. In order to describe the Jacobian $\mathbf{J}^{\mathfrak{I}}$, the normalization and the quasiconformality constants $\lambda_{\mathfrak{I}}, Q_{\mathfrak{I}}$ induced by a definition of energy $\mathfrak{I}$, it is crucial to understand $\mathfrak{I}$-minimal norms. By general symmetry reasons one might expect that $\mathfrak{I}$-minimal norms are particularly round. Our first result, essentially contained in [17], confirms this expectation for the Reshetnyak energy:

Lemma 3.2. *Let $\mathfrak{I} = I_+^2$ be the Reshetnyak definition of energy. A seminorm $s \in \mathfrak{S}_2$ is I_+^2-minimal if and only if s is isotropic in the sense of Section 2.2.*

Proof. For seminorms which are not norms the statement is clear. Thus we may assume that s is a norm. After rescaling, we may assume $I_+^2(s) = 1$. Hence $1 = \sup\{s(v) \mid v \in S^1\}$, and $\bar{D}$ is the largest Euclidean disc contained in the unit ball B of the norm s.

Assume that s is I_+^2-minimal and $\bar{D}$ is not the John ellipse of B. Then there exists an area increasing linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that B still contains the ellipse $A(D)$, hence $I_+^2(s \circ A) \leq 1$. Consider the map $T = \det(A)^{-\frac{1}{2}} \cdot A \in \mathrm{SL}_2$. Then $I_+^2(s \circ T) < 1$ since $\det(A) > 1$. This contradicts the assumption that s is I_+^2-minimal.

On the other hand, if s is isotropic, then $\bar{D}$ is the John ellipse of B. If $T \in \mathrm{SL}_2$, then $T(\bar{D})$ cannot be contained in the interior of B, and hence $I_+^2(s \circ T) \geq I_+^2(s)$. □

Now we can easily deduce:

Corollary 3.3. *For the Reshetnyak definition of energy $\mathfrak{I} = I_+^2$ the normalization constant $\lambda_{\mathfrak{I}}$ equals 1, the optimal quasiconformality constant $Q_{\mathfrak{I}}$ equals $\sqrt{2}$, and the induced definition of area $\mu^{\mathfrak{I}}$ is the inscribed Riemannian area μ^i.*

Proof. We have

$$\lambda_{\mathfrak{I}} = \frac{1}{\hat{\mathfrak{I}}(s_0)} = \frac{1}{\mathfrak{I}(s_0)} = 1$$

since s_0 is I_+^2-minimal. Isotropic seminorms are $\sqrt{2}$-quasiconformal by John's theorem. The supremum norm $s_\infty \in \mathfrak{S}_2$ is isotropic, hence I_+^2-minimal. For s_∞ the quasiconformality constant $\sqrt{2}$ is optimal.

In order to prove that the induced definition of area coincides with the inscribed Riemannian area μ^i, it suffices to evaluate the Jacobians on any I_+^2-minimal norm s. By homogeneity we may assume again that the John ellipse of the unit ball B of s is the unit disc $\bar{D}$. Then $\mathbf{J}^{\mathfrak{I}}(s) = 1 = \mathbf{J}^i(s)$. □

3.3 The Korevaar–Schoen energy and the Dirichlet area

Unfortunately, in the classical case of the Korevaar–Schoen energy $\mathfrak{I} = I^2$ we do not know much about the induced definition of area/Jacobian. We call this the Dirichlet definition of area/Jacobian and denote it by μ^D and $\mathbf{J}^D$, respectively. Only the normalization constant in this case is easy to determine.

Lemma 3.4. *For the Korevaar–Schoen energy $\mathfrak{I} = I^2$, the canonical Euclidean norm s_0 is I^2-minimal. The normalization constant $\lambda_{\mathfrak{I}}$ equals $\frac{1}{2}$.*

Proof. We have $I^2(s_0) = \frac{1}{\pi} \cdot 2\pi = 2$. Therefore, it suffices to prove the I^2-minimality of s_0. Since I^2 and s_0 are SO_2-invariant, it suffices to prove $I^2(s_0 \circ T) \geq I^2(s_0)$ for any symmetric matrix $T \in \mathrm{SL}_2$. In this case, one easily computes $I^2(s_0 \circ T) = \lambda_1^2 + \lambda_2^2$, where $\lambda_{1,2}$ are the eigenvalues of T. Under the assumption $\lambda_1 \cdot \lambda_2 = \det(T) = 1$ the minimum is achieved for $\lambda_1 = \lambda_2 = 1$. Hence s_0 is I^2-minimal. □

From the corresponding property of $\mathfrak{I} = I^2$, it is easy to deduce that for norms $s \neq s'$ the inequality $s \geq s'$ implies the strict inequality $\mathbf{J}^D(s) > \mathbf{J}^D(s')$, in contrast to the cases of inscribed Riemannian and Benson definitions of areas μ^i and m^*. In [17] it is shown that for $\mathfrak{I} = I^2$ the quasiconformality constant $Q_{\mathfrak{I}}$ in Lemma 3.1

can be chosen to be $2\sqrt{2}+\sqrt{6}$. However, the computation of $Q_{\mathcal{J}}$ in [17] and the above strict monotonicity statement show that this constant is not optimal.

Computing $\mathbf{J}^D$ on the supremum norm s_∞ it is possible to see that μ^D is different from the Busemann and Holmes–Thompson definitions of area. We leave the lengthy computation to the interested reader.

4 Main lemma and main theorems

4.1 Basic observations

Let $\mathcal{J}$ be a definition of energy and let $\mu^{\mathcal{J}}$ and $\mathbf{J}^{\mathcal{J}}$ be the corresponding definitions of area and Jacobian. Let $\lambda_{\mathcal{J}}$ be the normalization constant from the previous section.

Let X be a metric space, $\Omega \subset \mathbb{R}^2$ a domain and let $u \in W^{1,2}(\Omega, X)$ be a Sobolev map. Integrating the pointwise inequality (3.1) we deduce

$$\mathrm{Area}_{\mu^{\mathcal{J}}}(u) \le \lambda_{\mathcal{J}} \cdot E_{\mathcal{J}}(u). \tag{4.1}$$

Moreover, equality holds if and only if the approximate metric derivative $\operatorname{ap\,md} u_z$ is $\mathcal{J}$-minimal for almost all $z \in \Omega$. In case of equality, Lemma 3.1 implies that the map u is $Q_{\mathcal{J}}$-quasiconformal.

4.2 Main lemma

Conformal invariance of $\mathcal{J}$ together with the usual transformation rule [17, Lemma 4.9] has the following direct consequence: For any conformal diffeomorphism $\phi : \Omega' \to \Omega$ which is bi-Lipschitz and for any map $u \in W^{1,2}(\Omega, X)$, the composition $u \circ \phi$ is contained in $W^{1,2}(\Omega', X)$, and it has the same $\mathcal{J}$-energy as u.

The general transformation formula shows that for any definition of area μ, any bi-Lipschitz homeomorphism $\phi : \Omega' \to \Omega$, and any $u \in W^{1,2}(\Omega, X)$ the map $u \circ \phi \in W^{1,2}(\Omega', X)$ has the same μ-area as u.

Now we can state the main technical lemma, which appears implicitly in [17]:

Lemma 4.1. *Let $\mathcal{J}$, $\mu^{\mathcal{J}}$, $\lambda_{\mathcal{J}}$ be as above. Let X be a metric space and let $u \in W^{1,2}(D, X)$ be arbitrary. Then the following conditions are equivalent:*
(1) $\mathrm{Area}_{\mu^{\mathcal{J}}}(u) = \lambda_{\mathcal{J}} \cdot E_{\mathcal{J}}(u)$.
(2) *For almost every $z \in D$ the approximate metric derivative* $\operatorname{ap\,md} u_z$ *is an $\mathcal{J}$-minimal seminorm.*
(3) *For every bi-Lipschitz homeomorphism $\psi : D \to D$ we have $E_{\mathcal{J}}(u \circ \psi) \ge E_{\mathcal{J}}(u)$.*

Proof. We have already proven the equivalence of (1) and (2). If (1) holds, then (3) follows directly from the general inequality (4.1) and invariance of the Area_μ under diffeomorphisms.

It remains to prove the main part, namely that (3) implies (2). Thus assume (3) holds. The conformal invariance of $\mathcal{J}$ and the Riemann mapping theorem imply that for any other domain $\Omega \subset \mathbb{R}^2$ with smooth boundary and any bi-Lipschitz homeomorphism $\psi : \Omega \to D$ the inequality $E_{\mathcal{J}}(u \circ \psi) \ge E_{\mathcal{J}}(u)$ holds true. Indeed, we only need to compose ψ with a conformal diffeomorphism $F : D \to \Omega$, which is bi-Lipschitz since the boundary of Ω is smooth.

Assume now that (2) does not hold. Then it is possible to construct a bi-Lipschitz map ψ from a domain Ω to D such that $E_{\mathcal{J}}(u \circ \psi) < E_{\mathcal{J}}(u)$ in the same way as in the proof of [17, Theorem 6.2], to which we refer for some technical details. Here we just explain the major steps. First, we find a compact set $K \subset D$ of positive measure such that at no point $z \in K$ the approximate metric derivative $\operatorname{ap\,md} u_z$ is $\mathcal{J}$-minimal. Making K smaller we may assume that the map $z \mapsto \operatorname{ap\,md} u_z$ is continuous on K. By continuity, we find a Lebesgue point z of K, a map $T \in \mathrm{SL}_2$ and some $\epsilon > 0$ such that $\mathcal{J}(s \circ T) \le \mathcal{J}(s) - \epsilon$ for any seminorm s which arises as the approximate metric derivative $\operatorname{ap\,md} u_y$ at some point $y \in K \cap B_\epsilon(z)$.

We may assume without loss of generality that z is the origin 0 and that T is a diagonal matrix with two different eigenvalues $\lambda_1 > \lambda_2 = \frac{1}{\lambda_1} > 0$. Then we define a family of bi-Lipschitz homeomorphisms $\psi_r : \mathbb{R}^2 \to \mathbb{R}^2$ as follows. The map ψ_r coincides with T on the closed r-ball around 0. On the complement of this r-ball, the map ψ_r is the restriction of the holomorphic (hence conformal) map $f_r : \mathbb{C}^* \to \mathbb{C}$, defined by

$$f_r(z) = c \cdot z + r^2 \cdot d \cdot z^{-1},$$

where the constants $c, d \in \mathbb{C}$ are given by $c = \frac{1}{2}(\lambda_1 + \lambda_2)$ and $d = \frac{1}{2}(\lambda_1 - \lambda_2)$. Then the map f_r coincides with T on the r-circle around 0. This map ψ_r is bi-Lipschitz on $\mathbb{R}^2$ (and smooth outside of the r-circle around 0). Moreover, the map ψ_r preserves the $\mathfrak{I}$-energy of the map u on the complement of the r-ball, due to the conformality of f_r and the conformal invariance of $\mathfrak{I}$. Finally, by construction of T, the map ψ_r decreases the $\mathfrak{I}$-energy of u by some positive amount (at least $\frac{1}{2}\epsilon\pi r^2$), if r is small enough.

Thus $E_{\mathfrak{I}}(u \circ \psi_r) < E_{\mathfrak{I}}(u)$ for r small enough. This provides a contradiction and finishes the proof. □

4.3 Formulation of the main theorems

The proof of the following theorem is postponed to Section 5.

Theorem 4.2. *Let $\mathfrak{I}$ be a quasi-convex definition of energy. Then the corresponding definition of area $\mu^{\mathfrak{I}}$ is quasi-convex as well.*

Theorem 4.2 together with Theorem 4.3 below generalize Theorem 1.2. Together with Corollary 3.3 it shows that μ^i is quasi-convex, cf. Remark 1.2.

Before turning to the main theorem stating the connection of energy and area minimizers, we recall an important step in the solution of the Plateau problem ([17, Proposition 7.5], [16]): Let Γ be a Jordan curve in a proper metric space X. Assume that the sequence of maps $w_i \in \Lambda(\Gamma, X)$ has uniformly bounded Reshetnyak energy $E_+^2(w_i)$. Then there exist conformal diffeomorphisms $\phi_i : D \to D$ such that the sequence $w_i' = w_i \circ \phi \in \Lambda(\Gamma, X)$ converges in L^2 to a map $\bar{w} \in \Lambda(\Gamma, X)$. Note that for any quasi-convex definition of area μ or energy $\mathfrak{I}$, we have in this case ([17, Theorem 5.4]):

$$\mathrm{Area}_\mu(\bar{w}) \le \liminf \mathrm{Area}_\mu(w_i) \quad \text{and} \quad E_{\mathfrak{I}}(\bar{w}) \le \liminf E_{\mathfrak{I}}(w_i). \tag{4.2}$$

The proof of the following theorem will rely on Theorem 4.2.

Theorem 4.3. *Let $\mathfrak{I}$ be a quasi-convex definition of energy. Let Γ be a Jordan curve in a proper metric space X. Any map $u \in \Lambda(\Gamma, X)$ with minimal $\mathfrak{I}$-energy in $\Lambda(\Gamma, X)$ has minimal $\mu^{\mathfrak{I}}$-area in $\Lambda(\Gamma, X)$. Moreover, u is $Q_{\mathfrak{I}}$-quasiconformal.*

Proof. Let $u \in \Lambda(\Gamma, X)$ with minimal $\mathfrak{I}$-energy among all maps $v \in \Lambda(\Gamma, X)$ be given. Then $E_{\mathfrak{I}}(u) \le E_{\mathfrak{I}}(u \circ \phi)$ for any bi-Lipschitz homeomorphism $\phi : D \to D$. Due to Lemma 4.1, $\mathrm{Area}_{\mu^{\mathfrak{I}}}(u) = \lambda_{\mathfrak{I}} \cdot E_{\mathfrak{I}}(u)$. Moreover, by Lemma 3.1, almost all approximate derivatives of u are $Q_{\mathfrak{I}}$-quasiconformal. This proves the last statement.

Assume that the map u does not minimize the $\mu^{\mathfrak{I}}$-area and take another element $v \in \Lambda(\Gamma, X)$ with $\mathrm{Area}_{\mu^{\mathfrak{I}}}(v) < \mathrm{Area}_{\mu^{\mathfrak{I}}}(u)$. Consider the set Λ_0 of elements $w \in \Lambda(\Gamma, X)$ with $\mathrm{Area}_{\mu^{\mathfrak{I}}}(w) \le \mathrm{Area}_{\mu^{\mathfrak{I}}}(v)$. We take a sequence $w_n \in \Lambda_0$ such that $E_{\mathfrak{I}}(w_n)$ converges to the infimum of the $\mathfrak{I}$-energy on Λ_0. Due to (2.2), the Reshetnyak energy of all maps w_n is bounded from above by a uniform constant. Using the observation preceding Theorem 4.3, we find some $\bar{w} \in \Lambda(\Gamma, X)$ which satisfies (4.2). Here we have used the quasi-convexity of $\mu^{\mathfrak{I}}$, given by Theorem 4.2.

Thus, $\bar{w}$ is contained in Λ_0 and minimizes the $\mathfrak{I}$-energy in Λ_0. In particular, $E_{\mathfrak{I}}(\bar{w} \circ \phi) \ge E_{\mathfrak{I}}(\bar{w})$, for any bi-Lipschitz homeomorphism $\phi : D \to D$. Applying Lemma 4.1 to the map $\bar{w}$, we deduce

$$\lambda_{\mathfrak{I}} \cdot E_{\mathfrak{I}}(u) = \mathrm{Area}_{\mu^{\mathfrak{I}}}(u) > \mathrm{Area}_{\mu^{\mathfrak{I}}}(v) \ge \mathrm{Area}_{\mu^{\mathfrak{I}}}(\bar{w}) = \lambda_{\mathfrak{I}} \cdot E_{\mathfrak{I}}(\bar{w}).$$

This contradicts the minimality of $E_{\mathfrak{I}}(u)$. □

4.4 Regularity of energy minimizers

The regularity of energy minimizers is now a direct consequence of [17]. Recall that a Jordan curve $\Gamma \subset X$ is a *chord-arc curve* if the restriction of the metric to Γ is bi-Lipschitz equivalent to the induced intrinsic metric.

A map $u : D \to X$ is said to satisfy *Lusin's property* (N) if for any subset S of D with area 0 the image $u(S)$ has zero 2-dimensional Hausdorff measure.

Theorem 4.4. *Let X be a proper metric space which satisfies a uniformly local quadratic isoperimetric inequality. Let $\mathfrak{I}$ be a quasi-convex definition of energy and let Γ be a Jordan curve in X such that the set $\Lambda(\Gamma, X)$ is not empty. Then there exists a minimizer u of the $\mathfrak{I}$-energy in $\Lambda(\Gamma, X)$. Any such minimizer has a unique locally Hölder continuous representative which extends to a continuous map on $\bar{D}$. Moreover, u is contained in the Sobolev space $W^{1,p}_{\mathrm{loc}}(D, X)$ for some $p > 2$ and satisfies Lusin's property (N). If the curve Γ is a chord-arc curve, then u is Hölder continuous on $\bar{D}$.*

Proof. The existence of a minimizer u of the $\mathfrak{I}$-energy in $\Lambda(\Gamma, X)$ is a consequence of [17, Theorem 5.4 and Proposition 7.5, see also Theorem 7.6].

Any map u minimizing the $\mathfrak{I}$-energy in $\Lambda(\Gamma, X)$ is quasiconformal and minimizes the $\mu^{\mathfrak{I}}$-area in $\Lambda(\Gamma, X)$, by Theorem 4.3. The result now follows from [17, Theorems 8.1, 9.2 and 9.3]. □

4.5 Optimal regularity

We are going to provide the proof of Theorem 1.4. Thus, let $u \in W^{1,2}(D, X)$ be as in Theorem 1.4. Then for any bi-Lipschitz homeomorphism $\psi : D \to D$ we have $\mathrm{Area}_\mu(u \circ \psi) = \mathrm{Area}_\mu(u)$ and therefore $E_+^2(u \circ \psi) \geq E_+^2(u)$. Applying Lemma 4.1 and Lemma 3.2, we see that u is infinitesimally isotropic in the following sense.

Definition 4.1. A map $u \in W^{1,2}(D, X)$ is infinitesimally isotropic if for almost every $z \in D$ the approximate metric derivative of u at z is an isotropic seminorm.

Theorem 1.4 is thus an immediate consequence of the following theorem.

Theorem 4.5. *Let Γ be a Jordan curve in a metric space X. Assume that X satisfies the (C, l_0, μ)-isoperimetric inequality and let $u \in \Lambda(\Gamma, X)$ be an infinitesimally isotropic map having minimal μ-area in $\Lambda(\Gamma, X)$. Then u has a locally α-Hölder continuous representative with $\alpha = q(\mu) \cdot \frac{1}{4\pi C}$.*

Proof. Due to [17], u has a unique continuous representative. For any subdomain Ω of D we have

$$E_+^2(u|_\Omega) = \mathrm{Area}_{\mu^i}(u|_\Omega) \tag{4.3}$$

by Lemma 4.1. Looking into the proof of the Hölder continuity of u in [17, Proposition 8.7], we see that the quasiconformality factor Q of u (which, as we know, is bounded by $\sqrt{2}$) comes into the game only once. Namely, this happens in [17, estimate (40) in Lemma 8.8], where the inequality $E_+^2(u|_\Omega) \leq Q^2 \cdot \mathrm{Area}_\mu(u|_\Omega)$ appears for open balls $\Omega \subset D$.

Using (4.3) together with (2.3) we can replace [17, estimate (40) in Lemma 8.8] by

$$E_+^2(u|_\Omega) = \mathrm{Area}_{\mu^i}(u|_\Omega) \leq q(\mu)^{-1} \cdot \mathrm{Area}_\mu(u|_\Omega).$$

Hence we can replace the factor Q^2 in the proof of [17, Proposition 8.7] by the factor $q(\mu)^{-1}$. Leaving the rest of that proof unchanged, we get $\alpha = q(\mu) \cdot \frac{1}{4\pi C}$ as a bound for the Hölder exponent of u. □

5 Quasi-convexity of $\mu^{\mathfrak{I}}$

This section is devoted to the

Proof of Theorem 4.2. Assume on the contrary that the definition of energy $\mathfrak{I}$ is quasi-convex, but that $\mu^{\mathfrak{I}}$ is not quasi-convex. Consider a finite-dimensional normed vector space Y, a linear map $L : \mathbb{R}^2 \to Y$ and a sequence of smooth embeddings $\psi_m : \bar{D} \to Y$ as in Section 2.7 such that the following holds true. The maps ψ_m coincide with L on the boundary circle S^1, they are K-bi-Lipschitz with a fixed constant K, and they converge uniformly to the restriction of L to $\bar{D}$. Finally, for some $\epsilon > 0$ and all $m > 0$, we have

$$\mathrm{Area}_{\mu^{\mathfrak{I}}}(L|_D) \geq \mathrm{Area}_{\mu^{\mathfrak{I}}}(\psi_m) + \epsilon.$$

We will use this sequence to obtain a contradiction to the semi-continuity of $E_{\mathfrak{I}}$. The idea is to modify ψ_m by (almost) homeomorphisms, so that the new maps satisfy equality in the main area-energy inequality (4.1). We explain this modification in a slightly more abstract context of general bi-Lipschitz discs.

The first observation is a direct consequence of the fact that the diameter of a simple closed curve in $\mathbb{R}^2$ equals the diameter of the corresponding Jordan domain.

Lemma 5.1. *Let Z be a metric space which is K-bi-Lipschitz to the disc $\bar{D}$ and let $u : \bar{D} \to Z$ be any homeomorphism. Then for any Jordan curve $\gamma \subset \bar{D}$ and the corresponding Jordan domain $J \subset \bar{D}$ we have*

$$\mathrm{diam}(u(J)) \leq K^2 \cdot \mathrm{diam}(u(\gamma)).$$

By continuity, the same inequality holds true for any uniform limit of homeomorphisms from $\bar{D}$ to Z, the class of maps we are going to consider now more closely. Let again the space Z be K-bi-Lipschitz to the unit disc, let us fix three distinct points p_1, p_2, p_3 on S^1 and three distinct points x_1, x_2, x_3 on the boundary circle Γ of Z. Let $\Lambda_0(Z)$ denote the set of all continuous maps $u : \bar{D} \to Z$, which send p_i to x_i, which are uniform limits of homeomorphisms from $\bar{D}$ to Z, and whose restrictions to D are contained in the Sobolev space $W^{1,2}(D, Z)$.

As uniform limits of homeomorphisms, any map $u \in \Lambda_0(Z)$ has the whole set Z as its image. When applied to all circles γ contained in D, the conclusion of Lemma 5.1 shows that any $u \in \Lambda_0(Z)$ is K^2-pseudomonotone in the sense of [18]. Fixing a bi-Lipschitz homeomorphism $\psi : \bar{D} \to Z$, we see that $\psi^{-1} \circ u : \bar{D} \to \bar{D}$ is pseudomonotone as well. Using [18], we deduce that $\psi^{-1} \circ u$ satisfies Lusin's property (N), for any $u \in \Lambda_0(Z)$. Hence, any $u \in \Lambda_0(Z)$ satisfies Lusin's property (N) as well. See also [14, Theorem 2.4].

Lemma 5.2. *For all elements $u \in \Lambda_0(Z)$ the value $\mathrm{Area}_{\mu^{\mathfrak{I}}}(u)$ is independent of the choice of u.*

Proof. Fix again the bi-Lipschitz homeomorphism $\psi : \bar{D} \to Z$ and consider the map $v = \psi^{-1} \circ u$. Since v is a uniform limit of homeomorphisms, any fiber of v is a cell-like set ([9, p. 97]), in particular, any such fiber is connected. Applying the area formula to the continuous Sobolev map $v : D \to \bar{D}$ which satisfies Lusin's property (N) (cf. [14]), we see that for almost all $z \in D$ the preimage $v^{-1}(z)$ has only finitely many points. By the connectedness of the fibers, we see that almost every fiber $v^{-1}(z)$ has exactly one point. Now we see

$$\mathrm{Area}_{\mu^{\mathfrak{I}}}(u) = \int_D \mathbf{J}^{\mathfrak{I}}(\mathrm{ap\,md}\, u_z)\, dz = \int_D |\det(d_z v)| \mathbf{J}^{\mathfrak{I}}(\mathrm{md}\, \psi_{v(z)})\, dz.$$

The area formula for the Sobolev map $v : D \to D$ (cf. [14]) gives us

$$\mathrm{Area}_{\mu^{\mathfrak{I}}}(u) = \int_D \mathbf{J}^{\mathfrak{I}}(\mathrm{md}\, \psi_y)\, dy = \mathrm{Area}_{\mu^{\mathfrak{I}}}(\psi). \qquad \square$$

The next lemma is essentially taken from [13]:

Lemma 5.3. *For any $C > 0$, the set $\Lambda_0^C(Z)$ of all elements u in $\Lambda_0(Z)$ with $E_+^2(u) \leq C$ is equi-continuous.*

Proof. The equi-continuity of the restrictions of u to the boundary circle S^1 is part of the classical solution of the Plateau problem, see [17, Proposition 7.4]. By the Courant–Lebesgue lemma [17, Lemma 7.3], for any $\epsilon > 0$ there is some $\delta = \delta(\epsilon, C)$ such that for any $x \in \bar{D}$ and any $u \in \Lambda_0^C(Z)$ there is some $\sqrt{\delta} > r > \delta$ such that $\partial B_r(x) \cap \bar{D}$ is mapped by u to a curve of diameter $\leq \epsilon$.

If $B_\delta(x)$ does not intersect the boundary circle S^1, then $u(B_\delta(x))$ has diameter $\leq K^2 \cdot \epsilon$ by Lemma 5.1. On the other hand, if $B_\delta(x)$ intersects S^1, then we see that the image of the intersection of $B_\delta(x)$ with S^1 has diameter bounded as well by some $\epsilon' > 0$ depending only on δ and going to 0 with δ, due to the equi-continuity of the restrictions $u|_{S^1}$. We may assume $\epsilon = \epsilon'$. Then the Jordan curve consisting of the corresponding parts of $\partial B_\delta(x)$ and boundary S^1 has as its image a curve of diameter at most 2ϵ. Thus using Lemma 5.1, we see that the ball $B_\delta(x)$ is mapped onto a set of diameter $\leq 2K^2 \cdot \epsilon$. $\square$

The proof above shows that the modulus of continuity of any $u \in \Lambda_0^C(Z)$ depends only on the constants C, K, the boundary circle $\Gamma \subset Z$ and the choice of the fixed points $x_i \in \Gamma$.

Corollary 5.4. *There is a map $u \in \Lambda_0(Z)$ with minimal $\mathcal{I}$-energy in $\Lambda_0(Z)$. This element u satisfies*

$$\mathrm{Area}_{\mu^{\mathcal{I}}}(u) = \lambda_{\mathcal{I}} \cdot E_{\mathcal{I}}(u).$$

Proof. Take a sequence $u_n \in \Lambda_0(Z)$ whose $\mathcal{I}$-energies converge to the infimum of $\mathcal{I}$ on $\Lambda_0(Z)$. By (2.2), E_+^2 is bounded by a multiple of $\mathcal{I}$. Therefore, we can apply Lemma 5.3 and deduce that the sequence u_n is equi-continuous. By Arzela–Ascoli, we find a map $u : \bar{D} \to Z$ as a uniform limit of a subsequence of the u_n. This map u is a uniform limit of uniform limits of homeomorphisms, hence u itself is a uniform limit of homeomorphisms. Moreover, $u(p_i) = x_i$ for $i = 1, 2, 3$. Finally, the map is contained in $W^{1,2}(D, X)$ as an L^2-limit of Sobolev maps with uniformly bounded energy, hence $u \in \Lambda_0(Z)$. Since $\mathcal{I}$ is quasi-convex, we have $E_{\mathcal{I}}(u) \leq \lim_{n\to\infty} E_{\mathcal{I}}(u_n)$, see [17, Theorem 5.4]. Therefore, u has minimal $\mathcal{I}$-energy in $\Lambda_0(Z)$.

If $\phi : \bar{D} \to \bar{D}$ were a bi-Lipschitz homeomorphism with $E_{\mathcal{I}}(u \circ \phi) < E_{\mathcal{I}}(u)$, we would consider a Möbius map $\phi_0 : \bar{D} \to \bar{D}$ such that $\phi \circ \phi_0$ fixes the points p_i. Then the map $u' := u \circ \phi \circ \phi_0$ is in $\Lambda_0(Z)$ and has the same $\mathcal{I}$-energy as $u \circ \phi$, due to the conformal invariance of $\mathcal{I}$. This would contradict the minimality of $E_{\mathcal{I}}(u)$ in $\Lambda_0(Z)$. Hence such a homeomorphism ϕ cannot exist and we may apply Lemma 4.1, to obtain the equality $\mathrm{Area}_{\mu^{\mathcal{I}}}(u) = \lambda_{\mathcal{I}} \cdot E_{\mathcal{I}}(u)$. □

Now it is easy to use ψ_n to obtain a contradiction to the quasi-convexity of $\mathcal{I}$. Denote by Z_n the image $\psi_n(\bar{D})$ and by Z the ellipse $L(\bar{D})$. By construction, all Z_n and Z are K-bi-Lipschitz to $\bar{D}$ and share the same boundary circle. We denote it by Γ and fix the same triple x_1, x_2, x_3 in Γ for all Z_n and Z.

Consider a map $v_n \in \Lambda_0(Z_n)$ with minimal $\mathcal{I}$-energy in $\Lambda_0(Z_n)$. By Corollary 5.4, such a v_n exists and satisfies $\mathrm{Area}_{\mu^{\mathcal{I}}}(v_n) = \lambda_{\mathcal{I}} \cdot E_{\mathcal{I}}(v_n)$. Moreover, by Lemma 5.3 and the subsequent observation, the maps v_n are equi-continuous. Finally, by Lemma 5.2, we have $\mathrm{Area}_{\mu^{\mathcal{I}}}(v_n) = \mathrm{Area}_{\mu^{\mathcal{I}}}(\psi_n)$.

The images of the maps $v_n : \bar{D} \to Z_n \to Y$ are contained in a compact set. Hence, by Arzela–Ascoli after choosing a subsequence, the maps v_n uniformly converge to a map $v : \bar{D} \to Z$. Moreover, identifying Z_n with Z by some uniformly bi-Lipschitz homeomorphisms point-wise converging to the identity of Z, we see that the limiting map v can be represented as a uniform limit of homeomorphisms from $\bar{D}$ to Z. Since the v_n have uniformly bounded energies, the limit map v lies in the Sobolev class $W^{1,2}(D, Z)$. Thus, by construction, $v \in \Lambda_0(Z)$. Finally, by the semi-continuity of $\mathcal{I}$, we must have $E_{\mathcal{I}}(v) \leq \liminf_{n\to\infty} E_{\mathcal{I}}(v_n)$.

Taking all inequalities together we get for large n:

$$\begin{aligned}\mathrm{Area}_{\mu^{\mathcal{I}}}(v) &= \mathrm{Area}_{\mu^{\mathcal{I}}}(L|_D)\\ &\geq \mathrm{Area}_{\mu^{\mathcal{I}}}(\psi_n) + \epsilon\\ &= \lambda_{\mathcal{I}} \cdot E_{\mathcal{I}}(v_n) + \epsilon\\ &\geq \lambda_{\mathcal{I}} \cdot E_{\mathcal{I}}(v) + \frac{1}{2}\epsilon.\end{aligned}$$

But this contradicts the main inequality (4.1) and finishes the proof of Theorem 4.2. □

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