Asymptotic power utility-based pricing and hedging

**Jan Kallsen · Johannes Muhle-Karbe ·
Richard Vierthauer**

Abstract Kramkov and Sîrbu (Ann. Appl. Probab., 16:2140–2194, 2006; Stoch. Proc. Appl., 117:1606–1620, 2007) have shown that first-order approximations of power utility-based prices and hedging strategies for a small number of claims can be computed by solving a mean-variance hedging problem under a specific equivalent martingale measure and relative to a suitable numeraire. For power utilities, we propose an alternative representation that avoids the change of numeraire. More specifically, we characterize the relevant quantities using semimartingale characteristics similarly as in Černý and Kallsen (Ann. Probab., 35:1479–1531, 2007) for mean-variance hedging. These results are illustrated by applying them to exponential Lévy processes and stochastic volatility models of Barndorff-Nielsen and Shephard type (J. R. Stat. Soc. B, 63:167–241, 2001). We find that asymptotic utility-based hedges are virtually independent of the investor's risk aversion. Moreover, the price adjustments compared to the Black–Scholes model turn out to be almost linear in the investor's risk aversion, and surprisingly small unless very high levels of risk aversion are considered.

Keywords Utility-based pricing and hedging · Incomplete markets ·
Mean-variance hedging · Numeraire · Semimartingale characteristics

JEL Classification G11 · G13

J. Kallsen · R. Vierthauer
Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel,
Westring 383, 24118 Kiel, Germany
e-mail: kallsen@math.uni-kiel.de

R. Vierthauer
e-mail: vierthauer@math.uni-kiel.de

J. Muhle-Karbe (✉)
Departement Mathematik, ETH Zürich, Rämistrasse 101, 8092 Zurich, Switzerland
e-mail: johannes.muhle-karbe@math.ethz.ch

J. Muhle-Karbe
Swiss Finance Institute, Geneva, Switzerland

1 Introduction

In incomplete markets, derivative prices cannot generally be based on perfect replication. A number of alternatives have been suggested in the literature, relying, e.g., on superreplication, mean-variance hedging, calibration of parametric families, utility-based concepts, or ad-hoc approaches. This paper focuses on utility indifference prices as studied by Hodges and Neuberger [17] and many others. They make sense for over-the-counter trades of a fixed quantity of contingent claims. Suppose that a client approaches a potential seller in order to buy q European-style contingent claims maturing at T. The seller is supposed to be a utility maximizer with given preference structure. She will enter into the contract only if her maximal expected utility is increased by the trade. The utility indifference price is the lowest acceptable premium for the seller. If the trade is made, the seller's optimal position in the underlyings changes due to the presence of the option. This adjustment in the optimal portfolio process is called utility-based hedging strategy for the claim. Both the utility indifference price and the corresponding utility-based hedging strategy are typically hard to compute even if relatively simple incomplete market models are considered. A reasonable way out for practical purposes is to consider approximations for small q, i.e., the limiting structure for small numbers of contingent claims. Extending earlier work on the limiting price, Kramkov and Sîrbu [32,33] show that first-order approximations of the utility indifference price and the utility-based hedging strategy can be expressed in terms of a Galtchouk–Kunita–Watanabe (GKW) decomposition of the claim after changing both the numeraire and the underlying probability measure.

From a slightly different perspective one may say that Kramkov and Sîrbu [32,33] relate utility indifference pricing and hedging asymptotically to some mean-variance hedging problem. In this representation, the L^2-distance between payoff and terminal wealth of approximating portfolios needs to be considered relative to both a new numeraire and a new probability measure.

This differs from related results for exponential utility (see [3,28,34]), where no numeraire change is necessary. In the present study, we show that the numeraire change can also be avoided for power utilities, which constitute the most popular and tractable ones on the positive real line, i.e., in the setup of [32,33]. This allows to examine directly how the dynamics of the underlying change to account for utility-based rather than mean-variance hedging, and also allows to apply directly a number of explicit resp. numerical results from the literature. The key idea is to consider an equivalent mean-variance hedging problem relative to the original numeraire but under yet another probability measure. More specifically, the solution of [32,33] for a contingent claim H corresponds to a quadratic hedging problem of the form

$$\min_{c,\varphi} E_{Q^\$}\left(\left(\frac{c + \varphi \bullet S_T - H}{N_T}\right)^2\right) \qquad (1.1)$$

with some numeraire process N and some martingale measure $Q^\$$. If we define a new measure P^ϵ via

$$\frac{dP^\epsilon}{dQ^\$} := \frac{1/N_T^2}{E_{Q^\$}(1/N_T^2)},$$

the mean-variance hedging problem (1.1) can evidently be rewritten as

$$E_{Q^\$}(1/N_T^2) \min_{c,\varphi} E_{P^\epsilon}\left((c + \varphi \bullet S_T - H)^2)\right), \qquad (1.2)$$

where we minimize again over some a set of initial endowments c and trading strategies φ. Replacing (1.1) by (1.2) constitutes the key idea underlying our approach. For a related transition in the quadratic hedging literature compare [16] with and [6,39,42] without numeraire change. Since the stock is not a martingale in the reformulation (1.2), the Galtchouk-Kunita-Watanabe decomposition does not lead to the solution. Instead, representations as in [42] or, more generally, [6] can be used to obtain concrete formulas, which are provided in Theorem 4.7 of this paper. On a rigorous mathematical level, we do not consider mean-variance hedging problems because the expression in Theorem 4.7 is the solution to such a hedging problem only under additional regularity which does not hold in general. Instead, we show in a more direct fashion that the solution of [32,33] can be expressed as in Theorem 4.7.

In order to illustrate the applicability of our results and shed light on the role of the investor's risk aversion for power utility-based pricing and hedging, we consider exponential Lévy processes and the stochastic volatility model of Barndorff-Nielsen and Shephard [2] as examples. For these processes, all technical assumptions can be verified directly in terms of the model parameters. Moreover, results for the related mean-variance hedging problem (cf. [5,18,27,30]) can be adapted to obtain first-order approximations to utility-based prices and hedging strategies explicitly up to some numerical integrations. Using parameters estimated from an equity index time series, we find that the asymptotic utility-based hedging strategies are virtually independent of the investor's risk aversion, which holds exactly for exponential investors. Moreover, the risk premia per option sold turn out to be almost linear in the investor's (absolute) risk aversion, which again holds exactly for exponential utilities. Hence, these examples suggest very similar pricing and hedging implications of both exponential and power utilities: risk aversion barely influences the optimal hedges, and enters linearly into the first-order risk-premia. Similarly as in [14,13] in the context of basis risk, we find that the price adjustments are negligibly small for the levels of risk aversion typically considered in the literature. In particular, surprisingly high levels of risk aversion are needed to obtain bid- and ask prices below and above the Black–Scholes price, respectively.

The remainder of the paper is organized as follows. After briefly recalling the general theory of power utility-based pricing and hedging in Sect. 2, we review the asymptotic results of Kramkov and Sîrbu [32,33]. As a byproduct we derive a feedback formula for the utility-based hedging strategy. Subsequently, we develop our alternative representation in Sect. 4. Throughout, we explain how to apply the general theory to exponential Lévy processes and the stochastic volatility model of Barndorff-Nielsen and Shephard [2]. A concrete numerical example is considered in Sect. 6. Finally, the appendix summarizes notions and results concerning semimartingale calculus for the convenience of the reader.

Unexplained notation is generally used as in the monograph of Jacod and Shiryaev [21]. In particular, for a semimartingale X, we denote by $L(X)$ the predictable X-integrable processes and by $\varphi \bullet X$ the stochastic integral of $\varphi \in L(X)$ with respect to X. We write $\mathscr{E}(X)$ for the stochastic exponential of a semimartingale X and denote by $\mathscr{L}(Z) := \frac{1}{Z_-} \bullet Z$ the *stochastic logarithm* of a semimartingale Z satisfying $Z, Z_- \neq 0$. For semimartingales X and Y, $\langle X, Y \rangle$ represents the predictable compensator of $[X, Y]$, provided that the latter is a special semimartingale (cf. [20, p. 37]). Finally, we write c^{-1} for the *Moore-Penrose pseudoinverse* of a matrix or matrix-valued process c (cf. [1]) and denote by E_d the identity matrix on $\mathbb{R}^d$.

2 Utility-based pricing and hedging

Our mathematical framework for a frictionless market model is as follows. Fix a terminal time $T > 0$ and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, P)$ in the sense of [21, I.1.2].

For ease of exposition, we assume that $\mathscr{F}_T = \mathscr{F}$ and $\mathscr{F}_0 = \{\varnothing, \Omega\}$ up to null sets, i.e., all $\mathscr{F}_0$-measurable random variables are almost surely constant.

We consider a securities market which consists of $d + 1$ assets, a bond and d stocks. As is common in Mathematical Finance, we work in *discounted* terms. This means we suppose that the bond has constant value 1 and denote by $S = (S^1, \ldots, S^d)$ the *discounted price process* of the d stocks in terms of multiples of the bond. The process S is assumed to be an $\mathbb{R}^d$-valued semimartingale.

Example 2.1 1. Throughout this article, we will illustrate our results by considering one-dimensional *exponential Lévy models*. This means that $d = 1$ and $S = S_0 \mathscr{E}(X)$ for a constant $S_0 > 0$ and a Lévy process X with Lévy-Khintchine triplet (b^X, c^X, F^X) relative to some truncation function h on $\mathbb{R}$. We write

$$\psi^X(z) = zb^X + \frac{1}{2}z^2 c^X + \int (e^{zx} - 1 - zh(x))F^X(dx)$$

for the corresponding *Lévy exponent*, i.e., the function $\psi^X : i\mathbb{R}^d \to \mathbb{C}$ such that $E(e^{zX_t}) = \exp(t\psi^X(z))$. When considering exponential Lévy models, we will always assume $S > 0$, which is equivalent to $\Delta X > -1$ resp. the support of F^X being concentrated on $(-1, \infty)$.

2. We will also consider the stochastic volatility model of Barndorff-Nielsen and Shephard [2] (henceforth *BNS model*). Here $d = 1$ and the return process X driving $S = S_0 \mathscr{E}(X)$ is modelled as

$$dX_t = \mu y_t dt + \sqrt{y_t} dW_t, \quad X_0 = 0,$$

for a constant $\mu \in \mathbb{R}$, a standard Brownian motion W, and an independent Lévy-driven Ornstein-Uhlenbeck process y. The latter is given as the solution to the SDE

$$dy_t = -\lambda y_t dt + dZ_t, \quad y_0 > 0,$$

with some constant $\lambda > 0$ and an increasing Lévy process Z with Lévy-Khintchine triplet $(b^Z, 0, F^Z)$ relative to a truncation function h on $\mathbb{R}$.

Self-financing *trading strategies* are described by $\mathbb{R}^d$-valued predictable stochastic processes $\varphi = (\varphi^1, \ldots, \varphi^d)$, where φ^i_t denotes the number of shares of security i held at time t. We consider an investor whose preferences are modelled by a *power utility function* $u(x) = x^{1-p}/(1-p)$ with constant relative risk aversion $p \in \mathbb{R}_+ \backslash \{0, 1\}$. Given an initial endowment $v > 0$, the investor solves the *pure investment problem*

$$U(v) := \sup_{\varphi \in \Theta(v)} E(u(v + \varphi \bullet S_T)), \tag{2.1}$$

where the set $\Theta(v)$ of *admissible strategies* for initial endowment v is given by

$$\Theta(v) := \{\varphi \in L(S) : v + \varphi \bullet S \geq 0\}.$$

To ensure that the optimization problem (2.1) is well-posed, we make the following two standard assumptions.

Assumption 2.2 There exists an *equivalent local martingale measure*, i.e., a probability measure $Q \sim P$ such that S is a local Q-martingale.

Assumption 2.3 The maximal expected utility in the pure investment problem (2.1) is finite, i.e., $U(v) < \infty$.

Example 2.4 1. In a univariate exponential Lévy model $S = S_0\mathscr{E}(X) > 0$, Assumption 2.2 is satisfied if X is neither a.s. decreasing nor a.s. increasing. In this case, by [37, Corollary 3.7], Assumption 2.3 holds if and only if $\int_{\{|x|>1\}} x^{1-p} F^X(dx) < \infty$, i.e., if and only if the return process X has finite $(1-p)$-th moments.

2. By [26, Theorem 3.3], Assumptions 2.2 and 2.3 are always satisfied in the BNS model if the investor's risk aversion p is bigger than 1. For $p \in (0, 1)$, they hold provided that

$$\int_1^\infty \exp\left(\frac{1-p}{2p}\mu^2 \frac{1-e^{-\lambda T}}{\lambda} z\right) F^Z(dz) < \infty, \tag{2.2}$$

i.e., if sufficiently large exponential moments of the driving Lévy process Z exist.

In view of [31, Theorem 2.2], Assumptions 2.2 and 2.3 imply that the supremum in (2.1) is attained for some strategy $\widehat{\varphi} \in \Theta(v)$ with strictly positive wealth process $v + \widehat{\varphi} \bullet S$. By Assumption 2.2 and [21, I.2.27], $v + \widehat{\varphi} \bullet S_-$ is strictly positive as well and we can write

$$v + \widehat{\varphi} \bullet S = v\mathscr{E}(-\widetilde{a} \bullet S)$$

for the *optimal number of shares per unit of wealth*

$$-\widetilde{a} := \frac{\widehat{\varphi}}{v + \widehat{\varphi} \bullet S_-},$$

which is independent of the initial endowment v for power utility. Finally, [31, Theorem 2.2] also establishes the existence of a *dual minimizer*, i.e., a strictly positive supermartingale $\widehat{Y}$ with $\widehat{Y}_T = \mathscr{E}(-\widetilde{a} \bullet S)_T^{-p}$ such that $(v + \varphi \bullet S)\widehat{Y}$ is a supermartingale for all $\varphi \in \Theta(v)$ and $(v + \widehat{\varphi} \bullet S)\widehat{Y}$ is a true martingale. Alternatively, one can represent this object in terms of the *opportunity process* $L := L_0\mathscr{E}(K) := \mathscr{E}(-\widetilde{a} \bullet S)^p \widehat{Y}$ of the power utility maximization problem (cf. [6,26] for motivation and more details).

The optimal strategy $\widehat{\varphi}$ as well as the joint characteristics of the assets and the opportunity process L satisfy a semimartingale *Bellman equation* (cf. [36, Theorem 3.2]). In concrete models, this sometimes allows to determine $\widehat{\varphi}$ and L by making an appropriate ansatz.

Example 2.5 1. Let $S = S_0\mathscr{E}(X) > 0$ for a non-monotone Lévy process X with finite $(1-p)$-th moments. Then it follows from [37, Lemma 5.1] that there exists a unique maximizer $\widehat{\eta}$ of

$$\mathfrak{g}(\eta) = \eta b^X - \frac{p}{2}\eta^2 c^X + \int \left(\frac{(1+\eta x)^{1-p} - 1}{1-p} - \eta h(x)\right) F^X(dx),$$

over the set $\mathscr{C}^0 = \{\eta \in \mathbb{R} : F^X(x \in \mathbb{R} : \eta x < -1) = 0\}$ of fractions of wealth invested into stocks that lead to nonnegative wealth processes. By [37, Theorem 3.2], the optimal number of shares per unit of wealth is given by

$$-\widetilde{a} = \widehat{\eta}/S_-,$$

with corresponding wealth process $v\mathscr{E}(-\widetilde{a} \bullet S) = v\mathscr{E}(\widehat{\eta}X)$ and opportunity process

$$L_t = \exp(a(T-t)), \quad \text{where } a = (1-p)\max_{\eta \in \mathscr{C}^0} \mathfrak{g}(\eta).$$

2. By [26, Theorem 3.3], it is also optimal to hold a constant fraction of wealth in stocks in the BNS model, namely $\widehat{\eta} = \mu/p$ (provided that the conditions of Example 2.4 are satisfied). The optimal number of shares per unit of wealth is then given by $-\widetilde{a} = \widehat{\eta}/S_-$ with corresponding wealth process $v\mathscr{E}(\widehat{\eta}X)$, and opportunity process

$$L_t = \exp(\alpha_0(t) + \alpha_1(t)v_t),$$

for

$$\alpha_1(t) = \frac{1-p}{2p}\mu^2\frac{1-e^{-\lambda(T-t)}}{\lambda}, \quad \alpha_0(t) = \int\limits_t^T \psi^Z(\alpha_1(s))ds,$$

where ψ^Z denotes the Lévy exponent of Z.

In addition to the traded securities, we now also consider a non-traded European contingent claim with maturity T and payment function H, which is an $\mathscr{F}_T$-measurable random variable. Following [32,33], we assume that H can be superhedged by some admissible strategy as, e.g., for European puts and calls.

Assumption 2.6 $|H| \leq w + \varphi \bullet S_T$ for some $w \in (0, \infty)$ and $\varphi \in \Theta(w)$.

If the investor sells q units of H at time 0, her terminal wealth should be sufficiently large to cover the payment $-qH$ due at time T. This leads to the following definition (cf. [9,19] for more details).

Definition 2.7 A strategy $\varphi \in \Theta(v)$ is called *maximal* if the terminal value $v + \varphi \bullet S_T$ of its wealth process is not dominated by that of any other strategy in $\Theta(v)$. An arbitrary strategy φ is called *acceptable* if its wealth process can be written as

$$v + \varphi \bullet S = v' + \varphi' \bullet S - (v'' + \varphi'' \bullet S)$$

for some $v', v'' \in \mathbb{R}_+$ and $\varphi', \varphi'' \in L(S)$ such that $v' + \varphi' \bullet S \geq 0$, $v'' + \varphi'' \bullet S \geq 0$ and, in addition, φ'' is maximal. For $v \in (0, \infty)$ and $q \in \mathbb{R}$ we denote by

$$\Theta^q(v) := \{\varphi \in L(S) : \varphi \text{ is acceptable}, v + \varphi \bullet S_T - qH \geq 0\},$$

the set of acceptable strategies whose terminal value dominates qH.

Remark 2.8 Given Assumption 2.2, we have $\Theta(v) = \Theta^0(v)$ by [9, Theorem 5.7] combined with [23, Lemma 3.1 and Proposition 3.1] .

Let an initial endowment of $v \in (0, \infty)$ be given. If the investor sells q units of H for a price of $x \in \mathbb{R}$ each, her initial position consists of $v + qx$ in cash as well as $-q$ units of the contingent claim H. Hence $\Theta^q(v + qx)$ represents the natural set of admissible trading strategies for utility functions defined on $\mathbb{R}_+$. The maximal expected utility the investor can achieve by dynamic trading in the market is then given by

$$U^q(v + qx) := \sup_{\varphi \in \Theta^q(v+qx)} E(u(v + qx + \varphi \bullet S_T - qH)).$$

Definition 2.9 Fix $q \in \mathbb{R}$. A number $\pi^q \in \mathbb{R}$ is called *utility indifference price* of H if

$$U^q(v + q\pi^q) = U(v). \tag{2.3}$$

Existence of indifference prices does not hold in general for power utility. However, a unique indifference price π^q always exists if the number q of contingent claims sold is sufficiently small or, conversely, if the initial endowment v is sufficiently large.

Lemma 2.10 *Suppose Assumptions 2.2, 2.3 and 2.6 hold. Then a unique indifference price exists for sufficiently small q. More specifically, (2.3) has a unique solution π^q if $q < \frac{v}{2w}$, respectively if $q < \frac{v}{w}$ and $H \geq 0$, where w denotes the initial endowment of the superhedging strategy for H from Assumption 2.6.*

Proof First notice that $g_v^q : x \mapsto U^q(v + qx)$ is concave and strictly increasing on its effective domain. By [31, Theorem 2.1], $g_v^q(x) \leq U(v + qx + qw) < \infty$ for all $x \in \mathbb{R}$. For $H \geq 0$ and $q < \frac{v}{w}$ we have $g_v^q(x) > -\infty$ for $x > w - \frac{v}{q}$. In particular, g_v^q is continuous and strictly increasing on $(w - \frac{v}{q}, \infty)$ and in particular on $[0, w]$ by [40, Theorem 10.1]. By $H \geq 0$ we have $g_v^q(0) \leq U(v)$. Moreover, Assumption 2.6 implies $g_v^q(w) \geq U(v)$. Hence there exists a unique solution $\pi^q \in [0, w]$ to $g_v^q(x) = U(v)$. Similarly, for general H and $q < \frac{v}{2w}$, the function g_v^q is finite, continuous and strictly increasing on an open set containing $[-w, w]$. Moreover, $g_v^q(-w) \leq U(v)$ and $g_v^q(w) \geq U(v)$. Hence there exists a unique $\pi^q \in (-w, w)$ such that $g_v^q(\pi^q) = U(v)$. This proves the assertion. $\qquad\square$

We now turn to optimal trading strategies in the presence of random endowment. Their existence has been established by [8] resp. [19] in the bounded resp. general case.

Theorem 2.11 *Fix $q \in \mathbb{R}$ satisfying the conditions of Lemma 2.10 and suppose Assumptions 2.2, 2.3 and 2.6 are satisfied. Then there exists $\varphi^q \in \Theta^q(v + q\pi^q)$ such that*

$$E(u(v + q\pi^q + \varphi^q \bullet S_T - qH)) = U^q(v + q\pi^q).$$

Moreover, the corresponding optimal value process $v + q\pi^q + \varphi^q \bullet S$ is unique.

Proof This follows from [19, Theorem 2 and Corollary 1] because the proof of Lemma 2.10 shows that $(v + q\pi^q, q)$ belongs to the interior of $\{(x, r) \in \mathbb{R}^2 : \Theta^r(x) \neq \varnothing\}$. $\qquad\square$

Without contingent claims, the investor will trade according to the strategy $\widehat{\varphi}$, whereas she will invest into φ^q if she sells q units of H for π^q each. Hence, the difference between both strategies represents the action the investors needs to take in order to compensate for the risk of selling q units of H. This motivates the following notion:

Definition 2.12 The trading strategy $\varphi^q - \widehat{\varphi}$ is called *utility-based hedging strategy*.

3 The asymptotic results of Kramkov and Sîrbu

We now give a brief exposition of some of the deep results of [32,33] concerning the existence and characterization of first-order approximations of utility-based prices and hedging strategies in the following sense.

Definition 3.1 Real numbers π^0 and π' are called *marginal utility-based price* resp. *risk premium* per option sold if

$$\pi^q = \pi^0 + q\pi' + o(q)$$

for $q \to 0$, where π^q is well-defined for sufficiently small q by Lemma 2.10. A trading strategy $\varphi' \in L(S)$ is called *marginal utility-based hedging strategy* if there exists $v' \in \mathbb{R}$ such that

$$\lim_{q \to 0} \frac{(v + q\pi^q + \varphi^q \bullet S_T) - (v + \widehat{\varphi} \bullet S_T) - q(v' + \varphi' \bullet S_T)}{q} = 0 \qquad (3.1)$$

in P-probability and $(v' + \varphi' \bullet S)\widehat{Y}$ is a martingale for the dual minimizer $\widehat{Y}$ of the pure investment problem.

Remark 3.2 [32, Theorems A.1, 8, and 4] show that for power utility functions, a trading strategy φ' is a marginal utility-based hedging strategy in the sense of Definition 3.1 if and only if it is a marginal hedging strategy in the sense of [33, Definition 2].

The asymptotic results of [32,33] are derived subject to two technical assumptions.

Assumption 3.3 The following process is σ-*bounded*:

$$S^\$:= \left(\frac{1}{\mathscr{E}(-\widetilde{a} \bullet S)}, \frac{S}{\mathscr{E}(-\widetilde{a} \bullet S)} \right).$$

The reader is referred to [32] for more details on σ-bounded processes as well as for sufficient conditions that ensure the validity of this assumption. In our concrete examples, we have the following:

Lemma 3.4 *1. Let $S = S_0 \mathscr{E}(X) > 0$ for a non-monotone Lévy process X with finite $(1 - p)$-th moments. Then Assumption 3.3 holds if the optimizer $\widehat{\eta}$ from Example 2.5 lies in the* interior *of the set $\mathscr{C}^0$ of fractions of wealth in stocks leading to nonnegative wealth processes.*

2. Assumption 3.3 is automatically satisfied if the stock price S is continuous. In particular, it holds in the BNS model.

Proof First consider Assertion 1. In view of [32, Lemma 8], it suffices to show that $S^\$$ is bounded by a predictable process. If $\widehat{\eta} \geq 0$, there exists $\overline{\eta} \in \mathscr{C}^0$ with $\widehat{\eta} < \overline{\eta}$; hence $\Delta X > -1/\overline{\eta}$ by definition of $\mathscr{C}^0$. Consequently, $\widehat{\eta} \Delta X \geq -\widehat{\eta}/\overline{\eta} > -1$ and thus

$$\left| \frac{1}{\mathscr{E}(-\widetilde{a} \bullet S)} \right| = \frac{1}{|\mathscr{E}(\widehat{\eta} X)|} = \frac{1}{|1 + \widehat{\eta} \Delta X|} \frac{1}{|\mathscr{E}(\widehat{\eta} X)_-|} \leq \frac{\overline{\eta}}{(\overline{\eta} - \widehat{\eta})|\mathscr{E}(\widehat{\eta} X)_-|} \quad \text{a.s.,}$$

which shows that the first component of $S^\$$ is bounded by a predictable process and hence σ-bounded. Likewise, if $\widehat{\eta} < 0$, there exists $\underline{\eta} \in \mathscr{C}^0$ with $\underline{\eta} < \widehat{\eta}$. Then $\Delta X < -1/\underline{\eta}$ and in turn $\widehat{\eta} \Delta X > -\widehat{\eta}/\underline{\eta} > -1$. Hence it follows as above that $|1/\mathscr{E}(-\widetilde{a} \bullet S)|$ is bounded by a predictable process. The assertion for the second component of $S^\$$ follows similarly.

If the stock price process is continuous, both S and $\mathscr{E}(-\widetilde{a} \bullet S)$ are predictable. Hence Assertion 2 follows immediately from [32, Lemma 8]. $\square$

Since $\mathscr{E}(-\widetilde{a} \bullet S) \widehat{Y}$ is a martingale with terminal value $\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}$, we can define an equivalent probability measure $Q^\$ \sim P$ via

$$\frac{dQ^\$}{dP} := \frac{\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}}{C_0}, \quad C_0 := E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}).$$

Let $\mathscr{H}_0^2(Q^\$)$ be the space of square-integrable $Q^\$$-martingales starting at 0 and set

$$\mathscr{M}_\$^2 := \left\{ M \in \mathscr{H}_0^2(Q^\$) : M = \varphi \bullet S^\$ \text{ for some } \varphi \in L(S^\$) \right\}. \tag{3.2}$$

Assumption 3.5 There exists a constant $w^\$ \geq 0$ and a process $M^\$ \in \mathscr{M}_\$^2$, such that

$$|H^\$| \leq w^\$ + M_T^\$$$

for

$$H^\$:= \frac{H}{\mathscr{E}(-\widetilde{a} \bullet S)_T}.$$

Assumption 3.5 means that the claim under consideration can be superhedged with portfolios as in (3.2). Note that this is again evidently satisfied for European puts and calls.

Remark 3.6 By [32, Remark 1], Assumption 3.5 implies that Assumption 2.6 holds. In particular, it ensures that indifference prices and utility-based hedging strategies exist for sufficiently small q if the pure investment problem is well-posed, i.e., if Assumptions 2.2 and 2.3 are also satisfied.

In the proof of [32, Lemma 1] it is shown that the process

$$V_t^\$:= E_{Q^\$}\left(H^\$ | \mathscr{F}_t\right), \quad t \in [0, T]$$

is a square-integrable $Q^\$$-martingale. Hence it admits a decomposition

$$V^\$ = E_{Q^\$}\left(H^\$\right) + \xi \bullet S^\$ + N^\$ = \frac{1}{C_0} E\left(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-p} H\right) + \xi \bullet S^\$ + N^\$, \quad (3.3)$$

where $\xi \bullet S^\$ \in \mathscr{M}_\$^2$ and $N^\$$ is an element of the orthogonal complement of $\mathscr{M}_\$^2$ in $\mathscr{H}_0^2(Q^\$)$. Note that this decomposition coincides with the classical *Galtchouk–Kunita–Watanabe* decomposition if $S^\$$ itself is a square-integrable martingale. The following theorem is a reformulation of the results of [32,33] applied to power utility, and also contains a feedback representation of the utility-based hedging strategy in terms of the original numeraire.

Theorem 3.7 *Suppose Assumptions 2.2, 2.3, 3.3, and 3.5 hold. Then the marginal utility-based price π^0 and the risk premium π' exist and are given by*

$$\pi^0 = \frac{1}{C_0} E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-p} H), \quad \pi' = \frac{p}{2v} E_{Q^\$}((N_T^\$)^2).$$

A marginal-utility-based hedging strategy ϕ' is given in feedback form as the solution of the stochastic differential equation

$$\phi' = (\widetilde{a}, E_d + \widetilde{a} S_-^\top)\xi - \left(\pi^0 + \phi' \bullet S_-\right)\widetilde{a},$$

with ξ from (3.3), and where E_d denotes the identity matrix on $\mathbb{R}^d$.

Proof The first two assertions follow immediately from [32, Theorems A.1, 8, and 4] adapted to the present notation. For the third, [33, Theorem 2] and [32, Theorems A.1, 8, and 4] yield

$$\lim_{q \to 0} \frac{(v + q\pi^q + \varphi^q \bullet S_T) - (v + \widehat{\varphi} \bullet S_T) - q\mathscr{E}(-\widetilde{a} \bullet S)_T(\pi^0 + \xi \bullet S_T^\$)}{q} = 0. \quad (3.4)$$

because the process $X_T'(x)$ from [33, Equation (23)] coincides with $\mathscr{E}(-\widetilde{a} \bullet S)$ for power utility. Set

$$\xi^0 := \pi^0 + \xi \bullet S^\$ - \xi^\top S^\$ = \pi^0 + \xi \bullet S_-^\$ - \xi^\top S_-^\$.$$

Then we have $(\xi^0, \xi^2, \ldots, \xi^{d+1}) \in L((\mathscr{E}(-\widetilde{a} \bullet S), S))$ and

$$\pi^0 + (\xi^0, \xi^2, \ldots, \xi^{d+1}) \bullet (\mathscr{E}(-\widetilde{a} \bullet S), S) = \mathscr{E}(-\widetilde{a} \bullet S)(\pi^0 + \xi \bullet S^\$) \quad (3.5)$$

by [14, Proposition 2.1]. The predictable sets $D_n := \{|\widetilde{a}| \leq n, |S_-| \leq n, |(\xi^0, \xi)| \leq n\}$ increase to $\Omega \times \mathbb{R}_+$, the predictable process $(\widetilde{a}, E_d + \widetilde{a} S_-^\top)\xi 1_{D_n}$ is bounded, and we have

$$((\widetilde{a}, E_d + \widetilde{a} S_-^\top)\xi 1_{D_n}) \bullet S$$
$$= ((\mathscr{E}(-\widetilde{a} \bullet S)_- \xi^\top S_-^\$ \widetilde{a} + (\xi^2, \ldots, \xi^{d+1}))1_{D_n}) \bullet S$$
$$= ((\xi^0, \xi^2, \ldots, \xi^{d+1})1_{D_n}) \bullet (\mathscr{E}(-\widetilde{a} \bullet S), S) + (\mathscr{E}(-\widetilde{a} \bullet S)_-(\pi^0 + \xi \bullet S_-^\$)1_{D_n}) \bullet (\widetilde{a} \bullet S)$$
$$= 1_{D_n} \bullet ((\xi^0, \xi^2, \ldots, \xi^{d+1}) \bullet (\mathscr{E}(-\widetilde{a} \bullet S), S) + (\mathscr{E}(-\widetilde{a} \bullet S)_-(\pi^0 + \xi \bullet S_-^\$)) \bullet (\widetilde{a} \bullet S)).$$

By [23, Lemma 2.2] and (3.5), this implies $(\widetilde{a}, E_d + \widetilde{a} S_-^\top)\xi \in L(S)$ as well as

$$\pi^0 + ((\widetilde{a}, E_d + \widetilde{a} S_-^\top)\xi) \bullet S = \mathscr{E}(-\widetilde{a} \bullet S)(\pi^0 + \xi \bullet S^\$)$$
$$+ (\mathscr{E}(-\widetilde{a} \bullet S)_-(\pi^0 + \xi \bullet S_-^\$)) \bullet (\widetilde{a} \bullet S).$$

Hence $\mathscr{E}(-\widetilde{a} \bullet S)(\pi^0 + \xi \bullet S^\$)$ solves the stochastic differential equation

$$G = \pi^0 + ((\widetilde{a}, E_d + \widetilde{a}S_-^\top)\xi) \bullet S - G_- \bullet (\widetilde{a} \bullet S). \tag{3.6}$$

By [20, (6.8)] this solution is unique. Since we have shown $(\widetilde{a}, E_d + \widetilde{a}S_-^\top)\xi \in L(S)$ above, it follows as in the proof of [6, Lemma 4.9] that ϕ' is well-defined. $\pi^0 + \phi' \bullet S$ also solves (3.6), hence we obtain

$$\mathscr{E}(-\widetilde{a} \bullet S)(\pi^0 + \xi \bullet S^\$) = \pi^0 + \phi' \bullet S.$$

In view of (3.4), the process $\pi^0 + \phi' \bullet S$ therefore satisfies (3.1), so that ϕ' is indeed a marginal utility-based hedge in the sense of Definition 3.1.   □

Remark 3.8 If the dual minimizer $\widehat{Y}$ is a martingale and hence—up to the constant C_0—the density process of the *q-optimal martingale measure* Q_0 with respect to P, the generalized Bayes formula yields $V_t^\$ = E_{Q_0}(H|\mathscr{F}_t)/\mathscr{E}(-\widetilde{a} \bullet S)_t$. In particular, the marginal utility-based price of the claim H is given by its expectation $\pi^0 = E_{Q_0}(H)$ under Q_0 in this case.

The computation of the optimal strategy $\widehat{\varphi}$ and the corresponding dual minimizer $\widehat{Y}$ in the pure investment problem 2.1 has been studied extensively in the literature. In particular, these objects have been determined explicitly in a variety of Markovian models using stochastic control theory resp. martingale methods. Given $\mathscr{E}(-\widetilde{a} \bullet S)$, the computation of π^0 can then be dealt with using integral transform methods or variants of the Feynman-Kac formula. Consequently, we suppose from now on that $\widehat{\varphi}$ and π^0 are known and focus on how to obtain π' and φ'.

As reviewed above, [32,33] show that φ' and π' can be obtained by calculating the generalized Galtchouk–Kunita–Watanabe decomposition (3.3). Since $S^\$$ is generally only a $Q^\$$-supermartingale, this is typically very difficult. If however, $S^\$$ happens to be a square-integrable $Q^\$$-martingale, (3.3) coincides with the classical Galtchouk-Kunita–Watanabe decomposition. By [11], this shows that ξ represents the mean-variance optimal hedging strategy for the claim H hedged with $S^\$$ under the measure $Q^\$$ and $E_{Q^\$}((N_T^\$)^2)$ is given by the corresponding minimal expected squared hedging error in this case. Moreover, ξ and $E_{Q^\$}((N_T^\$)^2)$ can then be characterized in terms of semimartingale characteristics.

Assumption 3.9 $S^\$$ is a square-integrable $Q^\$$-martingale.

For exponential Lévy models, this assumption satisfied if the budget constraint $\mathscr{C}^0$ is "not binding" for the optimal fraction $\widehat{\eta}$ of stocks and if, in addition, the driving Lévy process is square-integrable. For the BNS model it is only a matter of integrability.

Lemma 3.10 *1. Let $S = S_0\mathscr{E}(X) > 0$ for a non-monotone Lévy process X with finite second moments. Then Assumption 3.9 is satisfied if the optimizer $\widehat{\eta}$ of the pure investment problem lies in the interior of $\mathscr{C}^0$.*

2. Let $S = S_0\mathscr{E}(X)$, where (y, X) is a BNS model. If $p > 1$ or (2.2) holds, then $S^\$$ is a $Q^\$$-martingale.

Proof If $\widehat{\eta}$ lies in the interior of $\mathscr{C}^0$, it follows from [36, Proposition 5.12] that the dual optimizer $Y = L\mathscr{E}(-\widetilde{a} \bullet S)^{-p}$ is a local martingale. Since it is also the exponential of a Lévy process (cf. [37, Sect. 6]), it is in a fact a true martingale. Thus it is—up to normalization with $1/L_0$—the density process of the q-optimal martingale measure by [36, Remark 5.18]. Combined with [21, Proposition III.3.8], this yields that $S^\$$ is a $Q^\$$-martingale and it remains to show that $S^\$$ is square-integrable. By Propositions 6.2, 6.3, and 6.4, the process

$S^\$ = (1/\mathscr{E}(\widehat{\pi} X), S_0 \mathscr{E}(X)/\mathscr{E}(\widehat{\pi} X))$ is the stochastic exponential $(\mathscr{E}(R^1), \mathscr{E}(R^2))$ of a semi-martingale R with local $Q^\$$-characteristics

$$\left(0, \begin{pmatrix} \widehat{\eta}^2 & -\widehat{\eta}(1-\widehat{\eta}) \\ -\widehat{\eta}(1-\widehat{\eta}) & (1-\widehat{\eta})^2 \end{pmatrix}\right), G \mapsto \int 1_G \left(\frac{-\widehat{\eta}x}{1+\widehat{\eta}x}, \frac{(1-\widehat{\eta})x}{1+\widehat{\eta}x}\right) (1+\widehat{\eta}x)^{1-p} F^X(dx)\right),$$

relative to the truncation function $h(x) = x$ on $\mathbb{R}^2$. This truncation function can be used because R is $Q^\$$-locally a square-integrable martingale. By [21, Propositions II.2.29 and III.6.35], this holds because X is square-integrable, $|1/(1+\widehat{\eta}\Delta X)|$ is bounded (cf. the proof of Lemma 3.4) and hence $\int x_1^2 F^{R,\$}(dx) < C \int x^2 F^X(dx) < \infty$ and $\int x_2^2 F^{R,\$}(dx) < C \int x^2 F^X(dx) < \infty$ for some constant $C \in \mathbb{R}_+$ (cf. [21, Theorem II.1.8]). As the $Q^\$$-characteristics of R are deterministic, R is a $Q^\$$-Lévy process by [21, Corollary II.4.19] and a square-integrable martingale by [21, Proposition I.4.50]. Therefore $S^\$ = (\mathscr{E}(R^1), \mathscr{E}(R^2))$ is a square-integrable martingale as well by [35, Lemma A.1.(x)]. This proves Assertion 1.

Assertion 2 is shown in the proof of [26, Theorem 3.3]. □

The square-integrability of $S^\$$ in the BNS model is discussed in Remarks 4.3 and 5.7 below. Given Assumption 3.9, we have the following representation.

Lemma 3.11 *Suppose Assumptions 2.2, 2.3, 3.3, 3.5 and 3.9 hold. Denote by $\widetilde{c}^{(S^\$, V^\$)\$}$ the modified second $Q^\$$-characteristic of $(S^\$, V^\$)$ with respect to some $A \in \mathscr{A}_{\mathrm{loc}}^+$ (cf. Appendix 6). Then*

$$\xi = (\widetilde{c}^{S^\$\$})^{-1} \widetilde{c}^{S^\$, V^\$\$}, \tag{3.7}$$

$$E_{Q^\$}((N_T^\$)^2) = E_{Q^\$}\left((\widetilde{c}^{V^\$\$} - (\widetilde{c}^{S^\$, V^\$\$})^\top (\widetilde{c}^{S^\$\$})^{-1} \widetilde{c}^{S^\$, V^\$\$}) \bullet A_T\right).$$

Proof Since $S^\$$ is a square integrable $Q^\$$-martingale by Assumption 3.9, the claim follows from [6, Theorems 4.10 and 4.12] applied to the martingale case. □

4 An alternative representation

We now develop our alternative representation of power utility-based prices and hedging strategies. As explained in the introduction, they can—morally speaking—be represented as the solution to a mean-variance hedging problem relative to the original numeraire, but subject to yet another probability measure $P^{\EUR} \sim P$. Given Assumption 3.9, the latter can be defined as follows:

$$\frac{dP^{\EUR}}{dP} := \frac{\mathscr{E}(-\widetilde{a} \bullet S)_T^{-1-p}}{C_1}, \quad C_1 := E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-1-p}).$$

Remark 4.1 If we write the density process of $P^{\EUR}$ with respect to P as $L^{\EUR} \mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}/C_1$ for a semimartingale $L^{\EUR} > 0$ with $L_T^{\EUR} = 1$, the local joint P-characteristics of S and $K^{\EUR} := \mathscr{L}(L^{\EUR})$ relative to some truncation function (h_1, h_2) on $\mathbb{R}^d \times \mathbb{R}$ satisfy

$$\int\limits_{\{|x|>1\}} (1+x_2)(1-\widetilde{a}^\top x_1)^{-1-p} F^{(S,K^{\EUR})}(dx) < \infty, \tag{4.1}$$

and solve

$$0 = b^{K^{\EUR}} + (1+p)\widetilde{a}^\top b^S + (1+p)\widetilde{a}^\top c^{S,K^{\EUR}} + \frac{(p+1)(p+2)}{2}\widetilde{a}^\top c^S \widetilde{a}$$

$$+ \int \left((1+x_2)(1-\widetilde{a}^\top x_1)^{-1-p} - 1 - h_2(x_2) - (1+p)\widetilde{a}^\top h_1(x_1)\right) F^{(S,K^{\EUR})}(dx), \tag{4.2}$$

by [23, Lemma 3.1] and Propositions 6.3, 6.2. Conversely, if a strictly positive semimartingale $L^{\euro} = L_0^{\euro}\mathscr{E}(K^{\euro})$ satisfies $L_T^{\euro} = 1$ and (4.1), (4.2), then $L^{\euro}\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p}/C_1$ is a σ-martingale and the density process of $P^{\euro}$ if it is a true martingale.

In concrete models, the drift condition (4.2) often allows to determine $L^{\euro}$ by making an appropriate parametric ansatz. For exponential Lévy models and the BNS model, this leads to the following results.

Example 4.2   1. For exponential Lévy models as in Example 4.2, plugging the ansatz $a^{\euro}t$ with $a^{\euro} \in \mathbb{R}$ for $K^{\euro}$ into (4.2) yields

$$a^{\euro} = (1+p)\widehat{\eta}b^X - \frac{(p+1)(p+2)}{2}\widehat{\eta}^2 c^X$$
$$- \int \left((1+\widehat{\eta}x)^{-1-p} - 1 + (1+p)\widehat{\eta}h(x)\right) F^X(dx).$$

This expression is well-defined because the integrand is of order $O(x^2)$ for small x and bounded on the support of F^X by the proof of Lemma 3.4 and [21, Theorem II.1.8]. One then easily verifies that $L_t^{\euro} = \exp(a^{\euro}(T-t))$. Indeed, the strictly positive σ-martingale $L^{\euro}\mathscr{E}(\widehat{\eta}X)$ is a true martingale because it is also the stochastic exponential of a Lévy process.

2. For the BNS model, one has to make a more general ansatz for $L_t^{\euro}$. Choosing $\exp(\alpha_0^{\euro}(t) + \alpha_1^{\euro}(t)y_t)$ with smooth functions $\alpha_0^{\euro}, \alpha_1^{\euro}$ satisfying $\alpha_0^{\euro}(T) = \alpha_1^{\euro}(T) = 0$ as in [26], insertion into 4.2 leads to

$$\alpha_1^{\euro}(t) = \frac{(1+p)(2-p)}{2p^2}\mu^2\frac{1-e^{-\lambda(T-t)}}{\lambda}, \quad \alpha_0^{\euro}(t) = \int\limits_t^T \psi^Z(\alpha_1^{\euro}(s))ds.$$

Then, (4.1) is satisfied and $\exp(\alpha_0^{\euro}(t) + \alpha_1^{\euro}(t)y)\mathscr{E}(\widehat{\eta}X)^{-1-p}$ is a σ-martingale if

$$\int\limits_{\{|x|>1\}} e^{\alpha_1^{\euro}(t)z}F^Z(dz) < \infty \quad \text{for all } t \in [0,T]. \tag{4.3}$$

(4.3) automatically holds for $p \geq 2$, because $\alpha_1^{\euro} \leq 0$ in this case. For $p \in (0,2)$, (4.3) is satisfied if

$$\int\limits_1^{\infty} \exp\left(\frac{(1+p)(2-p)}{2p^2}\mu^2\frac{1-e^{-\lambda T}}{\lambda}z\right) F^Z(dz) < \infty. \tag{4.4}$$

In either case, the true martingale property of the exponentially affine σ-martingale $\exp(\alpha_0^{\euro}(t) + \alpha_1^{\euro}(t)y)\mathscr{E}(\widehat{\eta}X)^{-1-p}$ follows from [24, Corollary 3.9]. This shows that $L^{\euro}$ is indeed given by $\exp(\alpha_0^{\euro}(t) + \alpha_1^{\euro}(t)y)$.

Remark 4.3 Part 2 of Example 4.2 shows that in the BNS model the first component of $S^{\euro}$ is square-integrable if $p \geq 2$ or (4.4) holds. Hence the measure $P^{\euro}$ is well defined with density process $L^{\euro}\mathscr{E}(\widehat{\eta}X)^{-1-p}$ in either case.

As motivated in the introduction, the measures $P^{\euro}$ and $Q^{\$}$ are linked as follows.

Lemma 4.4 *Suppose Assumptions 2.2, 2.3 and 3.9 hold. Then the process*

$$L_t^{\$} := E_{P^{\euro}}\left(\frac{\mathscr{E}(-\widetilde{a}\bullet S)_T^2}{\mathscr{E}(-\widetilde{a}\bullet S)_t^2}\,\middle|\,\mathscr{F}_t\right), \quad 0 \leq t \leq T,$$

satisfies $L_T^\$ = 1$ *and the density process of* $Q^\$$ *with respect to* P^ϵ *is given by*

$$E_{P^\epsilon}\left(\frac{dQ^\$}{dP^\epsilon}\bigg|\mathscr{F}_t\right) = \frac{C_1}{C_0}L_t^\$ \mathscr{E}(-\tilde{a}\bullet S)_t^2 = \frac{L_t^\$ \mathscr{E}(-\tilde{a}\bullet S)_t^2}{L_0^\$}.$$

In particular, $L^\$, L_-^\$ > 0$ *and the stochastic logarithm* $K^\$:= \mathscr{L}(L^\$)$ *is well-defined.*

Proof The first part of the assertion is trivial, whereas the second follows from $dQ^\$/dP^\epsilon = \frac{C_1}{C_0}\mathscr{E}(-\tilde{a}\bullet S)_T^2$. Since $\mathscr{E}(-\tilde{a}\bullet S), \mathscr{E}(-\tilde{a}\bullet S)_- > 0$, [21, I.2.27] yields $L^\$, L_-^\$ > 0$ and hence the third part of the assertion by [21, II.8.3]. □

Remark 4.5 $L^\$$ is linked to the opportunity process L of the pure investment problem and the process L^ϵ from Remark 4.1 via

$$L_0^\$ \mathscr{E}(K^\$) = L^\$ = \frac{L}{L^\epsilon} = \frac{L_0 \mathscr{E}(K)}{L_0^\epsilon \mathscr{E}(K^\epsilon)},$$

by the generalized Bayes' formula, $L_T = L_T^\epsilon = 1$, and because $L\mathscr{E}(-\tilde{a}\bullet S)^{1-p}$ as well as $L^\epsilon \mathscr{E}(-\tilde{a}\bullet S)^{-1-p}$ are martingales.

In our examples, this leads to the following.

Example 4.6 1. Suppose $S = S_0\mathscr{E}(X) > 0$ for a non-monotone Lévy process with finite second moments. Then $L^\$ = \exp((a - a^\epsilon)(T - t))$ and $K_t^\$ = (a^\epsilon - a)t$ for a and a^ϵ as in Examples 2.5 resp. 4.2.

2. Let $S = S_0\mathscr{E}(X) > 0$ for a BNS model satisfying (4.4) if $p < 2$ and, additionally, (2.2) if $p < 1$. Then $L_t^\$ = \exp(\alpha_0^\$(t)) + \alpha_1^\$(t)y_t)$ and, by Itô's formula,

$$K_t^\$ = \alpha_0^\$(t) - \alpha_0^\$(0) + \alpha_1^\$(t)y_t - \alpha_1^\$(0)y_0 + \sum_{s\leq t}\left(e^{\alpha_1^\$(s)\Delta Z_s} - 1 - \alpha_1^\$(s)\Delta Z_s\right),$$

for $\alpha_i^\$ = \alpha_i - \alpha_i^\epsilon, i = 0, 1$, with α_i and α_i^ϵ as in Examples 2.5 resp. 4.2.

Now define

$$V_t := \mathscr{E}(-\tilde{a}\bullet S)_t V_t^\$ = \frac{E(\mathscr{E}(-\tilde{a}\bullet S)_T^{-p}H|\mathscr{F}_t)}{L_t\mathscr{E}(-\tilde{a}\bullet S)_t^{-p}}, \quad 0 \leq t \leq T,$$

which coincides with the conditional expectation under the q-optimal martingale measure Q_0, if the latter exists. Denote by

$$\left(\begin{pmatrix} b^{S\epsilon} \\ b^{V\epsilon} \\ b^{K^\$\epsilon} \end{pmatrix}, \begin{pmatrix} c^{S\epsilon} & c^{S,V\epsilon} & c^{S,K^\$\epsilon} \\ c^{V,S\epsilon} & c^{V\epsilon} & c^{V,K^\$\epsilon} \\ c^{K^\$,S\epsilon} & c^{K^\$,V\epsilon} & c^{K^\$\epsilon} \end{pmatrix}, F^{(S,V,K^\$)\epsilon}, A\right)$$

P^ϵ-differential characteristics of the semimartingale $(S, V, K^\$)$ and define

$$\tilde{c}^{S\star} := \frac{1}{1 + \Delta A^{K^\$}}\left(c^{S\epsilon} + \int(1 + x_3)x_1 x_1^\top F^{(S,V,K^\$)\epsilon}(dx)\right),$$

$$\tilde{c}^{S,V\star} := \frac{1}{1 + \Delta A^{K^\$}}\left(c^{S,V\epsilon} + \int(1 + x_3)x_1 x_2 F^{(S,V,K^\$)\epsilon}(dx)\right),$$

$$\tilde{c}^{V\star} := \frac{1}{1 + \Delta A^{K^\$}}\left(c^{V\epsilon} + \int(1 + x_3)x_2^2 F^{(S,V,K^\$)\epsilon}(dx)\right),$$

where $K^\$ = K_0^\$ + A^{K^\$} + M^{K^\$}$ denotes an arbitrary $P^{\in}$-semimartingale decomposition of $K^\$$. We then have the following representation of the marginal utility-based hedging strategy φ' and the risk premium π' in terms of semimartingale characteristics, which is the main result of this paper.

Theorem 4.7 *Suppose Assumptions 2.2, 2.3, 3.3, 3.5 and 3.9 hold. Then $\tilde{c}^{S\star}, \tilde{c}^{S,V\star}, \tilde{c}^{V\star}$ are well-defined, the strategy φ' given in feedback form as the solution of the stochastic differential equation*

$$\varphi' = (\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} - \left(\pi^0 + \varphi' \bullet S_- - V_-\right)\tilde{a}$$

is a marginal utility-based hedge, and the corresponding risk premium is

$$\pi' = \frac{pC_1}{2vC_0} E_{P^{\in}}\left(\left(\left(\tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^{\top}(\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star}\right) L^\$\right) \bullet A_T\right).$$

Remark 4.8 As is customary for mean-variance optimal hedges [6,42], the strategy φ' is described in "feedback form", i.e., it is computed as the solution of a stochastic differential equation involving its past trading gains $\varphi' \bullet S_-$, which reduces to a simple recursive formula in discrete time (cf., e.g., [43, Theorem 2.4]). Alternatively, the corresponding linear stochastic differential equation for $\varphi' \bullet S$ can be solved [6, Corollary 4.11],

$$\varphi' \bullet S = \mathscr{E}(\tilde{a} \bullet S)\left(\frac{(\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} + (V_- - \pi^0)\tilde{a}}{\mathscr{E}(-\tilde{a} \bullet S)_-} \bullet \left(S + \frac{\tilde{a}}{1 - \tilde{a}^{\top}\Delta S} \bullet [S, S]\right)\right),$$

leading to a cumbersome but explicit expression for the hedge φ'.

In view of [6, Theorems 4.10 and 4.12], Theorem 4.7 states that the first-order approximations for φ^q and π^q can *essentially* be computed by solving the mean-variance hedging problem for the claim H under the (non-martingale) measure $P^{\in}$ relative to the original numeraire. However, this assertion only holds true *literally* if the dual minimizer $\widehat{Y}$ is a martingale and if the optimal strategy $\widehat{\varphi}$ in the pure investment problem is *admissible* in the sense of [6, Corollary 2.5], i.e., if $\widehat{\varphi} \bullet S_T \in L^2(P^{\in})$ and $(\widehat{\varphi} \bullet S)Z^Q$ is a $P^{\in}$-martingale for any absolutely continuous signed σ-martingale measure Q with density process Z^Q and $\frac{dQ}{dP^{\in}} \in L^2(P^{\in})$. More precisely, in this case the strategy $-\tilde{a}1_{]\!]\tau, T]\!]}\mathscr{E}(-\tilde{a}1_{]\!]\tau, T]\!]} \bullet S)_-$ is *efficient* on the stochastic interval $]\!]\tau, T]\!]$ in the sense of [6, Sect. 3.1] and $\tilde{a}$ is the corresponding *adjustment process* in the sense of [6, Definition 3.8]. By [6, Corollary 3.4] this in turn implies that $L^\$$ is the *opportunity process* in the sense of [6, Definition 3.3]. Hence it follows along the lines of [6, Lemma 3.15] that the *opportunity neutral measure* $P^\star$ with density process

$$Z^{P^\star} := \frac{L^\$}{L_0^\$\mathscr{E}(A^{K^\$})}$$

exists. By [6, Lemma 3.17 and Theorem 4.10], $\tilde{c}^{S\star}, \tilde{c}^{V\star}, \tilde{c}^{S,V\star}$ indeed coincide with the corresponding modified second characteristics of (S, V, K) under $P^\star$. Hence [6, Theorems 4.10 and 4.12] yield that relative to the probability measure $P^{\in}$, the process φ' represents a variance-optimal hedging strategy for H while the minimal expected squared hedging error of H is given by the $2C_0v/(pC_1)$-fold of π'. Moreover, $V_t = E_{Q_0}(H|\mathscr{F}_t)$ and in particular the marginal utility-based price $\pi^0 = E_{Q_0}(H)$ are given as conditional expectations under the *variance-optimal martingale measure* Q_0 with respect to $P^{\in}$, which coincides with the q-optimal martingale measure with respect to P.

Proof of Theorem 4.7 An application of Propositions 6.3 and 6.2 yields the P^ϵ-differential characteristics of the process $(S, V, \mathcal{E}(-\tilde{a} \bullet S), \mathcal{L}(\frac{C_1}{C_0}L^\$\mathcal{E}(-\tilde{a} \bullet S)^2))$. Now, since $\frac{C_1}{C_0}L^\$\mathcal{E}(-\tilde{a} \bullet S)^2$ is the density process of $Q^\$$ with respect to P^ϵ, the $Q^\$$-characteristics of $(S, V, \mathcal{E}(-\tilde{a} \bullet S))$ can be obtained with Proposition 6.4. Another application of Proposition 6.2 then allows to compute the $Q^\$$-characteristics of $(S^\$, V^\$)$.

Since $S^\$ \in \mathcal{H}^2(Q^\$)$ by Assumption 3.9 and $V^\$ \in \mathcal{H}^2(Q^\$)$ by the proof of [33, Lemma 1], the modified second characteristics $\tilde{c}^{V^\$\$}$, $\tilde{c}^{S^\$, V^\$\$}$ and $\tilde{c}^{S^\$\$}$ exist and are given by

$$\tilde{c}^{S^\$\$} = \frac{1 + \Delta A^{K^\$}}{\mathcal{E}(-\tilde{a} \bullet S)^2_-} \begin{pmatrix} \tilde{a}^\top \tilde{c}^{S\star} \tilde{a} & \tilde{a}^\top \tilde{c}^{S\star} R^\top \\ R \tilde{c}^{S\star} \tilde{a} & R \tilde{c}^{S\star} R^\top \end{pmatrix}, \tag{4.5}$$

$$\tilde{c}^{S^\$, V^\$\$} = \frac{1 + \Delta A^{K^\$}}{\mathcal{E}(-\tilde{a} \bullet S)^2_-} \begin{pmatrix} \tilde{a}^\top \\ R \end{pmatrix} \left(\tilde{c}^{S,V\star} + \tilde{c}^{S\star} \tilde{a} V_- \right), \tag{4.6}$$

$$\tilde{c}^{V^\$\$} = \frac{1 + \Delta A^{K^\$}}{\mathcal{E}(-\tilde{a} \bullet S)^2_-} \left(\tilde{c}^{V\star} + 2V_- \tilde{a}^\top \tilde{c}^{S,V\star} + V_-^2 \tilde{a}^\top \tilde{c}^{S\star} \tilde{a} \right) \tag{4.7}$$

for $R := E_d + S_- \tilde{a}^\top$. In particular it follows that $\tilde{c}^{V\star}$, $\tilde{c}^{S,V\star}$ and $\tilde{c}^{S\star}$ are well defined. By the definition of ξ in Eq. (3.7) and [1, Theorem 9.1.6] we have

$$\tilde{c}^{S^\$\$} \xi = \tilde{c}^{S^\$, V^\$\$}.$$

In view of Eqs. (4.6) and (4.5), this yields

$$\begin{pmatrix} \tilde{a}^\top \tilde{c}^{S\star} \tilde{a} & \tilde{a}^\top \tilde{c}^{S\star} R^\top \\ R \tilde{c}^{S\star} \tilde{a} & R \tilde{c}^{S\star} R^\top \end{pmatrix} \xi = \begin{pmatrix} \tilde{a}^\top \\ R \end{pmatrix} \left(\tilde{c}^{S,V\star} + \tilde{c}^{S\star} \tilde{a} V_- \right),$$

or equivalently, decomposed into the first and last d components,

$$\tilde{a}^\top \tilde{c}^{S\star} (\tilde{a}, R^\top) \xi = \tilde{a}^\top (\tilde{c}^{S,V\star} + \tilde{c}^{S\star} \tilde{a} V_-) \tag{4.8}$$

and

$$R \tilde{c}^{S\star} (\tilde{a}, R^\top) \xi = R(\tilde{c}^{S,V\star} + \tilde{c}^{S\star} \tilde{a} V_-). \tag{4.9}$$

By multiplying both sides of (4.8) with S_- from the left and subtracting the result from (4.9), this leads to

$$\tilde{c}^{S\star} (\tilde{a}, R^\top) \xi = \tilde{c}^{S,V\star} + \tilde{c}^{S\star} \tilde{a} V_-, \tag{4.10}$$

since $R - S_- \tilde{a}^\top = E_d$. By Theorem 3.7,

$$\phi' = (\tilde{a}, R^\top) \xi - (\pi^0 + \phi' \bullet S_-) \tilde{a}$$

defines a marginal utility-based hedging strategy. Let

$$\psi' := \phi' - ((\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} - (\pi^0 + \phi' \bullet S_- - V_-) \tilde{a}) = (\tilde{a}, R^\top) \xi - (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} - V_- \tilde{a}.$$

Then it follows from the definition of ψ' and (4.10) that

$$\tilde{c}^{S\star} \psi' = \tilde{c}^{S,V\star} + \tilde{c}^{S\star} V_- \tilde{a} - \tilde{c}^{S,V\star} - \tilde{c}^{S\star} V_- \tilde{a} = 0,$$

because $\tilde{c}^{S\star} (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} = \tilde{c}^{S,V\star}$ by [1, Theorem 9.1.6]. In particular, $(\psi')^\top \tilde{c}^{S\star} \psi' = 0$. Since $L^\$/L_0^\$ = \mathcal{E}(K^\$) > 0$ and hence $\Delta K^\$ > -1$ by [21, I.4.61], this implies

$$(\psi')^\top \tilde{c}^S \psi' = 0. \tag{4.11}$$

For $n \in \mathbb{N}$ define the predictable sets $D_n := \{|\psi'| \leq n\}$. By Proposition 6.3 and (4.11), we have $\tilde{c}^{\psi' 1_{D_n} \bullet S} = 0$ and hence $c^{\psi' 1_{D_n} \bullet S} = 0$ and $F^{\psi' 1_{D_n} \bullet S} = 0$. Together with Proposition 6.4, this implies that the local characteristics of $\psi' 1_{D_n} \bullet S$ under the equivalent local martingale measure Q from Assumption 2.2 vanish by [23, Lemma 3.1]. Hence $\psi' 1_{D_n} \bullet S = 0$ and it follows from [23, Lemma 2.2] that $\psi' \in L(S)$ with $\psi' \bullet S = 0$. Taking into account the definition of ψ', this shows

$$\phi' \bullet S = ((\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} - (\pi^0 - V_-)\tilde{a}) \bullet S - (\phi' \bullet S_-) \bullet (\tilde{a} \bullet S),$$

i.e., $\phi' \bullet S$ solves the feedback equation

$$G = ((\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} - (\pi^0 - V_-)\tilde{a}) \bullet S - G_- \bullet (\tilde{a} \bullet S). \tag{4.12}$$

Since $\psi' \in L(S)$ and $L(S)$ is a vector space, it follows that $(\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} \in L(S)$, too. As in the proof of [6, Lemma 4.9], this in turn yields that φ' is well-defined and in $L(S)$. Evidently, $\varphi' \bullet S$ also solves (4.12) and, since the solution is unique by [20, (6.8)], we obtain $\varphi' \bullet S = \phi' \bullet S$. Therefore φ' is a marginal utility-based hedging strategy.

We now turn to the risk premium π'. First notice that by [1, Theorem 9.1.6],

$$C^\$:= \tilde{c}^{V\$\$} - (\tilde{c}^{S\$,V\$\$})^\top \xi = \tilde{c}^{V\$\$} - (\tilde{c}^{S\$,V\$\$})^\top (\tilde{c}^{S\$\$})^{-1} \tilde{c}^{S\$,V\$\$} \geq 0,$$
$$C^\epsilon := \tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^\top (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} \geq 0.$$

Hence $C^\$ \bullet A$ is an increasing predictable process and, by Lemmas 3.11 and 6.5,

$$\begin{aligned}
E_{Q^\$}((N_T^\$)^2) &= E_{Q^\$}(C^\$ \bullet A_T) \\
&= \frac{C_1}{C_0} E_{P^\epsilon}\left(L_-^\$ \mathscr{E}(-\tilde{a} \bullet S)_-^2 C^\$ \bullet A_T\right) \\
&= \frac{C_1}{C_0} E_{P^\epsilon}\left(L_-^\$ \mathscr{E}(-\tilde{a} \bullet S)_-^2 \bullet (\langle V^\$, V^\$\rangle_T^{Q^\$} - \langle V^\$, \xi \bullet S^\$\rangle_T^{Q^\$})\right).
\end{aligned}$$

Since we have shown $\phi' \bullet S = \varphi' \bullet S$ above, [15, Proposition 2.1] and the proof of Theorem 3.7 yield $\xi \bullet S^\$ = (\varphi'^0, \varphi') \bullet S^\$$ for $\varphi'^0 := \pi^0 + \varphi' \bullet S - \varphi' S$. Hence

$$\begin{aligned}
E_{Q^\$}((N_T^\$)^2) &= \frac{C_1}{C_0} E_{P^\epsilon}\left(L_-^\$ \mathscr{E}(-\tilde{a} \bullet S)_-^2 \bullet \left(\langle V^\$, V^\$\rangle_T^{Q^\$} - \langle V^\$, (\varphi'^0, \varphi') \bullet S^\$\rangle_T^{Q^\$}\right)\right) \\
&= \frac{C_1}{C_0} E_{P^\epsilon}\left(L_-^\$ \mathscr{E}(-\tilde{a} \bullet S)_-^2 \left(\tilde{c}^{V\$\$} - (\tilde{c}^{S\$,V\$\$})^\top (\varphi'^0, \varphi')\right) \bullet A_T\right).
\end{aligned}$$

After inserting $\tilde{c}^{V\$\$}$, $\tilde{c}^{S\$,V\$\$}$ from (4.7) resp. (4.6) and the definition of (φ'^0, φ'), this leads to

$$E_{Q^\$}((N_T^\$)^2) = \frac{C_1}{C_0} E_{P^\epsilon}\left(\left(1 + \Delta A^{K\$}\right) L_-^\$ C^\epsilon \bullet A_T\right). \tag{4.13}$$

Now notice that the definition of the stochastic exponential and [21, I.4.36] imply

$$L^\$ = \left(1 + \Delta A^{K\$} + \Delta M^{K\$}\right) L_-^\$.$$

By [21, I.4.49] the process $\Delta M^{K\$} \bullet (L_-^\$ C^\epsilon \bullet A)$ is a local martingale. If $(T_n)_{n\in\mathbb{N}}$ denotes a localizing sequence, this yields

$$\begin{aligned}
E_{P^\epsilon}(L^\$ C^\epsilon \bullet A_{T\wedge T_n}) &= E_{P^\epsilon}\left(\left(1 + \Delta A^{K\$} + \Delta M^{K\$}\right) L_-^\$ C^\epsilon \bullet A_{T\wedge T_n}\right) \\
&= E_{P^\epsilon}\left(\left(1 + \Delta A^{K\$}\right) L_-^\$ C^\epsilon \bullet A_{T\wedge T_n}\right),
\end{aligned}$$

and hence

$$E_{P^\epsilon}(L^\$ C^\epsilon \bullet A_T) = E_{P^\epsilon}\left(\left(1 + \Delta A^{K^\$}\right) L_-^\$ C^\epsilon \bullet A_T\right)$$

by monotone convergence. Combining this with (4.13), we obtain

$$E_{Q^\$}((N_T^\$)^2) = \frac{C_1}{C_0} E_{P^\epsilon}\left(\left(\tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^\top (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star}\right) L^\$ \bullet A_T\right).$$

In view of Theorem 3.7, this completes the proof. □

Remarks 1. The arguments used to show $\varphi' \bullet S = \phi' \bullet S$ in the proof of Theorem 4.7 also yield that one obtains a marginal utility-based hedging strategy if the *pure hedge coefficient* $(\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star}$ is replaced by any other solution ζ of $\tilde{c}^{S\star}\zeta = \tilde{c}^{S,V\star}$.

2. An inspection of the proof of Theorem 4.7 shows that the formulas for φ' and π' are independent of the specific semimartingale decomposition of $K^\$$ that is used. In particular, the not necessarily predictable term $1 + \Delta A^{K^\$ \epsilon}$ disappears in the formula for φ' by [1, Theorem 3.9]. If the semimartingale $K^\$$ is P^ϵ-special, one can choose the *canonical* decomposition [21, II.2.38]. By [21, II.2.29], this yields

$$\Delta A^{K^\$} = \Delta A \int x F^{K^\$ \epsilon}(dx).$$

If additionally $K^\$$ has no fixed times of discontinuity, [21, II.2.9] shows that A can be chosen to be continuous, which implies $\Delta A^{K^\$} = 0$.

3. For *continuous S*, our feedback representation of φ' coincides with [33, Theorem 3] because the modified second characteristic is invariant with respect to equivalent changes of measure for continuous processes.

5 Semi-explicit formulas in concrete models

We now discuss how Theorem 4.7 can be applied in our concrete examples to yield numerically tractable representations of power utility-based prices and hedging strategies.

5.1 Exponential Lévy models

For exponential Lévy models, Theorem 4.7 indeed leads to a mean-variance hedging problem. Consequently, semi-explicit formulas for the objects of interest are provided by the results of Hubalek et al. [18] for mean-variance hedging in exponential Lévy models.

To this end, we fix a univariate exponential Lévy model $S = S_0 \mathscr{E}(X) > 0$, with some non-monotone square-integrable Lévy process X. Its Lévy-Khintchine triplet relative to the truncation function $h(x) = x$ is denoted by (b^X, c^X, F^X). Finally, we suppose throughout that the optimal fraction $\widehat{\eta}$ for the pure investment problem lies in the interior of the admissible fractions of wealth in stock $\mathscr{C}^0$, implying that all assumptions of Sects. 3 and 4 are satisfied.

Remark 5.1 By [15, Lemma A.8], the stock price can also be written as the ordinary exponential $S = S_0 \exp(\tilde{X})$ of the Lévy process $\tilde{X}$ with Lévy-Khintchine triplet

$$b^{\tilde{X}} = b^X - \frac{1}{2}c^X + \int (\log(1+x) - x)F^X(dx), \quad c^{\tilde{X}} = c^X,$$

$$F^{\tilde{X}}(G) = \int 1_G(\log(1+x))F^X(dx) \quad \forall G \in \mathscr{B},$$

relative to $h(x) = x$.

Since the density process $(L^{\epsilon}/L_0^{\epsilon})\mathscr{E}(\widehat{\eta}X)^{-1-p}$ of P^{ϵ} with respect to P is an exponential Lévy process, Proposition 6.4 shows that X is also a Lévy process under P^{ϵ} with Lévy-Khintchine triplet $(b^{X,\epsilon}, c^{X,\epsilon}, F^{X,\epsilon})$ given by

$$\left(b^X - (1+p)\widehat{\eta}c^X - \int x(1 - (1+\widehat{\eta}x)^{-1-p})F^X(dx), c^X, (1+\widehat{\eta}x)^{-1-p}F^X(dx) \right),$$

relative to $h(x) = x$. This truncation function can be used because X is square-integrable under P^{ϵ} as well by [41, Corollary 25.8] and the proof of Lemma 3.4. Moreover, since the budget constraint $\mathscr{C}^0$ is "not binding," the first-order condition [37, Equation (6.3)] implies that the drift rate $b^{X,\epsilon}$ can also be written as

$$b^{X,\epsilon} = -\widehat{\eta}\left(c^X + \int \frac{x^2}{(1+\widehat{\eta}x)^{1+p}} F^X(dx) \right) = -\widehat{\eta}\widetilde{c}^{X,\epsilon}. \tag{5.1}$$

Lemma 5.2 *The optimal trading strategy $\widehat{\varphi} = -v\widetilde{a}\mathscr{E}(-\widetilde{a} \bullet S)$ in the pure investment problem is admissible in the sense of [6, Corollary 2.5].*

Proof In view of [42, Proposition 13], the mean-variance optimal hedge for the constant claim $H = v$ is $-v\widetilde{\lambda}\mathscr{E}(-\widetilde{\lambda} \bullet S)$ for $\widetilde{\lambda} = b^{X,\epsilon}/(S_-\widetilde{c}^{X,\epsilon})$. Hence $\widetilde{a} = \widetilde{\lambda}$ by (5.1); in particular, $\widehat{\varphi}$ is admissible in the sense of Schweizer [42] and therefore in the sense of Černý and Kallsen [6] as well by [6, Corollary 2.9]. $\qquad\square$

Together with the discussion at the end of Sect. 4, Lemma 5.2 immediately yields

Corollary 5.3 *Let H be a contingent claim satisfying Assumption 3.5. Then the marginal utility-based price π^0, the marginal utility-based hedging strategy φ', and the risk premium π' from Theorem 4.7 coincide with the mean-variance optimal initial capital, the mean-variance optimal hedge and the $p\exp((a^{\epsilon} - a)T)/2v$-fold of the minimal expected squared hedging error ε_{ϵ}^2 for H under P^{ϵ}.*

Corollary 5.3 implies that—in first-order approximation—power utility-based hedging corresponds to mean-variance hedging, but for a Lévy process with different drift and jump measure. If $\eta = 0$, which is equivalent to S being a martingale under the physical measure P, then $P^{\epsilon} = P$ and no adjustment is necessary. If $\widehat{\eta} > 0$ in the economically most relevant case of a positive drift, the stock price process is a P-submartingale, but turns into a supermartingale under P^{ϵ}. Moreover, negative jumps become more likely and positive jumps less likely, such that a negative skewness is amplified when passing from P to P^{ϵ}. The magnitude of these effects depends on the investor's risk aversion p. Note that as the latter becomes large, the P^{ϵ} dynamics of the return process converge to those under the minimal entropy martingale measure (cf., e.g., [12]). Hence, as risk aversion becomes large, asymptotic power utility-based pricing and hedging approaches its counterpart for exponential utility.

The above considerations apply to *any* contingent claim satisfying Assumption 3.5, i.e., which can be superhedged with respect to the numeraire given by the optimal wealth process in the pure investment problem. To obtain numerically tractable formulas, one has to make additional assumptions. For example, semi-explicit solutions to the mean-variance hedging problem for exponential Lévy models have been obtained in [18] using the *Laplace transform approach* put forward in [38]. The key assumption for this approach is the existence of an integral representation of the payoff function in the following sense.

Assumption 5.4 Suppose $H = f(S_T)$ for a function $f : (0, \infty) \mapsto \mathbb{R}$ such that

$$f(s) = \int_{R-i\infty}^{R+i\infty} l(z)s^z dz, \quad s \in (0, \infty),$$

for $l : \mathbb{C} \to \mathbb{C}$ such that the integral exists for all $s \in (0, \infty)$ and $R \in \mathbb{R}$ such that $E(S_T^R) < \infty$.

Most European options admit a representation of this kind, see, e.g., [18, Sect. 4].

Example 5.5 For a European call option with strike $K > 0$ we have $H = (S_T - K)^+$ and, for $s > 0$ and $R > 1$,

$$(s - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \frac{K^{1-z}}{z(z-1)} s^z dz.$$

By evaluating the formulas of Hubalek et al. [18] under P^ϵ, we obtain the following semi-explicit representations. They are expressed in terms of the Lévy exponent $\psi^{\tilde{X}\epsilon}$ of the log-price $\tilde{X}$ under P^ϵ.

Theorem 5.6 *For a contingent claim H satisfying Assumptions 3.5 and 5.4, the marginal utility-based price and a marginal utility-based hedging strategy are given by*

$$\pi(0) = V_0,$$
$$\varphi'_t = \xi_t - \left(V_0 + \varphi' \bullet S_{t-} - V_{t-} \right) \tilde{a},$$

with

$$\Psi(z) := \psi^{\tilde{X}\epsilon}(z) - \psi^{\tilde{X}\epsilon}(1) \frac{\psi^{\tilde{X}\epsilon}(z+1) - \psi^{\tilde{X}\epsilon}(z) - \psi^{\tilde{X}\epsilon}(1)}{\psi^{\tilde{X}\epsilon}(2) - 2\psi^{\tilde{X}\epsilon}(1)},$$

$$\tilde{a} := \frac{1}{S_{t-}} \frac{\psi^{\tilde{X}\epsilon}(1)}{\psi^{\tilde{X}\epsilon}(2) - 2\psi^{\tilde{X}\epsilon}(1)},$$

$$V_t := \int_{R-i\infty}^{R+i\infty} S_t^z e^{\Psi(z)(T-t)} l(z) dz,$$

$$\xi_t := \int_{R-i\infty}^{R+i\infty} S_{t-}^{z-1} \frac{\psi^{\tilde{X}\epsilon}(z+1) - \psi^{\tilde{X}\epsilon}(z) - \psi^{\tilde{X}\epsilon}(1)}{\psi^{\tilde{X}\epsilon}(2) - 2\psi^{\tilde{X}\epsilon}(1)} e^{\Psi(z)(T-t)} l(z) dz.$$

Moreover, the corresponding risk premium π' for H can be written as

$$\pi' = \frac{p \exp((a^\epsilon - a)T)}{2v} \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} J(z_1, z_1) l(z_1) l(z_2) dz_1 dz_2,$$

for a and a^ϵ as in Examples 2.5 and 4.2, respectively, and

$$k(z_1, z_2) := \Psi(z_1) + \Psi(z_2) - \frac{\psi^{\tilde{X}\epsilon}(1)^2}{\psi^{\tilde{X}\epsilon}(2) - 2\psi^{\tilde{X}\epsilon}(1)},$$

$$j(z_1, z_2) := \psi^{\tilde{X}\epsilon}(z_1 + z_2) - \psi^{\tilde{X}\epsilon}(z_1) - \psi^{\tilde{X}\epsilon}(z_2)$$
$$- \frac{(\psi^{\tilde{X}\epsilon}(z_1 + 1) - \psi^{\tilde{X}\epsilon}(z_1) - \psi^{\tilde{X}\epsilon}(1))(\psi^{\tilde{X}\epsilon}(z_2 + 1) - \psi^{\tilde{X}\epsilon}(z_2) - \psi^{\tilde{X}\epsilon}(1))}{\psi^{\tilde{X}\epsilon}(2) - 2\psi^{\tilde{X}\epsilon}(1)},$$

$$J(z_1, z_2) := \begin{cases} S_0^{z_1+z_2} j(z_1, z_2) \dfrac{e^{k(z_1,z_2)T} - e^{\psi^{\tilde{X}\epsilon}(z_1+z_2)T}}{k(z_1, z_2) - \psi^{\tilde{X}\epsilon}(z_1 + z_2)} & \text{if } k(z_1, z_2) \neq \psi^{\tilde{X}\epsilon}(z_1 + z_2), \\ S_0^{z_1+z_2} j(z_1, z_2) T e^{\psi^{\tilde{X}\epsilon}(z_1,z_2)T} & \text{if } k(z_1, z_2) = \psi^{\tilde{X}\epsilon}(z_1, z_2). \end{cases}$$

Proof See [18, Theorems 3.1 and 3.2].   □

5.2 BNS model

We now turn to the application of Theorem 4.7 to the BNS model with stochastic volatility. Throughout, we assume that the conditions of Examples 2.4 and 4.2 are satisfied, i.e., either $p \geq 2$ or sufficiently large exponential moments of the subordinator Z driving the variance process y exist. In the first case, we also suppose Z is integrable. By Proposition 6.4, the $P^{\in}$-dynamics of the variance process y and the return process X are given by

$$dy_t = -\lambda dt + dZ_t^{\in},$$
$$dX_t = (-\mu/p)y_t dt + \sqrt{y_t} dW_t.$$

Here μ and λ are the constant drift and mean reversion rates of the BNS model under P, W is a standard Brownian motion (under both P and $P^{\in}$), and $Z^{\in}$ is an *inhomogeneous $P^{\in}$-Lévy* process with characteristics

$$\left(b^Z + \int\limits_0^\infty z(e^{\alpha_1^{\in}(t)z} - 1) F^Z(dz), 0, e^{\alpha_1^{\in}(t)z} F^Z(dz) \right)$$

relative to the truncation function $h(z) = z$. Hence (y, X) is an *inhomogeneous* BNS model under $P^{\in}$. Note that as for exponential Lévy models, the drift rate of the return process changes its sign when moving from μ (under P) to $-\mu/p$ (under $P^{\in}$). The effect on the volatility process y depends on the sign of $\alpha_1^{\in}$, which is positive for $p < 2$ and negative for $p > 2$. If $p < 2$, i.e., for less risk-averse investors, the mean of $Z^{\in}$ (i.e., the average size of the positive volatility jumps) increases because jumps (in particular, large ones) become more likely under $P^{\in}$. For more risk averse investors with $p > 2$, the frequency of jumps is decreased under $P^{\in}$, which also leads to a decrease in the average value of volatility. Since $\alpha_1^{\in}(t)$ decreases resp. increases to 0 as $t \to T$ for $p < 2$ resp. $p > 2$, the deviation from the P-dynamics of Z is largest at the initial time $t = 0$ and tends to zero as $t \to T$. Finally, as the investor's risk aversion becomes large, the $P^{\in}$ dynamics of (y, X) again tend to their counterparts under the minimal entropy martingale measure corresponding to exponential utility, which was determined in [4].

With the $P^{\in}$-dynamics of S at hand, we can now provide a sufficient condition for the validity of Assumption 3.9 in the BNS model. More specifically, S is square-integrable under $P^{\in}$ by [24, Theorem 5.1] provided that

$$\int\limits_1^\infty \exp\left(\frac{1 - e^{-\lambda T}}{\lambda} \left(\frac{(1+p)(2-p)}{2p^2}\mu^2 + 2 - \frac{\mu}{p} \right) z \right) F^Z(dz) < \infty.$$

If, in addition, the conditions of Example 4.2 are satisfied, Assumption 3.9 holds.

With the $P^{\in}$-dynamics of S at hand, we now turn to the computation of semi-explicit representations for the marginal utility-based price π^0 (cf. Remark 3.8) as well as the utility-based hedge φ' and the risk premium π' from Theorem 4.7 for claims admitting an integral representation as in Assumption 5.4. The (inhomogeneous) BNS model is studied from the point of view of mean-variance hedging in [30]. As noted in the introduction, the formulas in Theorem 4.7 formally agree with such a problem under the appropriate probability measure $P^{\in}$. Therefore the calculations in [30] can be adapted to the present situation. In that paper, admissibility of the candidate solution $\tilde{a}$ to the pure investment problem under quadratic utility is not shown. Nevertheless, the results from [30] can be applied here because $\tilde{a}$ does not have to be admissible for the application of Theorem 4.7. Put differently, the calculations in [30] can be used without explicitly referring

to the quadratic hedging problem studied there. Below, we outline the necessary steps. This sketch could be turned into a rigorous proof, similarly as in [27, Theorems 4.1 and 4.2].

The *first step* is to determine the mean value process $V = E_{Q_0}(H|\mathscr{F}_t)$. Since the density process $L\mathscr{E}(\widehat{\eta}X)^{-p}$ of Q_0 with respect to P is the exponential of an inhomogeneous affine process (cf. [10,24] for more details), Proposition 6.4 shows that (y, X) is also an inhomogeneous affine process under Q_0. Using the integral representation for H, Fubini's theorem, and the affine transform formula for (y, X) (compare [10,24]) then leads to

$$V_t = \int_{R-i\infty}^{R+i\infty} S_t^z \exp\left(\Psi^0(t, T, z) + \Psi^1(t, T, z)y_t\right) l(z)dz, \qquad (5.2)$$

with

$$\Psi^1(t, T, z) = \frac{(1 - z)z}{2\lambda}(e^{-\lambda(T-t)} - 1),$$

$$\Psi^0(t, T, z) = \int_t^T \left(\psi^Z(\alpha_1(s) + \Psi^1(s, T, z)) - \psi^Z(\alpha_1(s))\right) ds.$$

In the *second step*, we turn to the marginal utility-based hedging strategy φ'. The representation (5.2) for V and the bilinearity of the predictable quadratic variation yields integral representations for the modified second $P^\in$-characteristics of (S, V), too, where the integrands can be computed using Proposition 6.2 (cf. the proof of [30, Theorem 3.3] for more details). Plugging these in Theorem 4.7 gives

$$\varphi' = \xi_t - (V_0 + \varphi' \bullet S_{t-} - V_{t-})\widetilde{a}_t,$$

with

$$\xi_t = \int_{R-i\infty}^{R+i\infty} zS_t^{z-1} \exp\left(\Psi^0(t, T, z) + \Psi^1(t, T, z)y_{t-}\right) l(z)dz,$$

for Ψ_0, Ψ_1 as above.

Remark 5.7 Provided that differentiation and integration can be interchanged, the *pure hedge coefficient* ξ_t in the BNS model is given by the derivative of V_t with respect to S_t. Hence the marginal utility-based hedging strategy is given as the sum of the *delta hedge* with respect to the marginal utility-based option price and a feedback term. This is a generic result in affine models with continuous asset prices and uncorrelated volatility processes, compare [30].

Finally, in a *third step*, it remains to consider the risk premium π' in Theorem 4.7. Plugging in the expression for L and $L^\in$, we find

$$\frac{pC_1}{2vC_0} = \frac{pL_0^\in}{2vL_0} = \frac{p}{2v} \exp\left(\int_0^T (\psi^Z(\alpha_1^\in(t)) - \psi^Z(\alpha_1(t)))dt + \frac{\mu^2(1 - e^{-\lambda T})}{p^2\lambda}y_0\right).$$

Hence it remains to compute the expectation in the formula for the risk premium π'. Here, (5.2) again leads to integral representations for $\widetilde{c}^{S\star}$, $\widetilde{c}^{S,V\star}$, $\widetilde{c}^{V\star}$. The product of $L^{\$}$ and the integrand once more turns out to be the exponential of an inhomogeneous affine process. Its expectation can therefore again be computed using the affine transform formula for (y, X) (cf. the proof of [30, Theorem 3.4] for more details). This leads to

$$\pi' = \frac{pC_1}{2vC_0} \int_0^T \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} J(t, z_1, z_2)l(z_1)l(z_2)dz_1 dz_2 dt,$$

for

$$\psi^{Z\epsilon}(t, u) = \psi^Z(u + \alpha_1^\epsilon(t)) - \psi^Z(\alpha_1^\epsilon(t)),$$

$$j(t, z_1, z_2) = \psi^{Z\epsilon}(t, \alpha_1^\$(t) + \Psi^1(t, T, z_1) + \Psi^1(t, T, z_2)) + \psi^{Z\epsilon}(t, \alpha_1^\$(t))$$
$$- \psi^{Z\epsilon}(t, \alpha_1^\$(t) + \Psi^1(t, T, z_1)) - \psi^{Z\epsilon}(t, \alpha_1^\$(t) + \Psi^1(t, T, z_2)),$$

$$g(z_1, z_1) = \frac{2\mu + p}{2p}(z_1 + z_2) - \frac{1}{2}(z_1 + z_2)^2,$$

and

$$\Upsilon^1(s, t, T, z_1, z_2) = (\alpha_1^\$(t) + \Psi^1(t, T, z_1) + \Psi^1(t, T, z_2))e^{\lambda(s-t)} + g(z_1, z_2)\frac{e^{\lambda(s-t)} - 1}{\lambda},$$

$$\Upsilon^0(s, t, T, z_1, z_2) = \int_s^t \psi^{Z\epsilon}(r, \Upsilon^1(r, t, T, z_1, z_2))dr,$$

$$J(t, z_1, z_2) = S_0^{z_1+z_2} j(t, z_1, z_2) \exp\left(\Upsilon^0(0, t, T, z_1, z_2) + \Upsilon^1(0, t, T, z_1, z_2)y_0\right)$$
$$\times \exp\left(\int_t^T \psi^{Z\epsilon}(s, \alpha_1^\$(s))ds + \Psi^0(t, T, z_1) + \Psi^0(t, T, z_2)\right).$$

If the volatility process y is chosen to be a Gamma-OU process, all expressions involving integrals of the characteristic exponent $\psi^{Z\epsilon}(t, u) = \psi^Z(u + \alpha_1^\epsilon(t)) - \psi^Z(\alpha_1^\epsilon(t))$ can be computed in closed form as well. More specifically, let y be a Gamma-OU process with mean reversion rate $\lambda > 0$ and stationary $\Gamma(a, b)$-distribution and let

$$m(s) := c_1\left(e^{-\lambda(\widetilde{t}-s)} - 1\right) + c_2 e^{-\lambda(\widetilde{t}-s)} + c_3, \quad \widetilde{t} \in [0, T],$$

for constants $c_1, c_2, c_3 \in \mathbb{C}$. Then if $m(s) \neq b, s \in [t, T]$ we have

$$\int_{t_1}^{t_2} \psi^Z(m(s))ds = \begin{cases} \frac{-a}{b+c_1-c_3}\left(\lambda(t_2 - t_1)(c_1 - c_3) - b \log\left(\frac{-b+m(t_1)}{-b+m(t_2)}\right)\right) & b \neq c_3 - c_1, \\ -\lambda a(t_2 - t_1) + \frac{ab}{c_1+c_2}\left(e^{\lambda(\widetilde{t}-t_2)} - e^{\lambda(\widetilde{t}-t_1)}\right) & b = c_3 - c_1 \end{cases}$$

for $0 \leq t_1 \leq t_2 \leq T$ and where log denotes the *distinguished logarithm* in the sense of [41, Lemma 7.6]. This follows by inserting the Lévy exponent $\psi^Z(u) = \frac{\lambda a u}{b-u}$, which is analytic on $\mathbb{C}\backslash\{b\}$, and integration using decomposition into partial fractions.

6 Numerical illustration

Mean-variance hedging for the BNS Gamma-OU stochastic volatility model is considered in [30]. Since the formulas in the previous section are of the same form, the numerical algorithm applied in [30] can also be used to explore this model from the point of view of utility-based pricing and hedging. Exponential Lévy processes could be treated analogously (compare [18]). Since the corresponding results are very similar, we omit them here.

As a concrete specification, we consider the discounted BNS-Gamma-OU model with parameters as estimated in [25] from a DAX time series, i.e.,

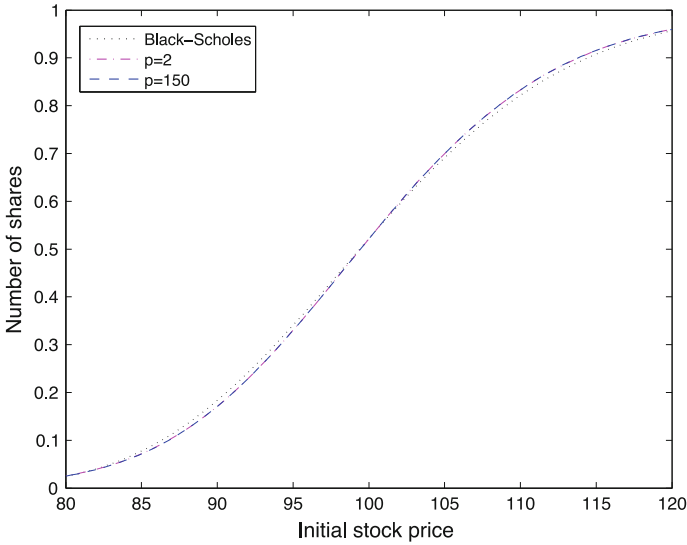$$\mu = 1.404, \quad \lambda = 2.54, \quad a = 0.848, \quad b = 17.5.$$

Fig. 1 Initial Black–Scholes hedge and initial utility-based BNS-hedges for $p = 2$, $p = 150$ and a European call with strike $K = 100$ and maturity $T = 0.25$
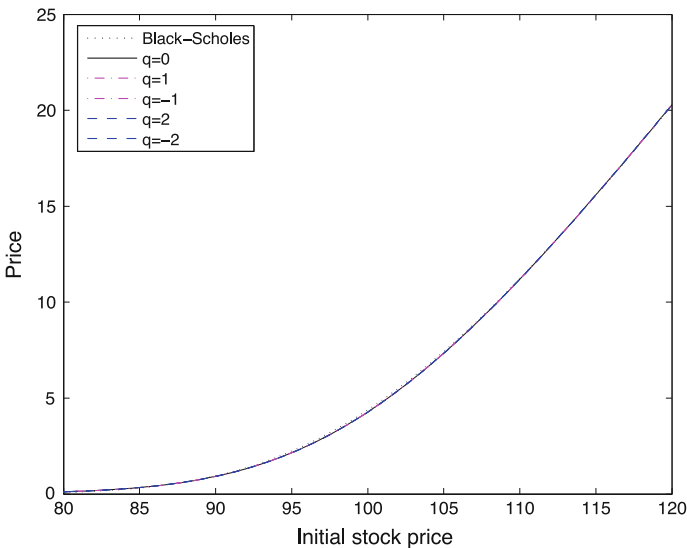

Fig. 2 Black–Scholes price and approximate indifference price $\pi(0) + q\pi'$ in the BNS model for $p = 2$ and a European call with strike $K = 100$ and maturity $T = 0.25$

We let $y_0 = 0.0485$ and put $v = 241$, which implies that indifference prices and utility-based hedging strategies exist for $S_0 \in [80, 120]$ and $q \in [-2, 2]$. By our above results, first-order approximations of the utility-indifference price and the utility-based hedging strategy exist for $p = 0.5, \dots, 150$ by Lemma 2.10 resp. Theorem 2.11. Moreover, Assumptions 3.5 and 5.4 hold for European call-options by Example 5.5. The formulas from Sect. 5.2 can now be evaluated using numerical quadrature, where we use $R = 1.2$.
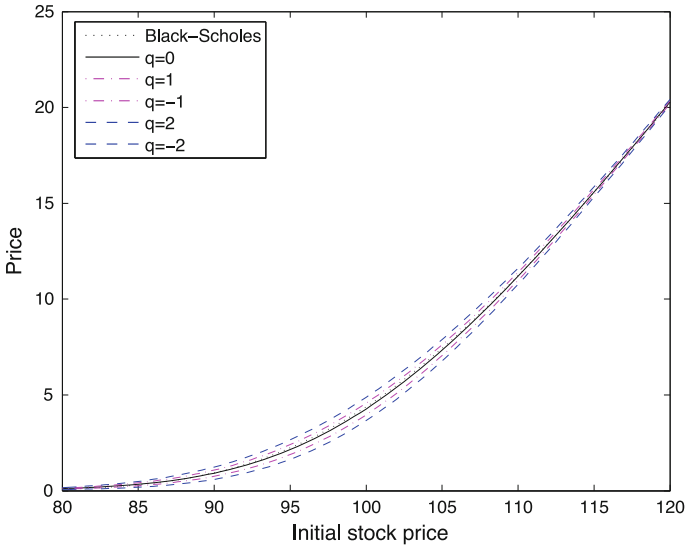
Fig. 3 Black–Scholes price and approximate indifference price $\pi(0) + q\pi'$ in the BNS model for $p = 150$ and a European call with strike $K = 100$ and maturity $T = 0.25$
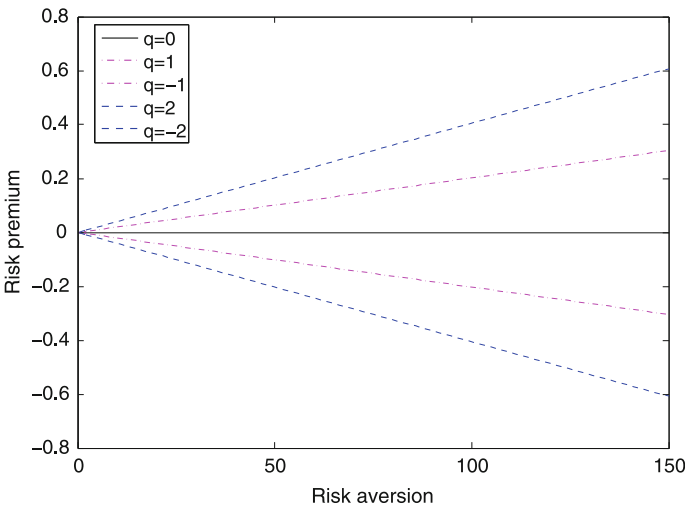

Fig. 4 Risk premia $q\pi'$ for q at-the-money European calls with strike $K = 100$ and maturity $T = 0.25$ in the BNS model for risk aversions $p = 0.5, \ldots, 150$

The initial hedges for $p = 2$ and $p = 150$ in Fig. 1 below cannot be distinguished by eye. Indeed, the maximal relative difference between the two strategies is 0.4 % for $80 \leq S_0 \leq 120$, which implies that the utility-based hedging strategy is virtually independent of the investor's risk aversion. Moreover, both strategies are quite close to the Black–Scholes hedging strategy, the maximal relative difference being about 8.9 %.

We now turn to utility-based pricing. First, note that in our specification the marginal utility-based price π^0 barely depends on the investor's risk aversion, and is almost

indistinguishable from its Black–Scholes counterpart. For a relative risk aversion of $p = 2$, the effect of the first-order risk adjustment is also very small (cf. Fig. 2). This resembles similar findings of [14,13] on utility-based pricing and hedging for basis risk.

In fact, much higher risk aversions as, e.g., $p = 150$ in Fig. 3 are required to obtain a bid price below and an ask-price above the Black–Scholes price for one option as a result of the first-order risk adjustment. For evidence supporting such high levels of risk aversion, cf., e.g., [22]. Finally, Fig. 4 depicts the dependence of the risk premium π' on the investor's relative risk aversion p, which turns out to be almost linear. Note that since π' is inversely proportional to the initial endowment v, this also implies that π' is virtually linear in the investor's absolute risk aversion p/v, which holds exactly for exponential utility (cf. [3,28,34]).

Appendix

In this appendix we summarize some basic notions regarding semimartingale characteristics (cf. [21] for more details). In addition, we state and prove an auxiliary result which is used in the proof of Theorem 4.7.

To any $\mathbb{R}^d$-valued semimartingale X there is associated a triplet (B, C, ν) of *characteristics*, where B resp. C denote $\mathbb{R}^d$- resp. $\mathbb{R}^{d \times d}$-valued predictable processes and ν a random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ (cf. [21, II.2.6]). The first characteristic B depends on a *truncation function* $h : \mathbb{R}^d \to \mathbb{R}^d$ such as $h(x) = x 1_{\{|x| \leq 1\}}$. Instead of the characteristics themselves, we typically use the following notion.

Definition 6.1 Let X be an $\mathbb{R}^d$-valued semimartingale with characteristics (B, C, ν) relative to some truncation function h on $\mathbb{R}^d$. In view of [21, II.2.9], there exist a predictable process $A \in \mathscr{A}_{\mathrm{loc}}^+$, an $\mathbb{R}^d$-valued predictable process b, an $\mathbb{R}^{d \times d}$-valued predictable process c and a transition kernel F from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$ such that

$$B_t = b \bullet A_t, \quad C_t = c \bullet A_t, \quad \nu([0, t] \times G) = F(G) \bullet A_t \quad \text{for } t \in [0, T], \ G \in \mathscr{B}^d,$$

where we implicitly assume that (b, c, F) is a good version in the sense that the values of c are non-negative symmetric matrices, $F_s(\{0\}) = 0$ and $\int (1 \wedge |x|^2) F_s(dx) < \infty$. We call (b, c, F, A) *local characteristics* of X.

If (b, c, F, A) denote local characteristics of some semimartingale X, we write

$$\tilde{c} := c + \int x x^\top F(dx)$$

and call $\tilde{c}$ the *modified second characteristic* of X provided that the integral exists. This notion is motivated by the fact that $\langle X, X \rangle = \tilde{c} \bullet A$ by [21, I.4.52] if the corresponding integral is finite. We write (b^X, c^X, F^X, A) and $\tilde{c}^X$ for the differential characteristics and the modified second characteristic of a semimartingale X. Likewise, the joint local characteristics of two semimartingales X, Y are denoted by

$$(b^{(X,Y)}, c^{(X,Y)}, F^{(X,Y)}, A) = \left(\begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \begin{pmatrix} c^X & c^{X,Y} \\ c^{Y,X} & c^Y \end{pmatrix}, F^{(X,Y)}, A \right)$$

and

$$\tilde{c}^{(X,Y)} = \begin{pmatrix} \tilde{c}^X & \tilde{c}^{X,Y} \\ \tilde{c}^{Y,X} & \tilde{c}^Y \end{pmatrix},$$

if the modified second characteristic of (X, Y) exists. The characteristics of a semimartingale X under some other measure $Q^\$$ are denoted by $(b^{X\$}, c^{X\$}, F^{X\$}, A)$. The following rules for the computation of characteristics are used repeatedly in the proofs of this paper.

Proposition 6.2 (C^2-function) *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b^X, c^X, F^X, A). Suppose that $f : U \to \mathbb{R}^n$ is twice continuously differentiable on some open subset $U \subset \mathbb{R}^d$ such that X, X_- are U-valued. Then the $\mathbb{R}^n$-valued semimartingale $f(X)$ has local characteristics $(b^{f(X)}, c^{f(X)}, F^{f(X)}, A)$, where*

$$b_t^{f(X),i} = \sum_{k=1}^d \partial_k f^i(X_{t-}) b_t^{X,k} + \frac{1}{2} \sum_{k,l=1}^d \partial_{kl} f^i(X_{t-}) c_t^{X,kl}$$
$$+ \int \left(\tilde{h}^i(f(X_{t-} + x) - f(X_{t-})) - \sum_{k=1}^d \partial_k f^i(X_{t-}) h^k(x) \right) F_t^X(dx),$$

$$c_t^{f(X),ij} = \sum_{k,l=1}^d \partial_k f^i(X_{t-}) c_t^{X,kl} \partial_l f^j(X_{t-}),$$

$$F_t^{f(X)}(G) = \int 1_G(f(X_{t-} + x) - f(X_{t-})) F_t^X(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G.$$

Here, ∂_k etc. denote partial derivatives and $\tilde{h}$ again the truncation function on $\mathbb{R}^n$.

Proof This follows immediately from [15, Corollary A.6]. □

Proposition 6.3 (*Stochastic integration*) *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b^X, c^X, F^X, A) and H an $\mathbb{R}^{n \times d}$-valued predictable process with $H^{j\cdot} \in L(X)$ for $j = 1, \ldots, n$. Then local characteristics of the $\mathbb{R}^n$-valued integral process $H \bullet X := (H^{j\cdot} \bullet X)_{j=1,\ldots,n}$ are given by $(b^{H\bullet X}, c^{H\bullet X}, F^{H\bullet X}, A)$, where*

$$b_t^{H\bullet X} = H_t b_t^X + \int (\tilde{h}(H_t x) - H_t h(x)) F_t^X(dx),$$

$$c_t^{H\bullet X} = H_t c_t^X H_t^\top,$$

$$F_t^{H\bullet X}(G) = \int 1_G(H_t x) F_t^X(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G.$$

Here $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the truncation function which is used on $\mathbb{R}^n$.

Proof [29, Lemma 3]. □

Let $P^\star \overset{\text{loc}}{\sim} P$ be a probability measure with density process Z. Local equivalence yields that Z and Z_- are strictly positive by [21, I.2.27]. Hence the *stochastic logarithm* $N := \mathscr{L}(Z) = \frac{1}{Z_-} \bullet Z$ is a well-defined semimartingale. For an $\mathbb{R}^d$-valued semimartingale X we now have the following result, which relates the local $P^\star$-characteristics of (X, N) to the local characteristics of (X, N) under P.

Proposition 6.4 *(Equivalent change of measure) Local $P^\star$-characteristics of the process* (X, N) *are given by* $(b^{(X,N)\star}, c^{(X,N)\star}, F^{(X,N)\star}, A)$, *where*

$$b^{(X,N)\star} = b^{(X,N)} + c^{(X,N),N} + \int h(x)x_{d+1}F^{(X,N)}(dx),$$

$$c^{(X,N)\star} = c^{(X,N)},$$

$$F^{(X,N)\star} = \int 1_G(x)(1+x_{d+1})F^{(X,N)}(dx) \quad \forall G \in \mathscr{B}^{d+1} \text{ with } 0 \notin G.$$

Proof [23, Lemma 5.1]. □

The following observation is needed in the proof of Theorem 4.7.

Lemma 6.5 *Let $Q \overset{\text{loc}}{\ll} P$ with density process Z. Then for any increasing, predictable process A with $A_0 = 0$ we have*

$$E_Q(A_T) = E_P(Z_- \bullet A_T).$$

Proof Since Z is a P-martingale and A is predictable and of finite variation, $A \bullet Z$ is a local P-martingale by [21, I.3.10 and I.4.34]. If $(T_n)_{n\in\mathbb{N}}$ denotes a localizing sequence, $A \bullet Z_{T \wedge T_n}$ is a martingale starting at 0. By [21, III.3.4 and I.4.49], this implies

$$E_Q(A_{T \wedge T_n}) = E_P(Z_{T \wedge T_n} A_{T \wedge T_n})$$
$$= E_P(Z_- \bullet A_{T \wedge T_n} + A \bullet Z_{T \wedge T_n})$$
$$= E_P(Z_- \bullet A_{T \wedge T_n}).$$

Hence monotone convergence yields $E_Q(A_T) = E_P(Z_- \bullet A_T)$ as claimed. □

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