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An Invariance Principle to Ferrari–Spohn Diffusions

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Abstract: We prove an invariance principle for a class of tilted 1 + 1-dimensional SOS models or, equivalently, for a class of tilted random walk bridges in $\mathbb{Z}_+$. The limiting objects are stationary reversible ergodic diffusions with drifts given by the logarithmic derivatives of the ground states of associated singular Sturm–Liouville operators. In the case of a linear area tilt, we recover the Ferrari–Spohn diffusion with log-Airy drift, which was derived in Ferrari and Spohn (Ann Probab 33(4):1302—1325, 2005) in the context of Brownian motions conditioned to stay above circular and parabolic barriers.

1. Introduction and Results

1.1. Physical motivations. We start with an informal description, in a restricted setting, of the effective interface model at the core of our study; a detailed description in the more general framework considered in the present work will be given in Sect. 1.3.

We consider a Gibbs random field $(X_i)_{1\leq i\leq N}$, with $X_i \in \mathbb{N}$ for all i, and effective Hamiltonian

$$H_\lambda = \sum_{i=1}^{N-1} \Phi(X_{i+1} - X_i) + \sum_{i=1}^{N} V_\lambda(X_i),$$

depending on a parameter $\lambda > 0$. Later, we shall allow rather general forms for the interaction Φ and for the external potential V_λ. For the purpose of this introductory section, let us however restrict the discussion to the physically very relevant case of $V_\lambda(x) = \lambda x$, and assume that Φ is symmetric and grows fast enough: for example, $\Phi(x) = x^2$, $\Phi(x) = |x|$ or $\Phi(x) = \infty \cdot \mathbf{1}_{|x|>R}$. Let us denote by $\mu_{N;\lambda}$ the corresponding Gibbs measure with boundary condition $X_1 = X_N = 0$.

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With this choice, this model can be interpreted as follows. The random variable X_i models the height of an interface above the site i. This interface separates an equilibrium phase (above the interface) and a layer of unstable phase (delimited by the interface and the wall located at height 0). The parameter λ corresponds to the excess free energy associated to the unstable phase.

Of course, when $\lambda = 0$, the distribution of X is just that of a random walk, conditioned to stay positive, with distribution of jumps given by

$$p_x = \frac{e^{-\Phi(x)}}{\sum_y e^{-\Phi(y)}}. \tag{1.1}$$

In particular, the field delocalizes as $N \to \infty$. When $\lambda > 0$, however, the field remains localized uniformly in N. We shall be mostly interested in the behavior as $\lambda \downarrow 0$ (say, either after letting $N \to \infty$, or by assuming that $N = N(\lambda)$ grows fast enough). In that case, one can prove that the typical width of the layer is of order $\lambda^{-1/3}$ and that the correlation along the interface is of order $\lambda^{-2/3}$ [1,14].

In the present work, we are interested in the scaling limit of the random field X as $\lambda \downarrow 0$, that is, in the limiting behavior of $x_\lambda(t) = \lambda^{1/3} X_{[\lambda^{-2/3} t]}$. As stated in Theorem A below, in the particular case considered here, the scaling limit is given by the diffusion on $(0, \infty)$ with generator

$$\frac{\sigma^2}{2}\frac{d^2}{dr^2} + \sigma^2 \frac{\varphi_0'}{\varphi_0}\frac{d}{dr},$$

where (see (1.1)) $\sigma^2 = \sum_x x^2 p_x$ and $\varphi_0 = \mathsf{Ai}(\sqrt[3]{\frac{2}{\sigma^2}}\, x - \omega_1)$ with $-\omega_1$ the first zero of the Airy function Ai. This diffusion was first introduced by Ferrari and Spohn in the context of Brownian motions conditioned to stay above circular and parabolic barriers [12].

This scaling limit should be common to a wide class of systems, of which the following are but a few examples:

- Critical prewetting in the 2d Ising model: behavior of the film of unstable negatively magnetized layer induced by $(-)$-boundary conditions, in the presence of a positive bulk magnetic field [18]; see Fig. 1.
- Interfacial adsorption at the interface between two equilibrium phases [13,17].
- Geometry of the top-most layer of the 2+1-dimensional SOS model above a wall [7].
- Island of activity in kinetically constrained models [5].

1.2. Limiting objects. Limiting objects are quantified in terms of Sturm–Liouville problems.

A Sturm–Liouville problem. The basic space we shall work with is $\mathbb{L}_2(\mathbb{R}_+)$. The notations $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle_2$ are reserved for the corresponding norm and scalar product. Given $\sigma > 0$ and a non-negative function $q \in \mathsf{C}^2(\mathbb{R}_+)$ which satisfies $\lim_{r\to\infty} q(r) = \infty$, consider the following family of singular Sturm–Liouville operators on $\mathbb{R}_+$:

$$\mathsf{L} = \mathsf{L}_{\sigma,q} = \frac{\sigma^2}{2}\frac{d^2}{dr^2} - q(r), \tag{1.2}$$

with zero boundary condition $\varphi(0) = 0$.

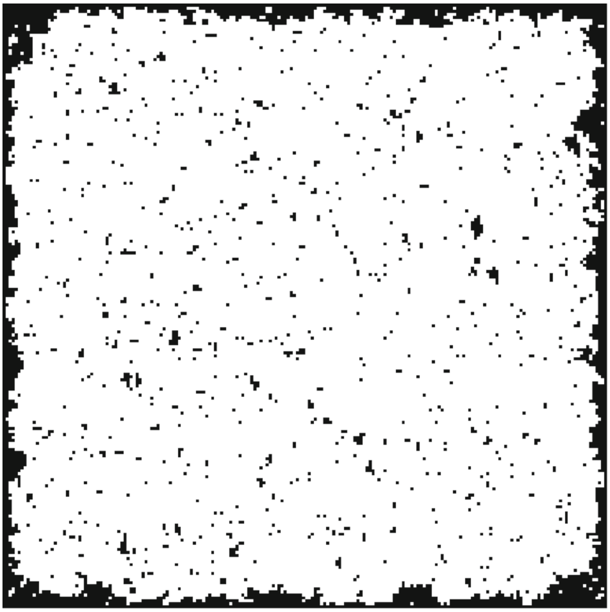

Fig. 1. A low-temperature two-dimensional Ising model in a box of sidelength $N = 200$ with negative boundary condition and a positive magnetic field of the form $h = c/N$. When c is above a critical threshold, the bulk of the system is occupied by a positively magnetized phase, while the walls are wet by a film of (unstable) negatively magnetized phase [16]. For a slightly different geometry, it was shown in [18] that this film has a width (along the walls) of order $h^{-1/3+o(1)}$ as $h \downarrow 0$

It is a classical result [10] that L possesses a complete orthonormal family $\{\varphi_i\}$ of simple eigenfunctions in $\mathbb{L}_2(\mathbb{R}_+)$ with eigenvalues

$$0 > -\zeta_0 > -\zeta_1 > -\zeta_2 > \dots; \ \lim \zeta_j = \infty. \tag{1.3}$$

The eigenfunctions φ_i are smooth and φ_i has exactly i zeroes in $(0, \infty)$, $i = 0, 1, \dots$.

The domain of (the closure of) L in $\mathbb{L}_2(\mathbb{R}_+)$ is

$$\mathcal{D}(\mathsf{L}) = \left\{ f = \sum_i a_i \varphi_i \ : \ \sum_i \zeta_i^2 a_i^2 < \infty \right\} \ \text{ and } \ \mathsf{L} f = -\sum_i \zeta_i a_i \varphi_i \text{ for } f \in \mathcal{D}. \tag{1.4}$$

Clearly, $\mathcal{D}(\mathsf{L})$ is dense in $\mathbb{L}_2(\mathbb{R}_+)$. Indeed, since any function $f \in \mathbb{L}_2$ can be written as $f = \sum_i a_i \varphi_i$, the linear space of all finite linear combinations $\mathcal{U} = \left\{ \sum_{i=0}^N a_i \varphi_i \right\} \subset \mathcal{D}(\mathsf{L})$ is dense in $\mathbb{L}_2$. For any function $f = \sum_i a_i \varphi_i \in \mathcal{D}(\mathsf{L})$, $\lim_{N \to \infty} \mathsf{L}\left(\sum_{i=0}^N a_i \varphi_i \right) = \mathsf{L} f$. In particular, $\mathcal{U}$ is a core for L.

If $f = \sum_i a_i \varphi_i \in \mathcal{D}$ and $\lambda > -\zeta_0$, one has

$$\|(\lambda \mathsf{I} - \mathsf{L}) f\|_2 = \left\| \sum_i (\lambda + \zeta_i) a_i \varphi_i \right\|_2 \geq (\lambda + \zeta_0) \|f\|_2, \tag{1.5}$$

which shows that $\mathsf{L} + \zeta_0 \mathsf{I}$ is dissipative.

Furthermore, for any $\lambda > -\zeta_0$, Range $(\lambda\mathsf{I} - \mathsf{L}) = \mathbb{L}_2(\mathbb{R}_+)$, and L has a compact resolvent $\mathsf{R}_\lambda = (\lambda\mathsf{I} - \mathsf{L})^{-1}$. Indeed,

$$(\lambda\mathsf{I} - \mathsf{L}) \sum_i \frac{a_i}{\lambda + \zeta_i} \varphi_i = \sum_i a_i \varphi_i \quad \text{and} \quad \mathsf{R}_\lambda \left(\sum_i a_i \varphi_i \right) = \sum_i \frac{a_i}{\lambda + \zeta_i} \varphi_i .$$

By the Hille–Yosida theorem, $\mathsf{L} + \zeta_0 \mathsf{I}$ generates a strongly continuous contraction semigroup T^t on $\mathbb{L}_2(\mathbb{R}_+)$. Explicitly,

$$\mathsf{T}^t \left(\sum_i a_i \varphi_i \right) = \sum_i \mathrm{e}^{-(\zeta_i - \zeta_0)t} a_i \varphi_i . \tag{1.6}$$

Ferrari–Spohn diffusions. Define

$$\mathsf{G}_{\sigma,q} \psi = \frac{1}{\varphi_0} (\mathsf{L} + \zeta_0)(\psi \varphi_0) = \frac{\sigma^2}{2} \frac{\mathrm{d}^2 \psi}{\mathrm{d}r^2} + \sigma^2 \frac{\varphi_0'}{\varphi_0} \frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{\sigma^2}{2\varphi_0^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\varphi_0^2 \frac{\mathrm{d}\psi}{\mathrm{d}r} \right) . \tag{1.7}$$

The sub-indices σ and q will be dropped whenever there is no risk of confusion. We shall say that $\mathsf{G}_{\sigma,q}$ is the generator of a Ferrari–Spohn diffusion on $(0, \infty)$. The diffusion itself is ergodic and reversible with respect to the measure $\mathrm{d}\mu_0(r) = \varphi_0^2(r)\mathrm{d}r$. In the sequel, we shall denote by $\mathsf{S}^t_{\sigma,q}$ the corresponding semigroup,

$$\mathsf{S}^t_{\sigma,q} \psi = \frac{1}{\varphi_0} \mathsf{T}^t (\psi \varphi_0), \tag{1.8}$$

and by $\mathbb{P}_{\sigma,q}$ the corresponding path measure.

1.3. Random walks with tilted areas. Let p_y be an irreducible random walk kernel on $\mathbb{Z}$. The probability of a finite trajectory $\mathbb{X} = (X_1, X_2, \ldots, X_k)$ is $\mathsf{p}(\mathbb{X}) = \prod_i p_{X_{i+1} - X_i}$. Let $\mathsf{u}, \mathsf{v} \in \mathbb{N}$ and $M, N \in \mathbb{Z}$ with $M \leq N$. Let $\mathcal{P}^{\mathsf{u},\mathsf{v}}_{M,N,+}$ be the family of trajectories starting at u at time M, ending at v at time N and staying positive during the time interval $\{M, \ldots, N\}$. For $N > 0$, we shall use shorthand notations $\mathcal{P}^{\mathsf{u},\mathsf{v}}_{N,+} = \mathcal{P}^{\mathsf{u},\mathsf{v}}_{-N,N,+}$ and $\hat{\mathcal{P}}^{\mathsf{u},\mathsf{v}}_{N,+} = \mathcal{P}^{\mathsf{u},\mathsf{v}}_{1,N,+}$.

Assumptions on p. Assume that

$$\sum_{\mathsf{z}} \mathsf{z} p_{\mathsf{z}} = 0 = \sum_{\mathsf{z}} \mathsf{z} p_{-\mathsf{z}} \quad \text{and } \mathsf{p} \text{ has finite exponential moments.} \tag{1.9}$$

In the sequel, we shall use the notation

$$\sigma^2 = \sum_{\mathsf{z}} \mathsf{z}^2 p_{\mathsf{z}} = \sum_{\mathsf{z}} \mathsf{z}^2 p_{-\mathsf{z}} = \mathbb{V}\mathrm{ar}_{\mathsf{p}}(X) < \infty. \tag{1.10}$$

The model. Let $\{V_\lambda\}_{\lambda>0}$ be a family of self-potentials, $V_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$. Given $\lambda > 0$, define the partition function

$$Z^{\mathsf{u},\mathsf{v}}_{N,+,\lambda} = \sum_{\mathbb{X}\in\mathcal{P}^{\mathsf{u},\mathsf{v}}_{N,+}} \mathrm{e}^{-\sum_{-N}^{N} V_\lambda(X_i)} \mathsf{p}(\mathbb{X}), \tag{1.11}$$

and, accordingly, the probability distribution

$$\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\mathbb{X}) = \frac{1}{Z^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}} \mathrm{e}^{-\sum_{-N}^{N} V_\lambda(X_i)} \mathsf{p}(\mathbb{X}). \tag{1.12}$$

The term $\sum_{-N}^{N} V_\lambda(X_i)$ represents a generalized (non-linear) area below the trajectory $\mathbb{X}$. It reduces to (a multiple of) the usual area when $V_\lambda(x) = \lambda x$. We make the following set of assumptions on V_λ:

Assumptions on V_λ. For any $\lambda > 0$, the function V_λ is continuous monotone increasing and satisfies

$$V_\lambda(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} V_\lambda(x) = \infty. \tag{1.13}$$

In particular, the relation

$$H_\lambda^2 V_\lambda(H_\lambda) = 1 \tag{1.14}$$

determines unambiguously the quantity H_λ. Furthermore, we make the assumptions that $\lim_{\lambda\to 0} H_\lambda = \infty$ and that there exists a function $q \in \mathsf{C}^2(\mathbb{R}^+)$ such that

$$\lim_{\lambda\to 0} H_\lambda^2 V_\lambda(rH_\lambda) = q(r), \tag{1.15}$$

uniformly on compact subsets of $\mathbb{R}_+$. Note that H_λ, respectively H_λ^2, is the spatial, respectively temporal, scale for the invariance principle which is formulated below in Theorem A.

Finally, we shall assume that there exist $\lambda_0 > 0$ and a (continuous non-decreasing) function $q_0 \geq 0$ with $\lim_{r\to\infty} q_0(r) = \infty$ such that, for all $\lambda \leq \lambda_0$,

$$H_\lambda^2 V_\lambda(rH_\lambda) \geq q_0(r) \quad \text{on } \mathbb{R}_+. \tag{1.16}$$

1.4. Main result. It will be convenient to think about $\mathbb{X}$ as being frozen outside $\{-N, \ldots, N\}$. In this way, we can consider $\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}$ as a distribution on the set of doubly infinite paths $\mathbb{N}^{\mathbb{Z}}$.

We set $h_\lambda = H_\lambda^{-1}$. The paths are rescaled as follows: For $t \in h_\lambda^2\mathbb{Z}$, define

$$x_\lambda(t) = h_\lambda X_{H_\lambda^2 t} = \frac{1}{H_\lambda} X_{H_\lambda^2 t}. \tag{1.17}$$

$x_\lambda(t)$ is then extended to $t \in \mathbb{R}$ by linear interpolation. In this way, we can talk about the induced distribution of $\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}$ on the space of continuous function $\mathsf{C}[-T, T]$, for any $T \geq 0$.

Theorem A. *Let* λ_N *be a sequence satisfying* $\lim_{N\to\infty}\lambda_N = 0$. *Assume, furthermore, that* $\lim_{N\to\infty} N h_{\lambda_N}^2 = +\infty$. *Set* $x_N(\cdot) = x_{\lambda_N}(\cdot)$. *Fix any* $c \in (0,\infty)$. *Then, as* $N \to \infty$, *the distributions of* $x_N(\cdot)$ *under* $\mathbb{P}_{N,+,\lambda}^{\mathsf{u},\mathsf{v}}$ *are, uniformly in* $\mathsf{u},\mathsf{v} \le c H_{\lambda_N}$, *weakly convergent to the stationary Ferrari–Spohn diffusion* $x_{\sigma,q}(\cdot)$ *on* $\mathbb{R}_+$ *with the generator* $\mathsf{G}_{\sigma,q}$ *specified in* (1.7).

Remark 1. The condition $\lim_{N\to\infty} N h_{\lambda_N}^2 = +\infty$ or, equivalently, $H_{\lambda_N}^2 \ll N$ has a transparent meaning: N is the size of the system, whereas $H_{\lambda_N}^2$ is the correlation length of the random walk with V_{λ_N}-area tilts. A precise statement of this sort is formulated in Proposition 5 in Sect. 3.4 below.

In the case $V_\lambda(x) = \lambda x$, the typical height $H_\lambda = \lambda^{-\frac{1}{3}}$, $q(r) = r$ and the ground state φ_0 is the rescaled Airy function:

$$\varphi_0 = \mathsf{Ai}(\chi r - \omega_1) \quad \text{and} \quad e_0 = \frac{\omega_1}{\chi}, \tag{1.18}$$

where $-\omega_1 = -2.33811\ldots$ is the first zero of Ai and $\chi = \sqrt[3]{\frac{2}{\sigma^2}}$. Indeed, for φ_0 defined as in (1.18),

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\mathsf{Ai}(r) = r\mathsf{Ai}(r) \;\Rightarrow\; \frac{\mathrm{d}^2}{\mathrm{d}r^2}\varphi_0(r) = \chi^2(\chi r - \omega_1)\varphi_0(r),$$

and one only needs to tune χ in order to adjust to the expression (1.2) for L.

2. Proofs

The proof is a combination of probabilistic estimates based on an early paper [14] and rather straightforward functional analytic considerations. We shall first express the partition functions $Z_{N,+,\lambda}^{\mathsf{u},\mathsf{v}}$ and, subsequently, the probability distributions $\mathbb{P}_{N,+,\lambda}^{\mathsf{u},\mathsf{v}}$ in terms of powers of a transfer operator T_λ. For each λ, the operator T_λ gives rise, via Doob's transform, to a stationary positive-recurrent Markov chain X^λ with path measure $\mathbb{P}_\lambda$. In the sequel, we shall refer to X^λ as to the *ground-state chain*. Following (1.17), the ground-state chains are rescaled as $x_\lambda(t) = h_\lambda X^\lambda_{H_\lambda^2 t}$.

The proof of Theorem A comprises three steps:

STEP 1. As $\lambda \to 0$, the finite-dimensional distributions of the rescaled ground-state chains x_λ converge to the finite-dimensional distributions of the Ferrari-Spohn diffusion $x_{\sigma,q}$. This is the content of Corollary 1 in Sect. 2.2.
STEP 2. Under our assumptions on p and on the family of potentials V_λ, the induced family of distributions $\mathbb{P}_\lambda$ is tight on $\mathsf{C}[-T,T]$ for any $T < \infty$. This is the content of Proposition 4 in Sect. 3.3.
STEP 3. Under the conditions of Theorem A, the following happens: For each $T \ge 0$, the variational distance between the induced distributions on $\mathsf{C}[-T,T]$ of $\mathbb{P}_{N,+,\lambda_N}^{\mathsf{u},\mathsf{v}}$ and of $\mathbb{P}_{\lambda_N}$ tends to zero as $N \to \infty$. This is the content of Corollary 2 in Sect. 3.4.

Remark on constants. $c_1, \nu_1, \kappa_1, c_2, \nu_2, \kappa_2, \ldots$ denote positive constants which may take different values in different Subsections, but are otherwise universal, in the sense that the corresponding bounds hold uniformly in the range of the relevant parameters.

2.1. The operator T_λ *and its Doob transform.* In the sequel, we shall make a slight abuse of notation and identify the spaces $\ell_p(\mathbb{N})$ with sub-spaces of $\ell_p(\mathbb{Z})$:

$$\ell_p(\mathbb{N}) = \left\{\phi \in \ell_p(\mathbb{Z}) \,:\, \phi(\mathsf{x}) = 0 \text{ for } \mathsf{x} \le 0\right\}.$$

For $\lambda > 0$, consider the operators $\tilde{\mathsf{T}}_\lambda$ defined by

$$\tilde{\mathsf{T}}_\lambda[\phi](\mathsf{x}) = \sum_{\mathsf{y}} p_{\mathsf{y}-\mathsf{x}} \mathrm{e}^{-\frac{1}{2}(V_\lambda(\mathsf{x})+V_\lambda(\mathsf{y}))}\phi(\mathsf{y}). \tag{2.1}$$

In terms of $\tilde{\mathsf{T}}_\lambda$, the partition function can be expressed as

$$\mathrm{e}^{\frac{1}{2}(V_\lambda(\mathsf{u})+V_\lambda(\mathsf{v}))} Z_{N,+,\lambda}^{\mathsf{u},\mathsf{v}} = \tilde{\mathsf{T}}_\lambda^{2N}[\mathbf{1}_\mathsf{v}](\mathsf{u}). \tag{2.2}$$

For each $\lambda > 0$, the operator $\tilde{\mathsf{T}}_\lambda$ is positive on $\ell_p(\mathbb{N})$ and compact from $\ell_p(\mathbb{N})$ to $\ell_q(\mathbb{N})$ for every $p, q \in [1, \infty]$ (we use $\ell_\infty(\mathbb{N})$ for the Banach space of functions $\phi \in \mathbb{R}^\mathbb{N}$ which tend to zero as $\mathsf{x} \to \infty$). Indeed, $\{\mathbf{1}_\mathsf{x}\}_{\mathsf{x}\in\mathbb{N}}$ is a basis of $\ell_p(\mathbb{N})$. Since, for $\mathsf{x}, \mathsf{y} > 0$,

$$\tilde{\mathsf{T}}_\lambda[\mathbf{1}_\mathsf{x}](\mathsf{y}) = \mathrm{e}^{-\frac{1}{2}(V_\lambda(\mathsf{x})+V_\lambda(\mathsf{y}))} p_{\mathsf{x}-\mathsf{y}},$$

it follows that

$$\tilde{\mathsf{T}}_\lambda[\mathbf{1}_\mathsf{x}] = \mathrm{e}^{-\frac{1}{2}V_\lambda(\mathsf{x})} \sum_{\mathsf{y}} p_{\mathsf{x}-\mathsf{y}} \mathrm{e}^{-\frac{1}{2}V_\lambda(\mathsf{y})} \mathbf{1}_\mathsf{y} \;\Rightarrow\; \|\tilde{\mathsf{T}}_\lambda[\mathbf{1}_\mathsf{x}]\|_p \le \mathrm{e}^{-\frac{1}{2}V_\lambda(\mathsf{x})} \sum_{\mathsf{y}} p_{\mathsf{x}-\mathsf{y}} \mathrm{e}^{-\frac{1}{2}V_\lambda(\mathsf{y})}. \tag{2.3}$$

Hence, by the second condition in (1.13) and by the assumption on exponential tails of p in (1.9), the closure $\{\tilde{\mathsf{T}}_\lambda[\mathbf{1}_\mathsf{x}]\}$ is compact in any $\ell_q(\mathbb{N})$ whenever $\lambda > 0$.

Since $\tilde{\mathsf{T}}_\lambda$ is a positive operator (and since, e.g., $\sum_n 2^{-n}\tilde{\mathsf{T}}_\lambda^n$ is strictly positive and still compact), the Krein–Rutman Theorem [15, Theorem 6.3] applies, and $\tilde{\mathsf{T}}_\lambda$ possesses a strictly positive leading eigenfunction ϕ_λ (the same for all $\ell_p(\mathbb{N})$ spaces, by compact embedding) of algebraic multiplicity one. Let E_λ be the corresponding leading eigenvalue. All the above reasoning applies to the adjoint operator $\tilde{\mathsf{T}}_\lambda^*$ with matrix entries

$$\tilde{\mathsf{T}}_\lambda^*(\mathsf{x}, \mathsf{y}) = \tilde{\mathsf{T}}_\lambda(\mathsf{y}, \mathsf{x}) = \mathrm{e}^{-\frac{1}{2}(V_\lambda(\mathsf{x})+V_\lambda(\mathsf{y}))} p_{\mathsf{x}-\mathsf{y}}. \tag{2.4}$$

Let ϕ_λ^* be the Krein–Rutman eigenfunction (with the very same leading eigenvalue E_λ) of $\tilde{\mathsf{T}}_\lambda^*$. As Theorem A indicates, the relevant spatial scale is given by $h_\lambda = H_\lambda^{-1}$. To fix notation, we normalize ϕ_λ and ϕ_λ^* as in (2.19) below, that is $h_\lambda \sum_x \phi_\lambda(x)^2 = h_\lambda \sum_x \left(\phi_\lambda^*(x)\right)^2 = 1$.

It will be convenient to work with the following normalized version T_λ of $\tilde{\mathsf{T}}_\lambda$: for $\mathsf{x}, \mathsf{y} \in \mathbb{N}$, let us introduce the kernel

$$\mathsf{T}_\lambda(\mathsf{x}, \mathsf{y}) = \frac{1}{E_\lambda}\tilde{\mathsf{T}}_\lambda(\mathsf{x}, \mathsf{y}) = \frac{1}{E_\lambda}\mathrm{e}^{-\frac{1}{2}(V_\lambda(\mathsf{x})+V_\lambda(\mathsf{y}))} p_{\mathsf{y}-\mathsf{x}}. \tag{2.5}$$

In this way, ϕ_λ and ϕ_λ^* are the principal positive Krein–Rutman eigenfunctions of T_λ and T_λ^* with eigenvalue 1.

Ground-state chains. Define

$$\pi_\lambda(\mathsf{x},\mathsf{y}) = \frac{1}{\phi_\lambda(\mathsf{x})}\mathsf{T}_\lambda(\mathsf{x},\mathsf{y})\phi_\lambda(\mathsf{y}) \quad \text{and} \quad \pi^*_\lambda(\mathsf{x},\mathsf{y}) = \frac{1}{\phi^*_\lambda(\mathsf{x})}\mathsf{T}^*_\lambda(\mathsf{x},\mathsf{y})\phi^*_\lambda(\mathsf{y}). \tag{2.6}$$

π_λ and π^*_λ are irreducible Markov kernels on $\mathbb{N}$. The corresponding chains are positively recurrent and have the invariant probability measure $\mu_\lambda(\mathsf{x}) = c_\lambda\phi^*_\lambda(\mathsf{x})\phi_\lambda(\mathsf{x})$. As we shall prove below in Theorem 2, $\lim_{\lambda\to 0}\frac{h_\lambda}{c_\lambda} = 1$.

Notice that

$$\mu_\lambda(\mathsf{x})\pi_\lambda(\mathsf{x},\mathsf{y}) = \mu_\lambda(\mathsf{y})\pi^*_\lambda(\mathsf{y},\mathsf{x}) = c_\lambda\phi^*_\lambda(\mathsf{x})\mathsf{T}_\lambda(\mathsf{x},\mathsf{y})\phi_\lambda(\mathsf{y}). \tag{2.7}$$

In the sequel, we shall denote by $\mathbb{P}_\lambda$ the stationary distribution on $\mathbb{N}^{\mathbb{Z}}$ of the (direct) ground-state chain which corresponds to π_λ.

Variational description of E_λ and μ_λ. Let us formulate a Donsker–Varadhan type formula for the principal eigenvalue E_λ of $\tilde{\mathsf{T}}_\lambda$ or, equivalently, for the eigenvalue 1 of T_λ.

Theorem 1.

$$1 = \sup_\mu \inf_{u\in\mathbb{U}_+}\sum_{\mathsf{x}\in\mathbb{N}}\mu(\mathsf{x})\frac{\mathsf{T}_\lambda[u]}{u}(\mathsf{x}). \tag{2.8}$$

Above, the first supremum is over probability measures on $\mathbb{N}$, and

$$\mathbb{U}_+ = \{u = v\phi_\lambda \,:\, 0 < \inf v \le \sup v < \infty\}. \tag{2.9}$$

Remark 2. Eventually, our proof of Theorem A will not rely on the variational formula (2.8). Theorem 1 and its consequence, Proposition 1, are formulated and proved in their own right, but also because they elucidate the type of variational convergence behind Theorem 2 below.

Proof. As before, set $h_\lambda = H_\lambda^{-1}$ and consider the following functional:

$$\mathcal{F}_\lambda(\mu) = \frac{1}{h_\lambda^2}\sup_{u\in\mathbb{U}_+}\sum_{\mathsf{x}\in\mathbb{N}}\mu(\mathsf{x})\frac{(1-\mathsf{T}_\lambda)u}{u}(\mathsf{x}). \tag{2.10}$$

The coefficient h_λ^2 plays no role in the proof, it just fixes the proper scaling. The claim of Theorem 1 will follow once we show that

$$\mathcal{F}_\lambda(\mu_\lambda) = 0 \quad \text{and} \quad \mathcal{F}_\lambda(\mu) > 0 \quad \text{whenever } \mu \ne \mu_\lambda. \tag{2.11}$$

Taking $u \equiv \phi_\lambda$, we readily infer that $\mathcal{F}_\lambda$ is non-negative. In order to check the first statement in (2.11), we need to verify that

$$\sum_{\mathsf{x}\in\mathbb{N}}\mu_\lambda(\mathsf{x})\frac{(1-\mathsf{T}_\lambda)u}{u}(\mathsf{x}) \le 0, \tag{2.12}$$

whenever $u = v\phi_\lambda$ and $v \not\equiv 1$. Let us write v as $v = \mathrm{e}^h$. Then, using the notation (2.6),

$$\sum_{\mathsf{x}\in\mathbb{N}}\mu_\lambda(\mathsf{x})\frac{(1-\mathsf{T}_\lambda)u}{u}(\mathsf{x}) = \sum_{\mathsf{x}\in\mathbb{N}}\left(1-\mathrm{e}^{-h}\pi_\lambda\mathrm{e}^h\right)\mu_\lambda(\mathsf{x}) \le \left(1-\mathrm{e}^{-\langle\mu_\lambda,h\rangle+\langle\mu_\lambda,\pi_\lambda h\rangle}\right) = 0,$$

where we used again Jensen's inequality and the invariance of μ_λ: $\mu_\lambda\pi_\lambda = \mu_\lambda$.

If $\mu = g^2\mu_\lambda$ and g is bounded away from 0 and ∞, then, taking $u = g\phi_\lambda$,

$$\sum_{\mathsf{x}\in\mathbb{N}} \mu(\mathsf{x})\frac{(1-\mathsf{T}_\lambda)u}{u}(\mathsf{x}) = \frac{1}{2}\sum_{\mathsf{x},\mathsf{y}} \hat{\pi}_\lambda(\mathsf{x},\mathsf{y})(g(\mathsf{x})-g(\mathsf{y}))^2\mu_\lambda(\mathsf{x}), \tag{2.13}$$

where $\hat{\pi}_\lambda$ is the symmetrized kernel,

$$\hat{\pi}_\lambda(\mathsf{x},\mathsf{y}) = \frac{1}{2}(\pi_\lambda(\mathsf{x},\mathsf{y}) + \pi_\lambda^*(\mathsf{x},\mathsf{y})). \tag{2.14}$$

By (2.7),

$$\mathcal{F}_\lambda(g^2\mu_\lambda) \geq \frac{c_\lambda}{4}\sum_{\mathsf{x},\mathsf{y}}\left(\frac{g(\mathsf{x})-g(\mathsf{y})}{h_\lambda}\right)^2(\phi_\lambda^*(\mathsf{x})\phi_\lambda(\mathsf{y}) + \phi_\lambda^*(\mathsf{y})\phi_\lambda(\mathsf{x})). \tag{2.15}$$

We claim that (2.15) still holds when g is not bounded away from 0 and ∞. This will follow if we show that there exists a sequence $u_\ell \in \mathbb{U}_+$ such that

$$\begin{aligned}\limsup_{\ell\to\infty}\sum_{\mathsf{x}\in\mathbb{N}} \mu_\lambda(\mathsf{x})g^2(\mathsf{x})\frac{\mathsf{T}_\lambda[u_\ell\phi_\lambda](\mathsf{x})}{u_\ell(\mathsf{x})\phi_\lambda(\mathsf{x})} &\leq \sum_{\mathsf{x}\in\mathbb{N}} \mu_\lambda(\mathsf{x})g^2(\mathsf{x})\frac{\mathsf{T}_\lambda[g\phi_\lambda](\mathsf{x})}{g(\mathsf{x})\phi_\lambda(\mathsf{x})} \\ &= c_\lambda\sum_{\mathsf{x}\in\mathbb{N}}\phi_\lambda^*(\mathsf{x})g(\mathsf{x})\mathsf{T}_\lambda[g\phi_\lambda](\mathsf{x}). \end{aligned}\tag{2.16}$$

Assume that g is not bounded away from zero and consider $g_n = g \vee \frac{1}{n}$. Then,

$$\sum_{\mathsf{x}\in\mathbb{N}} \mu_\lambda(\mathsf{x})g^2(\mathsf{x})\frac{\mathsf{T}_\lambda[g_n\phi_\lambda](\mathsf{x})}{g_n(\mathsf{x})\phi_\lambda(\mathsf{x})} \leq c_\lambda\sum_{\mathsf{x}\in\mathbb{N}}\phi^*(\mathsf{x})g(\mathsf{x})\mathsf{T}_\lambda[g_n\phi_\lambda](\mathsf{x}).$$

By a monotone convergence argument, the right-hand side above converges (as $n \to \infty$) to $c_\lambda\sum_{\mathsf{x}\in\mathbb{N}}\phi^*(\mathsf{x})g(\mathsf{x})\mathsf{T}_\lambda[g\phi_\lambda](\mathsf{x})$. If, in addition, g is not bounded above, then consider $g_{n,M} = g_n \wedge M \in \mathbb{U}_+$. Define $A_M = \{\mathsf{x} : g(\mathsf{x}) > M\}$. Then,

$$\sum_{\mathsf{x}\in\mathbb{N}} \mu_\lambda(\mathsf{x})g^2(\mathsf{x})\frac{\mathsf{T}_\lambda[g_{n,M}\phi_\lambda](\mathsf{x})}{g_{n,M}(\mathsf{x})\phi_\lambda(\mathsf{x})} \leq \sum_{\mathsf{x}\notin A_M} \mu_\lambda(\mathsf{x})g^2(\mathsf{x})\frac{\mathsf{T}_\lambda[g_n\phi_\lambda](\mathsf{x})}{g_n(\mathsf{x})\phi_\lambda(\mathsf{x})} + \sum_{\mathsf{x}\in A_M}\mu_\lambda(\mathsf{x})g^2(\mathsf{x}).$$

Since $\lim_{M\to\infty}\sum_{\mathsf{x}\in A_M}\mu_\lambda(\mathsf{x})g^2(\mathsf{x}) = 0$, the approximation procedure goes through as claimed in (2.16). □

As a byproduct, we obtain the following

Proposition 1. *The functional $\mathcal{F}_\lambda$ is convex and lower-semicontinuous. It has a unique minimum:*

$$\mathcal{F}_\lambda(\mu) = 0 \;\Leftrightarrow\; \mu = \mu_\lambda. \tag{2.17}$$

Furthermore, μ_λ is a quadratic minimum in the sense of (2.15).

2.2. Convergence of eigenfunctions, invariant measures and semigroups. It will be convenient to think about T_λ and π_λ as acting on the rescaled spaces $\ell_2(\mathbb{N}_\lambda)$, where

$$\mathbb{N}_\lambda = h_\lambda \mathbb{N} \quad \text{and the scalar product is} \quad \langle u, v\rangle_{2,\lambda} = h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} u(\mathsf{r})v(\mathsf{r}). \tag{2.18}$$

Accordingly, we rescale ϕ_λ and ϕ_λ^* in such a way that

$$\|\phi_\lambda\|_{2,\lambda} = \|\phi_\lambda^*\|_{2,\lambda} = 1. \tag{2.19}$$

We use the same notation $\mu_\lambda = c_\lambda \phi_\lambda \phi_\lambda^*$ for the rescaled probability measure on $\mathbb{N}_\lambda$. In other words, the constants c_λ are defined via

$$\frac{1}{c_\lambda} = \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \phi_\lambda(\mathsf{r})\phi_\lambda^*(\mathsf{r}) \quad \text{or} \quad \frac{h_\lambda}{c_\lambda} = \langle \phi_\lambda, \phi_\lambda^*\rangle_{2,\lambda}, \tag{2.20}$$

where ϕ_λ and ϕ_λ^* are the principal eigenfunctions satisfying the normalization condition (2.19).

Remark 3. As in the case of $\ell_p(\mathbb{N})$, with a slight abuse of notation, we shall identify $\ell_2(\mathbb{N}_\lambda)$ with a closed linear sub-space of $\ell_2(\mathbb{Z}_\lambda)$, where $\mathbb{Z}_\lambda = h_\lambda\mathbb{Z}$. Namely,

$$\ell_2(\mathbb{N}_\lambda) = \{u \in \ell_2(\mathbb{Z}_\lambda) \,:\, u(r) = 0 \text{ for all } r \le 0\}. \tag{2.21}$$

In this way, if k_λ is a kernel on $\mathbb{Z}_\lambda$, then the operators

$$u(\cdot) \mapsto \mathbf{1}_{\cdot\in\mathbb{N}_\lambda} \sum_{\mathsf{s}\in\mathbb{Z}_\lambda} k_\lambda(\mathsf{s}-\cdot)u(\mathsf{s}) \quad \text{and} \quad u(\cdot) \mapsto \mathbf{1}_{\cdot\in\mathbb{N}_\lambda} \sum_{\mathsf{s}\in\mathbb{Z}_\lambda} k_\lambda(\cdot-\mathsf{s})(u(\mathsf{s})-u(\cdot)),$$

can be considered as operators on $\ell_2(\mathbb{N}_\lambda)$. Accordingly,

$$\sum_{\mathsf{s},\mathsf{r}\in\mathbb{Z}_\lambda} k_\lambda(\mathsf{s}-\mathsf{r})(u(\mathsf{s})-u(\mathsf{r}))u(\mathsf{r})$$

is a quadratic form on $\ell_2(\mathbb{N}_\lambda)$.

In the sequel, we shall write $p_\lambda(\mathsf{r}) = p_{H_\lambda \mathsf{r}}$ for the rescaled random walk kernel on $\mathbb{Z}_\lambda$.

Convergence of Hilbert spaces. Let us fix a map $\rho_\lambda : \mathbb{L}_2(\mathbb{R}_+) \to \ell_2(\mathbb{N}_\lambda)$ with $\|\rho_\lambda\| \le 1$. The specific choice is not really important; for instance, we may define

$$\rho_\lambda u(\mathsf{r}) = \tfrac{1}{h_\lambda} \int_{\mathsf{r}-h_\lambda}^{\mathsf{r}} u(s)\mathrm{d}s. \tag{2.22}$$

Definition 1. Let us say that a sequence $u_\lambda \in \ell_2(\mathbb{N}_\lambda)$ converges to $u \in \mathbb{L}_2(\mathbb{R}_+)$, $u = \mathsf{s}\text{-}\lim u_\lambda$, if

$$\lim_{\lambda\to 0} \|u_\lambda - \rho_\lambda u\|_{2,\lambda} = 0. \tag{2.23}$$

We shall write $\lim_{\lambda\to 0}$ instead of $\mathsf{s}\text{-}\lim_{\lambda\to 0}$ whenever no ambiguity arises.

Compactness of eigenfunctions. The following two probabilistic estimates will be proved in Sect. 3.

Lemma 1. *Define* $\mathsf{e}_\lambda = -H_\lambda^2 \log E_\lambda$. *Then,*

$$0 < \liminf_{\lambda\to 0} \mathsf{e}_\lambda \le \limsup_{\lambda\to 0} \mathsf{e}_\lambda < \infty. \tag{2.24}$$

As we already noted, it follows from the compactness of T_λ that the eigenfunctions ϕ_λ and ϕ_λ^* belong to $\ell_p(\mathbb{N}_\lambda)$ for any $p \ge 1$ and $\lambda > 0$. The second probabilistic input is a tail estimate on ϕ_λ and ϕ_λ^*.

Lemma 2. *There exist positive constants* ν_1 *and* ν_2 *such that*

$$h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \phi_\lambda(\mathsf{r})\mathbf{1}_{\{\mathsf{r}>K\}} \le \nu_1 \mathrm{e}^{-\nu_2 K H_\lambda(\sqrt{V_\lambda(H_\lambda K)}\wedge 1)} \le \nu_1 \mathrm{e}^{-\nu_2 K(\sqrt{q_0(K)}\wedge H_\lambda)}, \tag{2.25}$$

uniformly in $K > 0$ *and* $\lambda \le \lambda_0$. *The same holds for* ϕ_λ^*. *In particular, both sequences* ϕ_λ *and* ϕ_λ^* *are bounded in* $\ell_1(\mathbb{N}_\lambda)$*:*

$$\limsup_{\lambda\to 0} h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \phi_\lambda(\mathsf{r}) < \infty \quad \text{and} \quad \limsup_{\lambda\to 0} h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \phi_\lambda^*(\mathsf{r}) < \infty. \tag{2.26}$$

Using the two lemmas above and Rellich's theorem (see, e.g., [2, Chapter 6]) on compact embeddings of the Sobolev spaces $\mathbb{H}^1[a,b]$ into $\mathbb{L}_2[a,b]$ for finite intervals $[a,b]$, we shall prove the following

Proposition 2. *Under our assumptions on* V_λ *and* p, *the sequence* ϕ_λ *is sequentially compact* (*in the sense of* s-*convergence as described above in* (2.23)).

Proof. The proof comprises two steps: We first show that we can restrict attention to the compactness properties of the functions ψ_λ defined in (2.27) below. We then check that the sequence ψ_λ satisfies the energy-type estimate (2.33), which enables a uniform control of both tails of ψ_λ and of their Sobolev norms over $\mathbb{R}_+$. In this way, sequential compactness follows by a standard diagonal argument.

STEP 1. In view of Lemma 2, rather than studying directly the functions ϕ_λ, we can instead study the convergence properties of the functions

$$\psi_\lambda(\mathsf{r}) = \mathrm{e}^{-\frac{1}{2}V_\lambda(H_\lambda \mathsf{r})}\phi_\lambda(\mathsf{r}). \tag{2.27}$$

Indeed, by (2.25), there exists a sequence $\delta_\lambda \to 0$ such that

$$\lim_{\lambda\to 0} h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \phi_\lambda^2(\mathsf{r})\mathbf{1}_{\{V_\lambda(H_\lambda \mathsf{r})>\delta_\lambda\}} = 0. \tag{2.28}$$

So, (2.28) implies that the norm of the difference $\|\phi_\lambda - \psi_\lambda\|_{2,\lambda}$ tends to zero, and hence $\phi_\lambda - \psi_\lambda$ tends to zero in the sense of Definition 1.

STEP 2. In terms of ψ_λ, the eigenvalue equation $\tilde{\mathsf{T}}_\lambda\phi_\lambda = E_\lambda\phi_\lambda$ reads as (recall Remark 3)

$$\sum_{\mathsf{s}\in\mathbb{Z}_\lambda} p_\lambda(\mathsf{s}-\mathsf{r})(\psi_\lambda(\mathsf{s}) - \psi_\lambda(\mathsf{r})) = (E_\lambda \mathrm{e}^{V_\lambda(H_\lambda \mathsf{r})} - 1)\psi_\lambda(\mathsf{r}). \tag{2.29}$$

Multiplying both sides by $-\psi_\lambda(\mathsf{r})$ and summing over r, we get

$$h_\lambda \sum_{\mathsf{r},\mathsf{s}\in\mathbb{Z}_\lambda} p_\lambda(\mathsf{s}-\mathsf{r})\Big(\frac{-\psi_\lambda(\mathsf{r})\psi_\lambda(\mathsf{s})+\psi_\lambda^2(\mathsf{r})}{h_\lambda^2}\Big) + h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \frac{E_\lambda e^{V_\lambda(H_\lambda \mathsf{r})}-1}{h_\lambda^2}\psi_\lambda^2(\mathsf{r}) = 0.$$

So, for the symmetrized kernel $\hat{p}_\lambda(\mathsf{z}) = (p_\lambda(\mathsf{z})+p_\lambda(-\mathsf{z}))/2$, we obtain

$$h_\lambda \sum_{\mathsf{r},\mathsf{s}\in\mathbb{Z}_\lambda} \hat{p}_\lambda(\mathsf{s}-\mathsf{r})\Big(\frac{\psi_\lambda(\mathsf{s})-\psi_\lambda(\mathsf{r})}{h_\lambda}\Big)^2 + h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \frac{E_\lambda \mathrm{e}^{V_\lambda(H_\lambda \mathsf{r})}-1}{h_\lambda^2}\psi_\lambda^2(\mathsf{r}) = 0. \tag{2.30}$$

In view of Lemma 1, we may assume that there exists $\bar{\mathsf{e}} < \infty$ such that, possibly going to a subsequence, the limit $\mathsf{e} = \lim_{\lambda\to 0}\mathsf{e}_\lambda$ exists and satisfies $\mathsf{e} < \bar{\mathsf{e}}$. So, we may assume that $E_\lambda \geq \mathrm{e}^{-\bar{\mathsf{e}}h_\lambda^2}$. Recall also our assumption (1.16) on the growth of V_λ. Let $\bar{\mathsf{r}} = \sup\{\mathsf{r} : q_0(\mathsf{r}) < \bar{\mathsf{e}}\}$. Then, (2.30) implies that

$$h_\lambda \sum_{\mathsf{r},\mathsf{s}\in\mathbb{Z}_\lambda} \hat{p}_\lambda(\mathsf{s}-\mathsf{r})\Big(\frac{\psi_\lambda(\mathsf{s})-\psi_\lambda(\mathsf{r})}{h_\lambda}\Big)^2 + h_\lambda \sum_{\mathsf{r}\geq\bar{\mathsf{r}}} q_0(\mathsf{r})\psi_\lambda^2(\mathsf{r}) \leq \bar{\mathsf{e}}\|\psi_\lambda\|_{2,\lambda}^2. \tag{2.31}$$

By construction, $\|\psi_\lambda\|_{2,\lambda}^2 \leq 1$ and, as we have already mentioned, (2.28) implies that actually $\lim_{\lambda\to 0}\|\psi_\lambda\|_{2,\lambda}^2 = 1$.

Furthermore, since p is an irreducible kernel, there exists $\delta > 0$ and a finite sequence of integer states $\mathsf{x}_0, \mathsf{x}_1, \dots, \mathsf{x}_n$ with $\hat{p}_{\mathsf{x}_i-\mathsf{x}_{i-1}} \geq \delta$, which connects $\mathsf{x}_0 = 0$ to $\mathsf{x}_n = 1$. Therefore,

$$\sum_{\mathsf{r},\mathsf{s}\in\mathbb{Z}_\lambda} \hat{p}_\lambda(s-r)\Big(\frac{\psi_\lambda(\mathsf{s})-\psi_\lambda(\mathsf{r})}{h_\lambda}\Big)^2 \geq \frac{\delta}{n^2}\sum_{\mathsf{r}\in\mathbb{N}_\lambda}\Big(\frac{\psi_\lambda(\mathsf{r})-\psi_\lambda(\mathsf{r}-h_\lambda)}{h_\lambda}\Big)^2, \tag{2.32}$$

where we use the elementary inequality

$$(z_0-z_1)^2+(z_1-z_2)^2+\cdots+(z_{n-1}-z_n)^2 \geq \frac{1}{n}(z_0-z_n)^2,$$

valid for all real z_i. The additional $1/n$ in the prefactor $1/n^2$ in (2.32) is due to the fact that each term $(\psi_\lambda(\mathsf{s})-\psi_\lambda(\mathsf{r}))^2$ is used in this way at most n times. Together with (2.31), this implies the existence of two finite positive constants c_1 and c_2 such that

$$c_1 h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda}\Big(\frac{\psi_\lambda(\mathsf{r})-\psi_\lambda(\mathsf{r}-h_\lambda)}{h_\lambda}\Big)^2 + h_\lambda\sum_{\mathsf{r}\geq\bar{\mathsf{r}}} q_0(\mathsf{r})\psi_\lambda^2(\mathsf{r}) \leq c_2. \tag{2.33}$$

This is the desired energy estimate, which holds for all $\lambda > 0$ small.

The rest of the proof is straightforward. Let Ψ_λ be the linear interpolation of ψ_λ: for $\mathsf{r}\in\mathbb{N}_\lambda\cup\{0\}$ and $t\in[0,1]$,

$$\Psi_\lambda(\mathsf{r}+th_\lambda) = (1-t)\psi_\lambda(\mathsf{r}) + t\psi_\lambda(\mathsf{r}+h_\lambda).$$

The relation (2.33) and $\lim_{\mathsf{r}\to\infty} q_0(\mathsf{r}) = \infty$ imply that $\lim_{n\to\infty}\|\Psi_\lambda \mathbf{1}_{\{r>n\}}\|_2 = 0$, uniformly in λ small. On the other hand, the very same (2.33) and Rellich's compact embedding theorem imply that, for any $n < \infty$, the family $\Psi_\lambda\mathbf{1}_{\{r\leq n\}}$ is subsequentially compact in $\mathbb{L}_2[0,n]$. Alternatively, (2.33) implies that the linear interpolations Ψ_λ are uniformly continuous on $[0, n]$ for each n fixed. We conclude that the family Ψ_λ is subsequentially compact in $\mathbb{L}_2(\mathbb{R}_+)$.

Remember how the map ρ_λ was defined in (2.22). Since Ψ_λ is the linear interpolation of ψ_λ, and since $\lim_{\lambda\to 0} h_\lambda = 0$, the energy estimate (2.33) evidently implies that

$$\lim_{\lambda\to 0} \|\psi_\lambda - \rho_\lambda \Psi_\lambda\|_{2,\lambda} = 0.$$

Hence, ψ_λ is subsequentially compact as well. □

Convergence of semigroups. Possibly going to a subsequence, we can assume that $\lim \mathsf{e}_\lambda = \mathsf{e}$. We shall rely on Kurtz's semigroup convergence theorem [11, Theorem I.6.5]: Define

$$\mathsf{L}_\lambda f(\mathsf{r}) = \frac{\mathsf{T}_\lambda - \mathsf{I}}{h_\lambda^2} f(\mathsf{r}). \tag{2.34}$$

The following two statements are equivalent:

(a) For any $u \in \mathcal{U}$, one can find a sequence $u_\lambda \in \ell_2(\mathbb{N}_\lambda)$ such that both $\lim_{\lambda\to 0} u_\lambda = u$ and $\lim_{\lambda\to 0} \mathsf{L}_\lambda u_\lambda = (\mathsf{L} + \mathsf{e})u$.

(b) If $\lim_{\lambda\to 0} f_\lambda = f$, then $\lim_{\lambda\to 0} \mathsf{T}_\lambda^{\lfloor H_\lambda^2 t\rfloor} f_\lambda = \mathrm{e}^{(\mathsf{L}+\mathsf{eI})t} f$.

The above equivalence holds provided that the operators T_λ are linear contractions (which is straightforward), and that $\mathrm{e}^{(\mathsf{L}+\mathsf{eI})t}$ is a strongly continuous semigroup with generator $\mathsf{L} + \mathsf{eI}$, but that's exactly how it was constructed, see (1.6). Recall that the core $\mathcal{U}$ consists of finite linear combinations of eigenfunctions φ_j. Equivalently, we might have considered $\mathcal{U}' = \mathsf{C}_0^2[0, \infty)$. Indeed, if χ_0 is a smooth function which is 1 on $(-\infty, 0]$ and 0 on $[1, \infty)$ and if $\chi_R(r) = \chi_0(r - R)$, then, for any j,

$$\lim_{R\to\infty} \chi_R\varphi_j = \varphi_j \quad \text{and} \quad \lim_{R\to\infty} \mathsf{L}_{\sigma,q}(\chi_R\varphi_j) = \mathsf{L}_{\sigma,q}\varphi_j = -\zeta_j\varphi_j. \tag{2.35}$$

Above, both convergences are pointwise and in $\mathbb{L}_2(\mathbb{R}_+)$. In order to check the second claim in (2.35), just note that

$$\mathsf{L}_{\sigma,q}(\chi_R\varphi_j) = -\zeta_j\varphi_j + \frac{\sigma^2}{2}(\varphi_j\chi_R'' + 2\varphi_j'\chi_R'),$$

and the conclusion follows, since both φ_j and φ_j' belong to $\mathbb{L}^2$.

Consider, therefore, $u \in \mathsf{C}_0^2[0, \infty)$. Define $u_\lambda(\mathsf{r}) = u(\mathsf{r})$. Clearly, $\lim_{\lambda\to 0} u_\lambda = u$. On the other hand (see Remark 3),

$$\begin{aligned} E_\lambda \mathrm{e}^{V_\lambda(H_\lambda \mathsf{r})}\mathsf{L}_\lambda u_\lambda(\mathsf{r}) &= \frac{1}{h_\lambda^2}\Big(\sum_{\mathsf{s}\in\mathbb{Z}_\lambda} p_\lambda(\mathsf{s}-\mathsf{r})\mathrm{e}^{\frac{V_\lambda(H_\lambda\mathsf{r})-V_\lambda(H_\lambda\mathsf{s})}{2}}u(\mathsf{s}) - E_\lambda\mathrm{e}^{V_\lambda(H_\lambda\mathsf{r})}u_\lambda(\mathsf{r})\Big) \\ &= \frac{1}{h_\lambda^2}\sum_{\mathsf{s}} p_\lambda(\mathsf{s}-\mathsf{r})\big(\mathrm{e}^{\frac{V_\lambda(H_\lambda\mathsf{r})-V_\lambda(H_\lambda\mathsf{s})}{2}}u(\mathsf{s}) - u(\mathsf{r})\big) + \frac{1 - E_\lambda\mathrm{e}^{V_\lambda(H_\lambda\mathsf{r})}}{h_\lambda^2}u(\mathsf{r}). \end{aligned} \tag{2.36}$$

Choose R such that $\mathrm{supp}(u) \in [0, R]$. Possibly going to a sub-sequence assume that $\mathsf{e} = \lim \mathsf{e}_\lambda$ exists. Then, by our assumptions on V_λ, the second term converges to $(\mathsf{e} - q(\mathsf{r}))u(\mathsf{r})$, uniformly in $r \in [0, R]$.

As for the first term in (2.36), note that, since $u_\lambda(\mathsf{s}) \equiv 0$ for $\mathsf{s} > R$ and $p_\lambda(\mathsf{s}-\mathsf{r}) \leq \mathrm{e}^{-cH_\lambda|\mathsf{s}-\mathsf{r}|}$, we may restrict attention to $\mathsf{r}, \mathsf{s} \leq R+1$. But then, again by our assumptions on V_λ, the quantity

$$|(V_\lambda(H_\lambda \mathsf{s}) - V_\lambda(H_\lambda \mathsf{r}))p_\lambda(\mathsf{s}-\mathsf{r})| = h_\lambda^2\,|(q(\mathsf{s}) - q(\mathsf{r}))\,p_\lambda(\mathsf{s}-\mathsf{r})| + \mathrm{o}(h_\lambda^2) = \mathrm{o}(h_\lambda^2).$$

Finally, by our assumptions (1.9) and (1.10) on the underlying random walk,

$$\lim_{\lambda\to 0}\frac{1}{h_\lambda^2}\sum_{\mathsf{s}\in\mathbb{Z}_\lambda} p_\lambda(\mathsf{s}-\cdot)(u(\mathsf{s}) - u(\cdot)) = \frac{\sigma^2}{2}u''(\cdot),$$

in the sense of Definition 1.

We have proved:

Proposition 3. *Under our assumptions on V_λ and p, the following convergence (in the sense of Definition 1), holds uniformly in t on compact subsets of $\mathbb{R}_+$: If $\lim_{k\to\infty}\mathsf{e}_{\lambda_k} = \mathsf{e}$ and $\lim_{k\to\infty} f_{\lambda_k} = f$, then*

$$\lim_{k\to\infty}\mathsf{T}_{\lambda_k}^{\lfloor H_{\lambda_k}^2 t\rfloor} f_{\lambda_k} = \mathrm{e}^{(\mathsf{L}+\mathsf{e}\mathsf{I})t} f. \tag{2.37}$$

Convergence of eigenvalues and eigenfunctions.

Theorem 2. *Under our assumptions on V_λ and p,*

$$\zeta_0 = \lim_{\lambda\to 0}\mathsf{e}_\lambda, \quad \varphi_0 = \lim_{\lambda\to 0}\phi_\lambda = \lim_{\lambda\to 0}\phi_\lambda^* \quad \textit{and} \quad \lim_{\lambda\to 0}\frac{c_\lambda}{h_\lambda} = 1. \tag{2.38}$$

Proof. By Lemma 1, the set $\{\mathsf{e}_\lambda\}$ is bounded and, by Proposition 2, the set $\{\phi_\lambda\}$ is sequentially compact. Let $\lambda_k \searrow 0$ be a sequence such that both $\mathsf{e} = \lim_{k\to\infty}\mathsf{e}_{\lambda_k}$ and $\varphi = \lim_{k\to\infty}\phi_{\lambda_k}$ exist. Then Proposition 3 implies that

$$\varphi = \mathrm{e}^{(\mathsf{L}+\mathsf{e}\mathsf{I})t}\varphi.$$

By compactness, $\|\varphi\|_2 = 1$. In other words, φ is a non-negative normalized $\mathbb{L}_2(\mathbb{R}_+)$-eigenfunction of L with eigenvalue $-\mathsf{e}$. Which means that $\varphi = \varphi_0$ and $\mathsf{e} = \zeta_0$. Exactly the same argument applies to ϕ_λ^*.

By construction (see (2.20)), $1 \equiv c_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda}\phi_\lambda(\mathsf{r})\phi_\lambda^*(\mathsf{r}) = \frac{c_\lambda}{h_\lambda}\langle\phi_\lambda,\phi_\lambda^*\rangle_{2,\lambda}$. Since, by the second assertion of (2.38), $\lim_{\lambda\to 0}\langle\phi_\lambda,\phi_\lambda^*\rangle_{2,\lambda} = \|\varphi_0\|_2^2 = 1$, the last claim of Theorem 2 follows as well. □

Convergence of finite-dimensional distributions. Recall our notations $\mathbb{P}_\lambda$ and $\mathbb{P}_{\sigma,q}$ for the path measures of the ground-state chain X_n and the Ferrari-Spohn diffusion $x(t)$. Recall also our rescaling of the ground-state chain: $x_\lambda(t) = h_\lambda X_{\lfloor H_\lambda^2 t\rfloor}$.

Corollary 1. *For any k, any $0 < s_1 < s_2 < \cdots < s_k$ and for any collection of bounded continuous functions $u_0, \ldots, u_k \in \mathsf{C}_\mathrm{b}(\mathbb{R}_+)$,*

$$\lim_{\lambda\to 0}\mathbb{E}_\lambda\{u_0(x_\lambda(0))u_1(x_\lambda(s_1))\cdots u_k(x_\lambda(s_k))\} = \mathbb{E}_{\sigma,q}\{u_0(x(0))u_1(x(s_1))\cdots u_k(x(s_k))\}. \tag{2.39}$$

Proof. Set $s_0 = 0$ and $t_i = s_i - s_{i-1}$. Since $\mu_\lambda = c_\lambda \phi_\lambda^* \phi_\lambda$ and in view of the expressions (2.6) for transition probabilities π_λ of the ground-state chain, the rightmost asymptotic relation in (2.38), and (1.8) for Ferrari–Spohn semigroups, the target formula (2.39) can be written as

$$\lim_{\lambda\to 0} h_\lambda \sum_{\mathsf{r}\in\mathbb{N}_\lambda} \phi_\lambda^*(\mathsf{r}) u_{\lambda,0}(\mathsf{r}) \mathsf{T}_\lambda^{\lfloor H_\lambda^2 t_1\rfloor}\big(u_{\lambda,1}\mathsf{T}_\lambda^{\lfloor H_\lambda^2 t_2\rfloor}\big(u_{\lambda,2}\cdots \mathsf{T}_\lambda^{\lfloor H_\lambda^2 t_k\rfloor}(u_{\lambda,k}\phi_\lambda)\cdots\big)\big)(\mathsf{r})$$
$$= \int_0^\infty \varphi_0(r) u_0(r) \mathsf{T}^{t_1}\big(u_1 \mathsf{T}^{t_2}\big(u_2\cdots \mathsf{T}^{t_k}(u_k\varphi_0)\cdots\big)\big)(r)\,\mathrm{d}r, \tag{2.40}$$

where $u_{\lambda,i}$ and u_i coincide on $\mathbb{N}_\lambda$. Theorem 2 implies that $\lim_{\lambda\to 0}\phi_\lambda = \varphi_0$ and $\lim_{\lambda\to 0}\phi_\lambda^* = \varphi_0$. Hence, by induction, (2.40) is a consequence of Proposition 3 and the following two elementary facts:

(a) If $\lim v_\lambda = v$ and $u_\lambda(\mathsf{r}) = u(\mathsf{r})$ with u being a bounded continuous function, then $\lim_{\lambda\to 0} v_\lambda u_\lambda = vu$.
(b) If $\lim u_\lambda = u$ and $\lim v_\lambda = v$, then $\lim_{\lambda\to 0}\langle u_\lambda, v_\lambda\rangle_{2,\lambda} = \langle u, v\rangle_2$. $\square$

3. Probabilistic Tools

The derivations of the probabilistic estimates given below are based on the techniques and ideas developed in [14]. Nevertheless, because our setting is slightly different and for completeness, we provide detailed proofs. In addition, one of the needed claims from [14] (Theorem 1.2 therein) contains a mistake, which we correct here.

Recall our notation $\hat{\mathcal{P}}_{N,+}^{\mathsf{u},\mathsf{v}} = \mathcal{P}_{1,N,+}^{\mathsf{u},\mathsf{v}}$. As before, given a path $\mathbb{X} = (X_1,\ldots,X_N)$, set $\mathsf{p}(\mathbb{X}) = \prod_{i=1}^{N-1} p_{X_{i+1}-X_i}$. Define $\hat{\mathcal{P}}_{N,+}^{\mathsf{u},\varnothing} = \cup_{\mathsf{v}\in\mathbb{Z}_+}\hat{\mathcal{P}}_{N,+}^{\mathsf{u},\mathsf{v}}$ and consider the partition functions

$$\hat{Z}_{N,+,\lambda}^{\mathsf{u},\varnothing} = \sum_{\mathbb{X}\in\hat{\mathcal{P}}_{N,+}^{\mathsf{u},\varnothing}} e^{-\sum_{i=1}^N V_\lambda(X_i)}\,\mathsf{p}(\mathbb{X}).$$

More generally, given any subset $\mathcal{C}\subset\hat{\mathcal{P}}_{N,+}^{\mathsf{u},\varnothing}$, we denote by

$$\hat{Z}_{N,+,\lambda}^{\mathsf{u},\varnothing}[\mathcal{C}] = \sum_{\mathbb{X}\in\mathcal{C}} e^{-\sum_{i=1}^N V_\lambda(X_i)}\,\mathsf{p}(\mathbb{X}),$$

the partition function restricted to paths satisfying the constraint $\mathcal{C}$.

3.1. Proof of Lemma 1. The proof will rely on the following identity which, exactly as (2.2), is straightforward from the very definition of $\tilde{\mathsf{T}}_\lambda$ in (2.1):

$$\hat{Z}_{N,+,\lambda}^{\mathsf{u},\varnothing} = \mathrm{e}^{-\frac{1}{2}V_\lambda(\mathsf{u})}\tilde{\mathsf{T}}_\lambda^N[f_\lambda](\mathsf{u}),$$

where $f_\lambda(x) = \mathrm{e}^{-\frac{1}{2}V_\lambda(x)}$. Since f_λ is positive,

$$\log E_\lambda = \lim_{N\to\infty}\frac{1}{N}\log\hat{Z}_{N,+,\lambda}^{\mathsf{u},\varnothing}, \tag{3.1}$$

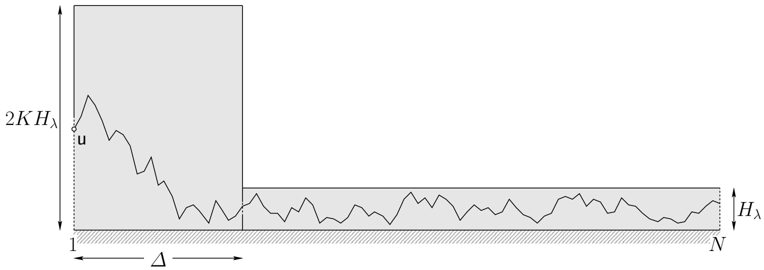

Fig. 2. The construction for the lower bound in the proof of Lemma 1

for all $\mathsf{u} \in \mathbb{Z}^+$. In particular, the claim of Lemma 1 will follow from lower and upper bounds on $\hat{Z}^{\mathsf{u},\varnothing}_{N,+,\lambda}$ for finite values of N and λ. In the sequel, we shall allow rather general values of the boundary condition u. Of course, to derive the claim of Lemma 1, we could as well take $\mathsf{u} = 0$.

We shall compare the tilted partition functions $\hat{Z}^{\mathsf{u},\varnothing}_{N,+,\lambda}$ and $\hat{Z}^{\mathsf{u},\varnothing}_{N,+,0}$. The latter equals to the probability that the random walk starting at u stays positive for first N steps of its life. This probability is evidently non-decreasing with u and, as is well known (see for instance [3]), it is of order N^{-1} for $\mathsf{u} = 1$. In particular,

$$\lim_{N\to\infty} \frac{1}{N} \log \hat{Z}^{\mathsf{u},\varnothing}_{N,+,0} = 0, \tag{3.2}$$

uniformly in $\mathsf{u} \in \mathbb{N}$.

Lower bound on $\hat{Z}^{\mathsf{u},\varnothing}_{N,+,\lambda}$ *and upper bound on* e_λ. We claim that there exist finite constants $\bar{\mathsf{e}}$ and c_1 such that, for any $K \geq 1$ fixed,

$$\hat{Z}^{\mathsf{u},\varnothing}_{N,+,\lambda} \geq e^{-\bar{\mathsf{e}} N H_\lambda^{-2} - c_1 K \sqrt{q(2K)}}\, \hat{Z}^{\mathsf{u},\varnothing}_{N,+,0}, \tag{3.3}$$

uniformly in λ small, $0 \leq \mathsf{u} \leq K H_\lambda$ and

$$N \gg \Delta = \Delta(K,\lambda) = \frac{K H_\lambda^2}{\sqrt{q(2K)}}. \tag{3.4}$$

In view of (3.1) and (3.2), this implies that $\mathsf{e}_\lambda = -H_\lambda^2 \log E_\lambda \leq \bar{\mathsf{e}}$ for all λ sufficiently small.

In order to check (3.3), we restrict the partition function to trajectories made of two pieces (see Fig. 2). The left part is used to bring the interface below H_λ; in the remaining piece, the interface remains inside a tube of height H_λ.

We consider the events[1]

$$\mathcal{D}_L = \Bigl\{ \mathbb{X} \in \hat{\mathcal{P}}^{\mathsf{u},\varnothing}_{N,+} \,:\, \max_{i\in\{1,\dots,\Delta\}} X_i \leq 2K H_\lambda,\ X_\Delta \in [\tfrac{1}{3}H_\lambda, \tfrac{2}{3}H_\lambda] \Bigr\},$$

$$\mathcal{D}_M = \Bigl\{ \mathbb{X} \in \hat{\mathcal{P}}^{\mathsf{u},\varnothing}_{N,+} \,:\, \max_{i\in\{\Delta,\dots,N\}} X_i \leq H_\lambda \Bigr\}.$$

Then

$$\hat{Z}^{\mathsf{u},\varnothing}_{N,+,\lambda} \geq e^{-\Delta V_\lambda(2KH_\lambda) - (N-\Delta)V_\lambda(H_\lambda)}\, \mathsf{p}(\mathcal{D}_L \cap \mathcal{D}_M \mid \hat{\mathcal{P}}^{\mathsf{u},\varnothing}_{N,+})\, \hat{Z}^{\mathsf{u},\varnothing}_{N,+,0}.$$

[1] Here and several times in the sequel, we assume numbers like Δ to be integers whenever it is desirable.

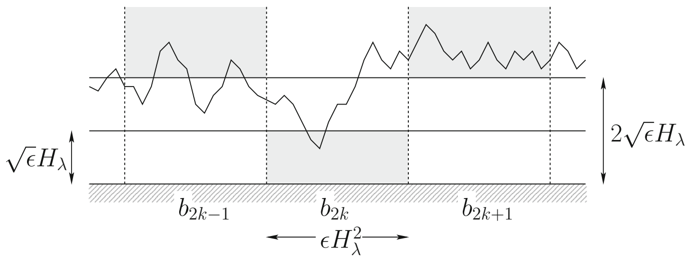

Fig. 3. The event $\mathcal{B}_k$ occurs if the path visits both *leftmost and rightmost shaded areas*. The event $\mathcal{C}_k$ occurs if, in addition, it also visits the third one

By the assumptions (1.14) and (1.15), $H_\lambda^2 V_\lambda(\mathsf{r} H_\lambda) < 2q(\mathsf{r})$ uniformly in $\mathsf{r} \in [0, 2K]$, for all λ sufficiently small. Hence, for such λ, the exponent in the right-hand side is bounded below by $e^{-2K\sqrt{q(2K)}-2NH_\lambda^{-2}}$.

It remains to estimate $\mathsf{p}(\mathcal{D}_L \cap \mathcal{D}_M \mid \mathcal{P}_{N,+}^{\mathsf{u},\varnothing})$. By the invariance principle (for a random walk conditioned to stay positive; see first [4, Theorem 1] and then [8, Theorem 1.1]),

$$\liminf_{N\to\infty} \mathsf{p}(\mathcal{D}_L \mid \mathcal{P}_{N,+}^{\mathsf{u},\varnothing}) \geq e^{-c_2 K\sqrt{q(2K)}},$$

for some absolute constant $c_2 > 0$, provided that λ be small enough.

On the other hand, letting $\mathcal{D}' = \{\sup_{0\leq i\leq H_\lambda^2} X_i \leq H_\lambda\} \cap \{X_{\lceil H_\lambda^2\rceil} \in [\frac{1}{3}H_\lambda, \frac{2}{3}H_\lambda]\}$, it follows from the Markov property that

$$\begin{aligned}
&\inf_{\ell\in[\frac{1}{3}H_\lambda,\frac{2}{3}H_\lambda]} \mathsf{p}(\mathcal{D}_M \mid X_L = \ell, X_i \geq 0\ \forall \Delta \leq i \leq N)\\
&\geq \left\{ \inf_{\ell\in[\frac{1}{3}H_\lambda,\frac{2}{3}H_\lambda]} \mathsf{p}(\mathcal{D}' \mid X_0 = \ell, X_i \geq 0\ \forall \Delta \leq i \leq H_\lambda^2)\right\}^{\lceil N/H_\lambda^2\rceil}\\
&\geq e^{-c_3 N H_\lambda^{-2}}.
\end{aligned}$$

Upper bound on $\hat{Z}_{N,+,\lambda}^{\mathsf{u},\varnothing}$ *and lower bound on* e_λ. We claim that there exist $\bar{\lambda} > 0$ and a positive constant $\underline{\mathsf{e}}$ such that

$$\hat{Z}_{N,+,\lambda}^{\mathsf{u},\varnothing} \leq e^{-\underline{\mathsf{e}}NH_\lambda^{-2}} \hat{Z}_{N,+,0}^{\mathsf{u},\varnothing}, \tag{3.5}$$

uniformly in $\mathsf{u} \geq 0$, $N \geq H_\lambda^2$ and $\lambda < \bar{\lambda}$.

Let us fix some small $\epsilon > 0$ (which does not have to be very small; one can optimize over it at the end of the proof). The idea behind the proof is that a typical trajectory has many disjoint segments of the length at least ϵH_λ^2, which are at a distance at least $\sqrt{\epsilon}H_\lambda$ from the wall.

We partition the interval $\{1, \ldots, N\}$ into N_λ disjoint intervals $b_1, \ldots, b_{N_\lambda}$ of length ϵH_λ^2 and, possibly, one additional shorter rightmost interval.

We say that the event $\mathcal{B}_k$ occurs if (see Fig. 3)

$$\max_{i\in b_{2k-1}} X_i > 2\sqrt{\epsilon}H_\lambda \quad \text{and} \quad \max_{i\in b_{2k+1}} X_i > 2\sqrt{\epsilon}H_\lambda. \tag{3.6}$$

Let us denote by G the number of indices k for which the event $\mathcal{B}_k$ occurs. It follows from the CLT that there exists $\kappa_1 > 0$ such that

$$\inf_{\mathsf{v},\mathsf{w}\geq 0} \mathsf{p}\Big(\max_{1\leq i\leq \epsilon H_\lambda^2} X_i > 2\sqrt{\epsilon} H_\lambda \;\Big|\; \mathcal{P}^{\mathsf{v},\mathsf{w}}_{1,\epsilon H_\lambda^2,+}\Big) > \kappa_1. \tag{3.7}$$

Observe that the events $\{\max_{i\in b_{2j-1}} X_i > 2\sqrt{\epsilon} H_\lambda\}$, $j = 1,\ldots, N_\lambda/2$, are conditionally independent given the trajectories in the intervals b_{2k}. As a result, (3.7) implies that there exists $\kappa_2 > 0$ such that

$$\mathsf{p}(G \leq \tfrac{1}{8}\kappa_1^2 N_\lambda \,|\, \hat{\mathcal{P}}^{\mathsf{u},\varnothing}_{N,+}) \leq e^{-\kappa_2 N_\lambda}, \tag{3.8}$$

uniformly in u.

Similarly, let us say that the event $\mathcal{C}_k$ occurs if $\mathcal{B}_k$ occurs and (see Fig. 3)

$$\min_{i\in b_{2k}} X_i < \sqrt{\epsilon} H_\lambda,$$

and let us denote by G' the number of indices such that $\mathcal{C}_k$ occurs.

The occurrence of $\mathcal{C}_k$ enforces a downward fluctuation at least as large as $\sqrt{\epsilon} H_\lambda$ on a time interval of length at most $3\epsilon H_\lambda^2$. The functional CLT [9, Theorem 2.4] implies that such an event has probability at most κ_3, for some $\kappa_3 < 1$, uniformly in λ small. This implies that there exists $\kappa_4 > 0$, such that

$$\mathsf{p}\Big(G' \geq \frac{1+\kappa_3}{2} g \;\Big|\; \hat{\mathcal{P}}^{\mathsf{u},\varnothing}_{N,+}; G = g\Big) \leq \mathrm{e}^{-\kappa_4 g}, \tag{3.9}$$

uniformly in u and g. Altogether, (3.8) and (3.9) yield

$$\mathsf{p}\Big(G - G' \leq \frac{1-\kappa_3}{8}\kappa_1^2 N_\lambda \;\Big|\; \hat{\mathcal{P}}^{\mathsf{u},\varnothing}_{N,+}\Big) \leq \mathrm{e}^{-\kappa_5 N_\lambda} \leq \mathrm{e}^{-\kappa_6 \epsilon^{-1} H_\lambda^{-2} N}. \tag{3.10}$$

The quantity $G - G'$ provides a lower bound on the number of disjoint intervals b_{2k} of length ϵH_λ^2 such that $\min_{i\in b_{2k}} X_i \geq \sqrt{\epsilon} H_\lambda$. Therefore,

$$\sum_{i=1}^N V_\lambda(X_i) \geq (G-G')\epsilon H_\lambda^2 V(\sqrt{\epsilon} H_\lambda) \geq (G-G')\epsilon q_0(\sqrt{\epsilon}) \geq \kappa_7 q_0(\sqrt{\epsilon}) H_\lambda^{-2} N, \tag{3.11}$$

whenever $G - G' \geq \kappa_7 N_\lambda$. Take $\kappa_7 = \frac{1-\kappa_3}{8}\kappa_1^2$. The conclusion (3.5) follows from (3.10) and (3.11). □

3.2. Proof of Lemma 2. We shall prove Lemma 2 only for ϕ_λ^*. The proof for ϕ_λ is a literal repetition for reversed walks. For the sake of notations, we shall think of T_λ in (2.5) as acting on non-rescaled spaces $\ell_2(\mathbb{N})$, with the norm

$$\langle u, v\rangle_{2,\lambda} = h_\lambda \sum_{\mathsf{r}\in\mathbb{N}} u(\mathsf{r}) v(\mathsf{r}).$$

Compare with (2.18).

Similarly, we shall think of ϕ_λ and ϕ_λ^* as of functions on $\mathbb{N}$. Recall the normalizing constant c_λ which was introduced in (2.20), and recall that $\mu_\lambda(\mathsf{x}) = c_\lambda \phi_\lambda(\mathsf{x})\phi_\lambda^*(\mathsf{x})$ is

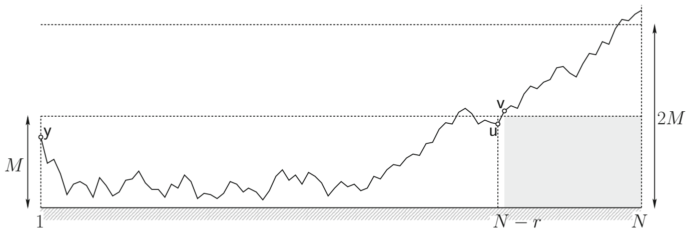

Fig. 4. The last exit decomposition in (3.14). After time $N-r$, the path cannot visit the *shaded area* and has to end up above level $2M$

the invariant measure of the positively recurrent chain on $\mathbb{N}$ with transition probabilities π_λ specified in (2.6). Define $g_M(\mathsf{x}) = \mathbf{1}_{\{x>M\}} = \sum_{\mathsf{x}>M}\mathbf{1}_{\mathsf{x}}$. Then,

$$\lim_{N\to\infty}\mathsf{T}_\lambda^N g_M(\cdot) = \phi_\lambda(\cdot)\lim_{N\to\infty}\pi_\lambda^N\left[\sum_{\mathsf{x}>M}\frac{\mathbf{1}_{\mathsf{x}}}{\phi_\lambda(\mathsf{x})}\right](\cdot) = \phi_\lambda(\cdot)\sum_{\mathsf{x}>M}\frac{\mu_\lambda(\mathsf{x})}{\phi_\lambda(\mathsf{x})}$$
$$= c_\lambda\phi_\lambda(\cdot)\sum_{\mathsf{x}>M}\phi_\lambda^*(\mathsf{x}), \tag{3.12}$$

for any $\lambda > 0$.

For $\mathsf{v} > M$ and $k \geq 0$ let $\mathcal{Q}_{k,+}^{\mathsf{v},M}$ be the family of k-step paths $\mathbb{X} = (\mathsf{x}_0,\dots,\mathsf{x}_k)$ which start at v, $\mathsf{x}_0 = \mathsf{v}$, stay above level M, and end up above level $2M$, $\mathsf{x}_k > 2M$. We employ the notation (see (2.5))

$$\mathsf{T}_\lambda^k\{\mathcal{Q}_{k,+}^{\mathsf{v},M}\} = \sum_{\mathbb{X}\in\mathcal{Q}_{k,+}^{\mathsf{v},M}}\prod_1^k\mathsf{T}_\lambda(\mathsf{x}_{i-1},\mathsf{x}_i). \tag{3.13}$$

By convention, $\mathsf{T}_\lambda^0\{\mathcal{Q}_{0,+}^{\mathsf{v},M}\} = \mathbf{1}_{\{\mathsf{v}>2M\}}$.

Let us fix $0 < \mathsf{y} < M$ and consider paths $\mathbb{X} \in \hat{\mathcal{P}}_{N,+}^{\mathsf{y},\emptyset}$ ending up above level $2M$, $X_N > 2M$. By the last exit decomposition from $\{1,\dots,M\}$ (see Fig. 4),

$$\mathsf{T}_\lambda^N g_{2M}(\mathsf{y}) = \sum_{\mathsf{u}\leq M}\sum_{\mathsf{v}>M}\sum_{r=1}^{N-1}\mathsf{T}_\lambda^{N-r-1}[\mathbf{1}_{\mathsf{u}}](\mathsf{y})\mathsf{T}_\lambda(\mathsf{u},\mathsf{v})\mathsf{T}_\lambda^{r-1}\{\mathcal{Q}_{r-1,+}^{\mathsf{v},M}\}$$
$$= \sum_{\mathsf{u}\leq M}\sum_{\mathsf{v}>M}\sum_{r=1}^{N-1}\mathsf{T}_\lambda^{N-r-1}[\mathbf{1}_{\mathsf{u}}](\mathsf{y})E_\lambda^{-1}p_{\mathsf{v}-\mathsf{u}}\mathsf{e}^{-\frac{V_\lambda(\mathsf{u})+V_\lambda(\mathsf{v})}{2}}\mathsf{T}_\lambda^{r-1}\{\mathcal{Q}_{r-1,+}^{\mathsf{v},M}\}. \tag{3.14}$$

Taking the limit $N\to\infty$, we infer from (3.12), (3.14) and positivity of ϕ_λ and c_λ that

$$\sum_{\mathsf{x}>2M}\phi_\lambda^*(\mathsf{x}) = \sum_{\mathsf{u}\leq M}\phi_\lambda^*(\mathsf{u})\sum_{\mathsf{v}>M}E_\lambda^{-1}p_{\mathsf{v}-\mathsf{u}}\mathsf{e}^{-\frac{V_\lambda(\mathsf{u})+V_\lambda(\mathsf{v})}{2}}\sum_{r\geq 0}\mathsf{T}_\lambda^r\{\mathcal{Q}_{r,+}^{\mathsf{v},M}\}. \tag{3.15}$$

Let us try to derive an upper bound on

$$\max_{\mathsf{u}\leq M}\sum_{\mathsf{v}>M}E_\lambda^{-1}p_{\mathsf{v}-\mathsf{u}}\mathsf{e}^{-\frac{V_\lambda(\mathsf{u})+V_\lambda(\mathsf{v})}{2}}\sum_{r\geq 0}\mathsf{T}_\lambda^r\{\mathcal{Q}_{r,+}^{\mathsf{v},M}\}. \tag{3.16}$$

For simplicity, we shall prove directly the second inequality in (2.25). The arguments rely on the lower bound $H_\lambda^2 V_\lambda(M) \geq q_0(h_\lambda M)$. If instead we keep track of the original quantity $H_\lambda^2 V_\lambda(M)$, then the first inequality in (2.25) will follow.

By Lemma 1 and by our assumptions on p and V_λ,

$$E_\lambda^{-1} p_{\mathsf{v}-\mathsf{u}} \mathrm{e}^{-\frac{V_\lambda(\mathsf{u})+V_\lambda(\mathsf{v})}{2}} \leq \exp\bigl\{-c_1 h_\lambda^2 \bigl(q_0(h_\lambda M) - 1\bigr) - c_2(\mathsf{v} - M)\bigr\}. \tag{3.17}$$

On the other hand, Lemma 1 and crude estimates on the values of the potential V above level M and of the hitting probability of the half-line $\{2M, 2M+1, \ldots\}$ by an r-step random walk which starts at v imply that, for $r > 1$,

$$\begin{aligned} \mathsf{T}_\lambda^r\{\mathcal{Q}_{r,+}^{\mathsf{v},M}\} &\leq E_\lambda^{-r} \mathrm{e}^{-r h_\lambda^2 q_0(h_\lambda M)} \mathsf{p}(X_r > 2M - \mathsf{v}) \\ &\leq \exp\Bigl\{-c_3 h_\lambda^2 r\bigl(q_0(h_\lambda M) - 1\bigr) - c_4 \frac{(2M-\mathsf{v})_+^2}{r} \wedge (2M - \mathsf{v})_+\Bigr\}. \end{aligned} \tag{3.18}$$

Indeed, the second term in the exponent on the right-hand side above follows from the exponential Markov inequality $\mathsf{p}(X_r > a) \leq \mathrm{e}^{-c_4 \frac{a_+^2}{r} \wedge a_+}$.

The right-hand sides of both (3.17) and (3.18) are already independent of u. Let us sum over $\mathsf{v} > M$. If $r = 0$, then v has to satisfy $\mathsf{v} > 2M$. Using (3.17),

$$\sum_{\mathsf{v} > 2M} E_\lambda^{-1} p_{\mathsf{v}-\mathsf{u}} \mathrm{e}^{-\frac{V_\lambda(\mathsf{u})+V_\lambda(\mathsf{v})}{2}} \leq \exp\bigl\{-c_5\bigl(h_\lambda^2\bigl(q_0(h_\lambda M) - 1\bigr) + M\bigr)\bigr\}. \tag{3.19}$$

For $r > 0$, we take advantage of both upper bounds (3.17) and (3.18) above:

$$\sum_{\mathsf{v} > M} \mathrm{e}^{-c_2(\mathsf{v}-M) - c_4 \frac{(2M-\mathsf{v})_+^2}{r} \wedge (2M-\mathsf{v})_+} \leq \mathrm{e}^{-c_6 \frac{M^2}{r} \wedge M}. \tag{3.20}$$

Putting things together for $M = H_\lambda K$, we conclude that the expression in (3.16) is bounded above by

$$\mathrm{e}^{-c_7 H_\lambda K} + \sum_{r \geq 1} \exp\Bigl\{-c_8\Bigl(h_\lambda^2 r q_0(K) + \frac{(H_\lambda K)^2}{r} \wedge H_\lambda K\Bigr)\Bigr\} \leq c_{10} \mathrm{e}^{-c_9 K(\sqrt{q_0(K)} \wedge H_\lambda)}, \tag{3.21}$$

uniformly in $K > 0$ and λ sufficiently small.

Coming back to (3.15), we infer: For any $K > 0$ fixed,

$$\sum_{\mathsf{x} > H_\lambda K} \phi_\lambda^*(\mathsf{x}) \leq c_{11} \mathrm{e}^{-c_{12} K \sqrt{q_0(K)}} \sum_{\mathsf{x}} \phi_\lambda^*(\mathsf{x}), \tag{3.22}$$

for all $\lambda < \lambda_0(K)$. Notice that $\sum_{\mathsf{x}} \phi_\lambda^*(\mathsf{x}) < \infty$ by compactness of T_λ.

Let us return to our basic rescaling (2.19) of ϕ_λ and ϕ_λ^* as unit norm elements of $\ell_2(\mathbb{N}_\lambda)$. The bound (3.22) can be rewritten as

$$h_\lambda \sum_{\mathsf{r} > K} \phi_\lambda^*(\mathsf{r}) \leq c_{11} \mathrm{e}^{-c_{12} K \sqrt{q_0(K)}} \|\phi_\lambda^*\|_{1,\lambda}. \tag{3.23}$$

Since $h_\lambda \sum_{\mathsf{r} \leq K} \phi_\lambda^*(\mathsf{r}) \leq \sqrt{K} \|\phi_\lambda^*\|_{2,\lambda} = \sqrt{K}$, we conclude that $\{\|\phi_\lambda^*\|_{1,\lambda}\}$ is a bounded sequence. The bound (2.26) and, in view of (3.23), also (2.25) follow. □

3.3. Tightness of $(x_\lambda, \mathbb{P}_\lambda)$. Fix any $T < \infty$ and consider the family of rescaled processes x_λ defined in (1.17). Precisely, x_λ is the linear interpolation of the rescaled stationary ergodic ground-state chain $X = X^\lambda$ with transition probabilities π_λ defined in (2.6) and invariant distribution μ_λ. With a slight abuse of notation, we shall continue to use $\mathbb{P}_\lambda$ for the induced distribution of $x_\lambda(\cdot)$ on $\mathsf{C}[-T, T]$.

Proposition 4. *The family* $(x_\lambda, \mathbb{P}_\lambda)$ *is tight on* $\mathsf{C}[-T, T]$.

Proof of Proposition 4. Recall that the invariant measure μ_λ at $\lambda > 0$ is given by $\mu_\lambda = c_\lambda \phi_\lambda \phi_\lambda^*$. Thus, by Theorem 2, the sequence $\{x_\lambda(0)\}$ is tight. It remains to show that, for each $\epsilon, \nu > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}_\lambda\Big(\max_{0\le t\le\delta} |x_\lambda(t) - x_\lambda(0)| > \epsilon\Big) \le \nu\delta. \tag{3.24}$$

For any event $\mathcal{A} \in \sigma(X_n \,:\, 0 \le n \le \delta H_\lambda^2)$, and for any $\mathsf{x}, \mathsf{y} \in \mathbb{N}$, let us define $\mathcal{A}_{\mathsf{x},\mathsf{y}} = \Big\{\mathbb{X} \subset \mathcal{A} \,:\, X_0 = \mathsf{x},\, X_{\delta H_\lambda^2} = \mathsf{y}\Big\}$. As in (3.13), we employ the notation

$$\mathsf{T}_\lambda^{\delta H_\lambda^2}\{\mathcal{A}_{\mathsf{x},\mathsf{y}}\} = \sum_{\mathbb{X}\in\mathcal{A}_{\mathsf{x},\mathsf{y}}} \prod_1^{\delta H_\lambda^2} \mathsf{T}_\lambda(\mathsf{x}_{i-1}, \mathsf{x}_i). \tag{3.25}$$

for the corresponding restricted partition function. In this way,

$$\mathbb{P}_\lambda(\mathcal{A}) = c_\lambda \sum_{\mathsf{x},\mathsf{y}\in\mathbb{N}} \phi_\lambda^*(\mathsf{x}) \mathsf{T}_\lambda^{\delta H_\lambda^2}\{\mathcal{A}_{\mathsf{x},\mathsf{y}}\}\phi_\lambda(\mathsf{y}). \tag{3.26}$$

Since $V_\lambda \ge 0$,

$$\mathsf{T}_\lambda^{\delta H_\lambda^2}\{\mathcal{A}_{\mathsf{x},\mathsf{y}}\} \le \mathrm{e}^{-\delta H_\lambda^2 \log E_\lambda} \mathsf{p}(\mathcal{A}_{\mathsf{x},\mathsf{y}}).$$

By Lemma 1, $\{-\mathsf{e}_\lambda = H_\lambda^2 \log E_\lambda\}$ is a bounded sequence. In the case of

$$\mathcal{A} = \Big\{\max_{0\le n\le\delta H_\lambda^2} |X_n - X_0| > \epsilon H_\lambda\Big\},$$

the upper bound on the probabilities

$$\mathsf{p}(\mathcal{A}_{\mathsf{x},\mathsf{y}}) \le \kappa_1 \frac{h_\lambda}{\sqrt{\delta}} \mathrm{e}^{-\kappa_2 \frac{\epsilon^2}{\delta} \wedge (\epsilon H_\lambda)} \tag{3.27}$$

holds uniformly in $\delta, \epsilon > 0$, $\mathsf{x}, \mathsf{y} \in \mathbb{N}$ and λ small. By Theorem 2, c_λ / h_λ is bounded. Putting things together, we infer that

$$\begin{aligned}\frac{1}{\delta}\mathbb{P}_\lambda\Big(\max_{0\le t\le\delta} |x_\lambda(t) - x_\lambda(0)| > \epsilon\Big) &\le \kappa_3 \mathrm{e}^{-\kappa_2 \frac{\epsilon^2}{\delta}\wedge(\epsilon H_\lambda) - \frac{3}{2}\log\delta} h_\lambda^2 \sum_{\mathsf{x},\mathsf{y}\in\mathbb{N}} \phi_\lambda^*(\mathsf{x})\phi_\lambda(\mathsf{y}) \\ &= \kappa_3 \mathrm{e}^{-\kappa_2 \frac{\epsilon^2}{\delta}\wedge(\epsilon H_\lambda) - \frac{3}{2}\log\delta} \|\phi_\lambda^*\|_{1,\lambda} \|\phi_\lambda\|_{1,\lambda},\end{aligned} \tag{3.28}$$

uniformly in $\delta, \epsilon > 0$ and λ small. Since $\|\phi_\lambda\|_{1,\lambda}$ and $\|\phi_\lambda^*\|_{1,\lambda}$ are bounded and since H_λ is bounded away from zero, we are home. □

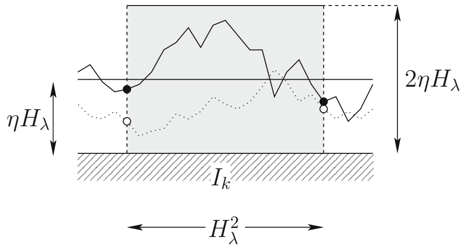

Fig. 5. The interval I_k is η-good if both paths X^1 and X^2 stay inside the *shaded area* and take values smaller than ηH_λ at the boundaries of the interval

3.4. Asymptotic ground-state structure of $\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}$. Let us fix $C > 0$ and $T > 1$. For $\lambda > 0$, $\mathsf{u},\mathsf{v} \le CH_\lambda$ and $N > 2TH_\lambda^2$, we are going to compare the restriction $\mathbb{P}^{\mathsf{u},\mathsf{v},T}_{N,+,\lambda}$ of $\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}$ to the σ-algebra

$$\mathcal{F}_{\lambda,T} = \sigma(X_i \,:\, -TH_\lambda^2 \le i \le TH_\lambda^2)$$

with the restriction $\mathbb{P}^T_\lambda$ of $\mathbb{P}_\lambda$ to $\mathcal{F}_{\lambda,T}$.

Proposition 5. *There exists* $c_1 > 0$ *and* $K = K(C,T) < \infty$ *such that*

$$\|\mathbb{P}^{\mathsf{u},\mathsf{v},T}_{N,+,\lambda} - \mathbb{P}^T_\lambda\|_{\mathrm{Var}} \le 2\mathrm{e}^{-c_1 N H_\lambda^{-2}}, \tag{3.29}$$

uniformly in λ *small,* $N > (T+K)H_\lambda^2$ *and* $\mathsf{u},\mathsf{v} \le CH_\lambda$. *Above,* $\|\cdot\|_{\mathrm{Var}}$ *is the total variational norm.*

As an immediate consequence, we deduce the following

Corollary 2. *Let* λ_N *be a sequence satisfying the assumptions of Theorem A. Let* $C, T < \infty$ *be fixed and assume that the sequences* $\mathsf{u}_N, \mathsf{v}_N \in \mathbb{N}$ *satisfy* $\mathsf{u}_N, \mathsf{v}_N \le CH_{\lambda_N}$. *Consider the sequence of processes* $x_N(\cdot) = x_{\lambda_N}(\cdot)$ *defined via linear interpolation from (1.17). With a slight abuse of notation, let* $\mathbb{P}^{\mathsf{u},\mathsf{v},T}_{N,+,\lambda}$ *and* $\mathbb{P}^T_{\lambda_N}$ *denote the induced distributions on* $\mathsf{C}[-T,T]$ *of* $\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda_N}$ *and, respectively, of the direct ground-state chain measure* $\mathbb{P}_{\lambda_N}$. *Then,*

$$\lim_{N\to\infty} \|\mathbb{P}^{\mathsf{u},\mathsf{v},T}_{N,+,\lambda_N} - \mathbb{P}^T_{\lambda_N}\|_{\mathrm{Var}} = 0. \tag{3.30}$$

Proof (Proof of Proposition 5). We shall use a coupling argument, considering two independent copies X^1 and X^2 of the process, with possibly different boundary conditions.

The first step is to show that we can typically find many pieces of tubes of length H_λ^2 and height of order H_λ inside which both paths are confined.

Let us first split the interval of length $2N+1$ into

$$m = \lfloor (2N+1)/H_\lambda^2 \rfloor$$

consecutive disjoint intervals $I_1, I_2, \ldots, I_m$, of length H_λ^2 (plus, possibly, a final interval of shorter length). We say that the interval I_k is η-*good* if (see Fig. 5)

$$\max_{i\in I_k} X_i^1 < 2\eta H_\lambda, \quad \max_{i\in I_k} X_i^2 < 2\eta H_\lambda$$

and the values of X_i; $i = 1, 2$, at the end-points of I_k are less than ηH_λ.

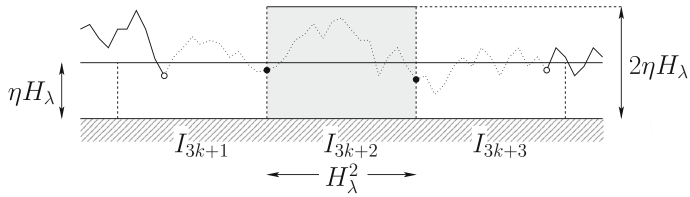

Fig. 6. When $\min_{i\in I_{3k+1}} X_i^1 < \eta H_\lambda$ and $\min_{i\in I_{3k+3}} X_i^1 < \eta H_\lambda$, there is a uniformly (in λ) positive probability that $\max_{i\in I_{3k+2}} X_i^1 < 2\eta H_\lambda$ while taking values smaller than ηH_λ at the boundary of I_{3k+2} (*black dots*). The *white vertices* correspond to the position of the path at times ℓ_k^1 and r_k^1

Lemma 3. *Given the realizations of the two paths X^1 and X^2, let us denote by M the number of η-good intervals of the form I_{3k+2}, $0 \le k < m/3$. Then, there exist η, $c_2 > 0$, $\rho > 0$ and $K_0 < \infty$ such that*

$$\mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(M < \rho \tfrac{m}{3}) \le e^{-c_2 N H_\lambda^{-2}},$$

uniformly in λ small, $0 \le \mathsf{u}, \mathsf{v} \le C H_\lambda$ and $N \ge K_0 H_\lambda^2$.

Proof. We first show that it is very unlikely for X^1 or X^2 to stay far away from the wall for a long time. Indeed, let us write B for the number of intervals I_k such that $\min_{i\in I_k} X_i > \eta H_\lambda$. Then, for any $\epsilon > 0$, there exists $\eta(C,\epsilon)$ such that for all $\eta > \eta(C,\epsilon)$,

$$\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(B > \epsilon m) \le e^{-c_3 N H_\lambda^{-2}}, \tag{3.31}$$

for some constant $c_3 > 0$, uniformly in $0 \le \mathsf{u}, \mathsf{v} \le C H_\lambda$. Indeed, on the event $B > \epsilon m$,

$$\sum_{i=-N}^{N} V_\lambda(X_i) \ge \epsilon m H_\lambda^2 V_\lambda(\eta H_\lambda) \ge \epsilon q_0(\eta)(2N H_\lambda^{-2} - 1),$$

which provides an upper bound on $Z^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}[B > \epsilon m]$.

Remark 4. A similar argument applies for the stationary measure $\mathbb{P}_\lambda$. This means that we may derive our target (3.29) for $\mathbb{P}_\lambda(\cdot \mid X_{-N}, X_N \le \eta H_\lambda)$ instead of deriving it for $\mathbb{P}_\lambda$ itself.

The claim (3.31) then follows by using the lower bound (3.3) on the partition function (and taking η large enough).

Let us say that the triple $(I_{3k+1}, I_{3k+2}, I_{3k+3})$ is *potentially η-good* if (see Fig. 6)

$$\min_{i\in I_{3k+1}} X_i^1 < \eta H_\lambda,\quad \min_{i\in I_{3k+1}} X_i^2 < \eta H_\lambda,\quad \min_{i\in I_{3k+3}} X_i^1 < \eta H_\lambda,\quad \min_{i\in I_{3k+3}} X_i^2 < \eta H_\lambda.$$

Let us denote by $\tilde{M}$ the number of potentially η-good triples. We deduce from (3.31) that, for any $\epsilon > 0$, we can find η such that

$$\mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\tilde{M} \le (1-\epsilon)\tfrac{m}{3}) \le e^{-c_4 N H_\lambda^{-2}},$$

for some $c_4 > 0$.

Now, given a potentially η-good triple $(I_{3k+1}, I_{3k+2}, I_{3k+3})$, let

$$\ell_k^1 = \min\left\{i \in I_{3k+1} \,:\, X_i^1 < \eta H_\lambda\right\},\ r_k^1 = \max\left\{i \in I_{3k+3} \,:\, X_i^1 < \eta H_\lambda\right\}.$$

Conditionally on $X^1_{\ell_k^1}$ and $X^1_{r_k^1}$, the probability that $X_i^1 \le 2\eta H_\lambda$ for all $\ell_k^1 < i < r_k^1$ and that both walks sit below ηH_λ at the end-points of I_{3k+2} is bounded away from zero, uniformly in λ. Indeed, uniformly in $\mathsf{x}, \mathsf{y} < \eta H_\lambda$, $n \le 3H_\lambda^2$ and $1 \le k < m \le n$,

$$\begin{aligned}&\mathbb{P}^{\mathsf{x},\mathsf{y}}_{n,+,\lambda}\big(\max_i X_i^1 \le 2\eta H_\lambda;\, X_k^1 < \eta H_\lambda;\, X_m^1 < \eta H_\lambda\big)\\ &\quad\ge e^{-6q(2\eta)}\, \mathbb{P}^{\mathsf{x},\mathsf{y}}_{n,+,0}\big(\max_i X_i^1 \le 2\eta H_\lambda;\, X_k^1 < \eta H_\lambda;\, X_m^1 < \eta H_\lambda\big),\end{aligned}$$

since $nV_\lambda(2\eta H_\lambda) \le 6q(2\eta)$ (and $\hat{Z}^{\mathsf{x},\mathsf{y}}_{n,+,\lambda} \le 1$). That the latter probability is bounded below is a consequence of the invariance principle.

The claim of the lemma now follows easily, since, conditionally on the pieces of paths between r_{k-1}^1 and ℓ_k^1, these events are independent (and since the same argument can be made independently for X^2). □

Now that we know that we can find $O(NH_\lambda^{-2})$ η-good intervals, the main observation is that, inside each such interval, there is a uniformly positive probability that the two paths meet. Let us make this more precise:

Definition. For $\frac{n}{3} \le m \le n$, let $\mathcal{R}^{\mathsf{x},\mathsf{y}}_{n,m}[\eta]$ be the set of paths $\mathbb{X} = (X_1, \dots, X_m)$ satisfying $X_1 = \mathsf{x}$, $X_m = \mathsf{y}$ and $0 < X_i < 2\eta\sqrt{n}$, $i = 1, \dots, n$. We shall employ the short-hand notation $\mathcal{R}^{\mathsf{x},\mathsf{y}}_{n}[\eta] = \mathcal{R}^{\mathsf{x},\mathsf{y}}_{n,n}[\eta]$; the argument η will often be dropped when no ambiguity arises. We also set $\mathcal{R}^{\mathsf{x},\mathsf{y},+}_{n,m} = \mathcal{R}^{\mathsf{x},\mathsf{y}}_{n,m}[\infty]$.

Proposition 6. *There exists $\eta_0 > 0$ such that the following happens: For every $\eta \ge \eta_0$, one can find $n_0 = n_0(\eta) \in \mathbb{N}$ and $p = p(\eta) > 0$ such that*

$$\mathbb{P}^{\mathsf{x},\mathsf{y}}_{n,+,0} \otimes \mathbb{P}^{\mathsf{z},\mathsf{w}}_{n,+,0}(\exists i \,:\, X_i^1 = X_i^2 \,|\, \mathcal{R}^{\mathsf{x},\mathsf{y}}_n \times \mathcal{R}^{\mathsf{z},\mathsf{w}}_n) \ge p, \tag{3.32}$$

uniformly in $n \ge n_0$ and $\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{w} \in (0, \eta\sqrt{n}] \cap \mathbb{N}$.

Proposition 6 is a statement about random walks with transition probabilities p. We relegate the proof to the Appendix and proceed with the proof of Proposition 5.

First of all pick $n = H_\lambda^2$ and note that, in view of Assumption (1.15), the following happens: For any $\mathsf{x}, \mathsf{y} \le \eta H_\lambda$ and any path $\mathbb{X} \in \mathcal{R}^{\mathsf{x},\mathsf{y}}_n[\eta]$, the value of the potential satisfies

$$0 \le \sum_1^n V_\lambda(X_i) \le nV_\lambda(2\eta\sqrt{n}) = H_\lambda^2 V_\lambda(2\eta H_\lambda) \le 2q(2\eta), \tag{3.33}$$

for all λ sufficiently small. In fact, (3.33) was precisely the reason to introduce the notion of η-good intervals. An immediate consequence of (3.32) and (3.33) is that

$$\mathbb{P}^{\mathsf{x},\mathsf{y}}_{H_\lambda^2,+,\lambda} \otimes \mathbb{P}^{\mathsf{z},\mathsf{w}}_{H_\lambda^2,+,\lambda}(\exists i \,:\, X_i^1 = X_i^2 \,|\, \mathcal{R}^{\mathsf{x},\mathsf{y}}_{H_\lambda^2} \times \mathcal{R}^{\mathsf{z},\mathsf{w}}_{H_\lambda^2}) \ge p\mathrm{e}^{-4q(2\eta)}, \tag{3.34}$$

for all λ sufficiently small. The formula (3.34) provides a uniform lower bound on probability of coupling inside an η-good interval.

Consider the product measures $\mathbb{P}^{0,0}_{N,+,\lambda_N} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}$. Let $\mathcal{M}$ be the event that the paths X^1 and X^2 meet both on the left and on the right of the segment $[-T, T]$. It follows from Lemma 3 and (3.34) that there exist $K = K(C, T)$ and $c_5 > 0$ such that

$$\mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\mathcal{M}) \geq 1 - e^{-c_5 N H_\lambda^{-2}}, \tag{3.35}$$

uniformly in λ small, $\mathsf{u}, \mathsf{v} \leq C H_\lambda$ and $N > (K + T) H_\lambda$.

For $\ell < -T H_\lambda^2, r > T H_\lambda^2$ and $\mathsf{x}, \mathsf{y} \in \mathbb{N}$, let $\mathcal{M}^{\mathsf{x},\mathsf{y}}_{\ell,r} \subset \mathcal{M}$ be the event that ℓ is the leftmost meeting point of X^1, X^2, and $X^1_\ell = X^2_\ell = \mathsf{x}$, whereas r is the rightmost meeting point of X^1, X^2, and $X^1_r = X^2_r = \mathsf{y}$. In this notation, $\mathcal{M}$ is the disjoint union, $\mathcal{M} = \cup \mathcal{M}^{\mathsf{x},\mathsf{y}}_{\ell,r}$.

Let $\mathcal{A} \in \mathcal{F}_{\lambda,T}$. Then,

$$\begin{aligned}
\mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\mathcal{A}) &= \mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\Omega \times \mathcal{A}) \\
&= \mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\Omega \times \mathcal{A}; \mathcal{M}^c) + \sum_{\substack{\ell,r \\ \mathsf{x},\mathsf{y}}} \mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\Omega \times \mathcal{A}; \mathcal{M}^{\mathsf{x},\mathsf{y}}_{\ell,r}).
\end{aligned}$$

However,

$$\mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\Omega \times \mathcal{A}; \mathcal{M}^{\mathsf{x},\mathsf{y}}_{\ell,r}) = \mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\mathcal{A} \times \Omega; \mathcal{M}^{\mathsf{x},\mathsf{y}}_{\ell,r}).$$

Therefore,

$$\left| \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\mathcal{A}) - \mathbb{P}^{0,0}_{N,+,\lambda}(\mathcal{A}) \right| \leq \mathbb{P}^{0,0}_{N,+,\lambda} \otimes \mathbb{P}^{\mathsf{u},\mathsf{v}}_{N,+,\lambda}(\mathcal{M}^c),$$

which, in view of Remark 4 and (3.35), implies (3.29). □

A. Proof of Proposition 6

We shall employ here our original notation p for the path measure of the underlying random walk. Our argument is based on the second moment method, which is put to work using the following input from [6,9]:

Bounds on probabilities of random walks to stay positive. There exists η_0, such that for any $\eta \geq \eta_0$ the following happens: One can find $n_0 = n_0(\eta)$, $c_1 = c_1(\eta)$ and $c_2 = c_2(\eta)$ such that

$$c_1 \frac{\mathsf{xy}}{n^{3/2}} \leq \mathsf{p}(\mathcal{R}^{\mathsf{x},\mathsf{y}}_{n,m}[\eta]) \leq c_2 \frac{\mathsf{xy}}{n^{3/2}}, \tag{A.1}$$

uniformly $n \geq n_0$, $\frac{n}{3} \leq m \leq n$ and $1 \leq \mathsf{x}, \mathsf{y} \leq \eta\sqrt{n}$. Indeed, in view of the invariance principle for random walk bridges [9, Theorem 2.4], the restriction $X_i \leq 2\eta\sqrt{n}$ may be removed from $\mathcal{R}^{\mathsf{x},\mathsf{y}}_{n,m}[\eta]$ in the following sense: There exists $c = c(\eta) \in [1, \infty)$, such that

$$1 \leq \frac{\mathsf{p}(\mathcal{R}^{\mathsf{x},\mathsf{y},+}_{n,m})}{\mathsf{p}(\mathcal{R}^{\mathsf{x},\mathsf{y}}_{n,m}[\eta])} \leq c(\eta), \tag{A.2}$$

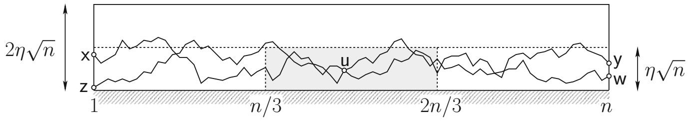

Fig. 7. The decomposition of $\mathsf{p}\otimes\mathsf{p}(\mathcal{N}\,|\,\mathcal{R}_n^{\mathsf{x},\mathsf{y}}\times\mathcal{R}_n^{\mathsf{z},\mathsf{w}})$

uniformly in $n\geq n_0$, $\frac{n}{3}\leq m\leq n$ and $\mathsf{x},\mathsf{y}\leq\eta\sqrt{n}$. Hence, the two-sided inequality (A.1) can be verified along the lines of the proof of Theorem 4.3 in [6], where a stronger asymptotic statement was derived for a more restricted range of parameters.

Let $1\leq\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w}\leq\eta\sqrt{n}$ and consider now the product measure,

$$\mathsf{p}\otimes\mathsf{p}\left(\cdot\,|\,\mathcal{R}_n^{\mathsf{x},\mathsf{y}}[\eta]\times\mathcal{R}_n^{\mathsf{z},\mathsf{w}}[\eta]\right).$$

Let $\mathcal{N}$ be the number of intersections of the two replicas X^1 and X^2 during the time interval $[\frac{n}{3},\frac{2n}{3}]$ below level $\eta\sqrt{n}$. Precisely,

$$\mathcal{N}=\#\left\{\ell\in[\frac{n}{3},\frac{2n}{3}]\,:\,X_\ell^1=X_\ell^2\leq\eta\sqrt{n}\right\}=\sum_{\ell=\frac{n}{3}}^{\frac{2n}{3}}\sum_{\mathsf{u}\leq\eta\sqrt{n}}\mathbf{1}_{\{X_\ell^1=X_\ell^2=\mathsf{u}\}}. \tag{A.3}$$

Lower bound on the expectation $\mathsf{p}\otimes\mathsf{p}(\mathcal{N}\,|\,\mathcal{R}_n^{\mathsf{x},\mathsf{y}}\times\mathcal{R}_n^{\mathsf{z},\mathsf{w}})$. The expectation

$$\mathsf{p}\otimes\mathsf{p}(\mathcal{N}\,|\,\mathcal{R}_n^{\mathsf{x},\mathsf{y}}\times\mathcal{R}_n^{\mathsf{z},\mathsf{w}})=\sum_{\ell=\frac{n}{3}}^{\frac{2n}{3}}\sum_{\mathsf{u}\leq\eta\sqrt{n}}\Phi_n(\ell,\mathsf{u};\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w}),$$

where (see Fig. 7)

$$\Phi_n(\ell,\mathsf{u};\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w})=\frac{\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{x},\mathsf{u}})\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{z},\mathsf{u}})\mathsf{p}(\mathcal{R}_{n,n-\ell}^{\mathsf{u},\mathsf{y}})\mathsf{p}(\mathcal{R}_{n,n-\ell}^{\mathsf{u},\mathsf{w}})}{\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{x},\mathsf{y}})\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{z},\mathsf{w}})}.$$

By (A.1),

$$\Phi_n(\ell,\mathsf{u};\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w})\geq c_3\frac{\mathsf{u}^4}{n^3},$$

uniformly in all the arguments in question. Consequently,

$$\mathsf{p}\otimes\mathsf{p}(\mathcal{N}\,|\,\mathcal{R}_n^{\mathsf{x},\mathsf{y}}\times\mathcal{R}_n^{\mathsf{z},\mathsf{w}})\geq c_4(\eta)\sqrt{n}, \tag{A.4}$$

also uniformly in $\mathsf{x},\ldots,\mathsf{w}\leq\eta\sqrt{n}$.

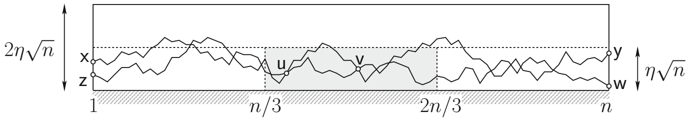

Fig. 8. The decomposition of $\mathsf{p}\otimes\mathsf{p}(\mathcal{N}^2 \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}})$

Upper bound on the expectation $\mathsf{p}\otimes\mathsf{p}(\mathcal{N}^2 \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}})$. The expectation

$$\mathsf{p}\otimes\mathsf{p}(\mathcal{N}^2 \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}}) \leq \sum_{\substack{\ell,m=\frac{n}{3} \\ \ell\leq m}}^{\frac{2n}{3}} \sum_{\mathsf{u},\mathsf{v}\leq\eta\sqrt{n}} \Psi_n(\ell,\mathsf{u};m,\mathsf{v};\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w}),$$

where (see Fig. 8)

$$\Psi_n(\ell,\mathsf{u};m,\mathsf{v};\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w}) = \frac{\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{x},\mathsf{u}})\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{z},\mathsf{u}})\mathsf{p}_{m-\ell}(\mathsf{u},\mathsf{v})^2\mathsf{p}(\mathcal{R}_{n,n-m}^{\mathsf{v},\mathsf{y}})\mathsf{p}(\mathcal{R}_{n,n-m}^{\mathsf{v},\mathsf{w}})}{\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{x},\mathsf{y}})\mathsf{p}(\mathcal{R}_{n,\ell}^{\mathsf{z},\mathsf{w}})}. \tag{A.5}$$

Above, $\mathsf{p}_r(\mathsf{u},\mathsf{v})$ is a short-hand notation for $\mathsf{p}(X_r = \mathsf{v} \mid X_0 = \mathsf{u})$. The inequality in (A.5) is due to the fact that we ignore the positivity condition on the interval $\ell,\ldots,m$.

By (A.1),

$$\Psi_n(\ell,\mathsf{u};m,\mathsf{v};\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w}) \leq c_6(\eta)\frac{\mathsf{p}_{m-\ell}(\mathsf{u},\mathsf{v})^2}{n},$$

uniformly in all the arguments in question. Consequently,

$$\sum_{\substack{\ell,m=\frac{n}{3} \\ \ell\leq m}}^{\frac{2n}{3}} \sum_{\mathsf{u},\mathsf{v}\leq\eta\sqrt{n}} \Psi_n(\ell,\mathsf{u};m,\mathsf{v};\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w}) \leq c_7\sqrt{n}\sum_{r=0}^{\frac{n}{3}}\sum_{\mathsf{v}} \mathsf{p}_r(0,\mathsf{v})^2.$$

The double sum on the right-hand side above is equal to the expectation of the number of intersections of two independent p-walks during the first $\frac{n}{3}$ steps of their life. It is bounded above by $c_8\sqrt{n}$. We conclude that

$$\mathsf{p}\otimes\mathsf{p}(\mathcal{N}^2 \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}}) \leq c_9 n. \tag{A.6}$$

The lower and upper bounds (A.4) and (A.6) imply the existence of $\nu = \nu(\eta) > 0$, such that

$$\mathsf{p}\otimes\mathsf{p}(\mathcal{N}^2 \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}}) \leq \nu[\mathsf{p}\otimes\mathsf{p}(\mathcal{N} \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}})]^2, \tag{A.7}$$

uniformly in $\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{w} \leq \eta\sqrt{n}$. Set $\mathcal{E} = \mathsf{p}\otimes\mathsf{p}(\mathcal{N} \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}})$. By the Paley–Zygmund inequality,

$$\mathsf{p}\otimes\mathsf{p}(\mathcal{N} > \alpha\mathcal{E} \mid \mathcal{R}_n^{\mathsf{x},\mathsf{y}} \times \mathcal{R}_n^{\mathsf{z},\mathsf{w}}) \geq \frac{(1-\alpha)^2}{\nu}, \tag{A.8}$$

for every $\alpha \leq 1$. (3.32) follows.

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