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Estimates on the zeros of E_2

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Abstract We improve the results from El Basraoui (Proc Amer Math Soc 138(7):2289–2299, 2010) about the Eisenstein series $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$. In particular we show that there exists exactly one (simple) zero in each Ford circle and give an approximation to its location.

Keywords Modular forms · Eisenstein series · Equivariant forms

Mathematics Subject Classification 11F11

1 Introduction

Let $\mathcal{M}_k(\Gamma)$ be the space of holomorphic modular forms of weight k for the full modular group $\Gamma = PSL(2,\mathbb{Z})$. It is well known that $\mathcal{M}_k(\Gamma)$ has dimension $\frac{k}{12} + O(1)$ and a modular form $f \in \mathcal{M}_k$ has $\frac{k}{12} + O(1)$ inequivalent zeros in a fundamental domain $\Gamma\backslash\mathbb{H}$.

For the cuspidal Hecke eigenforms, it is a consequence of the recent proof of the holomorphic Quantum Unique Ergodicity (QUE) by Holowinsky and Soundararajan [5] that the zeros

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are uniformly distributed. More precisely, for a sequence $\{f_k\}$ of cuspidal Hecke eigenforms of weight k we have that as $k \to \infty$ the zeros of f_k become equidistributed with respect to the normalized hyperbolic measure $\frac{3}{\pi}\frac{dxdy}{y^2}$.

In contrast to this in the case of Eisenstein series, it was conjectured by Rankin in 1968 and proved by Rankin and Swinnerton-Dyer [7] that all the zeros, in the standard fundamental domain, of the series

$$E_k(\tau) = \frac{1}{2}\sum_{(c,d)=1}(c\tau + d)^{-k}$$

lie on the geodesic arc $\{z \in \mathbb{H} | |z| = 1, 0 \leq \mathrm{Re}(z) \leq 1/2\}$ and as $k \to \infty$ they become uniformly distributed on this unit arc. A similar result for the cuspidal Poincare series was proved by Rankin [6]. For generalizations of these results to other Fuchsian groups and to weakly holomorphic modular functions see [1,2], among many others. For some recent work on the zeros of holomorphic Hecke cusp forms that lie on the geodesic segments of the standard fundamental domain see [4].

Next we turn our attention to the location of the zeros of the non-modular Eisenstein series of weight 2

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},\; q = e^{2\pi i\tau}.$$

As it is well known $E_2(\tau)$ is not modular but it is a quasi-modular form and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ satisfies the transformation property

$$E_2(\gamma\tau)(c\tau + d)^{-2} = E_2(\tau) + \frac{6}{\pi i}\frac{c}{(c\tau + d)}. \tag{1}$$

The zeros of E_2 has already been investigated in [3]. El Basraoui and Sebbar showed that there are infinitely many non-equivalent zeros of E_2 and two zeros are equivalent if and only if one is a $\mathbb{Z}$-translate of the other. Of particular interest is the unique zero on the imaginary axis $x = 0$ and the unique zero on $x = \frac{1}{2}$. Those two zeros also occur in [9]. They were computed by Cohen:

$$\tau_0 = 0.523521700017999266800534404806109769 68\ldots i$$

$$\tau_{\frac{1}{2}} = \frac{1}{2} + 0.13091903039676244690411482601971302060\ldots i.$$

If

$$\mathcal{F} := \left\{z \in \mathbb{H} |\, |z| > 1,\; |\mathrm{Re}(z)| < \tfrac{1}{2}\right\} \cup \left\{z \in \mathbb{H} |\, |z| \geq 1,\; -\tfrac{1}{2} \leq \mathrm{Re}(z) \leq 0\right\}$$

is the standard (strict) fundamental domain, El Basraoui and Sebbar also show that there are infinitely many $SL_2(\mathbb{Z})$-translates of $\mathcal{F}$ that contain a zero and infinitely many that do not contain a zero.

In this note we improve these results. More precisely for $(a, c) = 1$ recall that the associated Ford circle is the circle on the upper half plane with center $\frac{a}{c} + \frac{1}{2c^2}i$ and radius $\frac{1}{2c^2}$. Then we have

Theorem 1.1 *Inside each Ford circle there is a unique (simple) zero of E_2 and E_2 has no other zeros on the upper half plane.*

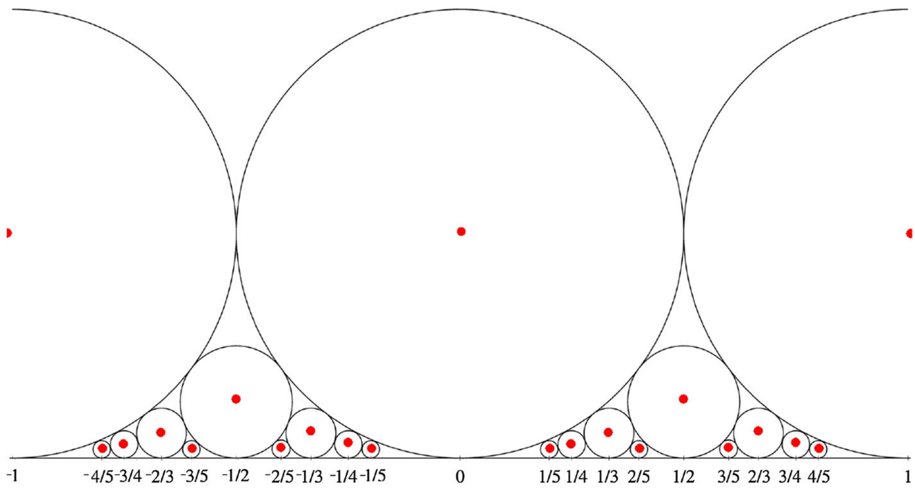

Fig. 1 Ford circles and zeros of E_2

Moreover in the Ford circle at the cusp $\frac{a}{c} \in \mathbb{Q}$ the zero τ of E_2 satisfies the following approximation:

$$0.000075\frac{1}{c^2} < \left|\tau - \frac{a}{c} - \frac{\pi}{6c^2}i\right| < 0.0000777\frac{1}{c^2}. \tag{2}$$

Since $\pi/6$ is close to 1/2, the zeros are almost at the center of each Ford circle and in fact when plotted on a not small enough scale as in Fig. 1, the zeros seem to have remarkable uniformity on the upper half plane.

Throughout the paper we use the variable τ for the location of the zeros of $E_2.z$ is used for general points in $\mathbb{H}$, often corresponding to a point in the fundamental domain.

Theorem 1.1 restricts the possible location of the zeros of E_2 to a very thin annular region in each Ford circle. For the images of these zeros inside the standard fundamental domain $\mathcal{F}$ we have

Theorem 1.2 *Let $\tau = \gamma z_{\frac{a}{c}}$ be the unique zero of E_2 around $\frac{a}{c}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c > 0$ and $z_{\frac{a}{c}} \in \mathcal{F}$ then we have*

$$0.00027 < \left|z_{\frac{a}{c}} - \left(-\frac{d}{c} + \frac{6}{\pi}i\right)\right| < 0.00029. \tag{3}$$

We may rephrase Theorem 1.1 as saying that around each finite cusp there is a unique zero of E_2. i.e. given $(a, c) = 1$ with $c > 0$, there is a unique matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $\gamma\mathcal{F}$ contains a zero.

While preparing this manuscript we learned that Wood and Young [11] proved a quantitatively equivalent result about the locations of the zeros of E_2 simultaneously and independently of us.

Theorems 1.1 and 1.2 are proved in the next section and are based on mapping properties of the function

$$f(z) := z - \frac{6i}{\pi E_2(z)}. \tag{4}$$

and its inverse.

It is possible to give further improvements on the approximations in the above theorems by studying the mapping properties of f in more detail and expanding f^{-1} in Fourier or Taylor series. The Fourier series is studied in Sect. 3. As a corollary to Theorems 3.5 and 3.6 it is possible to give an explicit and fast converging series approximation to $z_{a/c}$. For example we obtain the following improvement on Theorem 1.2.

Theorem 1.3 *Let $z_{\frac{a}{c}} \in \mathcal{F}$ be the $SL_2(\mathbb{Z})$-translate of the zero of E_2 around $\frac{a}{c}$ inside $\mathcal{F}$. Then we have the following stronger estimate:*

$$0.000000449 \leq \left| z_{\frac{a}{c}} - \left(-\frac{d}{c} + \frac{6i}{\pi} + \frac{144i}{\pi} e^{-12} e^{2\pi i(-d/c)} \right) \right| \leq 0.000000454.$$

The zeros for small a/c, in particular the zeros near $1/c$ were first considered by [3]. These zeroes are best approximated by Taylor series which are studied in Sect. 4. In the last two sections we also provide some numerical data on the Fourier and Taylor expansions.

2 Proofs of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 are based on the mapping properties of the function $f(z)$ defined in (4).

The function f appears in [10] and some of the properties shown here have been proved there. We will see in this section that f is analytic in the fundamental domain, and even on $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0.53\}$. Let z be in the fundamental domain $\mathcal{F}$, and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then based on the transformation formula (1), one has the fundamental relation

$$E_2(\gamma z) = 0 \iff f(z) = -d/c. \tag{5}$$

Therefore we are interested in the solutions of the equation

$$f(z) = -d/c \qquad z \in \mathcal{F}.$$

Here $d, c \in \mathbb{Z}$. We start by describing the set of all $z \in \mathcal{F}$ such that $f(z) \in \mathbb{R}$. Let $f = u + iv$, where $u = u(x, y)$ and $v = v(x, y)$ are $\mathbb{R}$-valued functions and

$$A = \{z \in \overline{\mathcal{F}} \mid v(z) = 0\}. \tag{6}$$

As a simple application of the implicit function theorem it will be proved below that A is the graph of a function $\phi : [-1/2, 1/2] \to (0, \infty)$. More precisely if $x + iy \in \mathcal{F}$ and $v(x, y) = 0$ then $y = \phi(x)$.

In view of the relation (5) the possible values $c, d \in \mathbb{Z}$ that arise from a zero of E_2 are those for which $-d/c \in f(A)$. Let $\tau = \gamma z$ be a zero of E_2 with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathcal{F}$. Then we will show that $|d/c| \leq 1/2$ and conversely any such $-d/c \in [-1/2, 1/2]$ arises from a zero of $E_2(z)$.

Namely we have

Theorem 2.1 $f(A) = [-1/2, 1/2]$.

The first claim in Theorem 1.1 now follows immediately from

Corollary 2.2 *If $z \in \mathcal{F}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $E_2(\gamma z) = 0$ then $|d/c| \leq 1/2$.*

In particular Corollary 2.2 uniquely determines the fundamental domain around $\frac{a}{c}$ which contains the zero of E_2.

The more precise location of zeros of E_2 given in (2) and (3) follow from bounds for E_2 and its derivative. To prove Theorem 2.1 we will use the implicit function theorem. We are interested in $v(x, y) = 0$. We need the following estimates.

Lemma 2.3 *Let $z = x + iy \in \mathbb{H}$ and let $q = e^{2\pi i z}$. We have the following estimates.*

$$|E_2(z) - 1| \leq \frac{24|q|}{(1 - |q|)^3},$$

$$23.68|q| < |E_2(z) - 1| \quad \textit{for } y \geq \frac{\sqrt{3}}{2},$$

$$|E_2'(z)| \leq 48\pi \frac{|q|(1 + |q|)}{(1 - |q|)^5}. \tag{7}$$

In particular for $y > 0.53$, using (7), we have $|E_2(z) - 1| < 1$ and so $E_2(z) \neq 0$ in this region and hence f in (4) is well defined.

Proof We have

$$|E_2(z) - 1| \leq 24 \sum_{n=1}^{\infty} \frac{n|q|^n}{1 - |q|^n} \leq \frac{24}{1 - |q|} \sum_{n=1}^{\infty} n|q|^n = \frac{24|q|}{(1 - |q|)^3}.$$

The claim for the lower bound for $E_2(z)$ and $E_2'(z)$ follows along the same lines:

$$\begin{aligned} |E_2(z) - 1| &\geq 24 \frac{|q|}{1 + |q|} - 24 \frac{1}{1 - |q|} \sum_{n=2}^{\infty} n|q|^n \\ &\geq 24|q| \frac{1 - 5|q|}{(1 + |q|)(1 - |q|)^3} > 23.68|q|, \text{ for } y \geq \frac{\sqrt{3}}{2}. \end{aligned}$$

□

Lemma 2.4 *If* $\operatorname{Im}(z) \geq 1$ *then*

$$|f'(z) - 1| < 0.6.$$

Proof Since

$$|f'(z) - 1| = \frac{6|E_2'(z)|}{\pi |E_2(z)|^2}$$

the result follows from Lemma 2.3 and the bounds (valid for $\operatorname{Im}(z) \geq 1$)

$$|E_2(z)| \geq |E_2(i \operatorname{Im}(z))| \geq |E_2(i)|$$

and

$$|E_2'(z)| \leq |E_2'(i \operatorname{Im}(z))| \leq |E_2'(i)|.$$

□

We will also need the following

Lemma 2.5 *For $x \in [-1/2, 1/2]$ and $y \in [\sqrt{3}/2, 1]$ we have*

$$\operatorname{Im}(f(x + iy)) < -0.39 < 0.$$

Proof For any x we have

$$\mathrm{Re}\,(E_2(x+iy)) \geq 1 - \frac{24e^{-2\pi y}}{(1-e^{-2\pi y})^3}$$

and

$$|E_2(x+iy)| \leq 1 + \frac{24e^{-2\pi y}}{(1-e^{-2\pi y})^3}.$$

Therefore for $y \in [\sqrt{3}/2, 1]$ we have

$$\mathrm{Im}(f(x+iy)) = y - \frac{6\,\mathrm{Re}\,(E_2(x+iy))}{\pi|E_2(x+iy)|^2} < 1 - \frac{6(1-t)}{\pi(1+t)^2}$$

where

$$t = \frac{24e^{-2\pi\sqrt{3}/2}}{(1-e^{-2\pi\sqrt{3}/2})^3}.$$

The lemma follows from the numerical estimate $t < 0.1054$. □

Since $\partial_y v = \partial_x u = \mathrm{Re}\left(f'(z)\right)$ we also have

Lemma 2.6 *If* $y \geq 1$ *then* $\partial_y v(x, y) > 0$.

Proof By Lemma 2.4,

$$|f'(z) - 1| < 0.6.$$

So $\partial_y v(z) = \mathrm{Re}\left(f'(z)\right) > 0.4$. □

From Lemma 2.6 it follows that for each $x \in [-1/2, 1/2]$ there can be only one y such that $v(x+iy) = 0$. By Lemma 2.5, it is also clear that such a y exists since $\lim_{y\to\infty} v(x+iy) = \infty$. Therefore there is a function $\phi : [-1/2, 1/2] \to (0, \infty)$ such that $x+iy \in \mathcal{F}$ and $v(x, y) = 0$ implies $y = \phi(x)$.

Moreover this ϕ is differentiable by the implicit function theorem, and we have $\phi'(x) = -\partial_x v/\partial_y v$.

Let now the arc A be as in (6). We know from [3], Proposition 3.1–3.2 that there are points $z_{1/2}$ and $z_{-1/2}$ on both of the vertical boundaries of $\mathcal{F}$, where f takes the values $-1/2$ and 1/2. They correspond to the zeros of E_2 with real parts 1/2 and $-1/2$.

By the intermediate value theorem it follows that f restricted to the arc A, (which is the same as u since v vanishes on A) takes all values between $-1/2$ and 1/2. This shows that there is at least one zero in each Ford circle. It is left to show that there is only one in each Ford circle. To show this, we need to show that $f(A) = u(A) = [-1/2, 1/2]$. We will prove this by showing that the function defined by $x \mapsto u(x, \phi(x))$ is monotonically increasing on the interval $[-1/2, 1/2]$.

Proposition 2.7 *Let* $f = u + iv$ *and* $\phi : [-1/2, 1/2] \to (1, \infty)$ *the function such that*

$$\{z \in \mathcal{F} \mid v(z) = 0\} = \{x + i\phi(x) | x \in [-1/2, 1/2]\}$$

For $|x| < 1/2$ *we have*

$$\frac{d}{dx} u(x, \phi(x)) > 0.$$

Proof We need to show that

$$\frac{d}{dx}u(x,\phi(x)) = \partial_x u(x,\phi(x) + \partial_y u(x,\phi(x))\phi'(x) > 0.$$

By implicit differentiation of $v(x,\phi(x)) = 0$ we have $\phi'(x) = -\partial_x v/\partial_y v$. Therefore

$$\frac{d}{dx}u(x,\phi(x)) = \frac{\partial_x u \partial_y v - \partial_y u \partial_x v}{\partial_y v} = \frac{(\partial_x u)^2 + (\partial_y u)^2}{\partial_y v} = \frac{|f'|^2}{\partial_y v} > 0.$$

□

This also finishes the proof of Theorem 2.1 and hence the first claim in Theorem 1.1.

The next Proposition on the other hand proves the estimates (2) and (3) about the location of zeros in Theorems 1.1 and 1.2.

Proposition 2.8 *If* $E_2(\gamma z) = 0, z = x + iy \in \mathcal{F}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with* $c > 0$ *and* $\tau = \gamma z$ *then* $1.909 < y < 1.911$ *and*

$$0.000144 < |E_2(z) - 1| < 0.000149, \tag{8}$$

$$0.00027 < \left|z - \left(-\frac{d}{c} + \frac{6}{\pi}i\right)\right| < 0.00029, \tag{9}$$

$$0.0000750\frac{1}{c^2} < \left|\tau - \left(\frac{a}{c} + \frac{\pi}{6c^2}i\right)\right| < 0.0000777\frac{1}{c^2}. \tag{10}$$

Proof We first note that since $y \geq \frac{\sqrt{3}}{2}$ on $\overline{\mathcal{F}}$, using (7) we have

$$|E_2(z) - 1| \leq \frac{24}{\left(1 - e^{-\pi\sqrt{3}}\right)^3}|q| < 24.32|q| < 0.106. \tag{11}$$

And similarly

$$23.68|q| < |E_2(z) - 1|. \tag{12}$$

We next recall that $E_2(\gamma z) = 0$ gives $f(z) = -d/c$ which in return implies $z - \frac{6i}{\pi E_2(z)} = \frac{-d}{c}$. Hence using estimate (11) we get:

$$\begin{aligned}\left|z - \left(-\frac{d}{c} + \frac{6}{\pi}i\right)\right| &= \frac{6}{\pi}\frac{|E_2(z) - 1|}{|E_2(z)|} \\ &< \frac{6}{\pi}\frac{0.106}{(1 - 0.106)} < 0.226.\end{aligned} \tag{13}$$

This in turn means that $y > \frac{6}{\pi} - 0.226 > 1.68$ which in return gives a better estimate for (11) and also in return for (13). Repeating this procedure twice again already gives the nice estimates:

$$\begin{aligned} y \geq 1.909, \quad |q| \leq 6.18 \times 10^{-6}, \\ |E_2(z) - 1| < 24\frac{1}{(1 - |q|)^3}|q| < 0.000149, \\ \left|z - \left(-\frac{d}{c} + \frac{6}{\pi}i\right)\right| < 0.00029. \end{aligned} \tag{14}$$

The estimates also show that $y < \frac{6}{\pi} + 0.00029 < 1.911$ resp. [using (12)] $|E_2(z) - 1| > 0.000144$, so

$$0.00027 < \left| z - \left(-\frac{d}{c} + \frac{6}{\pi} i \right) \right|.$$

In particular, we also see that Eq. (9) holds. On the other hand using $\tau = \gamma^{-1} z$ and $E_2(\gamma z) = 0$ we have

$$E(z) = E(\gamma^{-1}\tau) = \frac{6}{\pi i}(-c)(-c\tau + a)$$

and

$$\tau = \frac{a}{c} + \frac{\pi}{6c^2} i + \frac{\pi}{6c^2} i (E_2(z) - 1).$$

So

$$\left| \tau - \frac{a}{c} - \frac{\pi}{6c^2} i \right| < \frac{\pi}{6c^2} \left| E_2(z) - 1 \right| < 0.0000777 \frac{1}{c^2},$$

where we used (14). Similarly we get the other inequality in (10):

$$0.0000750 \frac{1}{c^2} < \frac{\pi}{6c^2} \left| E_2(z) - 1 \right| < \left| \tau - \frac{a}{c} - \frac{\pi}{6c^2} i \right|.$$

□

3 Fourier expansion

Let $f(z)$ be defined as in (4). As shown in [10] f is equivariant, i.e.

$$f(\gamma z) = \gamma(f(z)) \text{ for all } \gamma \in SL_2(\mathbb{Z}).$$

Consequently $f^{-1}(w)$ is also equivariant. In particular

$$G(w) := f^{-1}(w) - w = \sum_{n=0}^{\infty} c_n e^{2\pi i n w}$$

is 1-periodic and possesses a Fourier expansion, defined in some half-plane $\{w \mid \mathrm{Im}(w) > c\}$.

In this section we will determine the Fourier coefficients of G and the maximal possible half-plane mentioned above. This will also lead to the proof Theorem 1.3.

To this end we will first examine the mapping properties of f more closely than in the previous section.

By Lemmas 2.5 and 2.6 we have that the inverse of f (and hence also the Fourier expansion) exist in the half-plane $\{w \mid \mathrm{Im}(w) > -0.39\}$.

Let

$$\mathcal{C} = \{e^{i\theta} \mid \theta \in [\pi/3, 2\pi/3]\}$$

and

$$M = \{e^{i\theta} \mid \theta \in [\pi/3, 2\pi/3]\} + \mathbb{Z},$$

its periodic extension. M divides $\mathbb{C}$ in to two connected components. Let U denote the upper component. Similarly let V denote the upper component of $\overline{M}$. We use $\overline{M}$ to denote the conjugate of M. For the closure (resp. interior) of a set U we use the notation $cl(U)$ (resp. $int(U)$). The following stronger result holds:

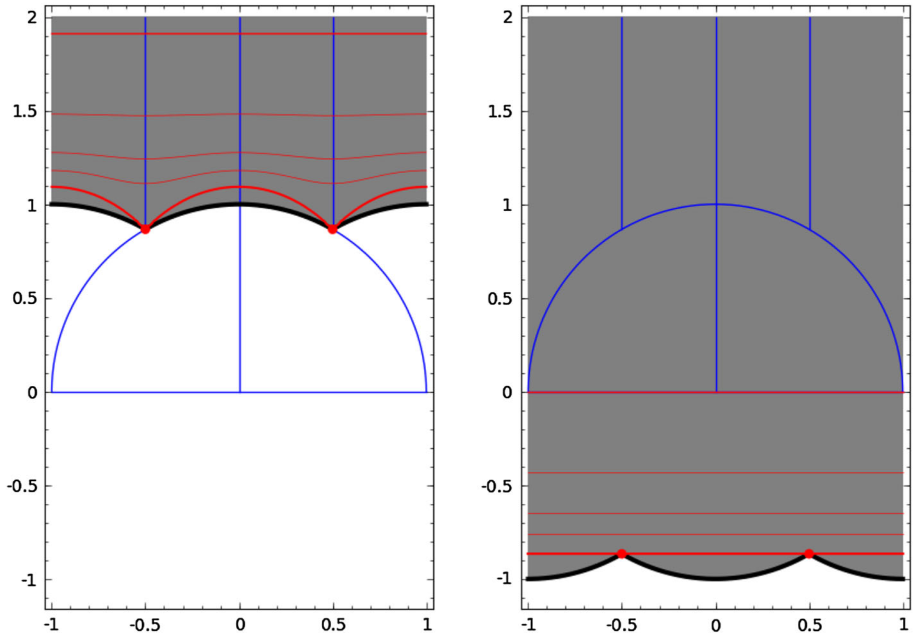

Fig. 2 Domain U and range V of f, together with preimages of $\{\mathrm{Im}(w) = c\}$, where $c = -\left(1 - \frac{1}{n}\right)\frac{\sqrt{3}}{2}$ for $n = 1, 2, 4, 8, \infty$

Theorem 3.1 *f maps U biholomorphically onto V. The map extends to a homeomorphism on the boundaries.*

For the proof of Theorem 3.1 we first need to understand the image of the boundary $\partial U = M$ (Fig. 2).

Proposition 3.2 *The map f is a homeomorphism between ∂U and $f(\partial U) = \partial V = \overline{\partial U}$.*

Proof Since f is equivariant we have for $|z| = 1$, $\mathrm{Im}\, z > 0$

$$f(-\overline{z}) = f(-1/z) = -1/f(z).$$

However $f(-\overline{z}) = -\overline{f(z)}$ for any $z \in \mathbb{H}$ and so the image of the arc $\mathcal{C} = \{e^{i\theta} \mid \theta \in [\pi/3, 2\pi/3]\}$ lies on the unit circle. An easy calculation determines that $f(z) = \overline{z}$ for $z = e^{\pi i/3}, e^{\pi i/2}$ and $e^{2\pi i/3}$. It follows that $f(\mathcal{C})$ contains $\overline{\mathcal{C}}$.

To show that it equals $\overline{\mathcal{C}}$ we first note that by an identity of Ramanujan, the derivative of f satisfies

$$f'(z) = 1 - \frac{E'(z)}{E_2^2(z)} = \frac{E_4(z)}{E_2^2(z)}. \tag{15}$$

Next we consider the function

$$g(z) = -i \log f(e^{iz})$$

where the principal branch of log is used. By Lemma 2.5 this is defined in a complex neighborhood of the real interval $[\pi/3, 2\pi/3]$ and by the above argument g takes real values on $[\pi/3, 2\pi/3]$, $g(\pi/3) = -\pi/3$ and $g(2\pi/3) = -2\pi/3$. Also note that $g'(z) = \frac{f'(e^{iz})e^{iz}}{f(e^{iz})}$ is real (again using the fact that g is real on $[\pi/3, 2\pi/3]$) and non-zero if $z \in (\pi/3, 2\pi/3)$ and so g is monotonically decreasing on $[\pi/3, 2\pi/3]$ and therefore bijective. This implies that $f : \mathcal{C} \to f(\mathcal{C})$ is homeomorphism.

Finally by $f(z+1) = f(z) + 1$ we also have that $f(\partial U) = f(M) = \overline{M}$ and the map is a homeomorphism. □

Lemma 3.3 *If $z_n \in U$ and $f(z_n)$ is a convergent sequence then z_n has a convergent subsequence.*

Proof Let $w_n = f(z_n) \in f(U)$ denote a sequence which converges to w. Since the function $f(z) - z = \frac{12}{2\pi i}\frac{1}{E_2(z)}$ is bounded on U, the sequence $w - z_n = (w - w_n) + (f(z_n) - z_n)$ is also bounded. □

Lemma 3.4 *We have $f(U) \subset V$.*

Proof Since $V = int(cl(V))$ and $f(U)$ is open, it is enough to show that $f(U) \subset cl(V)$.

Suppose there exists a $z \in U$ such that $w = f(z) \notin cl(V)$. Let $t_0 = \sup\{t \in [0, \infty) \mid w - it \in f(U)\}$. Then t_0 is finite since $\operatorname{Im} f(z) = \operatorname{Im} z + \operatorname{Im}(f(z) - z)$ is bounded from below on $cl(U)$. It is also positive, since f is open.

Therefore there exists a sequence $0 < t_n < t_0$, such that $t_n \to t_0$ and $w - t_n i = f(z_n)$ for some $z_n \in U$. By Lemma 3.3 we may assume that $z_n \to z \in cl(U)$.

If $z \in \partial U$ then $f(z) = w - t_0 i \in \partial V$ by Proposition 3.2, contradicting that w is not in $cl(V)$. Therefore $w - t_0 i \in f(U)$, but by the definition of t_0, $(w - t_0 i) - \epsilon i \notin f(U)$ for all $\epsilon > 0$. This gives a contradiction to the fact that f is an open map. □

Proof of Theorem 3.1 By Lemma 3.4 and since f is open the image $f(U)$ is an open subset of $int(cl(V)) = V$. Also if $z_n \in U$, $f(z_n) \to w \in V$ then Lemma 3.3 and Proposition 3.2 shows that $w \in f(U)$. It follows that $cl(f(U)) \cap V = f(U)$, so the image $f(U)$ is a closed subset of V as well. Since V is connected $f(U) = V$. Finally V is simply connected, U is connected and the map f is a surjective covering map since $f'(z) \neq 0$ on U by (15). Therefore f is a bijection. □

As a corollary of Theorem 3.1 we obtain the following theorem.

Theorem 3.5 *Let $G(w) = f^{-1}(w) - w$. Then $G(w)$ is holomorphic in $\left\{w \mid \operatorname{Im}(w) > -\frac{\sqrt{3}}{2}\right\}$, and extends continuously to the boundary. Moreover the Fourier coefficients c_n in $G(w) = \sum_{n=0}^{\infty} c_n e^{2\pi i w}$ satisfy the bound*

$$|c_n| \le 3e^{-\pi\sqrt{3}n}. \tag{16}$$

Proof. By Theorem 3.1 and the arguments at the beginning of the section we know that $G(w) = f^{-1}(w) - w = \sum_{n=0}^{\infty} c_n q^n$, where the Fourier expansion converges on the (closed) half-plane $\left\{\operatorname{Im}(w) \ge -\frac{\sqrt{3}}{2}\right\}$.

Hence

$$c_n = \int_0^1 G\left(x - i\frac{\sqrt{3}}{2}\right) e^{-2\pi i n(x - i\frac{\sqrt{3}}{2})} dx. \tag{17}$$

Note that

$$\max_{x\in[0,1]}\left|G\left(x-i\frac{\sqrt{3}}{2}\right)\right| \leq \max_{x\in[0,1]}\left(\left|f^{-1}\left(x-i\frac{\sqrt{3}}{2}\right)\right|+\left|x-\frac{\sqrt{3}}{2}i\right|\right)$$
$$\leq |0.5+1.911i|+\left|0.5-\frac{\sqrt{3}}{2}i\right| \leq 3,$$

where we used the fact that f^{-1} is equivariant and that $f^{-1}\left(\left\{\mathrm{Im}(w)=-\frac{\sqrt{3}}{2}\right\}\right)$ lies below $f^{-1}(\mathbb{R})$. The latter can be used to bound the former by using the estimates in Proposition 2.8. (16) now follows from a basic estimate of the absolute value of (17). □

Our next theorem gives an explicit formula for the Fourier coefficients c_n.

Theorem 3.6 *Let r_n be defined as the coefficients of the formal power series*

$$\sum_{n\geq 0} r_n q^n = -12+\log\left(\frac{\left(q\exp\left(\frac{12}{E_2(q)}-12\right)\right)^{-1}}{q}\right) \in \mathbb{Q}[[q]].$$

Then the Fourier coefficients of

$$G(w) = f^{-1}(w)-w = \sum_{n=0}^{\infty} c_n q^n$$

satisfy

$$c_n = \frac{1}{2\pi i} r_n e^{-12n}. \tag{18}$$

Proof Let $w = f(z)$ and consider

$$s = \exp(2\pi i w - 12) = \exp(2\pi i f(z) - 12) = q\exp(g(q))$$

where $q = \exp(2\pi i z)$ and

$$g(q) = \frac{-288\sum_{n=1}^{\infty}\sigma(n)q^n}{1-24\sum_{n=1}^{\infty}\sigma(n)q^n}.$$

Note that as a formal power series $g(q) = O(q)$ in $\mathbb{Q}[[q]]$ and so $\exp(g(q))$ has rational coefficients. Also since $s = q + O(q^2)$ there is an inverse function still given by rational coefficients

$$q = \sum_{n=1}^{\infty} a_n s^n.$$

This series converges on some disc $|s| < c$ where it gives for $f(z) = w$

$$e^{2\pi i z} = e^{-12} e^{2\pi i w}\sum_{n=0}^{\infty} a_{n+1} s^n$$

where $s = e^{-12+2\pi i w}$ is as above.

Finally since

$$e^{2\pi iz} = e^{2\pi iw} e^{2\pi iG(w)}$$

we have

$$e^{2\pi iG(w)} = e^{-12} \sum_{n=0}^{\infty} a_{n+1} s^n.$$

Since $a_1 \neq 0$ the right hand side has a continuous logarithm on a (maybe even smaller) disc. If the formal logarithm is used

$$2\pi iG(w) = -12 + \sum_{m=1}^{\infty} \frac{(-1)^m}{a_1^{m+1}(m+1)} \left(\sum_{n=1}^{\infty} a_{n+1} s^n \right)^m + 2\pi ki$$

for some integer k. A simple comparison for $\mathrm{Re}(w) = 0$ shows that $k = 0$. Note that when returning to the variable w the above formal computations make sense in some upper half plane, and by the uniqueness of Fourier coefficients we have

$$G(w) = \frac{1}{2\pi i} \left(-12 + \sum_{n=1}^{\infty} r_n s^n \right) = \frac{6i}{\pi} + \sum_{n=1}^{\infty} c_n e^{2\pi inw}$$

where $r_n \in \mathbb{Q}$ and where $c_n = \frac{r_n e^{-12n}}{2\pi i}$. □

Corollary 3.7 *Let $z_{\frac{a}{c}} := f^{-1}\left(\frac{-d}{c}\right) \in \mathcal{F}$ be the $SL_2(\mathbb{Z})$-translate of the zero of E_2 around $\frac{a}{c}$ inside $\mathcal{F}$. For $N \geq 1$ we have the following estimate:*

$$\left| z_{\frac{a}{c}} - \left(-\frac{d}{c} + \frac{6i}{\pi} + \sum_{n=1}^{N} r_n e^{-12n} e^{2\pi in\left(\frac{-d}{c}\right)} \right) \right| \leq 0.014 e^{-\sqrt{3}\pi N}.$$

Proof Since $z_{\frac{a}{c}} = G\left(\frac{-d}{c}\right) + \left(-\frac{d}{c}\right)$ Theorems 3.5 and 3.6 gives the estimate:

$$\left| z_{\frac{a}{c}} - \left(-\frac{d}{c} + \frac{6i}{\pi} + \sum_{n=1}^{N} r_n e^{-12n} e^{2\pi in\left(\frac{-d}{c}\right)} \right) \right| \leq \sum_{n=N+1}^{\infty} |c_n|$$
$$\leq 3e^{-\pi\sqrt{3}(N+1)} \frac{1}{1 - e^{-\pi\sqrt{3}}} \leq 0.014 e^{-\pi\sqrt{3}N}.$$

□

The coefficients r_n of the formal power series

$$\sum_{n\geq 0} r_n q^n = -12 + \log \left(\frac{\left(q \exp\left(\frac{12}{E_2(q)} - 12 \right) \right)^{-1}}{q} \right)$$

in Theorem 3.6 can be computed explicitly and we have

$$\sum_{n\geq 0} r_n q^n = -12 - 288q + 75168q^2 - 29321856q^3 + 13541649696q^4 + O(q^5)$$

(Note that higher index coefficients no longer all lie in $\mathbb{Z}$)

From this the constants c_n can be computed explicitly, for example $c_0 = \frac{6i}{\pi}$ and $c_1 = 144\frac{e^{-12}}{\pi}i$ etc.

In particular we can approximate the $SL_2(\mathbb{Z})$-translates of the zeros of E_2 arbitrarily well by explicit expressions. For example taking $N = 3$ in Corollary 3.7 gives the following improvement of Theorem 1.2 thereby proving Theorem 1.3:

$$z_{\frac{a}{c}} = \frac{-d}{c} + \frac{6i}{\pi} + \frac{144i}{\pi} e^{-12} e^{2\pi i\left(\frac{-d}{c}\right)} + \eta, \tag{19}$$

where

$$\begin{aligned} 0.000000449 &\leq \left|75168\frac{e^{-24}}{2\pi i}\right| - \left|-29321856\frac{e^{-36}}{2\pi i}\right| - 0.014e^{-\pi\sqrt{3}\cdot 3} \\ &\leq |\eta| \\ &\leq \left|75168\frac{e^{-24}}{2\pi i}\right| + \left|-29321856\frac{e^{-36}}{2\pi i}\right| + 0.014e^{-\pi\sqrt{3}\cdot 3} \leq 0.000000454. \end{aligned}$$

Calculations of the Fourier coefficients by numerical integration show perfect agreement with the computations from Theorem 3.6. The numerical results for c_n, $n \geq 0$ were calculated using `sage` [8]:

$$\begin{aligned} c_0 &= 1.90985931710274402922660516047017234441351574888547738497i \\ c_1 &= 0.00028162994902227980400370919939063856289594529890275357i \\ c_2 &= -0.00000045163288929282012635455207577614911204985274433204i \\ c_3 &= 0.00000000108245596925811696405920054080771200657423178116i \\ c_4 &= -0.00000000000307154282808538137128721799597366123291772057i \\ c_5 &= 0.00000000000000957094344711630129941209014246199967789040i \\ c_6 &= -0.00000000000000003165503372449709626701121359204401518804i \\ c_7 &= 0.00000000000000000010911333723210259127123321555755721374i \\ c_8 &= -0.00000000000000000000387695756899896759723043971850164 87i \\ c_9 &= 0.000000000000000000000140991650336337176376718140570072i \\ c_{10} &= -0.00000000000000000000000522238601930043508518075691844i \end{aligned}$$

The code is available at http://sites.google.com/site/jjermann2/research/e2_zeros/.

See Fig. 3 for a picture of $G(w)$, the error term in the approximation $f^{-1}(w) \approx w$, for w corresponding to zeros of height 100:

The closed curve $G(\mathbb{R}) = G\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ is already extremely well approximated by the first two terms $c_0 + c_1 \exp(2\pi i w)$. Indeed estimate (3) for $w = \frac{d}{c}$ (in fact for any $w \in \mathbb{R}$) and $G\left(-\frac{d}{c}\right) = f^{-1}\left(-\frac{d}{c}\right) + \frac{d}{c}$ shows:

$$0.00027 < \left|G(w) - \frac{6i}{\pi}\right| < 0.00029, \quad w \in \mathbb{R}.$$

This good approximation follows from the exponential decay of the Fourier coefficients c_n. Indeed the bounds are very close to the absolute value of the next Fourier coefficient $|c_1| = 0.00281\ldots$. These circular bounds are also visible in Fig. 3.

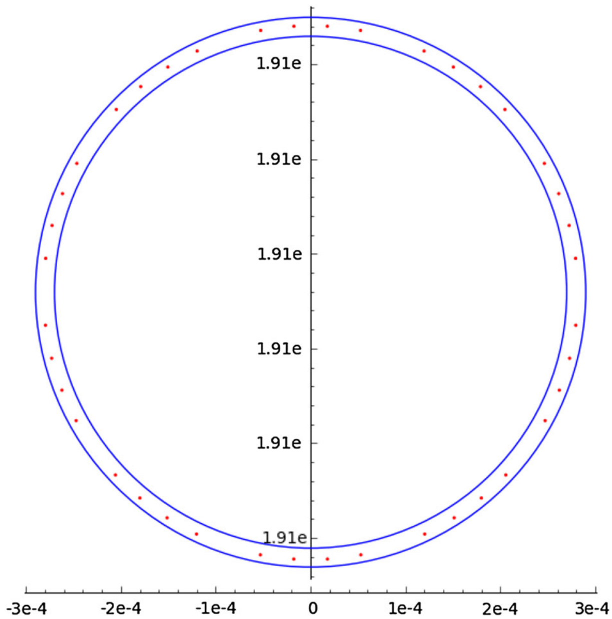

Fig. 3 $G(w)$ for $w = \frac{d}{100} \in \mathbb{Q} \cap \left[-\frac{1}{2}, \frac{1}{2}\right]$ with $(d, 100) = 1$

4 Taylor expansion

We can also examine the Taylor expansion $F(w) = \sum_{n=0}^{\infty} b_n w^n$ around $w = 0$. By Theorem 3.1 it has radius of convergence 1 and converges on the boundary.

By Lagrange inversion theorem the Taylor coefficients can be calculated in terms of the Taylor coefficients of $f(z)$ at z_0, where z_0 is the $SL_2(\mathbb{Z})$-translate in the strict fundamental domain of the zero of E_2 on the imaginary axis.

Note that $f(z)$ is a rational function in $E_2(z)$ and z. Its derivatives can be calculated as rational functions in $E_2(z)$, $E_4(z)$, $E_6(z)$ and z. Therefore the Taylor expansion at z_0 can be expressed as a rational function in $X := E_4(z_0)$, $Y := E_6(z_0)$, $Z := E_2(z_0)$ and z_0. By the transformation property (1) we have $z_0 = -\frac{12}{2\pi i Z}$, so in fact we get a rational function in X, Y and Z.

For the first coefficient we have $b_0 = \left(\frac{6i}{\pi}\right)\frac{1}{Z}$. For the remaining coefficients ($n > 0$) calculations seem to indicate that b_n is of the following shape:

$$b_n = (-1)^n \left(\frac{12}{2\pi i}\right)^{1-n} X^{-2n+1} a_n(X, Y, Z),$$

where $a_n(X, Y, Z) \in \mathbb{Q}[X, Y, Z]$ is a weighted homogenous polynomial of degree $10n - 6$, where $\deg(X) := 4$, $\deg(Y) := 6$, $\deg(Z) := 2$. The degree of the (weighted homogenous) rational function $b_n(X, Y, Z)$ is $2(n - 1)$.

Here is a list of the first few polynomials $a_n(X, Y, Z)$ for $n > 0$:

$$a_1 = Z^2$$
$$a_2 = XZ^5 + X^2Z^3 - 2YZ^4$$
$$a_3 = X^2Z^8 - 2X^3Z^6 - 4XYZ^7 + X^4Z^4 - 4X^2YZ^5 + 8Y^2Z^6$$

$$
\begin{aligned}
a_4 = {} & X^3Z^{11} - 5X^4Z^9 - 6X^2YZ^{10} - 9X^5Z^7 + 20X^3YZ^8 \\
& + 20XY^2Z^9 + X^6Z^5 - 6X^4YZ^6 + 24X^2Y^2Z^7 - 40Y^3Z^8 \\
a_5 = {} & X^4Z^{14} - 8X^5Z^{12} - 8X^3YZ^{13} + \frac{66}{5}X^6Z^{10} \\
& + 56X^4YZ^{11} + 36X^2Y^2Z^{12} - 20X^7Z^8 + 104X^5YZ^9 \\
& - \frac{836}{5}X^3Y^2Z^{10} - 112XY^3Z^{11} + X^8Z^6 - 8X^6YZ^7 \\
& + 48X^4Y^2Z^8 - 160X^2Y^3Z^9 + 224Y^4Z^{10}.
\end{aligned}
$$

The numerical evaluations of those polynomials ($Z = E_2(z_0)$, etc.) give for Taylor coefficients b_n :

$$
\begin{aligned}
b_0 &= 1.910140496498270982037654535798483091377748703 0i \\
b_1 &= 0.9982361219015924374815710878280361648431190825 \\
b_2 &= -0.0055236842011260453610739166397990326586651337i \\
b_3 &= 0.0114948915027420831631325909381504156370306 7326 \\
b_4 &= 0.0178252611095229253133162329291348589135823077i \\
b_5 &= -0.0218243134639575211728774441381952550676521991 \\
b_6 &= -0.0216634844629385759461618124642355350591382590i \\
b_7 &= 0.0173461622715009362175946164162223267332088060 \\
b_8 &= 0.0104172812309514952250501361120571266678673695i \\
b_9 &= -0.0029825116383882005761911965146832408919333302.
\end{aligned}
$$

Figure 4. shows the approximations arising from the Taylor approximations of order 6 and 8. In general the Taylor polynomial is inferior to the Fourier approximation. However for the zeros $z_{1/c}$ of E_2 near $1/c$ found in [3], the Taylor polynomials give a fast converging asymptotic series in c:

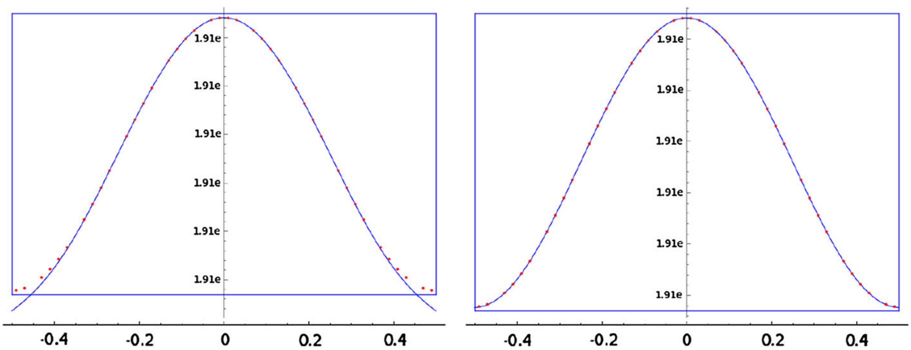

Fig. 4 Taylor approximation of $F(w)$ for $N = 6$ and 8 together with zeros of height 100 and the basic bounds from Eq. (3)

Proposition 4.1 *Let* $c \geq 2$, $z_{\frac{1}{c}} = f^{-1}\left(-\frac{1}{c}\right)$. *Then*

$$\left| z_{\frac{1}{c}} - \sum_{n=0}^{N} \frac{b_n}{(-c)^n} \right| < 3c^{-(N+1)} \frac{1}{1-\frac{1}{c}} \leq 6c^{-(N+1)}.$$

Proof The proposition follows from the following estimate:

$$|b_n| \leq \max_{|w| \leq 1} \left| f^{-1}(w) \right| < 3. \tag{20}$$

To prove the estimate (20) let $|w| \leq 1$ and (using $f(z+1) = f(z) + 1$)

$$z = f^{-1}(w) \in \{z = x + iy \in \mathcal{F} + \mathbb{Z} \mid |x| \leq 1\}.$$

If by contradiction $|z| \geq 3$ then:

$$|w| = |f(z)| \geq |z| - \frac{6}{\pi} \frac{1}{1 - |E_2(z) - 1|} > |z| - 2 \geq 1,$$

where we used e.g. estimate (7) to bound $(1 - |E_2(z) - 1|)\, \pi > 3$. This gives a contradiction, so $|z| = |f^{-1}(w)| < 3$. □

For example

$$\left| z_{\frac{1}{c}} - b_0 + \frac{b_1}{c} \right| < \frac{6}{c^2}.$$

Note that $b_0 = z_0 = -1/\tau_0$ for the unique zero τ_0 of real part 0 and we have $|b_0 - 6i/\pi| > 0.00027$. Also note b_1 is close but not equal to 1. These explain the limitations of the approximation $z_{\frac{1}{c}} \approx \frac{6i}{\pi} - \frac{1}{c}$ (Fig. 3).

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