Locally homogeneous aspherical Sasaki manifolds

Oliver Baues [a,*], Yoshinobu Kamishima [b]

[a] *Department of Mathematics, University of Fribourg, Chemin du Musée 23, CH-1700 Fribourg, Switzerland*
[b] *Department of Mathematics, Josai University, Keyaki-dai 1-1, Sakado, Saitama 350-0295, Japan*

A B S T R A C T

Let G/H be a contractible homogeneous Sasaki manifold. A compact locally homogeneous aspherical Sasaki manifold $\Gamma\backslash G/H$ is by definition a quotient of G/H by a discrete uniform subgroup $\Gamma \leq G$. We show that a compact locally homogeneous aspherical Sasaki manifold is always quasi-regular, that is, $\Gamma\backslash G/H$ is an S^1-Seifert bundle over a locally homogeneous aspherical Kähler orbifold. We discuss the structure of the isometry group $\mathrm{Isom}\,(G/H)$ for a Sasaki metric of G/H in relation with the pseudo-Hermitian group $\mathrm{Psh}\,(G/H)$ for the Sasaki structure of G/H. We show that a Sasaki Lie group G, when $\Gamma\backslash G$ is a compact locally homogeneous aspherical Sasaki manifold, is either the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ or a modification of a Heisenberg nilpotent Lie group with its natural Sasaki structure. We also show that any compact regular aspherical Sasaki manifold with solvable fundamental group is finitely covered by a Heisenberg manifold and its Sasaki structure may be deformed to a locally homogeneous one. In addition, we classify all aspherical Sasaki homogeneous spaces for semisimple Lie groups.

1. Introduction

Let M be a smooth contact manifold with contact form ω. Suppose that there exists a complex structure J on the contact bundle $\ker\omega$ and that the Levi form $d\omega \circ J$ is a positive definite Hermitian form. Then $\{\omega, J\}$ is called a *pseudo-Hermitian structure* on M and $\{\ker\omega, J\}$ is a CR-structure as well. The pair $\{\omega, J\}$ assigns a Riemannian metric g to M, where

$$g = \omega \cdot \omega + d\omega \circ J. \tag{1.1}$$

There are two typical, closely related, Lie groups on $(M, \{\omega, J\})$. The group of pseudo-Hermitian transformations of M is denoted by

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* Corresponding author.
E-mail addresses: oliver.baues@unifr.ch (O. Baues), kami@tmu.ac.jp (Y. Kamishima).

$$\mathrm{Psh}\,(M) = \{h \in \mathrm{Diff}\,(M) \mid h^*\omega = \omega, \ h_* \circ J = J \circ h_* \text{ on } \ker \omega\}.$$

As usual $\mathrm{Isom}\,(M)$ denotes the isometry group of (M, g). Obviously

$$\mathrm{Psh}\,(M) \leq \mathrm{Isom}\,(M)\,.$$

Assume that the Reeb field $\mathcal{A}$ for ω generates a one-parameter group T of holomorphic transformations on a CR-manifold $(M, \{\ker \omega, J\})$, that is,

$$T \leq \mathrm{Psh}\,(M)\,.$$

Then $(M, \{\omega, J\})$ is said to be a *standard pseudo-Hermitian manifold*. In this case, the vector field $\mathcal{A}$ is a Killing field of unit length with respect to g, and the Riemannian manifold (M, g) is also called a *Sasaki manifold* equipped with *Sasaki metric g* and *structure field $\mathcal{A}$*. If $\mathcal{A}$ is a complete vector field with a global flow T which acts freely and properly on M, $(M, \{g, \mathcal{A}\})$ is said to be a *regular Sasaki manifold*. Note that the Sasaki metric structure $(M, \{g, \mathcal{A}\})$ determines the standard pseudo-Hermitian structure $(M, \{\omega, J\})$ uniquely.

The pseudo-Hermitian group $\mathrm{Psh}\,(M)$ and isometry group $\mathrm{Isom}\,(M)$ of a Sasaki manifold are closely related. Since the Reeb vector field $\mathcal{A}$ is determined by ω alone, we have

$$h_*\mathcal{A} = \mathcal{A}\,, \text{ for all } h \in \mathrm{Psh}\,(M)\,.$$

Therefore, the Reeb flow T belongs to the center of $\mathrm{Psh}\,(M)$, that is,

$$\mathrm{Psh}\,(M) = C_{\mathrm{Psh}\,(M)}(T)\,. \tag{1.2}$$

Similarly, if $C_{\mathrm{Isom}\,(M)}(T)$ denotes the centralizer of T in $\mathrm{Isom}\,(M)$, using (1.1),

$$\mathrm{Psh}\,(M) = C_{\mathrm{Isom}\,(M)}(T)$$

follows easily, as well.

In general, the group $\mathrm{Isom}\,(M)$ acts on the set of Sasaki structures $\{g, \mathcal{A}\}$ with fixed metric g. Furthermore, if (M, g) is not isometrically covered by a round sphere, the set of Sasaki structures with metric g either consists of two elements $\{\mathcal{A}, -\mathcal{A}\}$, or M is a three-Sasaki manifold, admitting three linear independent Sasaki structures for g. In the latter case, M is compact with finite fundamental group. For these results, see [26,20,27]. Thus, unless M is compact with finite fundamental group, a complete Sasaki manifold always satisfies

$$\mathrm{Isom}\,(M) = \mathrm{Psh}^{\pm}\,(M) = \{h \in \mathrm{Isom}\,(M) \mid h^*\mathcal{A} = \pm\mathcal{A}\}\,.$$

Call a Sasaki manifold M a *homogeneous Sasaki manifold* if $\mathrm{Psh}\,(M)$ acts transitively on M. Accordingly, a homogeneous space G/H is called a *homogeneous Sasaki manifold* if G/H is a Sasaki manifold and the action of G factors over $\mathrm{Psh}\,(G/H)$. Note that any homogeneous Sasaki manifold is also a regular Sasaki manifold.

1.1. Locally homogeneous aspherical Sasaki manifolds

In the following we shall usually assume that G acts effectively on G/H and thereby identify G with a closed subgroup of $\mathrm{Psh}\,(G/H)$ whenever suitable.

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A *locally homogeneous* Sasaki manifold is a quotient space

$$M = \Gamma \backslash G / H$$

of a homogeneous Sasaki manifold G/H by a discrete subgroup Γ of G. The manifold M is called *aspherical* if its universal cover X is *contractible*. In this paper we take up the structure of compact *locally homogeneous aspherical* Sasaki manifolds M.

Setting the stage for the main structure result on compact *locally homogeneous aspherical* Sasaki manifolds, we note the following facts:

Let $X = G/H$ be a homogeneous Sasaki manifold. Then the Reeb flow T on X is isomorphic to the real line $\mathbb{R}$ or the circle group S^1 and it is acting freely and properly on X. Moreover, the homogeneous pseudo-Hermitian structure on X induces a unique *homogeneous* Kähler structure on the quotient manifold

$$W = X/T$$

such that the projection map $X \to W$ is a principal bundle projection which is pseudo-Hermitian (that is, $X \to W$ is horizontally holomorphic and horizontally isometric). With this structure the homogeneous Kähler manifold W will be called the *Kähler quotient* of X.

There is a well established theory of homogeneous Kähler manifolds and of the structure of their isometry groups [12]. As a cornerstone of the present paper we use this theory to develop the structure of compact locally homogeneous Sasaki manifolds. Recall that a Lie group is called *unimodular* if its Haar measure is biinvariant. Any Lie group G which admits a uniform lattice Γ is unimodular. Thus, in our context, homogeneous Sasaki spaces and Kähler manifolds of unimodular Lie groups are of particular importance. The structure theory of such spaces already plays a major role in the papers [1,16]. As a starting point of the present work we refine and also detail the approach therein to study contractible homogeneous Sasaki spaces of unimodular Lie groups and their corresponding Kähler quotients.

1.1.1. Structure of locally homogeneous aspherical Sasaki manifolds

Let us first introduce some additional notation. For any Kähler manifold W, we denote $\mathrm{Isom}_h^{\pm}(W)$ the subgroup of $\mathrm{Isom}(W)$ that consists of isometries which are either holomorphic or anti-holomorphic. Furthermore, $\mathrm{Isom}_h(W)$ denotes the subgroup of holomorphic (or Kähler-) isometries of W.

The main structure result on locally homogeneous aspherical Sasaki manifolds and their isometry groups is stated in the following two results:

Theorem 1. *Let $X = G/H$ be a contractible homogeneous Sasaki manifold of a unimodular Lie group G. Then the following hold:*

(1) *The Kähler quotient W of X is a product of a unitary space $\mathbb{C}^k$ with a bounded symmetric domain D.*
(2) *The Reeb flow T is isomorphic to the real line and it is a normal subgroup of $\mathrm{Isom}(X)$. There exists an induced homomorphism*

$$\phi : \mathrm{Isom}(X) \to \mathrm{Isom}_h^{\pm}(W) ,$$

which is onto and maps $\mathrm{Psh}(X)$ onto $\mathrm{Isom}_h(W)$ with kernel T.
(3) *There exists an anti pseudo-Hermitian involution τ of X such that*

$$\mathrm{Isom}(G/H) = \mathrm{Psh}^{\pm}(G/H) = \mathrm{Psh}(G/H) \rtimes \langle \tau \rangle .$$

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(4) *The identity component of the pseudo-Hermitian group of X satisfies*

$$\mathrm{Psh}\,(G/H)^0 = \mathrm{Isom}\,(G/H)^0 = (\mathcal{N} \rtimes \mathrm{U}(k)) \cdot S\,,$$

where $\mathcal{N}$ is a $2k+1$-dimensional Heisenberg Lie group and S is a normal semisimple Lie subgroup which covers the identity component

$$S_0 = \mathrm{Isom}\,(D)^0$$

of the isometry group of the symmetric bounded domain D. Moreover, S has infinite cyclic center Λ, and

$$S \cap \mathcal{N} = S \cap T = \Lambda\,.$$

Building on Theorem 1 we can deduce:

Corollary 1. *Let $M = \Gamma\backslash G/H$ be a compact locally homogeneous aspherical Sasaki manifold. Then the coset space $\Gamma\backslash G/H$ admits an S^1-bundle over a locally homogeneous aspherical Kähler orbifold*

$$S^1 \;\longrightarrow\; \Gamma\backslash G/H \;\longrightarrow\; \phi(\Gamma)\backslash W\,, \tag{1.3}$$

in which S^1 induces the Reeb field. In particular, the Sasaki manifold M is quasi-regular.

Remark 1.1. The bundle in (1.3) is called a *Seifert fibering*. Here, some finite covering space $\Gamma_0\backslash G/H$, with $\Gamma_0 \leq \Gamma$ a finite index subgroup, is a non-trivial S^1-bundle over a Kähler manifold $\phi(\Gamma_0)\backslash W$. Note, in addition, that for any Sasaki manifold $M = \Gamma\backslash G/H$ as above, $\mathrm{Psh}\,(\Gamma\backslash G/H)^0$ contains the flow of the Reeb field. This flow is a compact one-parameter group S^1 acting almost freely on M and it is giving rise to the bundle (1.3). Moreover, since the Sasaki structure on M arises from a connection form, the Kähler class of $\phi(\Gamma_0)\backslash W$ represents the characteristic class of the circle bundle.

We further remark:

(5) When the anti-holomorphic isometry τ of X from Theorem 1 normalizes Γ, we get $\mathrm{Isom}\,(\Gamma\backslash G/H) = \mathrm{Psh}\,(\Gamma\backslash G/H) \rtimes \mathbb{Z}_2$, otherwise we have $\mathrm{Isom}\,(\Gamma\backslash G/H) = \mathrm{Psh}\,(\Gamma\backslash G/H)$.

Let $\mathcal{N}$ denote the $2n+1$-dimensional Heisenberg group with its natural Sasaki metric. Using (5) above we also get:

(6) There exists a compact locally homogeneous aspherical Riemannian manifold

$$M = \pi\backslash\mathcal{N}\,,$$

whose metric is locally a Sasaki metric (that is, it is induced from the left-invariant Sasaki metric on $\mathcal{N}$). But M with metric g is not a Sasaki manifold itself.

1.1.2. The case of solvable fundamental group

We suppose that the fundamental group of the compact aspherical manifold M is virtually solvable. In this case, if M supports a locally homogeneous Sasaki structure, then Theorem 1 implies that M is finitely covered by a *Heisenberg manifold*

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$$\Delta \backslash \mathcal{N} \,,$$

where $\Delta \leq \mathcal{N}$ is a uniform discrete subgroup of $\mathcal{N}$. Moreover, M is a non-trivial circle bundle over a compact flat Kähler manifold, which in turn is finitely covered by a complex torus $\mathbb{C}^k/\Lambda$. As a matter of fact, *any* compact aspherical Kähler manifold is biholomorphic to a flat Kähler manifold (see [4, Theorem 0.2] and the references therein). As a consequence, any regular Sasaki manifold M is of the above type as well, and it admits a locally homogeneous Sasaki structure:

Corollary 2. *Let M be a regular compact aspherical Sasaki manifold with virtually solvable fundamental group. Then the following hold:*

(1) *The manifold M is a circle bundle over a Kähler manifold that is biholomorphic to a flat Kähler manifold.*
(2) *A finite cover of M is diffeomorphic to a Heisenberg manifold.*

Moreover, the Sasaki structure on M can be deformed (via regular Sasaki structures) to a locally homogeneous Sasaki structure.

1.2. Contractible Sasaki Lie groups and compact quotients

We call a Lie group G a *Sasaki group* if it admits a left-invariant Sasaki structure. Equivalently, G acts simply transitively by pseudo-Hermitian transformations on a Sasaki manifold X.

A prominent example of a Sasaki Lie group is the $2n + 1$-dimensional *Heisenberg Lie group* $\mathcal{N}$. The Lie group $\mathcal{N}$ arises as a non-trivial central extension of the form

$$\mathbb{R} \to \mathcal{N} \to \mathbb{C}^n \,,$$

and a natural Sasaki structure on $\mathcal{N}$ is obtained by a left-invariant connection form which is associated to this central extension.

More generally, we shall introduce a family of simply connected $2n + 1$-dimensional solvable Sasaki Lie groups

$$\mathcal{N}(k,l) \,, \ k+l = n,$$

called *Heisenberg modifications*. These groups are deformations of $\mathcal{N}$ in $\mathcal{N} \rtimes T^k$, where $T^k \leq \mathrm{U}(n)$ is a compact torus (cf. Definition 7.7).

Another noteworthy contractible Lie group which is Sasaki is

$$\widetilde{\mathrm{SL}(2,\mathbb{R})} \,,$$

the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. Indeed, take *any* left-invariant metric g on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with the additional property that g is also right-invariant by the one-parameter subgroup $\widetilde{\mathrm{SO}(2,\mathbb{R})}$. Then the Riemannian submersion map

$$\widetilde{\mathrm{SL}(2,\mathbb{R})} \ \to \ \widetilde{\mathrm{SL}(2,\mathbb{R})}/\widetilde{\mathrm{SO}(2,\mathbb{R})} = \mathbb{H}_{\mathbb{C}}^1$$

is defined and it is a principal bundle with group $\widetilde{\mathrm{SO}(2,\mathbb{R})} = \mathbb{R}$ over a Riemannian homogeneous space $\mathbb{H}_{\mathbb{C}}^1$ of constant negative curvature. The metric g defines a unique left-invariant connection form ω, which satisfies

(1.1) and has the property that the Reeb field is left-invariant and tangent to the subgroup $\widetilde{\mathrm{SO}(2,\mathbb{R})}$. The isomorphism classes of Sasaki structures thus obtained are parametrized by the curvature of the base.

In general, a simply connected unimodular Lie group that admits a left-invariant Sasaki structure is isomorphic to either of the Lie groups $\mathrm{SU}(2)$, N or the universal covering group of $\mathrm{SL}(2,\mathbb{R})$, see [16, Theorem 4] and also [8, Theorem 5]. (In [1, Theorem 2.1] a more precise statement is obtained by describing left-invariant Sasaki structures up to a modification process.)

As an application of our methods we *explicitly* state the classification of contractible unimodular Sasaki Lie groups as follows:

Theorem 2. *Let G be a unimodular contractible Sasaki Lie group. Then as a Sasaki Lie group G is isomorphic to either $\mathcal{N}(k,l)$ or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with one of the left invariant Sasaki structures as introduced above. (That is, G admits a pseudo-Hermitian isomorphism to either $\mathcal{N}(k,l)$ or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with a standard Sasaki structure.)*

Remark 1.2. As introduced above the family of all Sasaki Lie groups $\mathcal{N}(k,l)$ is in one to one correspondence with the set of isomorphism classes of flat Kähler Lie groups. Compare Section 7.2.2. For a discussion of the structure of flat Kähler Lie groups, see for example [13] or [3].

Remark 1.3. When dropping the assumption of contractibility, the compact group $\mathrm{SU}(2)$ appears as another unimodular Sasaki Lie group. This group is fibering over the projective line $\mathrm{P}^1\mathbb{C}$, and the example is dual to the Sasaki Lie group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The two groups are known to be the only simply connected semisimple Lie groups which admit a left-invariant Sasaki structure, cf. [8, Theorem 5].

Any Lie group G which admits a discrete uniform subgroup Δ must be unimodular, and if such G admits the structure of a Sasaki Lie group then the quotient manifold

$$\Delta \backslash G$$

inherits the structure of a compact locally homogeneous Sasaki manifold.

Thus, combining Theorem 2 with Corollary 1 we obtain:

Corollary 3. *Every compact locally homogeneous aspherical Sasaki manifold which is of the form*

$$\Delta \backslash G$$

is either a Seifert manifold, which is an S^1-bundle over a hyperbolic two-orbifold, or it is a Seifert manifold which is an S^1-bundle over a flat Kähler manifold (which is a complex torus bundle over a complex torus).

1.3. Sasaki homogeneous spaces of semisimple Lie groups

Here we consider the question which semisimple Lie groups act transitively by pseudo-Hermitian transformations on a contractible (or, more generally, aspherical) Sasaki manifold. The classification of such groups and of the corresponding homogeneous spaces is contained in Theorem 3 following below.

Let D be a bounded symmetric domain, equipped with its natural Bergman Riemannian metric. Then its isometry group

$$S_0 = \mathrm{Isom}\,(D)^0$$

is a semisimple Lie group which is called a group of Hermitian type, and

$$D = S_0/K_0$$

is a Riemannian symmetric space with respect to this metric, and a homogeneous Kähler manifold, as well. Moreover, K_0 is a maximal compact subgroup of S_0.

Theorem 3. *For any symmetric bounded domain $D = S_0/K_0$, there exists a unique semisimple Lie group S with infinite cyclic center, which is covering S_0, and gives rise to a contractible Sasaki homogeneous space*

$$X_S = S/K$$

with Kähler quotient D. Moreover, any contractible homogeneous Sasaki manifold of a semisimple Lie group is of this type.

Addendum: In the theorem, K is a maximal compact subgroup of S, and $S \leq \mathrm{Psh}\,(X_S)$ is acting faithfully on X_S. The Kähler quotient D is a homogeneous Kähler manifold whose complex structure is biholomorphic to a bounded symmetric domain. It carries an invariant symmetric Kähler Riemannian metric, which is unique up to scaling on irreducible factors of the homogeneous space D.

As a consequence of Theorem 3 any Lie group of Hermitian type acts as a transitive group of isometries on an aspherical Sasaki space:

Corollary 4. *For any semisimple Lie group S_0 of Hermitian type, there exists a unique Sasaki homogeneous space*

$$Y = S_0/K_1\,,$$

where Y is a circle bundle over the symmetric bounded domain $D = S_0/K_0$.

Note that, in Theorem 3 and Corollary 4 the Sasaki structure on X, respectively Y, is unique up to the choice of an S_0-invariant (and also symmetric) Kähler metric on D.

The paper is organized as follows. Starting in Section 2, we collect and explain some useful basic facts on regular Sasaki manifolds, including the Boothby-Wang fibration and the join construction.

In Section 3 we discuss the lifting of Kähler isometries and the role of gauge transformations in the Boothby-Wang fibration of a contractible Sasaki manifold.

We use these facts to show that every contractible homogeneous Kähler manifold determines a *unique* contractible homogeneous Sasaki manifold. Also the associated presentations of a homogeneous Sasaki manifold by transitive groups of pseudo-Hermitian transformation are discussed in Section 4.

Section 5 is devoted to the study of homogeneous contractible Kähler manifolds of unimodular Lie groups. Their classification is derived from the Dorfmeister-Nakajima fundamental holomorphic fiber bundle of a homogeneous Kähler manifold.

The structure of locally homogeneous aspherical Sasaki manifolds is picked up in Section 6. We establish in Corollary 1 that a compact locally homogeneous aspherical Sasaki manifold is always quasi-regular over a compact orbifold which is modeled on a homogeneous contractible Kähler manifold. The relevant global results are summarized in Theorem 1 and its proof. We also give the proof of Corollary 2 in Section 6.4.1, see in particular Proposition 6.10.

In Section 7 we turn our interest to the classification problem for global model spaces of locally homogeneous Sasaki manifolds: in particular, we classify contractible Sasaki Lie groups and contractible Sasaki homogeneous spaces of semisimple Lie groups. In the course, we prove Theorem 2 and Theorem 3.

In Section 8 we construct further explicit examples of locally homogeneous aspherical Sasaki manifolds.

Refer to [25], [7], [11] for background on Sasaki metric structures in general.

2. Preliminaries

Let $X = (X, \{\omega, J\})$ be a Sasaki manifold with Reeb flow T.

2.1. Regular Sasaki manifolds

The Sasaki manifold X is called regular if the Reeb flow T is complete and T acts freely and properly on X. In this situation, either $T = \mathbb{R}$, or $T = S^1$ is a circle group. Moreover,

$$W = X/T$$

is a smooth manifold and X is a principal bundle over W with group T.

Example 2.1. Let $X = G/H$ be a homogeneous Sasaki manifold. Then X is regular. (See [8] and Section 4 below.)

For the following, see [8]:

Proposition 2.2 (Boothby-Wang fibration). *Let X be a regular Sasaki manifold with Reeb flow T. Then there is an associated principal bundle*

$$T \to X \xrightarrow{q} W$$

over a Kähler manifold (W, Ω, J) such that the induced isomorphism

$$q_* : \ker \omega \to TW$$

is holomorphic and the Kähler form on the base is satisfying the equation

$$q^* \Omega = d\omega \ . \tag{2.1}$$

Furthermore, there is a natural induced homomorphism

$$\mathrm{Psh}\,(X) \xrightarrow{\phi} \mathrm{Isom}\,_h(W) \tag{2.2}$$

with kernel T, which is satisfying $q \circ \tilde{h} = \phi(\tilde{h}) \circ q$, for all $\tilde{h} \in \mathrm{Psh}\,(X)$.

With the above conditions satisfied, we call (W, Ω, J) the *Kähler quotient* of the regular Sasaki manifold X. Also we let

$$\mathrm{Isom}_h(W) = \mathrm{Isom}\,(W, \Omega, J)$$

denote the group of holomorphic isometries of the Kähler quotient W.

Proof of Proposition 2.2. The projection q induces an isomorphism

$$q_* : \ker \omega \longrightarrow TW$$

at each point. Since ω is invariant under T, ω induces a well defined 2-form Ω on W such that

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$$d\omega(\mathcal{X}, \mathcal{Y}) = \Omega(q_* \mathcal{X}, q_* \mathcal{Y}),$$

for all horizontal vector fields $\mathcal{X}, \mathcal{Y} \in \ker \omega$ that are horizontal lifts. As

$$\iota_{\mathcal{A}} \, d\omega = 0,$$

it follows that $d\omega = q^* \Omega$ and so $d\Omega = 0$. Since the Reeb flow T is holomorphic on $\ker \omega$, using J on $\ker \omega$, q_* induces a well defined almost complex structure $\hat{J}$ on W such that Ω is $\hat{J}$-invariant. Since J is integrable (that is, $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$ for the eigenvalue decomposition $\ker \omega \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$), $\hat{J}$ becomes a complex structure on W. Hence Ω is a Kähler form on the complex manifold $(W, \hat{J})$. To simplify notation, from now on, the same symbol J is used for the complex structure on W, for which we require that q is a *holomorphic map* on $\ker \omega$, that is, the induced isomorphism $q_* : \ker \omega \to TW$ satisfies $q_* \circ J = J \circ q_*$.

Since it is commuting with the principal bundle action of T, which is arising from the Reeb flow, each holomorphic isometry

$$\tilde{h} \in \mathrm{Psh}\,(X) = C_{\mathrm{Psh}\,(X)}(T)$$

induces a diffeomorphism $h : W \to W$, such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \tilde{h}\ } & X \\
q \downarrow & & q \downarrow \\
W & \xrightarrow{\ h\ } & W
\end{array}
\tag{2.3}
$$

is commutative. (We briefly verify that $h^* \Omega = \Omega$ and $h_* \circ J = J \circ h_*$ on W: Indeed, as $\tilde{h}^* d\omega = d\omega$, it follows by (2.1) that $q^*(h^* \Omega) = \tilde{h}^* q^* \Omega = q^* \Omega$. This shows $h^* \Omega = \Omega$. Since $q_* \tilde{J} = J q_*$ on $\ker \omega$, using (2.3) it follows $h_* J q_*(\mathcal{Y}) = h_* q_* \tilde{J}(\mathcal{Y}) = q_* \tilde{h}_* \tilde{J}(\mathcal{Y}) = q_* \tilde{J} \tilde{h}_*(\mathcal{Y}) = J h_* q_*(\mathcal{Y})$, for all vector fields $\mathcal{Y} \in \ker \omega$, which are horizontal lifts for a vector field on W. So $h_* \circ J = J \circ h_*$ on W.) Thus h is a holomorphic isometry of W.

Further any lift $\tilde{h} \in \mathrm{Psh}\,(X)$ of h is unique up to composition with an element of the Reeb flow: Indeed, suppose that $h = \mathrm{id}_W$. Since T acts transitively on the fibers, after composition with an element of T, we may assume that there exists a fixed point $x \in X$ for $\tilde{h}$. Moreover, since $\tilde{h}_* \mathcal{A} = \mathcal{A}$, the differential of $\tilde{h}$ at x is the identity of $T_x X$. Now every isometry $\tilde{h}$ of the Riemannian manifold X is determined by its one-jet at one point x. Hence, $\ker \phi = T$. $\quad\square$

2.2. Holomorphic and anti-holomorphic isometries

For any Sasaki manifold X with Reeb field $\mathcal{A}$, we briefly recall the interaction of

$$\mathrm{Psh}^{\pm}\,(X) = \{h \in \mathrm{Isom}\,(X) \mid h^* \mathcal{A} = \pm \mathcal{A}\}$$

with the pseudo-Hermitian structure of X. For any pseudo-Hermitian structure $\{\omega, J\}$, the structure $\{-\omega, -J\}$ is called the *conjugate structure*. Then the group of isometries $\mathrm{Psh}^{\pm}\,(X)$ permutes the pseudo-Hermitian structure of X and its conjugate:

Lemma 2.3 *(Sasaki isometries). Let X be any Sasaki manifold and let $h \in \mathrm{Isom}\,(X)$ satisfy $h_* \mathcal{A} = \pm \mathcal{A}$, where $\mathcal{A}$ is the Reeb field of X. Then $h^* \omega = \pm \omega$ and $h_* J = \pm J h_*$ on $\ker \omega$.*

Proof. For any $\mathcal{X} \in \ker \omega$, the equation $g(h_*\mathcal{A}, h_*\mathcal{X}) = g(\mathcal{A}, \mathcal{X})$ shows

$$0 = g(\mathcal{X}, \mathcal{A}) = \omega(\mathcal{X})\omega(\mathcal{A}) + d\omega(J\mathcal{X}, \mathcal{A}) = \omega(\mathcal{X})$$
$$= g(h_*\mathcal{X}, h_*\mathcal{A}) = \pm g(h_*\mathcal{X}, \mathcal{A}) = \pm\omega(h_*\mathcal{X}) \ .$$

In particular, h_* maps $\ker \omega$ onto itself. As

$$h^*\omega(\mathcal{A} + \mathcal{X}) = \omega(\pm\mathcal{A}) = \pm\omega(\mathcal{A} + \mathcal{X}),$$

we deduce that $h^*\omega = \pm\omega$. Next for any $\mathcal{X}, \mathcal{Y} \in \ker \omega$,

$$d\omega(J\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) = g(h_*\mathcal{X}, h_*\mathcal{Y}) = d\omega(Jh_*\mathcal{X}, h_*\mathcal{Y})$$
$$= d\omega(h_*(h_*^{-1}Jh_*)\mathcal{X}, h_*\mathcal{Y}) = h^*d\omega((h_*^{-1}Jh_*)\mathcal{X}, \mathcal{Y})$$
$$= \pm d\omega((h_*^{-1}Jh_*)\mathcal{X}, \mathcal{Y}).$$

By the non-degeneracy of the Levi form $d\omega \circ J$ it follows that

$$h_*J = \pm Jh_* \text{ on } \ker \omega. \quad \square \tag{2.4}$$

2.3. Join of regular Sasaki manifolds

We describe in detail a natural procedure which explicitly constructs a new Sasaki manifold from a pair of given regular Sasaki manifolds. This corresponds to a variant of the join construction as is discussed in [10] for the compact case. In our context we apply the join in the construction of homogeneous Sasaki manifolds.

2.3.1. Sasaki immersions

Let X, Y be regular Sasaki manifolds with pseudo-Hermitian structures $\{\omega, J\}$, $\{\eta, I\}$, respectively. Also, let $\mathcal{A}$, $\mathcal{B}$ denote the respective Reeb vector fields on X, Y. An immersion of manifolds

$$\iota : Y \to X$$

such that

i) the Reeb vectorfield $\mathcal{A}$ is tangent to the image $\iota(Y) \subseteq X$ and
ii) the tangent bundle of $\iota(Y)$ satisfies $J T\iota(Y) \subseteq T\iota(Y)$

is called a *Sasaki immersion* if

iii) $\{\eta, I\} = \iota^*\{\omega, J\}$

is satisfied. That is, for a Sasaki immersion, $\{\eta, I\}$ is obtained by pullback of $\{\omega, J\}$. Let $q : X \to W$ and $p : Y \to V$ denote the respective Kähler quotients. Then the Sasaki immersion ι induces a unique *Kählerian immersion*

$$j : V \to W \ ,$$

such that $j \circ p = q \circ \iota$. Note also that j determines the Sasaki immersion ι *uniquely* up to composition with an element of the Reeb flow T.

2.3.2. The join construction and Sasaki immersions

Let

$$(X_i, \{\omega_i, J_i\}), \ i = 1, 2 \ ,$$

be regular Sasaki manifolds with Reeb flows

$$T_i = \{\phi_{i,t}\}_{t \in \mathbb{R}} \ .$$

Furthermore, let (W_i, Ω_i) denote the Kähler quotients of X_i, and

$$q_i : X_i \to W_i$$

the corresponding Boothby-Wang fibrations. Now consider

$$\bar{T} = T_1 \times T_2 = \{(\phi_{1,s}, \phi_{2,t})\}_{s,t \in \mathbb{R}}$$

and define $\Delta = \{(\phi_{1,t}, \phi_{2,-t})\}_{t \in \mathbb{R}}$ as the diagonal in $\bar{T}$. Then put

$$T = \bar{T}/\Delta \ .$$

Proposition 2.4 *(Join of Sasaki manifolds X_1 and X_2). There exists a unique regular Sasaki manifold*

$$X = X_1 * X_2$$

with Reeb flow T and Kähler quotient

$$q : \ X_1 * X_2 \ \to \ W = (W_1 \times W_2, \Omega_1 \times \Omega_2) \ ,$$

which admits Sasaki immersions $\iota_{X_i} : X_i \to X_1 * X_2$ such that the diagram

$$
\begin{array}{ccc}
X_i & \xrightarrow{\ \iota_{X_i}\ } & X_1 * X_2 \\
\downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q} \\
W_i & \xleftarrow[\mathsf{pr}_i]{} & W_1 \times W_2
\end{array}
\tag{2.5}
$$

is commutative $(i = 1, 2)$.

Proof. Observe that, via the product action, $\bar{T} = T_1 \times T_2$ acts properly and freely on $X_1 \times X_2$ with quotient map

$$\bar{q} = q_1 \times q_2 : X_1 \times X_2 \ \to \ W = W_1 \times W_2 \ .$$

Define another quotient map

$$\mathsf{p} : X_1 \times X_2 \ \to \ X := (X_1 \times X_2)/\Delta \ , \tag{2.6}$$

and let

$$q : X \to W$$

be the induced map such that $\bar{q} = q \circ \mathsf{p}$.

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Let $\mathrm{pr}_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, denote the projection maps. Define

$$\bar{\omega} = \mathrm{pr}_1^* \omega_1 + \mathrm{pr}_2^* \omega_2 \,,$$

and consider the Kähler form $\Omega = \Omega_1 \times \Omega_2$ on W. By construction,

$$\bar{q}^* (\Omega) = d\bar{\omega} \,.$$

Next let $\bar{\mathcal{A}}_i$ denote the canonical lifts of the Reeb fields $\mathcal{A}_i$ to $X_1 \times X_2$, where $\bar{\mathcal{A}}_i$ is tangent to the factor X_i, respectively. The one-parameter groups generated by these vector fields are contained in the abelian Lie group $\bar{T}$. In particular, these vector fields are Δ-invariant. Let $\mathcal{V}_\Delta$ denote the one-dimensional distribution on $X_1 \times X_2$, which is spanned by the vector field $\bar{\mathcal{A}}_1 - \bar{\mathcal{A}}_2$. Then $\mathcal{V}_\Delta$ is vertical (tangent to the fibers) with respect to the quotient map p in (2.6) induced by the action of Δ. Therefore, both vector fields $\bar{\mathcal{A}}_i$ project to the same vector field $\mathcal{A}$ on X.

Note that $\bar{\omega}$ is a $\bar{T}$-invariant one-form which vanishes on $\mathcal{V}_\Delta$. Therefore, there exists on X a unique induced one form

$$\omega = \omega_1 * \omega_2 \ \text{ satisfying } \mathsf{p}^* \omega = \bar{\omega}.$$

In particular, ω satisfies $q^* \Omega = d\omega$, where $\Omega = \Omega_1 \times \Omega_2$. It follows that ω is a contact form with Reeb field $\mathcal{A}$. The Reeb flow of ω is the one-parameter group

$$T = \bar{T}/\Delta \,.$$

Summarizing the construction, we note that ω is a connection form for the T-principal bundle $q : X \to W$ and it has curvature form Ω.

Let J_i denote the complex structures on $\ker \omega_i$ (canonically extended to tensors on X_i by declaring $J_i(\mathcal{A}_i) = 0$). Observe that the kernel of ω coincides with the projection of

$$\ker \mathrm{pr}_1^* \omega_1 \cap \ker \mathrm{pr}_2^* \omega_2$$

to (the tangent bundle of) X. Therefore, $\bar{J} = J_1 \times J_2$ goes down to an almost complex structure J on $\ker \omega$ such that

$$q : \ (X, \{\ker \omega, J\}) \ \to \ (W, J)$$

is a holomorphic CR-map. Since $(W, \{\Omega, J\})$ is Kähler and ω a connection form with curvature Ω, the almost CR-structure $\{\ker \omega, J\}$ is integrable, see [17, Theorem 2]. Since

$$d\omega \circ J = (q^* \Omega) \circ J$$

is positive, $\{\omega, J\}$ defines a pseudo-Hermitian structure on X. By the construction T acts by holomorphic transformations on X. This shows that $(X, \{\omega, J\})$ is a regular Sasaki manifold with Kähler quotient (W, Ω).

Choose a base point $(x_o, y_o) \in X_1 \times X_2$ and define immersions $\iota_i : X_i \to X$, $\iota_1(x) = q(x, y_o)$ and $\iota_2(y) = q(x_o, y)$. (Note that all such pairs of maps are equivalent by an element of T.) By the above construction, ι_i are Sasaki immersions, and, in fact, they determine the Sasaki structure $\{\omega, J\}$ on the manifold $X_1 * X_2$ uniquely, together with the condition that $\mathcal{A}$ is the Reeb field. $\square$

The join of Sasaki manifolds enjoys the following functorial property:

Proposition 2.5. *For any pair of Sasaki immersions $\tau_i : Y_i \to X_i$ with induced Kähler immersions $j_i : V_i \to W_i$, $i = 1, 2$, there exists a unique Sasaki immersion*

$$\tau = \tau_1 * \tau_2 : \ Y_1 * Y_2 \ \to \ X_1 * X_2$$

such that the associated diagram

$$\begin{array}{ccc} Y_1 * Y_2 & \xrightarrow{\ \tau\ } & X_1 * X_2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ V_1 \times V_2 & \xrightarrow{\ j_1 \times j_2\ } & W \end{array} \tag{2.7}$$

is commutative and $\iota_{X_i} \circ \tau_i$ and $\tau \circ \iota_{Y_i}$ coincide up to an element of T.

Proof. Since τ_i are Sasaki immersions, the product map

$$\bar{\tau} = \tau_1 \times \tau_2 : \ Y_1 \times Y_2 \to X_1 \times X_2$$

induces a map

$$\tau_1 * \tau_2 := Y_1 * Y_2 \to X_1 * X_2$$

with the required properties. $\quad \square$

This gives:

Corollary 2.6. *The join of X_1 and X_2 defines a natural homomorphism*

$$\mathrm{Psh}\,(X_1) \times \mathrm{Psh}\,(X_2) \ \to \ \mathrm{Psh}\,(X_1 * X_2)\,, \ (\phi_1, \phi_2) \mapsto \phi_1 * \phi_2$$

with kernel the diagonal group $\Delta = \{(\phi_{1,t}, \phi_{2,-t})\}_{t \in \mathbb{R}}$.

Proof. Indeed, by the construction in Proposition 2.5, $\phi_1 * \phi_2 \in \mathrm{Psh}\,(X)$ and the above map is a homomorphism with kernel Δ. $\quad \square$

We call the group

$$\mathrm{Psh}\,(X_1) * \mathrm{Psh}\,(X_2) \ = \ \left(\mathrm{Psh}\,(X_1) \times \mathrm{Psh}\,(X_2)\right) \big/ \Delta$$

the *join* of the groups $\mathrm{Psh}\,(X_i)$. By the above, the join of $\mathrm{Psh}\,(X_i)$ identifies with a subgroup of $\mathrm{Psh}\,(X_1 * X_2)$.

Corollary 2.7. *Let X_1 and X_2 be homogeneous Sasaki manifolds. Then the join of groups $\mathrm{Psh}\,(X_1) * \mathrm{Psh}\,(X_2)$ is acting transitively by pseudo-Hermitian transformations on the Sasaki manifold $X_1 * X_2$. In particular, $X_1 * X_2$ is a homogeneous Sasaki manifold.*

Proof. The Kähler quotient $W_1 \times W_2$ of $X_1 * X_2$ is a homogeneous Kähler manifold for the group $G = G_1 \times G_2$, where G_i denotes the Boothby-Wang image of $\mathrm{Psh}\,(X_i)$ in $\mathrm{Isom}\,_h(W_i)$. Since G is also the Boothby-Wang image of $\mathrm{Psh}\,(X_1) * \mathrm{Psh}\,(X_2)$, and the latter also contains the Reeb-flow T, it follows that $\mathrm{Psh}\,(X_1) * \mathrm{Psh}\,(X_2)$ acts transitively on $X_1 * X_2$. $\quad \square$

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3. Pseudo-Hermitian group Psh (X) of a regular Sasaki manifold with vanishing Kähler class

Suppose that $(X, \{\omega, J\})$ is a regular Sasaki manifold with Reeb flow T isomorphic to the real line $\mathbb{R}$. Then the Boothby-Wang fibration Proposition 2.2 gives a principal bundle

$$\mathbb{R} \longrightarrow X \xrightarrow{\ q\ } W$$

over the Kähler quotient $W = (W, \Omega, J)$. Here the group $\mathbb{R} = \{\varphi_t\}_{t \in \mathbb{R}}$ of the principal bundle is generated by the Reeb field and the Kähler form on the base is satisfying the equation

$$q^*\Omega = d\omega \ . \tag{3.1}$$

Choose a smooth section $s : W \to X$ of q such that the bundle X is equivalent to the trivial bundle by a bundle map

$$f : \mathbb{R} \times W \longrightarrow X$$

which is defined by

$$f(t, w) = \varphi_t \, s(w) \ .$$

We thus have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{R} \times W & \xrightarrow{\ f\ } & X \\
\text{pr} \searrow & & \swarrow q \\
& W &
\end{array}
\quad , \quad q \circ s = \mathrm{id}_W \ . \tag{3.2}
$$

Declare a one-form θ on W by putting

$$\theta = s^*\omega \ . \tag{3.3}$$

Note then that $d\theta = \Omega$ from (3.1). In particular, *the Kähler form Ω on W is exact.*

Next extend θ to a translation invariant one-form on $\mathbb{R} \times W$ by declaring

$$\omega_0 = dt + \mathrm{pr}^*\theta, \ \ \text{so that } d\omega_0 = \mathrm{pr}^*\Omega \text{ holds.} \tag{3.4}$$

Noting $f(0, w) = s(w) = s \circ \mathrm{pr}(0, w)$, we have

$$\mathrm{pr}^*\theta_{|_{\{0\} \times W}} = \left((s \circ \mathrm{pr})^*\omega\right)_{|_{\{0\} \times W}} = f^*_{|_{\{0\} \times W}}\omega.$$

Since both forms $f^*\omega$ and ω_0 are translation invariant, we conclude that

$$f^*\omega = \omega_0 \ . \tag{3.5}$$

Then an almost complex structure $\tilde{J}$ on $\ker \omega_0$ is defined by

$$\mathrm{pr}_* \circ \tilde{J} = J \circ \mathrm{pr}_* . \tag{3.6}$$

By construction, the isomorphism $f_* : \ker \omega_0 \to \ker \omega$ is holomorphic, that is,

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$$f_* \circ \tilde{J} = J \circ f_* .$$

In particular, $\tilde{J}$ is a complex structure on $\ker \omega_0$. Summarizing the above we obtain:

Proposition 3.1. *Identifying the regular Sasaki manifold X with $\mathbb{R} \times W$ via f, the pseudo-Hermitian structure $\{\omega, J, \mathcal{A}\}$ corresponds to $\{\omega_0, \tilde{J}, \frac{\partial}{\partial t}\}$ on the trivial bundle $\mathbb{R} \times W$, where ω_0 is defined as in* (3.4).

Existence of a compatible regular Sasaki manifold. Conversely, any *exact* Kähler form

$$\Omega = d\theta$$

on a complex manifold W arises as the curvature form of a *connection form* ω on the trivial principal bundle

$$X = \mathbb{R} \times W .$$

In fact, such ω with Reeb field $\mathcal{A} = \frac{\partial}{\partial t}$ is given by (3.4). As a consequence (employing [17, Theorem 2] to show the integrability of the almost CR-structure $\{\ker \omega, J\}$), there exists on X a pseudo-Hermitian structure

$$\{\omega, J, \mathcal{A}\} , \tag{3.7}$$

which has the Kähler manifold (W, Ω) as its Kähler quotient. We call such a pseudo-Hermitian structure *compatible with the Kähler manifold (W, Ω)*.

We remark now that, under a mild assumption on the Kähler manifold W, any compatible pseudo-Hermitian structure on X is essentially determined uniquely by the Kähler structure on W.

Proposition 3.2. *Suppose $H^1(W, \mathbb{R}) = \{0\}$. Then any two pseudo-Hermitian structures $\{\omega, J, \mathcal{A}\}$ and $\{\omega', J', \mathcal{A}\}$ on X, which are compatible with the Kähler manifold (W, Ω), are related by a gauge transformation for the principal bundle $q : X \to W$.*

Proof. By the compatibility assumption, we have $\omega' - \omega = q^* \eta$, for some closed one-form $\eta \in \Omega^1(W)$. Since $H^1(W, \mathbb{R}) = \{0\}$, there exists a function $\lambda : W \to \mathbb{R}$ such $\eta = d\lambda$. In the view of Proposition 3.1, we may assume that $X = \mathbb{R} \times W$ and $\omega = dt + q^* \theta$, where $d\theta = \Omega$. We define a gauge transformation G for the bundle q, by putting

$$G(t, w) = (t + \lambda(w), w) .$$

We then calculate $G^* \omega = G^* dt + q^* \theta = dt + d\, q^* \lambda + q^* \theta = \omega + q^* \eta = \omega'$. $\square$

Remark 3.3. For an analogous existence result for Sasaki manifolds in the more elaborate case of circle bundles over Hodge manifolds, see [8, Theorem 3], respectively [21].

3.1. Lifting of isometries from the Kähler quotient

We now prove a structure result for the group of holomorphic isometries $\mathrm{Psh}(X)$ of X if the Boothby-Wang fibration has contractible fiber $\mathbb{R}$. That is, let X be a regular Sasaki manifold with Boothby-Wang fibration

$$\mathbb{R} \to X \to W . \tag{3.8}$$

As before let

$$\operatorname{Isom}_h(W) = \operatorname{Isom}(W, \Omega, J)$$

denote the group of holomorphic isometries of the Kähler quotient W for X.

Proposition 3.4. *Assume that the first cohomology of the Kähler quotient W, arising in* (3.8)*, satisfies $H^1(W) = \{0\}$. Then the Boothby-Wang homomorphism* (2.2) *defines a natural exact sequence*

$$1 \longrightarrow \mathbb{R} \longrightarrow \operatorname{Psh}(X) \overset{\phi}{\longrightarrow} \operatorname{Isom}_h(W) \longrightarrow 1 \,. \tag{3.9}$$

In particular, $\operatorname{Psh}(X)$ *acts transitively on X if and only if* $\operatorname{Isom}_h(W)$ *acts transitively on W.*

Proof. In the view of Proposition 2.2 it is sufficient to show that ϕ is surjective. Indeed, since $H^1(W) = \{0\}$, Lemma 3.5 below shows that for any $h \in \operatorname{Isom}_h(W)$, there exists an isometry $\tilde{h} \in \operatorname{Psh}(X)$, which is a lift of h, that is, $\phi(\tilde{h}) = h$. $\square$

Proposition 3.4 is implied by the following basic lifting result for holomorphic and anti-holomorphic isometries of the Kähler quotient W:

Lemma 3.5. *Assume that $H^1(W) = \{0\}$, and let $h \in \operatorname{Isom}(W)$ satisfy $h^*\Omega = \mu\,\Omega$, where $\mu \in \{\pm 1\}$. Then there exists an isometry $\tilde{h} \in \operatorname{Psh}^\pm(X)$ such that $\tilde{h}$ induces h on W and satisfies $\tilde{h}^*\omega = \mu\,\omega$. If $\mu = 1$ then $\tilde{h} \in \operatorname{Psh}(X)$.*

Proof. We may assume $X = \mathbb{R} \times W$. Define $\tilde{h}'(t, w) = (t, h(w))$ to be the canonical lift of h. Then $\omega' = \mu \cdot (\tilde{h}')^*\omega$ defines another pseudo-Hermitian structure on X which is compatible with (W, Ω). By Proposition 3.2, there exists a gauge transformation $G : X \to X$ with $G^*\omega' = \omega$. Therefore,

$$\tilde{h} = G \circ \tilde{h}'$$

satisfies $\tilde{h}^*\omega = \mu \cdot \omega$, and it is an isometric lift of h for the metric $g = \omega \cdot \omega + d\omega \circ J$. It also follows $\tilde{h}^*\mathcal{A} = \mu\mathcal{A}$. Thus, $\tilde{h} \in \operatorname{Psh}^\pm(X)$. $\square$

4. Homogeneous Sasaki manifolds

Suppose that the Lie group G acts transitively by pseudo-Hermitian isometries on the Sasaki manifold X. Then

$$X = G/H$$

is called a homogeneous Sasaki manifold. Since X is also a complete Riemannian manifold with respect to the Sasaki metric g, the Reeb field $\mathcal{A}$ for X, which is a Killing field for the metric g, is a complete vector field. Let

$$T = \{\varphi_t\}_{t \in \mathbb{R}}$$

denote the 1-parameter group on G/H generated by the Reeb field.

4.1. Natural fibering over homogeneous Kähler manifold

Since T commutes with G, there exists a one-parameter subgroup

$$\mathsf{A} = \{a_t\}_{t \in \mathbb{R}} \ \leq N_G(H) \tag{4.1}$$

such that

$$\varphi_t(xH) = x a_t^{-1} H \ , \tag{4.2}$$

where $N_G(H)$ denotes the normalizer of H in G.

Proposition 4.1. T *is a closed subgroup in* $\mathrm{Psh}\,(G/H)$. *In particular,* T *is isomorphic to* S^1 *or* $\mathbb{R}$, *and it is acting properly on* G/H.

Proof. The Reeb field $\mathcal{A}$ is uniquely determined by the equations:

$$\omega(\mathcal{A}) = 1, \ \ \iota_{\mathcal{A}} d\omega = 0.$$

Let $D = \overline{\{\varphi_t\}}_{t \in \mathbb{R}} \leq \mathrm{Psh}\,(G/H)$ be the closure. As $L_g \varphi_t = \varphi_t L_g$ (for all $g \in G$) from (4.2), every element of D commutes with G. Thus every vector field $\mathcal{B}$ induced from one-parameter groups in D is left-invariant. In particular, $\omega(\mathcal{B})$ is constant. By the Cartan formula, it follows $\iota_{\mathcal{B}} d\omega = 0$. If $\omega(\mathcal{B}) \neq 0$, by uniqueness of the Reeb field, $\mathcal{B} = \mathcal{A}$ up to a constant multiple on G/H. When $\omega(\mathcal{B}) = 0$, the non-degeneracy of the Levi form $d\omega \circ J$ on $\ker \omega$ implies $\mathcal{B} = 0$ on G/H. This shows $D = \{\varphi_t\}_{t \in \mathbb{R}}$. $\ \square$

Lemma 4.2. T *acts freely on* G/H.

Proof. If $\varphi_{t_0}(x_0 H) = x_0 a_{t_0}^{-1} H = x_0 H$, for some $x_0 \in G$, then $a_{t_0} \in H$ and so $\varphi_{t_0}(xH) = xH$ (for all $x \in G$). Since T acts *effectively*, $\varphi_{t_0} = 1$. $\ \square$

In particular, any homogeneous Sasaki manifold $X = G/H$ is a regular Sasaki manifold (cf. [8]). Moreover, by Proposition 2.2 the Kähler quotient

$$W = (G/H)\big/ T$$

is a homogeneous Kähler manifold for G. That is, G is acting transitively by holomorphic isometries on W. We thus have:

Theorem 4.3 *(Boothby-Wang fibration [8]). Every homogeneous Sasaki manifold* $X = G/H$ *arises as a principal* T-*bundle over a homogeneous Kähler manifold* W *which takes the form:*

$$T \longrightarrow G/H \xrightarrow{\ \mathsf{q}\ } W = G/H\mathsf{A} \ . \tag{4.3}$$

Remark 4.4. If G/H is contractible, so is $G/H\mathsf{A}$, and in this case $T \cong \mathbb{R}$.

The following *existence and uniqueness result* for contractible homogeneous Sasaki manifolds is now a direct consequence of Section 3:

Corollary 4.5 *(Contractible homogeneous Sasaki manifolds). Let* (W, Ω, J) *be a homogeneous Kähler manifold which is contractible. Then there exists a contractible homogeneous Sasaki manifold* $(X, \{\omega, J\})$ *which has Kähler quotient* (W, Ω, J). *Moreover, with these properties, the Boothby-Wang fibration* (4.3) *for* X *has fiber* $\mathbb{R}$, *and* X *is uniquely defined up to a pseudo-Hermitian isometry.*

Proof. Indeed, we may choose on the trivial principal bundle $X = \mathbb{R} \times W$, the pseudo-Hermitian structure (3.7), which has Reeb field $\mathcal{A} = \frac{\partial}{\partial t}$ and Kähler quotient (W, Ω, J). By Proposition 3.4, $(X, \{\omega, J, \mathcal{A}\})$ is a homogeneous Sasaki manifold. Let $(X', \{\omega', J', \mathcal{A}'\})$ be another contractible Sasaki manifold which has (W, Ω, J) as a Kähler quotient. Then the Boothby-Wang fibration for X' has fiber $\mathbb{R}$, and, by Proposition 3.1, there exists a pseudo-Hermitian isometry from X' to $(X, \{\omega', J', \mathcal{A}\})$. By Proposition 3.2, the latter admits a pseudo-Hermitian isometry to $(X, \{\omega, J, \mathcal{A}\})$ which is given by a gauge transformation of the bundle X. This implies the claimed uniqueness. $\square$

4.2. Pseudo-Hermitian presentations of W

Let X be a homogeneous Sasaki manifold with group G and W its Kähler quotient. We describe now the types of homogeneous presentations

$$W = G/H\mathsf{A}$$

which can arise in the associated Boothby-Wang fibration (4.3). For this we assume that

$$G \leq \operatorname{Psh}(X)$$

is a closed subgroup. In particular, G is acting faithfully on X. With this assumption the stabilizer H is always compact, since G is a closed group of isometries for X.

Lemma 4.6. *Let* Δ *denote the kernel of the induced G-action on the Kähler quotient W of X. Then the following hold:*

(1) $H\mathsf{A} = H \rtimes \mathsf{A}$ *decomposes as a semi-direct product.*
(2) $\Delta \leq H\mathsf{A}$, *and,* $\bar{L} = H\mathsf{A}/\Delta$ *is compact.*
(3) $\Delta = T \cap G$, *in particular,* Δ *is central in G.*
(4) *If A is non-compact then the projection homomorphism* $\pi_{\mathsf{A}} : H\mathsf{A} \to \mathsf{A}$ *maps Δ injectively to a closed subgroup of A.*
(5) *If A is normal in G then A is central in G.*

Proof. Since T acts freely on G/H, we infer from (4.2) that $\mathsf{A} \cap H = \{1\}$. This implies that

$$H\mathsf{A} = H \rtimes \mathsf{A}$$

is a semi-direct product, proving (1). Let

$$\pi_{\mathsf{A}} : H\mathsf{A} \to \mathsf{A}$$

denote the projection homomorphism. Since H is compact, the homomorphism π_{A} is proper. Therefore, the image $\bar{G}$ of G in $\operatorname{Isom}(G/H\mathsf{A})$ is closed and acts properly on $W = G/H\mathsf{A} = \bar{G}/\bar{L}$. We deduce that $\bar{L} = H\mathsf{A}/\Delta$ is a compact subgroup of $\bar{G} = G/\Delta$. Thus, (2) holds.

Since the homomorphism ϕ in (2.2) which maps G to $\bar{G}$ has kernel T,

$$\Delta = G \cap T \, ,$$

where the intersection is taken in $\mathrm{Psh}\,(X)$. Recall that T is central in $\mathrm{Psh}\,(X)$. Therefore, Δ is central in G. Hence, (3).

Next, consider $C = \ker \pi_{\mathsf{A}} \cap \Delta = H \cap \Delta$. Assuming that A is a vector group, C is the unique maximal compact subgroup of Δ. Since Δ is normal in G, so is C. Since C is also a subgroup of H and G/H is effective, we deduce that $C = \{1\}$. This shows that Δ is isomorphic to the closed subgroup $\pi_{\mathsf{A}}(\Delta) \leq \mathsf{A}$, proving (4).

Finally, assume that A is normal in G. Then the left-multiplication orbits of A on G/H coincide with the orbits of T. That is, for all $g \in G$:

$$T \cdot gH = g \cdot \mathsf{A}H = \mathsf{A} \cdot gH \, .$$

In particular, the left-action of A on G/H (which is by pseudo-Hermitian isometries) induces the trivial action on the Kähler-quotient W by the fibration sequence (4.3). That is, $\mathsf{A} \leq \Delta$ and by (3), $\mathsf{A} \leq T$. This implies that $\mathsf{A} = T$ is central in G. □

Two principal cases are arising, according to whether Δ is a continuous group or Δ is a discrete subgroup of G. Recall first that either $\mathsf{A} = S^1$ or $\mathsf{A} = \mathbb{R}$. Then we have:

Case I ($\Delta = \mathsf{A}$, T is contained in G). We suppose here that A can be chosen to be a normal subgroup in G. By (5) of Lemma 4.6, it follows that the isometries induced by the left-action of A are contained in the kernel of the homomorphism $\phi : \mathrm{Psh}\,(X) \to \mathrm{Isom}\,_h(W)$, which is just T. Since A is a non-trivial connected (one-dimensional) group, this implies

$$T = \mathsf{A} = \Delta$$

as subgroups of $\mathrm{Psh}\,(X)$. Then the fibration (4.3) turns into a principal bundle of homogeneous spaces of the form

$$\mathsf{A} \longrightarrow G/H \xrightarrow{\ \mathsf{q}\ } W = (G/\mathsf{A})\,/H = \bar{G}/\bar{H} \, , \tag{I}$$

where $\bar{H} = H$ and the group $\bar{G}$ is described by an exact sequence of groups

$$1 \longrightarrow \mathsf{A} = \mathbb{R} \longrightarrow G \xrightarrow{\ \phi\ } \bar{G} \longrightarrow 1 \, . \tag{I$'$}$$

Case II ($\mathsf{A} = \mathbb{R}$, $\Delta = \mathbb{Z}$). We are assuming that $\mathsf{A} \cong \mathbb{R}$ (for example, if G/H is contractible). By Lemma 4.6 (4), the central subgroup Δ of G is either infinite cyclic (and discrete) or Δ is a closed one-parameter subgroup in $H\mathsf{A}$ which is projecting surjectively onto A. Since Δ is contained in T, and T is one dimensional, we deduce $\Delta = T$, in the latter case. This situation was already described in Case I above.

So for case II, $\Delta = T \cap G$ is infinite cyclic and central in G. Moreover, $\Delta \leq H\mathsf{A}$ and by Lemma 4.6 (4) the map π_{A} is projecting Δ injectively onto a discrete lattice $\mathcal{Z}$ in A. Denote with $\bar{\mathsf{A}}$ the image of A in $\bar{G} = G/\Delta$. Then the Boothby-Wang fibration (4.3) can be written in the form

$$\mathsf{A} \longrightarrow G/H \xrightarrow{\ \mathsf{q}\ } W = \bar{G}/\bar{H}\bar{\mathsf{A}} \, , \tag{II}$$

where the group $\bar{G}$ is described by the exact sequence

$$1 \longrightarrow \Delta = \mathbb{Z} \longrightarrow G \overset{\phi}{\longrightarrow} \bar{G} \longrightarrow 1 \, . \tag{II$'$}$$

Recall also that $\bar{L} = \bar{H}\bar{\mathsf{A}}$ is a compact subgroup of $\bar{G}$, and $\bar{H}$ is a compact normal subgroup in $\bar{H}\bar{\mathsf{A}}$. Therefore, the simply connected one-parameter group A may be chosen in such a way that its quotient $\bar{\mathsf{A}}$ is a compact circle group, and the intersection $\bar{H} \cap \bar{A}$ is finite.

5. Homogeneous Kähler manifolds of unimodular groups

Let W be a homogeneous Kähler manifold. The fundamental conjecture for homogeneous Kähler manifolds (as proved by Dorfmeister and Nakajima [12]) asserts that W is a holomorphic fiber bundle over a homogeneous bounded domain D with fiber the product of a flat space $\mathbb{C}^k$ with a compact simply connected homogeneous Kähler manifold.

Recall that a Lie group G is called *unimodular* if its Haar measure is biinvariant. Let $\mathfrak{g}$ denote the Lie algebra of G. If G is connected, then G is unimodular if and only if the trace function over the adjoint representation of $\mathfrak{g}$ is zero.

Proposition 5.1. *Let W be a contractible homogeneous Kähler manifold that admits a connected unimodular subgroup*

$$G \leq \mathrm{Isom}_h(W)$$

which acts transitively on W. Then there exists a symmetric bounded domain D such

$$W = \mathbb{C}^k \times D$$

is a Kähler direct product.

A more general result for arbitrary homogeneous Kähler manifolds of unimodular Lie groups is stated in [1, Proposition 4.2] and Proposition 5.1 can be derived as a special case. For clarity and completeness of the exposition we give a detailed and self contained proof. The condition that a homogeneous Kähler manifold is contractible clarifies the existence of symmetric domain in the product decomposition and also simplifies the proof. This result will be used to determine isometry groups in the next section.

Proof of Proposition 5.1. For the proof we require some constructions which are developed in the proof of the fundamental conjecture as it is given in [12]. The first main step in the proof of the fundamental conjecture is to modify G in order to obtain a suitable connected transitive Lie group $\hat{G}$ with particular nice properties [12, Theorem 2.1]. By a modification procedure on the level of Lie algebras (as is described in [12, §2.4]), we obtain from the Kähler Lie algebra $\mathfrak{g}$ of G a quasi-normal Kähler Lie algebra $\hat{\mathfrak{g}}$. Moreover, it is shown that there exists a connected subgroup $\hat{G} \leq \mathrm{Isom}_h(W)$, which has Lie algebra $\hat{\mathfrak{g}}$ and acts transitively on W. As can be verified directly from [12, §2.4], the modified Lie algebra $\hat{\mathfrak{g}}$ preserves unimodularity of $\mathfrak{g}$ and also satisfies $\dim \hat{\mathfrak{g}} \leq \dim \mathfrak{g}$.

Therefore, from the beginning, we may assume that the connected unimodular transitive Lie group G of holomorphic isometries in question has a quasi-normal Lie algebra $\mathfrak{g}$. We can also replace G with its universal covering group, and we remark that K *is connected* ($W = G/K$ is simply connected, since we are assuming here that W is contractible). With these additional properties in place, according to [12, Theorem 2.5] combined with [12, §7], the following hold:

(1) There exists a closed connected normal abelian subgroup A of G, such that $G = AH$ is an almost semi-direct product.

(2) There exists a reductive subgroup $U \leq H$, with $K \leq U$, such that

$$D = H/U$$

is a bounded homogeneous domain and

$$U/K$$

is compact with finite fundamental group.

(3) Put $L = AU$. Then L is a closed subgroup of G and the map

$$W = G/K \to G/L = H/U = D$$

is a holomorphic fiber bundle with fiber $L/K = AU/K$.

We prove now that, if G is unimodular then *H is a unimodular Lie group*: For this recall from [12, Theorem 2.5] that A is tangent to a Kähler ideal $\mathfrak{a}$ of the Kähler algebra which belongs to W. (Recall that the Kähler algebra for G/K is $\mathfrak{g}$ together with an alternating two-form ρ which is representing the Kähler form on W.) Since K intersects A only trivially, the Kähler ideal $\mathfrak{a}$ is non-degenerate, that is, the restriction $\rho_\mathfrak{a}$ of the Kähler form ρ of $\mathfrak{g}$ to $\mathfrak{a}$ is non-degenerate. Since $\mathfrak{a}$ is abelian and ρ is a closed form on $\mathfrak{g}$, it follows that $\rho_\mathfrak{a}$ is invariant by the restriction of the adjoint representation of $\mathfrak{h}$ (respectively H). In particular, this restricted representation of H on $\mathfrak{a}$ is by unimodular maps. Since G is unimodular, it follows from the semi-direct product decomposition $G = AH$ that H is unimodular.

Let K_1 denote the maximal compact normal subgroup of H. Then the group $H' = H/K_1$ is unimodular. Moreover H' acts faithfully and transitively on $D = H/K = H'/K'$, $K' = K/K_1$. Hence, the bounded domain D has a transitive faithful unimodular group H' of isometries. By results of Hano [13, Theorem III, IV], H' must be semisimple and *D is a symmetric bounded domain.* We also conclude that there exists a semisimple subgroup $S \leq G$, which is of non-compact type, such that $H = K_1 S$ is an almost direct product and the homomorphism $S \to H'$ is a covering with finite kernel.

Contractibility of W further implies $U = K$. Therefore, in this case, the holomorphic bundle in (3) is of the form

$$G/K \to D = H/K \, ,$$

with fiber $A = \mathbb{C}^k$, and D is a symmetric bounded domain.

Finally the direct product decomposition follows: Note also that K_1 acts faithfully on $\mathbb{C}^k$ by Kähler isometries, and that S acts trivially on A, since it is of non-compact type. It follows that S is a normal subgroup of G. Therefore its tangent algebra must be orthogonal to A with respect to ρ. Since the Kähler algebra $\mathfrak{g}$ belonging to G is describing W, we conclude that there is an orthogonal product decomposition $W = \mathbb{C}^k \times D$. $\quad \square$

We also obtain:

Corollary 5.2. *Suppose that W is a contractible homogeneous Kähler manifold, and that there exists a discrete uniform subgroup in* $\mathrm{Isom}_h(W)$. *Then W is Kähler isometric to $\mathbb{C}^k \times D$, where D is a symmetric bounded domain.*

The following is obtained in the proof of Proposition 5.1:

Corollary 5.3. *Assume that W is a homogeneous Kähler manifold which admits a transitive unimodular group G. Then there exists a symmetric bounded domain D such that W is a holomorphic fiber bundle over D with fiber the product of a flat space $\mathbb{C}^k$ with a compact simply connected homogeneous Kähler manifold. Moreover, $\mathrm{Isom}_h(W)^0$ contains a covering group of the identity component of the holomorphic isometry group of D.*

Proof. In fact, in the proof of Proposition 5.1 it is established that D is symmetric with a semisimple transitive group S contained in the quasi normal modification $\bar{G}$ of G. It is also clear that S is normal in $\bar{G}$, and it is the maximal semisimple subgroup of non-compact type in $\bar{G}$ (in fact, in $\mathrm{Isom}_h(W)^0$), and S is covering $\mathrm{Isom}_h(D)^0$. $\square$

We recall that any symmetric bounded domain D admits an involutive anti-holomorphic isometry:

Proposition 5.4 *(Isometry group of symmetric bounded domain). Let D be a symmetric bounded domain with Kähler structure (Ω, J). If D is* irreducible *then*

$$\mathrm{Isom}(D) = \mathrm{Isom}_h^{\pm}(D) \ .$$

Moreover, for any D there exists an element $\bar{\tau} \in \mathrm{Isom}(D)$ such that

$$\bar{\tau}^2 = 1, \ \bar{\tau}^* \Omega = -\Omega, \ \bar{\tau}_* J = -J\bar{\tau}_*.$$

For the fact that every isometry of an irreducible bounded symmetric domain is either holomorphic or anti-holomorphic, see e.g. [18, Ch. VIII, Ex. B4]. For the existence of the anti-holomorphic involution $\bar{\tau}$, recall first that the metric on any symmetric bounded domain D is analytic (see [18]). Then the following holds:

Proposition 5.5. *Let W be a simply connected Kähler manifold with analytic Kähler metric. Then there exists an anti-holomorphic involutive isometry $\bar{\tau}$ of W.*

Proof. Since W is a complex manifold and the Kähler metric is Hermitian with respect to the complex structure, there exist local complex coordinates for W such that the metric can be written as

$$g_0 = 2 \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \ ,$$

where $g_{\alpha\bar{\beta}}$ is a Hermitian matrix, so that $g_{\alpha\bar{\beta}} = \overline{g_{\beta\bar{\alpha}}}$. In particular, the Kähler form Ω_0 is obtained as $\Omega_0 = -2i \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$.

Let $\tau_0 : \mathbb{C}^n \to \mathbb{C}^n$ be the complex conjugation map, that is,

$$\tau_0(z_1, z_2, \ldots, z_n) = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n).$$

Then τ_0 satisfies $\tau_0^* \Omega_0 = -\Omega_0$, $\tau_{0*} J_{\mathbb{C}} = -J_{\mathbb{C}} \tau_{0*}$. In particular, τ_0 defines a local anti-holomorphic isometry of W.

Since W is simply connected, we may use analytic continuation to extend τ_0 to an analytic map $\bar{\tau} : W \to W$. By the analyticity assumptions, $\bar{\tau}$ is an anti-holomorphic map and it is preserving the Kähler metric. Also it follows $\bar{\tau}^2 = \mathrm{id}_W$ by the local rigidity of analytic maps. Therefore, $\bar{\tau}$ is an involutive isometry of W. $\square$

Remark 5.6. Note that the holomorphic isometry group $\mathrm{Isom}_h(D)$ has finitely many connected components. Interestingly, even if D is irreducible $\mathrm{Isom}_h(D)$ is not necessarily connected [18, Ch. X, Ex. 8].

6. Locally homogeneous aspherical Sasaki manifolds

In this section X denotes a regular contractible Sasaki manifold.

6.1. Homogeneous Sasaki manifolds for unimodular groups

Since X is regular with Reeb flow isomorphic to the real line, Proposition 3.4 implies that the Reeb fibering

$$\mathbb{R} \to X \xrightarrow{q} W$$

gives rise to an exact sequence of groups

$$1 \to \mathbb{R} \to \mathrm{Psh}\,(X) \xrightarrow{\phi} \mathrm{Isom}_h(W) \to 1 \,, \tag{6.1}$$

where W is the Kähler quotient of X.

Proposition 6.1. *Let X be a homogeneous Sasaki manifold such that its Kähler quotient W is contractible. Suppose further that X admits a connected transitive unimodular subgroup*

$$G \leq \mathrm{Psh}\,(X) \,.$$

Then the following hold:

(1) *$W = \mathbb{C}^k \times D$ is the Kähler product of a flat space with a symmetric bounded domain D.*
(2) *If the Reeb flow of X is isomorphic to $\mathbb{R}$, then the pullback of*

$$\mathbb{C}^k \rtimes \mathrm{U}(k) \leq \mathrm{Isom}_h(W)$$

along the exact sequence (6.1) *is a normal subgroup*

$$\mathcal{N} \rtimes \mathrm{U}(k) \leq \mathrm{Psh}\,(X) \,,$$

where $\mathcal{N}$ is a $2k+1$-dimensional Heisenberg Lie group.

Proof. As the Reeb flow T is central in $\mathrm{Psh}\,(X)^0$, the associated Boothby-Wang homomorphism ϕ as in (6.1) maps the unimodular group G to

$$\bar{G} = \phi(G) \leq \mathrm{Isom}_h(W) \,.$$

Since also $\bar{G}$ is unimodular and transitive on the contractible Kähler manifold W, Proposition 5.1 states that $W = \mathbb{C}^k \times D$, where D is a symmetric bounded domain. This proves (1). It also follows that

$$\mathrm{Isom}_h(W) = \left(\mathbb{C}^k \rtimes \mathrm{U}(k)\right) \times \mathrm{Isom}_h(D) \,.$$

We may thus pull back the factor $\mathbb{C}^k \rtimes \mathrm{U}(k)$ by ϕ in the exact sequence (6.1). As pullback we obtain the subgroup $\mathcal{N} \rtimes \mathrm{U}(k) \leq \mathrm{Psh}\,(X)$, where $\mathcal{N}$ is the preimage of the translation group $\mathbb{C}^k$.

Assuming $T = \mathbb{R}$, we note that $\mathcal{N}$ is a central extension of the Reeb flow $\mathbb{R}$ by the abelian Lie group $\mathbb{C}^k$. We prove now that $\mathcal{N}$ is a $2k + 1$-dimensional Heisenberg Lie group by showing that its Lie algebra $\mathfrak{n}$ has one-dimensional center: Since $\mathcal{N}$ acts faithfully as a transformation group on X, we may identify $\mathfrak{n}$ with a subalgebra of pseudo-Hermitian Killing vector fields on X. This subalgebra contains the Reeb field $\mathcal{A}$ (tangent to the central one-parameter group $T = \mathbb{R}$) in its center. Now, since $\mathcal{N}$ is the pullback of $\mathbb{C}^k$, given any two vector fields $\mathcal{X}, \mathcal{Y} \in \mathfrak{n}$, we have

$$[\mathcal{X}, \mathcal{Y}] = \omega([\mathcal{X}, \mathcal{Y}])\mathcal{A} .$$

Using Lemma 6.2 below, we observe

$$q^* \Omega \left(\mathcal{X}, \mathcal{Y} \right) = d\omega(\mathcal{X}, \mathcal{Y}) = \omega([\mathcal{X}, \mathcal{Y}]) .$$

Since $(\mathbb{C}^k, \Omega)$ is Kähler, it follows that $d\omega$ defines a non-degenerate two-form on $\mathfrak{n}/\langle \mathcal{A} \rangle$. This shows that the Lie algebra $\mathfrak{n}$ has one-dimensional center $\mathcal{A}$. Therefore the Lie group $\mathcal{N}$ has one-dimensional center. So $\mathcal{N}$ is a Heisenberg group of dimension $2k + 1$. $\square$

A vector field on $\mathcal{X}$ with flow in $\mathrm{Psh}\,(X)$ will be called a *pseudo-Hermitian vector field*. The set of pseudo-Hermitian vector fields forms a subalgebra of the Lie algebra of Killing vector fields for the Sasaki metric g.

Lemma 6.2. *Let $\mathcal{X}, \mathcal{Y}$ be any two pseudo-Hermitian Killing vector fields on the Sasaki manifold X. Then*

$$q^* \Omega \left(\mathcal{X}, \mathcal{Y} \right) = d\omega(\mathcal{X}, \mathcal{Y}) = \omega([\mathcal{X}, \mathcal{Y}]) .$$

Proof. Since the flow of $\mathcal{X}$ preserves the contact form ω, we have

$$L_{\mathcal{X}}\, \omega = 0 .$$

(Here, $L_{\mathcal{X}}$ denotes the Lie derivative with respect to $\mathcal{X}$.) That is,

$$L_{\mathcal{X}}\, \omega(\mathcal{Z}) - \omega([\mathcal{X}, \mathcal{Z}]) = 0,$$

for all vector fields $\mathcal{Z}$ on X. We compute

$$
\begin{aligned}
0 &= L_{\mathcal{X}}\, \omega(\mathcal{Y}) - \omega([\mathcal{X}, \mathcal{Y}]) - L_{\mathcal{Y}}\, \omega(\mathcal{X}) + \omega([\mathcal{Y}, \mathcal{X}]) \\
&= L_{\mathcal{X}}\, \omega(\mathcal{Y}) - L_{\mathcal{Y}}\, \omega(\mathcal{X}) - \omega([\mathcal{X}, \mathcal{Y}]) + \omega([\mathcal{Y}, \mathcal{X}]) \\
&= d\omega(\mathcal{X}, \mathcal{Y}) - \omega([\mathcal{X}, \mathcal{Y}]) . \quad \square
\end{aligned}
$$

Let X be a contractible homogeneous Sasaki manifold with Kähler quotient $W = \mathbb{C}^k \times D$, where D is a symmetric bounded domain. Then

$$\mathrm{Isom}\,_h(W) = \left(\mathbb{C}^k \rtimes \mathrm{U}(k) \right) \times \mathrm{Isom}\,_h(D) . \tag{6.2}$$

Note further that $\mathrm{Isom}\,_h(D)^0 = S_0$ is the identity component of the holomorphic isometry group of a Hermitian symmetric space

$$D = S_0/H_0$$

of non-compact type. In particular, S_0 is semisimple of non-compact type [18, Ch. VIII, §7] and without center. Therefore (6.2) also gives:

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Proposition 6.3. $\mathrm{Psh}\,(X)$ *has finitely many connected components and*

$$\mathrm{Psh}\,(X)/\,\mathrm{Psh}\,(X)^0 = \mathrm{Isom}\,_h(D)/\,\mathrm{Isom}\,_h(D)^0 \ .$$

We add:

Proposition 6.4 *(Sasaki automorphism group). There exists a semisimple Lie group S of non-compact type, whose center Λ is infinite cyclic, and a $2k+1$ dimensional Heisenberg groups $\mathcal{N}$, such that there is an almost direct product decomposition*

$$\mathrm{Psh}\,(X)^0 \ = \ (\mathcal{N} \rtimes \mathrm{U}(k)) \cdot S \ .$$

Moreover, the Reeb flow T of X is the center of $\mathcal{N}$ and

$$T \cap S = (\mathcal{N} \rtimes \mathrm{U}(k)) \cap S = \Lambda \ (\cong \mathbb{Z}).$$

Proof. For the homogeneous Sasaki manifold X, the exact sequence of groups (6.1) associated to the Reeb fibering for X induces a central extension

$$1 \to T \to (\mathcal{N} \rtimes \mathrm{U}(k)) \cdot S \ \overset{\phi}{\to} \ (\mathbb{C}^k \rtimes \mathrm{U}(k)) \times S_0 \to 1 \ , \tag{6.3}$$

where the Reeb flow $T = \mathbb{R}$ maps to the center of $\mathcal{N}$. Here

$$S \leq \mathrm{Psh}\,(X)^0$$

is a semisimple normal subgroup of non-compact type, which is covering S_0 under ϕ. In particular, since S is a normal subgroup of $\mathrm{Psh}\,(X)^0$, it commutes with $\mathcal{N} \rtimes \mathrm{U}(k)$. (Note also that $\mathrm{U}(k)$ acts faithfully on $\mathcal{N}$ and maps to a maximal compact subgroup of $\mathrm{Aut}(\mathcal{N})$.)

The kernel Λ of the covering $S \to S_0$ is

$$\ker \phi \cap S = \mathbb{R} \cap S = (\mathcal{N} \rtimes \mathrm{U}(k)) \cap S \ .$$

Moreover, Λ is the center of S, since S_0 has trivial center. We claim that Λ is an infinite cyclic discrete subgroup and, in particular, it is a uniform subgroup in T. Indeed, in the light of Corollary 4.5, there exists a unique contractible homogeneous Sasaki manifold X_1 with Kähler quotient $\mathbb{C}^k$, and similarly a unique homogeneous Sasaki manifold X_2 with Kähler quotient D. Let $T_i \leq \mathrm{Psh}\,(X_i)$ denote the Reeb flow of X_i. Then (see Section 2.3, Corollary 2.7) the join $X_1 * X_2$ is a homogeneous Sasaki manifold with Kähler quotient $\mathbb{C}^k \times D$. According to the above, $\mathrm{Psh}\,(X_1) = \mathcal{N} \rtimes \mathrm{U}(k)$, and by Proposition 6.9 below $\mathrm{Psh}\,(X_2)^0 = T_2 \cdot S$, where S is a closed semisimple Lie subgroup covering S_0 with infinite cyclic kernel $\Lambda = Z(S)$, $T_2 \cap S = \Lambda$. It follows that $\mathrm{Psh}\,(X)^0 = \mathrm{Psh}\,(X_1) * \mathrm{Psh}\,(X_2)^0$ has the claimed properties. $\square$

6.2. Application to locally homogeneous Sasaki manifolds

We consider a compact aspherical Sasaki manifold of the form

$$M = \Gamma \backslash X \ ,$$

where X is a contractible Sasaki manifold and Γ is a torsion free discrete subgroup contained in $\mathrm{Psh}\,(X)$. If X is a homogeneous Sasaki manifold then M is called a *locally homogenous Sasaki manifold*.

Theorem 6.5. *Suppose that X is a contractible homogeneous Sasaki manifold and that X admits a discrete subgroup of isometries with*

$$\Gamma \backslash X$$

compact. Then:

(1) *The Kähler quotient of X is a Kähler product*

$$W = \mathbb{C}^k \times D$$

   of a flat space $\mathbb{C}^k$ with a symmetric bounded domain D.
(2) *Let T denote the Reeb flow of X. Then $\Gamma \cap T$ is a discrete uniform subgroup of T (in particular, $\Gamma \cap T$ is isomorphic to $\mathbb{Z}$).*
(3) *Let $\phi : \mathrm{Psh}\,(X) \to \mathrm{Isom}\,_h(W)$ be the Boothby-Wang homomorphism in (6.1). Then the subgroup*

$$\phi(\Gamma) \leq \mathrm{Isom}\,_h(W)$$

   is discrete and uniform.

Corollary 6.6. *Let $M = X/\Gamma$ be a compact locally homogeneous Sasaki manifold. Then M is a Sasaki manifold with compact Reeb flow $T = S^1$. Moreover, a finite covering space of M is a regular Sasaki manifold.*

Remark 6.7. Certain linear flows on the sphere give rise to *irregular* compact Sasaki manifolds, cf. [11, Chapters 2, 7].

For the preparation of the proof of Theorem 6.5 we shall recall some standard facts about:

Levi decomposition and uniform lattices. In general a *connected* Lie group G admits a Levi decomposition

$$\mathsf{G} = \mathsf{R} \cdot \mathsf{S} \, ,$$

where R is the solvable radical of G and S is a semisimple subgroup. Let K denote the maximal compact and *connected* normal subgroup of S, then put $\mathsf{S}_0 = \mathsf{G}/(\mathsf{RK})$. Note that S_0 is semisimple of non-compact type. We will need the following fact (see [28, Chapter 4, Theorem 1.7], for example):

Proposition 6.8. *Let Γ be a uniform lattice in G. Then the intersection $(\mathsf{RK}) \cap \Gamma$ is a uniform lattice in RK. In particular, in the associated exact sequence*

$$1 \longrightarrow \mathsf{RK} \longrightarrow \mathsf{G} \xrightarrow{\nu} \mathsf{S}_0 \longrightarrow 1, \tag{6.4}$$

the image $\nu(\Gamma)$ is a uniform lattice in the semisimple Lie group S_0.

Remark in addition the following: As the subgroup $\nu(\Gamma) \leq \mathsf{S}_0$ is discrete and uniform, and since S_0 has no compact normal connected subgroup, the image of $\nu(\Gamma)$ is a Zariski dense subgroup in the adjoint form of S_0 (by Borel's density theorem, cf. [24]). Consider any connected closed subgroup $\mathcal{G}$ of S_0, which contains $\nu(\Gamma)$. Then $\mathcal{G}$ is uniform and Zariski-dense. This implies that $\mathcal{G} = \mathsf{S}_0$.

Now we are ready for the

Proof of Theorem 6.5. Note that $\Gamma_0 = \Gamma \cap \mathrm{Psh}\,(X)^0$ is a discrete uniform subgroup of $\mathrm{Psh}\,(X)^0$ (compare [5, Lemma 2.3]). The existence of a lattice subgroup implies that $\mathrm{Psh}\,(X)^0$ is a *unimodular* Lie group, see e.g. [24, 1.9 Remark]. By Proposition 6.1, $W = \mathbb{C}^k \times D$, where D is a symmetric bounded domain and

$$\mathrm{Isom}\,_h(W) = \left(\mathbb{C}^k \rtimes \mathrm{U}(k)\right) \times \mathrm{Isom}\,_h(D)\,.$$

Since $S_0 = \mathrm{Isom}\,_h(D)^0$ is semisimple of non-compact type, we can apply Proposition 6.8 to $\mathrm{Psh}\,(X)^0$, to yield that the intersection $\Gamma \cap (\mathcal{N} \rtimes \mathrm{U}(k))$ is discrete uniform in $\mathcal{N} \rtimes \mathrm{U}(k)$. Then the Auslander-Bieberbach theorem [2] shows that, a fortiori, $\Gamma \cap \mathcal{N}$ is uniform in $\mathcal{N}$. As $\mathbb{R}$ is the center of the Heisenberg group $\mathcal{N}$, $\Gamma \cap \mathbb{R}$ is also uniform in $\mathbb{R}$ (cf. [24, Chapter II]). In particular, in the light of (6.4), this implies that $\phi(\Gamma)$ is a discrete uniform subgroup of $\mathrm{Isom}\,_h(W)$. $\square$

6.3. Sasaki homogeneous spaces over symmetric bounded domains

We assume now that the Kähler quotient of X is a symmetric bounded domain D. Let

$$S_0 = \mathrm{Isom}\,_h(D)^0$$

be the identity component of the group of holomorphic isometries of D, and

$$\phi : \mathrm{Psh}\,(X)^0 \to S_0$$

the Boothby-Wang homomorphism. Recall that S_0 is semisimple of non-compact type with trivial center. Moreover, we can write

$$D = S_0/K_0\,,$$

where K_0 is a maximal compact subgroup of S_0.

We prove that X is a Sasaki homogeneous space of a semisimple Lie group:

Proposition 6.9. *There exists a semisimple closed normal subgroup*

$$S \leq \mathrm{Psh}\,(X)^0$$

such that the restricted Boothby-Wang map

$$\phi : S \to S_0$$

is a covering with infinite cyclic kernel Λ, where Λ is the center of S. In particular, if $T = \mathbb{R}$ denotes the Reeb flow on X, then

$$\mathrm{Psh}\,(X)^0 = S \cdot \mathbb{R}, \ \text{with} \ S \cap \mathbb{R} = \Lambda \ (\cong \mathbb{Z})\,.$$

Moreover, the subgroup S of $\mathrm{Psh}\,(X)$ acts transitively on X.

Proof. Put $G = \mathrm{Psh}\,(X)^0$. Then G satisfies the exact sequence

$$1 \longrightarrow T = \mathbb{R} \longrightarrow G \overset{\phi}{\longrightarrow} S_0 \longrightarrow 1\,,$$

where the Reeb flow T is a central subgroup of G. By the Levi-decomposition theorem, the above exact sequence splits and

$$G = T \cdot S \, ,$$

where S is a covering group of S_0 under ϕ. Note that S is a normal subgroup of G, and $\ker \phi \cap S = T \cap S = Z(S)$ is the center of S, and a torsion-free abelian group.

Assume that $T \cap S = \{1\}$. In particular $S = S_0$ and $G = T \times S_0$. Then K_0 is also a maximal compact subgroup of G. Choose $x_o \in X$ such that $K_0 \, x_0 = x_0$. Then $S_0 \cdot x_0 = S_0/K_0$ and it follows that

$$X = \mathbb{R} \times S_0/K_0 \, .$$

Moreover, the Boothby-Wang fibering $q : X \to D$ corresponds to the projection onto the second factor. Let ω_0 be the contact form of the Sasaki structure on X. By Proposition 3.1 there exists a one-form θ on $D = S_0/K_0$ such that

$$\omega_0 = dt + q^*\theta \, .$$

Since ω_0 is invariant by $S = S_0$, this implies that $q^*\theta$ is invariant by S. Therefore also θ is invariant by S_0. In particular, the two form $\Omega = d\theta$ is an S_0-invariant exact form.

We can now apply a classical result of Koszul to Ω as follows. Let $\mathfrak{s}$ and $\mathfrak{k}$ denote the Lie algebras of S_0 and K_0, respectively. The S_0-invariant Kähler form Ω defines a cohomology class in the relative Lie algebra cohomology group $H^2(\mathfrak{s}, \mathfrak{k})$. Since $\mathfrak{s}$ is unimodular and $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{s}$, a result of Koszul [22] asserts that the cohomology ring $H^*(\mathfrak{s}, \mathfrak{k})$ satisfies Poincaré duality. Since Ω is a non-degenerate two-form, the class $[\Omega] \in H^2(\mathfrak{s}, \mathfrak{k})$ is non-zero. This contradicts $\Omega = d\theta$, for some S_0-invariant form θ on S_0/K_0. We conclude that $T \cap S = \{1\}$ is not possible.

Therefore, we have that $\ker \phi \cap S = T \cap S = \Lambda$ is isomorphic to $\mathbb{Z}^k$, $k \geq 1$. Since Λ is the center of S, there exists a closed k-dimensional subgroup B of S, $\mathsf{B} \cong \mathbb{R}^k$, containing Λ, and ϕ maps B to a toral subgroup $(S^1)^k$ contained in the center of K_0, cf. [18, Ch. VI, §1]. Let K be the maximal compact subgroup of S. We then have

$$1 + \dim D = \dim X \geq \dim S/K = k + \dim S_0/K_0 = k + \dim D \, .$$

Since $k \geq 1$, we deduce $k = 1$ and $X = S/K$. Hence, S acts transitively on X. Since $\mathbb{Z} \cong \ker \phi \cap S$ is an infinite cyclic discrete subgroup of T, it also follows that S is a closed subgroup of $\mathrm{Psh}\,(X)$, see [14, Theorem B]. $\quad\square$

6.4. Summary on locally homogeneous Sasaki manifolds

Most of the above is summarized in Theorem 1 in the introduction:

Proof of Theorem 1. Statement (1) about the Kähler quotient $W = X/T$ is established in (1) of Theorem 6.5.

We remark next that the Reeb flow T is normal in $\mathrm{Isom}\,(X)$. Indeed, since X is non-compact there can be only two Killing fields $\{\mathcal{A}, -\mathcal{A}\}$ which are Sasaki compatible with the metric g on X (cf. [26,20,27]). It follows that $\mathrm{Isom}\,(X) = \mathrm{Psh}^{\pm}\,(X)$. The properties of the homomorphism $\phi : \mathrm{Isom}\,(X) \to \mathrm{Isom}_h^{\pm}(W)$ are established in Proposition 3.4 and Lemma 3.5, proving (2).

Let $\bar{\tau} : W \to W$ be an anti-holomorphic involution (which exists by Proposition 5.4 and Note 7.5). Then by Lemma 3.5, there exists an anti pseudo-Hermitian and involutive lift $\tau : X \to X$. Now (3) follows.

Since $\mathrm{Isom}\,(X) = \mathrm{Psh}^{\pm}\,(X)$, we deduce that $\mathrm{Isom}\,(X)^0 = \mathrm{Psh}\,(X)^0$. Therefore part (4) is a consequence of Proposition 6.4.

Finally, let $\Gamma \backslash G/H$ be a locally homogeneous aspherical Sasaki manifold, and $X = G/H$. Then there is the exact sequence:

$$1 \longrightarrow \Gamma \longrightarrow N_{\mathrm{Isom}\,(X)}(\Gamma) \longrightarrow \mathrm{Isom}\,(\Gamma \backslash X) \longrightarrow 1.$$

Thus the claim (5) (stated below of Theorem 1) follows from (3). $\square$

Proof of Corollary 1. Assume that $\Gamma \backslash X$ is compact. As usual T denotes the Reeb flow for X. Then by (2) of Theorem 6.5, $\Gamma \cap T$ is an infinite cyclic group $\mathbb{Z}$. Put

$$S^1 = T/(\Gamma \cap T)\,.$$

According to (3) of Theorem 6.5, taking the quotient of $\Gamma \backslash X$ by S^1, this induces an S^1-bundle over a compact locally homogeneous aspherical Kähler orbifold of the form:

$$S^1 \longrightarrow \Gamma \backslash G/H \longrightarrow \phi(\Gamma) \backslash W\,.$$

Here S^1 induces the Reeb field of $\Gamma \backslash X$. This S^1-bundle is usually referred to as a *Seifert fibering* (cf. [23]). In particular, since $\mathrm{Isom}\,_h(W)$ is a linear Lie group, we can choose a torsionfree finite index normal subgroup of $\phi(\Gamma)$. Therefore, some finite cover of $\Gamma \backslash G/H$ becomes a regular Sasaki manifold. This proves Corollary 1. $\square$

6.4.1. Solvable fundamental group

Note (see [4, Theorem 0.2]) that every compact aspherical Kähler manifold N with virtually solvable fundamental group Γ is biholomorphic to a flat Kähler manifold $\mathbb{C}^k/\Gamma$ for some embedding of Γ into $\mathbb{C}^k \rtimes \mathrm{U}(k)$ as a discrete uniform subgroup. This shows, that the Kähler manifold N, in fact, admits a locally homogeneous (and flat) Kähler structure, with respect to its original complex structure. Based on this result we prove now the following (which is also implying Corollary 2 in the introduction):

Proposition 6.10. *Let M be a regular compact aspherical Sasaki manifold with virtually solvable fundamental group. Then the given Sasaki structure on M can be deformed (via regular Sasaki structures) to a locally homogeneous regular Sasaki structure.*

Proof. By the Boothby-Wang fibration result for compact regular Sasaki manifolds [8], M is a principal circle bundle $S^1 \to M \stackrel{q}{\to} N$ over a compact Kähler manifold $(N, \{\Omega, J\})$. Moreover, the Kähler class $[\Omega] \in H^2(N, \mathbb{R})$ is integral and it is the image of the characteristic class $c(q) \in H^2(N, \mathbb{Z})$ of the bundle. Let π denote the fundamental group of M. On the level of fundamental group the circle bundle gives rise to a central group extension

$$1 \to \mathbb{Z} \to \pi \to \Gamma \to 1 \tag{6.5}$$

such that its extension class in $H^2(\pi, \mathbb{Z}) \cong H^2(N, \mathbb{Z})$ also maps to $[\Omega]$. (In this context, the Seifert circle bundle M is said to realize the group extension (6.5).)

Since Γ is virtually solvable there exists a biholomorphic diffeomorphism $\Phi : \mathbb{C}^k/\Gamma \to (N, J)$. Since $\Lambda = \Gamma \cap \mathbb{C}^k$ is a finite index subgroup of Γ and a lattice in $\mathbb{C}^k$, we can construct an embedding $\pi \to \mathcal{N} \rtimes \mathrm{U}(k)$ such that $\Delta = \pi \cap \mathcal{N}$ is a uniform discrete subgroup in $\mathcal{N}$, and the embedding induces a compatible map of exact sequences from (6.5) to the defining exact sequence of the group $\mathrm{Psh}\,(\mathcal{N})$ which is of the form

$$1 \to \mathbb{R} \to \mathrm{Psh}\,(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(k) \to \mathbb{C}^k \rtimes \mathrm{U}(k)\,. \tag{6.6}$$

This constructs a locally homogeneous Sasaki structure on the quotient manifold $\mathcal{N}/\pi$ with Kähler quotient $\mathbb{C}^k/\Gamma$ and another Seifert circle bundle $S^1 \to \mathcal{N}/\pi \to \mathbb{C}^k/\Gamma$ which realizes the exact sequence (6.5).

By the rigidity for Seifert fiberings (cf. [23]) there exists an isomorphism of circle bundles $\Psi : \mathcal{N}/\pi \to M$ which induces the biholomorphic map Φ on the base spaces. This shows that the principal circle bundle $q : M \to N$ admits a compatible *locally homogeneous* Sasaki structure $(M, \{\omega', J'\})$ which is modeled on $\mathcal{N}$ and has Kähler quotient $(N, \{\Omega', J\})$, where Ω' is a flat (locally constant) Kähler form on (N, J).

Moreover, by the above remarks $[\Omega'] = [\Omega] \in H^2(N, \mathbb{R})$. Hence, we can write $\Omega' = \Omega + \theta$, where $\theta = J\partial\bar{\partial}\varphi$, for some potential function $\varphi : N \to \mathbb{R}$. (See [6, §11.C] for parametrization of the space of Kähler forms on the complex manifold (N, J), which is realizing the given Kähler class $[\Omega]$.) We may thus choose a continuous path of cohomologous Kähler forms $\Omega_t = \Omega + \theta_t$, $\theta_0 = 0$ and $\theta_1 = \theta$, that is joining Ω and Ω', e.g. $\theta_t = t\theta$. Since the forms θ_t are exact, we may lift to a continuous path of one-forms $\tau_t \in \Omega^1(N)$ which is satisfying $d\tau_t = \theta_t$.

Finally, let ω denote the connection form on the given circle bundle $q : M \to N$, which defines the given regular Sasaki structure with Kähler quotient $(N, \{\Omega, J\})$. Then it follows that the connection forms $\omega_t = \omega + q^*\theta_t$ give rise to a continuous family of regular Sasaki structures $\{\omega_t, J_t\}$ compatible with the circle bundle q and with Kähler quotients $(N, \{\Omega_t, J\})$. It follows that $(M, \{\omega_1, J_1\})$ and $(M, \{\omega', J'\})$ are Sasaki structures over the Kähler quotient $(N, \{\Omega', J\})$, with $(M, \{\omega', J'\})$ being locally homogeneous. The universal covering space X of M inherits the structure of a principal $\mathbb{R}$-bundle over the unitary space $\mathbb{C}^k$ with induced Sasaki structures from $\{\omega_1, J_1\}$ and $\{\omega', J'\}$. The latter one being homogeneous with group $\mathrm{Psh}\,(X) \cong \mathrm{Psh}\,(\mathcal{N})$. Proposition 3.2 shows that the induced structures on X are equivalent Sasaki structures. In particular, both are homogeneous Sasaki structures. This shows that $(M, \{\omega_1, J_1\})$ is a locally homogeneous Sasaki structure. $\square$

7. Classifications of homogeneous Sasaki spaces

In this section we tackle the classification problems for (1) aspherical Sasaki homogeneous spaces of semisimple Lie groups and (2) contractible Sasaki Lie groups up to equivalence.

7.1. Homogeneous Sasaki spaces of semisimple Lie groups

We call a connected semisimple Lie group S_0 of non-compact type a Lie group of *Hermitian type* if it is the identity component of the holomorphic isometry group of a symmetric bounded domain $D = S_0/K_0$.

Theorem 7.1. *Let X be a contractible Sasaki homogeneous space of a semisimple Lie group*

$$S \;\leq\; \mathrm{Psh}\,(X)^0\,.$$

Then S has infinite cyclic center and

$$X = S/K\,,$$

where K is a maximal compact subgroup of S. Moreover, S is covering a Lie group S_0 of Hermitian type, such that:

(1) *The Kähler quotient of X is the symmetric bounded domain*

$$D = S_0/K_0\,.$$

(2) *There exists a simply connected one parameter subgroup* $\mathsf{A} \leq S$, *contained in the centralizer of* K, *whose action on* X *induces the Reeb flow, and the Boothby-Wang fibration for* X *is of the form*

$$\mathsf{A} \to X = S/K \to D = X/\mathsf{A} = S/K\mathsf{A}\,.$$

(3) *If* T *denotes the Reeb flow for* X *then*

$$\mathrm{Psh}\,(X)^0 = T \cdot S\,,$$

and $T \cap S = \Lambda$ *is the center of* S.

Proof. Given a Sasaki metric on X which is homogeneous for the semisimple group $G = S$, the Boothby-Wang presentation of the Kähler quotient W must be of type (II) (cf. Section 4.2). That is, it is of the form

$$\mathsf{A} \to S/K \to W = S/K\mathsf{A} = S_0/K_0\,.$$

Moreover, $\mathsf{A} \leq S$ is a one-parameter subgroup centralizing K, and $S \to S_0$ is a covering homomorphism with infinite cyclic kernel Λ. In particular, $W = X/\mathsf{A}$ is contractible and it is a faithful Kähler homogeneous space of the semisimple Lie group $S_0 = S/\Lambda$. By Proposition 5.1, $W = D$ is Kähler isometric to a bounded symmetric domain D, and S_0 is the identity component of the isometry group of D. In particular, S_0 is a semisimple Lie group of Hermitian type, and $D = S_0/K_0$, where K_0 is maximal compact in S_0. Moreover, S_0 has trivial center. Therefore, the center of S coincides with the kernel Λ of $S \to S_0$, which is infinite cyclic. $\square$

The following complements Theorem 7.1 by showing that any symmetric bounded domain D is the Kähler quotient of a contractible Sasaki homogeneous space for a semisimple Lie group S:

Theorem 7.2. *For any symmetric bounded domain* $D = S_0/K_0$, *there exists a unique semisimple Lie group* S *with infinite cyclic center, which is covering* S_0 *and gives rise to a contractible Sasaki homogeneous space*

$$X_S = S/K$$

with Kähler quotient D.

Proof. Let X be the *unique* contractible Sasaki homogeneous space over D, which exists by Corollary 4.5. By Proposition 6.9, the maximal normal semisimple subgroup $S \leq \mathrm{Psh}\,(X)$ is acting transitively on X, and it is covering S_0 with infinite cyclic kernel. By Theorem 7.1 (3), any transitive semisimple Lie subgroup of $\mathrm{Psh}\,(X)$ coincides with S. $\square$

Dividing out the center of S gives rise to a homogeneous Sasaki manifold

$$Y_0 = X/\Lambda$$

whose Reeb flow is a circle group. This shows that any semisimple Lie group of Hermitian type is actually acting transitively on an associated Sasaki homogeneous space:

Corollary 7.3. *For any semisimple Lie group* S_0 *of Hermitian type, there exists a unique Sasaki homogeneous space*

$$Y_0 = S_0/K_1$$

with Kähler quotient $D = S_0/K_0$. In this situation, the following hold:

(1) *There exists a circle group $\bar{\mathsf{A}} \leq K_0$ such that $K_0 = \bar{\mathsf{A}} \times K_1$ is a maximal compact subgroup of S_0.*
(2) *The Reeb flow T_0 for the Sasaki space Y_0 is isomorphic to a circle group S^1 and*

$$\mathrm{Psh}\,(Y_0) \;\cong\; S_0 \times T_0 \,.$$

Moreover, every Sasaki homogeneous space with Kähler quotient D is a covering space of Y_0.

Proof. Consider the unique contractible Sasaki homogeneous space X over $D = S_0/K_0$. Then $X = S/K$, where the semisimple group S admits a covering $S \to S_0$ with kernel Λ, the center of S. By part (3) of Theorem 7.1, Λ is contained in the Reeb flow T for X. Therefore Λ is acting properly discontinuously and freely on X, and $Y_0 = X/\Lambda$ is a homogeneous Sasaki space for S_0, which has Reeb flow $T_0 = T/\Lambda = S^1$. Since X is the unique simply connected Sasaki homogeneous space with Kähler quotient D, any Sasaki homogeneous space over D is a quotient space of X, hence such a homogeneous space is covering Y_0. $\quad\square$

7.2. Sasaki Lie groups

A Lie group G is said to be a *Sasaki group* if G admits a left-invariant Sasaki structure (respectively, standard pseudo-Hermitian structure) $\{\omega, J\}$. Accordingly, any simply transitive pseudo-Hermitian action of G on a Sasaki space X determines a unique left-invariant Sasaki structure on G up to isomorphism. Two Sasaki Lie groups G and G' are considered to be *equivalent Sasaki Lie groups* if there exists an isomorphism $G \to G'$ which is a pseudo-Hermitian isometry. Two Sasaki Lie groups acting on X are equivalent if and only if they are conjugate subgroups of $\mathrm{Psh}\,(X)$.

7.2.1. Sasaki Heisenberg groups $\mathcal{N}$

Let X be the contractible homogeneous Sasaki manifold over $\mathbb{C}^k$. That is, we assume that the Reeb fibering for X is of the form

$$\mathbb{R} \;\to\; X \;\overset{q}{\to}\; \mathbb{C}^k \,.$$

By (2) of Proposition 6.1, the $2k$-dimensional Heisenberg group

$$\mathcal{N} \;\leq\; \mathrm{Psh}\,(X)$$

is the preimage of the translation subgroup $\mathbb{C}^k \leq \mathrm{Isom}\,_h(\mathbb{C}^k)$. Moreover, $\mathcal{N}$ acts simply transitively on X. Therefore, we get that $\mathcal{N}$ is a Sasaki Lie group, which as a space is isometric to X by a pseudo-Hermitian isometry. We also deduce that

$$\mathrm{Psh}\,(\mathcal{N}) = \mathrm{Psh}\,(X) = \mathcal{N} \rtimes \mathrm{U}(k)$$

is a connected Lie group. (Compare also [19], for example.)

We describe the standard Sasaki structure on $\mathcal{N}$ more explicitly as follows:

Example 7.4 *(Sasaki Heisenberg group $\mathcal{N}$)*. Let $\mathcal{N} = \mathbb{R} \times \mathbb{C}^k$ be the $2k+1$-dimensional Heisenberg group $(k \geq 0)$. We write the group law on $\mathcal{N}$ as

$$(t, z)(s, w) = (t + s - \mathrm{Im}(^t\bar{z}w), z + w). \tag{7.1}$$

The standard pseudo-Hermitian structure $\{\omega_0, J\}$ on $\mathcal{N}$ is given by the left-invariant contact one-form

$$\omega_0 = dt + \mathrm{Im}(^t\bar{z}dz) \,,$$

together with a left-invariant complex structure J, defined on $\ker\omega_0$ by the relation

$$\mathsf{q}_* \circ J = J_{\mathbb{C}} \circ \mathsf{q}_* \,.$$

Here $J_{\mathbb{C}}$ denotes the standard complex structure of $\mathbb{C}^n$, $q : \mathcal{N} \to \mathbb{C}^n$ is the natural projection. Then $g_0 = \omega_0 \cdot \omega_0 + d\omega_0 \circ J$ is the positive definite Sasaki metric on $\mathcal{N}$.

We calculate the isometry group of the Sasaki group $\mathcal{N}$ explicitly as follows:

Note 7.5 *(Isometry group of $\mathcal{N}$).* Consider the semidirect product group

$$\mathrm{Sim}(\mathcal{N}) = \mathcal{N} \rtimes (\mathrm{U}(k) \times \mathbb{R}^+) \,,$$

where $\mathrm{U}(k) \times \mathbb{R}^+$ is contained in $\mathrm{Aut}(\mathcal{N})$. The action of $(A, \lambda) \in \mathrm{U}(k) \times \mathbb{R}^+$ on $\mathcal{N}$ is given by:

$$(A, \lambda)\,(t, z) = (\lambda^2 t, \ \lambda A z) \,.$$

It follows that $(A, \lambda)^* \omega_0 = \lambda^2 \omega_0$. In particular, $\mathrm{U}(k)$ acts by strict contact transformations and holomorphically on the standard pseudo-Hermitian manifold $(\mathcal{N}, \{\omega_0, J\})$. That is, $\mathrm{U}(k)$ is a subgroup of $\mathrm{Psh}\,(\mathcal{N})$. Next define $\tau \in \mathrm{Aut}(\mathcal{N})$ by

$$\tau(t, z) = (-t, \bar{z}) \,. \tag{7.2}$$

Then $\tau^*\omega_0 = -\omega_0$ and $J \circ \tau_* = -\tau_* \circ J$. Thus

$$\langle \tau \rangle = \mathbb{Z}_2 \ \leq \ \mathrm{Psh}^{\pm}\,(\mathcal{N})$$

is contained in the isometry group of the Sasaki metric g_0, but does not belong to $\mathrm{Psh}\,(\mathcal{N})$. Observe further that

$$\mathrm{U}(k) \rtimes \langle \tau \rangle$$

is a maximal compact subgroup of the automorphism group $\mathrm{Aut}(\mathcal{N})$. We deduce:

$$\mathrm{Psh}\,(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(k) \ \text{ and } \ \mathrm{Isom}\,(\mathcal{N}) = \mathrm{Psh}^{\pm}\,(\mathcal{N}) = \mathrm{Psh}\,(\mathcal{N}) \rtimes \mathbb{Z}_2 \,. \tag{7.3}$$

(Recall also that by [29], the isometry group of any left-invariant Riemannian metric on $\mathcal{N}$ is contained in the group of affine transformations $\mathcal{N} \rtimes \mathrm{Aut}(\mathcal{N})$.)

We prove now that the Sasaki Lie group structure on the Heisenberg Lie group $\mathcal{N}$ is essentially unique:

Proposition 7.6. *Up to isomorphism of Sasaki Lie groups, there is a unique Sasaki structure on the Heisenberg Lie group $\mathcal{N}$.*

Proof. Suppose $(\mathcal{N}, \{\omega, J\})$ is a Sasaki Lie group of dimension $2k + 1$. In particular, the space $X = \mathcal{N}$ is a contractible homogeneous Sasaki manifold, on which the group $\mathcal{N}$ acts simply transitively. Via the Boothby-Wang homomorphism, $\mathcal{N}$ also acts transitively on the Kähler quotient $W = X/\mathbb{R}$. Since $\mathcal{N}$ is nilpotent, W must be flat (for example by [13]). So W is Kähler isometric to $\mathbb{C}^k$.

Then, as follows from Section 4.2, we must be in the situation Case II, where the Reeb flow T coincides with the center of $\mathcal{N}$. Therefore, the Boothby-Wang homomorphism for X maps $\mathcal{N}$ to an abelian simply transitive subgroup $\bar{\mathcal{N}}$ of isometries of unitary space $\mathbb{C}^k$. We conclude that this image group $\bar{\mathcal{N}}$ is actually the translation group $\mathbb{C}^k$, which is the unique abelian simply transitive subgroup of $\mathbb{C}^k \rtimes \mathrm{U}(k)$. Therefore, $\mathcal{N}$ is the normal subgroup of $\mathrm{Psh}\,(X)$ which is the preimage of $\mathbb{C}^k$. Now the Sasaki manifold X is determined uniquely by its Kähler quotient $\mathbb{C}^k$ (cf. Corollary 4.5) up to a pseudo-Hermitian isometry. By Proposition 6.4, $\mathcal{N}$ is the nilradical of $\mathrm{Psh}\,(X)$. Therefore, it is uniquely determined and characteristic in $\mathrm{Psh}\,(X)$. Since the space X is determined uniquely by $\mathbb{C}^k$, this constructs the left-invariant structure on $\mathcal{N}$ uniquely up to a pseudo-Hermitian isomorphism of Sasaki Lie groups. $\quad\square$

7.2.2. Heisenberg modifications $\mathcal{N}(k, l)$

We construct a family of simply connected Sasaki Lie groups which are modifications of the Heisenberg Sasaki group $\mathcal{N}$ introduced in Example 7.4.

Flat Kähler Lie groups. For this, let $\rho : \mathbb{C}^l \to \mathrm{U}(k)$ be a non-trivial homomorphism $(k + l = n)$. Then the semidirect product $\mathbb{C}^k \rtimes_\rho \mathbb{C}^l$ embeds in an obvious manner as a simply transitive subgroup

$$\mathbb{C}(k, l) \le \mathbb{C}^n \rtimes \mathrm{U}(n)$$

of the holomorphic isometry group of flat unitary space $\mathbb{C}^n$. Thus $\mathbb{C}(k, l)$ is a *flat Kähler group*, since it is acting simply transitively by holomorphic isometries on $\mathbb{C}^n$. (In fact, every flat Kähler Lie group contained in $\mathbb{C}^n \rtimes \mathrm{U}(n)$ is conjugate to some $\mathbb{C}(k, l)$, compare [13, Theorem II].) Note also that $k \ge 1$ and that the standard Kähler form of $\mathbb{C}^n$ is non-degenerate on $\mathbb{C}^k$.

Heisenberg modifications. Let X be *the* unique contractible Sasaki homogeneous space over $\mathbb{C}^n$. Consider the pull-back $\mathcal{N}(k, l)$ of $\mathbb{C}(k, l)$ in the central extension which is defining $\mathrm{Psh}\,(X)$ according to Proposition 6.4:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{Psh}\,(X) = \mathcal{N} \rtimes \mathrm{U}(n) & \overset{p}{\longrightarrow} & \mathbb{C}^n \rtimes \mathrm{U}(n) & \longrightarrow & 1 \\
& & \| & & \cup & & \cup & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{N}(k, l) & \overset{p}{\longrightarrow} & \mathbb{C}(k, l) & \longrightarrow & 1 \,.
\end{array}
\tag{7.4}
$$

In particular, such $\mathcal{N}(k, l)$ is a simply connected solvable Lie group (where $\mathcal{N}(n, 0) = \mathcal{N}$ is nilpotent). Moreover,

$$\mathcal{N}(k, l) \le \mathrm{Psh}\,(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(n)$$

acts simply transitively and by pseudo-Hermitian transformations on the Sasaki manifold $X = \mathcal{N}$. From this action, $\mathcal{N}(k, l)$ inherits a natural structure as a Sasaki Lie group.

Definition 7.7. Any Sasaki group of the form $\mathcal{N}(k, l) \le \mathrm{Psh}\,(\mathcal{N})$ as above is said to be a Heisenberg modification (of type (k, l)).

Remark 7.8. By definition, the groups $\mathcal{N}(k,l)$ are defined as preimage of Kähler Lie groups. The proof of Proposition 7.6 shows that the classification of groups $\mathcal{N}(k,l)$ up to isomorphism of Sasaki Lie groups amounts *exactly* to the classification of Kähler Lie groups $\mathbb{C}(k,l)$ up to isomorphism. For a discussion of the structure of flat Kähler Lie groups, see for example [3] and [13].

Also we note:

Lemma 7.9. *Let X be any contractible Sasaki manifold over a homogeneous Kähler manifold W. If $\mathbb{C}^n$ is the maximal flat factor of W then the preimage $\tilde{\mathcal{N}}$ in $\mathrm{Psh}\,(X)$ of a subgroup*

$$\mathbb{C}(k,l) \leq \mathbb{C}^n \rtimes \mathrm{U}(n)$$

under the homomorphism ϕ in the sequence (6.1) *is $\mathcal{N}(k,l)$.*

Proof. By Proposition 6.1 (2), the pullback of $\mathbb{C}^n \rtimes \mathrm{U}(n) \leq \mathrm{Isom}_h(W)$ to the group $\mathrm{Psh}\,(X)^0$ along the exact sequence (6.1) is $\mathcal{N} \rtimes \mathrm{U}(n)$. Therefore, the pullback $\tilde{\mathcal{N}}$ of $\mathbb{C}(k,l)$ satisfies the defining exact sequence (7.4) above. So $\tilde{\mathcal{N}} = \mathcal{N}(k,l)$. □

Proof of Theorem 2. Let G be a contractible unimodular Sasaki group. As follows from Theorem 4.3, there exists a one-parameter subgroup

$$\mathsf{A} \leq G$$

such that $W = G/\mathsf{A}$ is a homogeneous Kähler manifold for G.

If A is a normal subgroup in G (cf. case (I) of Section 4.2), then

$$\bar{G} = G/\mathsf{A}$$

is a Kähler group acting simply transitively on W, and A is, a fortiori, central in G. Hence, as G is unimodular, so is $\bar{G} = G/\mathsf{A}$. Therefore, Hano's theorem [13, Theorem II] implies that $W = \mathbb{C}^n$ is a flat Kähler space and that

$$\bar{G} = \mathbb{C}(k,l) \;\leq\; \mathbb{C}^n \rtimes \mathrm{U}(n)$$

is a meta-abelian Kähler group. Since G is simply connected, the Reeb flow T for the Sasaki manifold G is isomorphic to $\mathbb{R}$. By Lemma 7.9, this implies that, as a Sasaki Lie group,

$$G = \mathcal{N}(k,l) \;,$$

for some k, l, with $k + l = n$.

We may assume now that A is not normal in G. This is case (II) in Section 4.2. The presentation (II) for W is then a fiber bundle of the form

$$1 \;\longrightarrow\; \mathsf{A} \;\longrightarrow\; G \;\overset{\mathsf{q}}{\longrightarrow}\; W = \bar{G}/\bar{\mathsf{A}} \;\longrightarrow\; 1 \;,$$

where

$$\bar{G} = G/\mathcal{Z} \;,$$

with $\mathcal{Z}$ a discrete subgroup in the center of G, and $\bar{G}$ is acting faithfully on W. In particular, $\bar{G}$ is a unimodular group of Kähler isometries acting transitively on the contractible Kähler manifold W. By Proposition 6.1,

$$W = \mathbb{C}^n \times D,$$

where $D = S_0/K_0$ is a symmetric bounded domain. Therefore,

$$\bar{G} \leq (\mathbb{C}^n \rtimes \mathrm{U}(n)) \times S_0,$$

where $S_0 = \mathrm{Isom}\,(D)^0$ is a semisimple Lie group of hermitian type. Projecting $\bar{G}$ to S_0, with kernel

$$L = \bar{G} \cap (\mathbb{C}^n \rtimes \mathrm{U}(n))\ ,$$

the image of $\bar{G}$ in S_0 is a unimodular group, acting transitively on D. By Hano's theorem, the image of $\bar{G}$ must be semisimple. Therefore it is all of S_0. From the Levi-decomposition theorem, we infer that

$$\bar{G} = L \cdot S_0$$

is an almost semi-direct product. Therefore,

$$\dim \bar{G} = \dim L + \dim S_0 = \dim W + 1 \ \leq \ \dim L + (\dim S_0 - \dim K_0) + 1\ .$$

This implies $\dim K_0 \leq 1$.

Suppose first that D is non-trivial. Then we have that

$$D = \mathbb{H}_{\mathbb{C}}^1$$

is biholomorphic to the hyperbolic plane, S_0 is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ and K_0 is a circle group. It follows that the above kernel L of the projection $\bar{G} \to S$ acts simply transitively on the factor $\mathbb{C}^n$ of W. Hence, L is a flat Kähler Lie group, and therefore $L = \mathbb{C}(k,l)$. By Lemma 7.9, the preimage of L in $\mathrm{Psh}\,(G)$ under the Boothby-Wang homomorphism is a subgroup

$$\mathcal{N}(k,l) \leq \mathrm{Psh}\,(G)\ ,$$

which contains the Reeb flow T in its center. Since G is covering $\bar{G}$, G contains a subgroup

$$\tilde{L} = \mathcal{N}(k,l) \cap G$$

as a covering group of L. Therefore,

$$\mathcal{N}(k,l) = T \cdot \tilde{L}$$

is an almost semi-direct product. This is contradicting the fact that the extension class of the exact sequence in the bottom row of (7.4) is non-trivial (compare Lemma 6.2). The contradiction implies that the factor $\mathbb{C}^n$ must be trivial. Thus,

$$W = D = \mathbb{H}_{\mathbb{C}}^1$$

is a Kähler manifold of constant negative curvature. Hence,

$$G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$$

is the universal covering group of $S_0 = \mathrm{PSL}(2,\mathbb{R})$ with a standard Sasaki structure over $\mathbb{H}_{\mathbb{C}}^1$.

It remains to exclude the case that D is trivial. Suppose we have

$$W = \mathbb{C}^n = \bar{G}/\bar{\mathsf{A}} \, .$$

Since any reductive subgroup of isometries on $\mathbb{C}^n$ has a fixed point, the circle group $\bar{\mathsf{A}}$ must be a maximal reductive subgroup of $\bar{G}$. We deduce that $\bar{G}$ is a solvable Lie group with maximal compact subgroup $\bar{\mathsf{A}}$. Thus there exists a simply connected solvable normal subgroup $\bar{G}_0$ such that

$$\bar{G} = \bar{G}_0 \rtimes \bar{\mathsf{A}}$$

(see e.g. [5, Lemma 2.1]). It follows that $\bar{G}_0 = \mathbb{C}(k,l)$ is a flat Kähler Lie group. As above, this implies that

$$\mathcal{N}(k,l) = T \cdot (G \cap \mathcal{N}(k,l))$$

is an almost semi-direct product, which is not possible. Hence, the case D is trivial cannot occur, unless A is normal in G. $\quad\square$

8. Examples

We give further explicit examples of locally homogeneous aspherical Sasaki manifolds.

8.1. Sasaki manifolds modeled over complex hyperbolic spaces

The complex hyperbolic space is described as the homogeneous manifold

$$\mathbb{H}_{\mathbb{C}}^n = \mathrm{PU}(n,1)\big/\mathrm{U}(n) = \mathrm{SU}(n,1)\big/\,\mathrm{S}\left(\mathrm{U}(n) \times \mathrm{U}(1)\right) .$$

Consider the following diagram of principal bundle fiberings:

$$
\begin{array}{ccccc}
\mathbb{R} = \widetilde{\mathrm{U}(1)} & \longrightarrow & X = \widetilde{\mathrm{U}(n,1)}\big/\widetilde{\mathrm{U}(n)} & \xrightarrow{\tilde{P}} & \mathbb{H}_{\mathbb{C}}^n = \mathrm{PU}(n,1)/\mathrm{U}(n) \\
\Big\downarrow{/\mathbb{Z}} & & \Big\downarrow{/\mathbb{Z}} & & \| \\
S^1 = \mathrm{U}(1) & \longrightarrow & Y = \mathrm{U}(n,1)\big/\,\mathrm{U}(n) & \xrightarrow{P} & \mathbb{H}_{\mathbb{C}}^n = \mathrm{PU}(n,1)/\mathrm{U}(n)
\end{array} \, ,
$$

where the inclusions of $\widetilde{\mathrm{U}(n)}$, $\widetilde{\mathrm{U}(1)}$ arise from the standard embedding

$$\mathrm{U}(n) \times \mathrm{U}(1) \to \mathrm{U}(n,1) \, .$$

Remark 8.1. Denoting with $\pi : \widetilde{\mathrm{U}(n,1)} \to \mathrm{U}(n,1)$ the universal covering group of $\mathrm{U}(n,1)$, we declare connected subgroups

$$\widetilde{\mathrm{U}(n)} = \pi^{-1}(\mathrm{U}(n))^0 \, , \; \widetilde{\mathrm{SU}(n,1)} = \pi^{-1}(\mathrm{SU}(n,1))^0 \, .$$

Then $\widetilde{\mathrm{U}(n)}$ is a universal covering group for $\mathrm{U}(n)$, and the kernel $\mathcal{Z}$ $(\cong \mathbb{Z})$ of the latter covering is contained in the center of the group

$$\widetilde{\mathrm{U}(n,1)} \ .$$

This gives rise to the above (non-faithful) homogeneous presentation of the universal covering space X for Y in the diagram. It also follows that

$$\mathrm{Psh}\,(X)^0 = \widetilde{\mathrm{U}(n,1)}/\mathcal{Z} = \widetilde{\mathrm{SU}(n,1)} \cdot \widetilde{\mathrm{U}(1)} \ .$$

A pseudo-Hermitian structure $\{\omega, J\}$ on $Y = X/\mathbb{Z}$ is obtained as a *connection bundle* over $\mathbb{H}_{\mathbb{C}}^n$ such that $P^*\Omega = d\omega$, for the Kähler form Ω of $\mathbb{H}_{\mathbb{C}}^n$. Here S^1 becomes the Reeb flow for ω on Y, and

$$\mathrm{Psh}\,(Y)^0 = \mathrm{U}(n,1) \ .$$

The pseudo-Hermitian structure $(\tilde{\omega}, J)$ on X is a lift of ω. Note also that

$$Y = \mathrm{SU}(n,1)/\mathrm{SU}(n) \text{ and } X = \widetilde{\mathrm{SU}(n,1)}/\mathrm{SU}(n)$$

are *faithful* presentations as homogeneous Sasaki manifolds of simple Lie groups. Taking a torsionfree discrete uniform subgroup Γ of $\mathrm{SU}(n,1)$ (such a subgroup exists by [9], for example), gives rise to a *regular* locally homogeneous aspherical Sasaki manifold with Boothby-Wang fibering

$$S^1 \longrightarrow \Gamma \backslash \mathrm{SU}(n,1)/\mathrm{SU}(n) \longrightarrow Q \backslash \mathbb{H}_{\mathbb{C}}^n \ , \tag{8.1}$$

where $Q \leq \mathrm{PU}(n,1)$ is a torsionfree discrete uniform subgroup (isomorphic to Γ).

8.2. Join of locally homogeneous Sasaki manifolds

As above let

$$X_{\mathbb{C}} = \widetilde{\mathrm{SU}(n,1)}/\mathrm{SU}(n)$$

denote the contractible Sasaki homogeneous space over $\mathbb{H}_{\mathbb{C}}^n$. (Compare Section 8.1.) We may take the join (see Proposition 2.4) with the Sasaki Heisenberg group $\mathcal{N}$ to obtain a contractible homogeneous Sasaki manifold:

$$
\begin{array}{ccccc}
\mathbb{R} & \longrightarrow & X = (\mathcal{N} \times X_{\mathbb{H}_{\mathbb{C}}^n})/\Delta & \overset{\mathfrak{q}}{\longrightarrow} & \mathbb{C}^k \times \mathbb{H}_{\mathbb{C}}^n \\
& & \| & & \| \\
\mathbb{R} & \longrightarrow & \left(\mathcal{N} \cdot \widetilde{\mathrm{SU}(n,1)}\right)/\mathrm{SU}(n) & \overset{\mathfrak{q}}{\longrightarrow} & \mathbb{C}^k \times \mathrm{SU}(n,1)/\mathrm{S}\left(\mathrm{U}(n) \times \mathrm{U}(1)\right)
\end{array}
$$

A pseudo-Hermitian structure $\{\omega, J\}$ on

$$\mathcal{N} * X_{\mathbb{H}_{\mathbb{C}}^n} = (\mathcal{N} \times X_{\mathbb{H}_{\mathbb{C}}^n})/\Delta$$

is obtained as the quotient of $\omega_0 + \tilde{\omega}$, where ω_0 is the contact form on $\mathcal{N}$, $\tilde{\omega}$ on $X_{\mathbb{H}_{\mathbb{C}}^n}$ (see Proposition 2.4). Taking a suitable torsionfree discrete uniform subgroup π from

$$\mathrm{Psh}\,(X)^0 = (\mathcal{N} \rtimes \mathrm{U}(k)) * \mathrm{Psh}\,(X_{\mathbb{H}_{\mathbb{C}}^n}) = (\mathcal{N} \rtimes \mathrm{U}(k)) \cdot \widetilde{\mathrm{SU}(n,1)}$$

allows to construct a compact locally homogeneous aspherical Sasaki manifold over a product of compact Kähler manifolds:

$$S^1 \longrightarrow \pi \backslash (\mathcal{N} * X_{\mathbb{H}_{\mathbb{C}}^n}) \overset{\mathfrak{q}}{\longrightarrow} T_{\mathbb{C}}^k \times Q \backslash \mathbb{H}_{\mathbb{C}}^n \ .$$

8.3. Heisenberg Sasaki manifolds

Recall from the construction in (7.4) that the Sasaki Lie groups

$$\mathcal{N}(k,l)$$

are contained in the pseudo-Hermitian group $\mathrm{Psh}\,(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(k)$ of the Heisenberg Sasaki group $\mathcal{N}$. Therefore, taking quotients of $\mathcal{N}(k,l)$ by discrete uniform subgroups gives rise to:

Circle bundles over flat Kähler manifolds. Let Δ be a discrete uniform subgroup of $\mathcal{N}(k,l)$. Then

$$M = \Delta \backslash \mathcal{N}(k,l)$$

is a locally homogeneous $\mathcal{N}(k,l)$-manifold. Since $\mathcal{N}(k,l) \le \mathrm{Psh}\,(\mathcal{N})$ acts simply transitively on $\mathcal{N}$, $\Delta \le \mathrm{Psh}\,(\mathcal{N})$ acts properly discontinuously as a discrete group of holomorphic isometries on $\mathcal{N}$. Therefore

$$M = \mathcal{N} \big/ \Delta$$

is also quotient of $\mathcal{N}$ as a locally homogeneous manifold modeled on the homogeneous space $\mathcal{N}$. Moreover, the proof of Theorem 6.5 part (3), together with the exact sequence (7.4), show that Δ is a central extension of $p(\Delta)$, where $p(\Delta)$ is a uniform lattice in $\mathbb{C}(k,l)$. This gives rise to a circle bundle

$$S^1 \to \Delta \backslash \mathcal{N} \to p(\Delta) \backslash \mathbb{C}(k,l)\,,$$

where the Kähler solvmanifold $p(\Delta) \backslash \mathbb{C}(k,l)$ is a torus bundle over a torus, and it is finitely covered by a complex compact torus $T^n_{\mathbb{C}} = \mathbb{C}^n / \Lambda$, Λ isomorphic to $\mathbb{Z}^{2n}$ (compare [15]), where the Kähler metric on $T^n_{\mathbb{C}}$ is flat.

8.4. Locally homogeneous manifold $\pi \backslash \mathcal{N}$ which is not Sasaki

We explicitly construct an example of a Riemannian metric which is locally a Sasaki metric but does not admit a compatible structure vector field $\mathcal{A}$. (See also (6) in the introduction, following Remark 1.1.)

Example 8.2. Let

$$\Lambda = \mathbb{Z} \times (\mathbb{Z}^n + i\mathbb{Z}^n) \subseteq \mathcal{N} = \mathbb{R} \times \mathbb{C}^n$$

be the integral lattice in $\mathcal{N}$. Clearly, Λ is a subgroup and $\tau \Lambda = \Lambda$, where as in (7.2),

$$\tau(t,z) = (-t, \bar{z}).$$

Next put $\alpha_s = (0, (s,0,\ldots,0))$, $\mu = \alpha_{\frac{1}{2}} \tau$ and let

$$\pi = \langle \mu, \Lambda \rangle \le \mathcal{N} \rtimes \tau$$

be the group generated by μ and Λ. Since $\mu^2 = \alpha_1 \in \Lambda$ and $\mu \Lambda \mu^{-1} = \Lambda$, the group π satisfies an exact sequence

$$1 \to \Lambda \to \pi \to \mathbb{Z}_2 \to 1\,.$$

Since μ is of infinite order π must be torsionfree (see Lemma 8.3 below).

Lemma 8.3. π *is torsion-free.*

Proof. Recall that every non-trivial element of $\mathcal{N}$ has infinite order. Let $\gamma = \gamma_0 \tau$, where $\gamma_0 \in \mathcal{N}$. If γ has finite order, so has $\gamma^2 = \gamma_0 \gamma_0^\tau \in \mathcal{N}$. Thus $\gamma^2 = 1 \in \mathcal{N}$. Writing $\gamma_0 = (t, w)$, we have by (7.1) and (7.2) that

$$\gamma^2 = (t, w) \cdot (-t, \bar{w}) = (-\mathrm{Im}(^t \bar{w} \bar{w}), w + \bar{w}) = (0, 0) \ .$$

That is, γ is a torsion element, if and only if w is purely imaginary. Assuming now that $\gamma \in \pi$, we have $\gamma_0 = \lambda \alpha_{\frac{1}{2}}$, where $\lambda \in \Lambda$ is integral. This shows that the vector w for γ_0 has a non-trivial real part (in its first entry). Hence, γ is not a torsion element. So π is torsionfree, $\square$

Since π is without torsion, the quotient space

$$\pi \backslash \mathcal{N}$$

is a compact infra-nilmanifold. Since $\pi \leq \mathrm{Isom}\,(\mathcal{N}, g_0)$, for the Sasaki metric g_0 on $\mathcal{N}$ (as in Example 7.4), there is an induced Riemannian metric $\hat{g}_0$ on $\pi \backslash \mathcal{N}$, which is locally the same as the Sasaki metric g_0. But $(\pi \backslash \mathcal{N}, \hat{g}_0)$ never admits a compatible Sasaki structure. That is, there exists no pseudo-Hermitian structure $(\hat{\eta}, \hat{J}')$ on $\pi \backslash \mathcal{N}$ such that $\hat{g}_0 = \hat{\eta} \cdot \hat{\eta} + d\hat{\eta} \circ \hat{J}'$:

Lemma 8.4. *The infra-nilmanifold* $(\pi \backslash \mathcal{N}, \hat{g}_0)$ *does not admit any compatible Sasaki structure.*

Proof. Suppose $(\pi \backslash \mathcal{N}, \hat{g}_0)$ admits a Sasaki structure $(\hat{\eta}, \hat{J}')$ such that $\hat{g}_0 = \hat{\eta} \cdot \hat{\eta} + d\hat{\eta} \circ \hat{J}'$. Let η be a lift of $\hat{\eta}$ to $\mathcal{N}$, for which $g_0 = \eta \cdot \eta + d\eta \circ J'$ is a Sasaki metric on $\mathcal{N}$. Moreover,

(1) (η, J') is a standard pseudo-Hermitian structure on $\mathcal{N}$.
(2) $\pi \leq \mathrm{Psh}\,(\mathcal{N}, \{\eta, J'\}) \leq \mathrm{Isom}\,(\mathcal{N}, g_0) = \mathcal{N} \rtimes (\mathrm{U}(k) \rtimes \mathbb{Z}_2)$.
(3) With respect to the inclusion in (2), π maps onto $\mathbb{Z}_2$.

Let T' be the one-parameter group of the Reeb field for η. As T' is contained in the isometry group of g_0, and T' is connected, it follows that

$$T' \leq \mathcal{N} \rtimes \mathrm{U}(k)$$

by (2). In particular, T' normalizes $\mathcal{N}$. Since T is the lift of the Reeb flow on $\pi \backslash \mathcal{N}$, it centralizes π and $\pi \cap \mathcal{N}$ (also by (2)). Since $\pi \cap \mathcal{N}$ is discrete uniform in $\mathcal{N}$ (by the Auslander-Bieberbach theorem [2]), T' centralizes $\mathcal{N}$ by the Mal'cev unique extension property. Since $T' \leq \mathcal{N} \rtimes \mathrm{U}(k)$, this implies $T' = C(\mathcal{N}) = T$ is the one-parameter subgroup of the Reeb field for ω_0. As π centralizes T, it follows $\pi \leq \mathcal{N} \rtimes \mathrm{U}(k)$. This contradicts (3). $\square$

Therefore the compact locally homogeneous aspherical manifold $\pi \backslash \mathcal{N}$ admits a locally Sasaki metric but it is not a Sasaki manifold. In addition $\mathrm{Isom}\,(\pi \backslash \mathcal{N}, \hat{g}_0)$ is finite, and $\pi \backslash \mathcal{N}$ is an S^1-fibered infranil-manifold without any S^1-action.

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