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Ingo Sick

Importance of Tail of Proton Density

Or: How to Get the *rms*-Radius from (e, e) Data

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Abstract Triggered by the discrepancy between electron–proton scattering and muonic hydrogen results we scrutinize the determination of the charge *rms*-radius from (e, e) data. We find that the standard procedure of fitting parameterizations for the Sachs form factors $G(q)$ to the data is more uncertain than traditionally assumed. This is related to the (implicit) extrapolation from the q-range sensitive to finite size to momentum transfer $q = 0$ where the *rms*-radius is extracted. A reliable determination of the *rms*-radius requires a constraint on the large-r tail of the density, which can be imposed starting from our physics understanding of proton structure.

1 Introduction

The root-mean-square (*rms*) radii R (charge and magnetic) of the proton are important quantities to characterize the overall size of the proton. They have traditionally been determined via elastic electron–proton scattering at low momentum transfer q. These radii are needed to extract information from the highly precise measurement of the frequencies of transitions in atomic hydrogen. Recently, a measurement of the radius via the Lamb-shift in muonic hydrogen has also become available.

The comparison of the radii from the different probes reveals a major problem: while the charge-radius from electron scattering [7] of 0.886 ± 0.008 fm and the one from electronic hydrogen [4] of Parthey et al. are compatible within experimental errors, the radius from muonic hydrogen measured by Pohl et al. [5] of 0.8418 ± 0.0007 fm differs by many standard deviations. Many ideas to explain this discrepancy have been discussed in the literature, but no convincing solution has been found.

In this contribution, we take a closer look at the way how the radii are extracted from electron–proton scattering. We identify a shortcoming of the methods used in the past, and provide a better approach.

2 Radii from Electron Scattering

The *rms*-radii correspond to the slope of the Sachs form factors $G_e(q^2)$ and $G_m(q^2)$ at $q = 0$; with this definition the radii are the same as the ones employed in atomic hydrogen. This definition immediately highlights two problems:

1. The q^2=0 slope cannot be measured directly, as experiments cannot be performed at $q = 0$. In practice, the data between momentum transfers of 0.6–1.2 fm^{-1} are the ones that are sensitive to R [6], and the

I. Sick (✉)
Department of Physics, University of Basel, Basel, Switzerland
Tel: +41-61-2673741
Fax: +41-61-2673784
E-mail: Ingo.Sick@unibas.ch

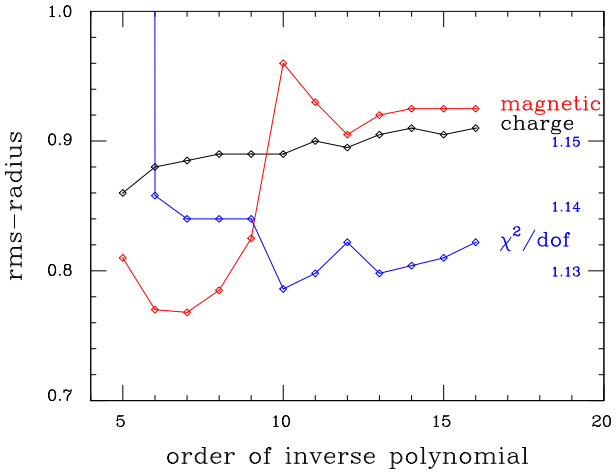

Fig. 1 Radii and χ^2 of the inverse polynomial fits of Bernauer et al. as a function of the order of the polynomial

slope of $G(q)$ at $q = 0$ is obtained from an (implicit) extrapolation, i.e. the value of the model-$G(q)$ fitted to the data.

2. Even if one could measure down to $q = 0$, experimental uncertainties would forbid extraction of an accurate radius; at low q, $G(q)$ is given by $1 - q^2R^2/6 + q^4\langle r^4\rangle/120 + \cdots$, so at very low q and in the presence of finite experimental errors one only measures the "1".

This extrapolation to $q = 0$ is particularly difficult for the proton, which has a form factor roughly described by the dipole parameterization $G(q) = 1/(1 + q^2c^2)^2$. The corresponding density (which, for the qualitative discussion that follows, is taken as the Fourier transform of $G(q)$) then is an exponential proportional to $e^{-r/c}$. Exponential densities have very long tails toward large radii r, and these tails affect the form factor at extremely low q, below the range where the data are sensitive to R. The density at $r > 3$ fm, for instance, still contributes 2 % to R. As the density at these large values of r is poorly known, the extrapolation via the model-$G(q)$ introduces a significant model dependence.

A reliable extrapolation to $q = 0$ is also hindered by the fact that some of the data, as for instance the recent very extensive set of ref. [2], have a floating normalization of the cross sections; this leads to much more freedom in the extrapolation of $G(q < q_{min})$.

3 Ambiguities of Past Determinations of *R*

We here illustrate the difficulties in determining R by discussing three recent analyses of e–p data.

Bernauer et al. [1,2] have analyzed their new set of e–p scattering data ($q_{max} \sim 5\,\mathrm{fm}^{-1}$) using various parameterizations, and the quoted results for the radii are quite similar. Among these parametrizations was the inverse polynomial form, $G(q) = 1/(1 + a_1q^2 + a_2q^4 + \cdots)$. As function of the order N of the polynomial used, Bernauer finds a curious behavior, see Fig. 1. While the χ^2 and the charge radius display the expected N-dependence, the magnetic radius jumps between $N = 7$ and $N = 10$ from 0.77 to 0.96 fm. Bernauer et al. chose $N = 7$ as the χ^2 was stabilized, but $N = 10$ actually produces the lowest χ^2.

An obvious question then is: how can the q^{20}-term that appears for $N = 10$ affect the *rms*-radius? Closer scrutiny reveals that the $N = 10$ form factor has a pole at $q > q_{max}$. This corresponds in r-space to a density that extends to extremely large r (i.e. $r = \infty$), see Fig. 2, left. And this large-r density affects the curvature and slope of $G(q)$ at momentum transfers below q_{min} (see Fig. 2, right). While the $N = 7$ fit looks more sensible, it also has a pole, and cannot be relied upon either.

Lorenz et al. [3] also employ the Bernauer data, but use a continued fraction parameterization of $G(q)$. The corresponding fits show systematic deviations from the data and, more importantly, have unphysical behavior (e.g. a peak) at $q > q_{max}$. The corresponding density again falls extremely slowly at large r, with undesirable effects upon R.

A 3-parameter Pade fit we have made ourselves to the Bernauer data (limited to the region $q \le q_{max} = 2\,\mathrm{fm}^{-1}$ which contains all data sensitive to R) demonstrates in the most extreme fashion the problem: although

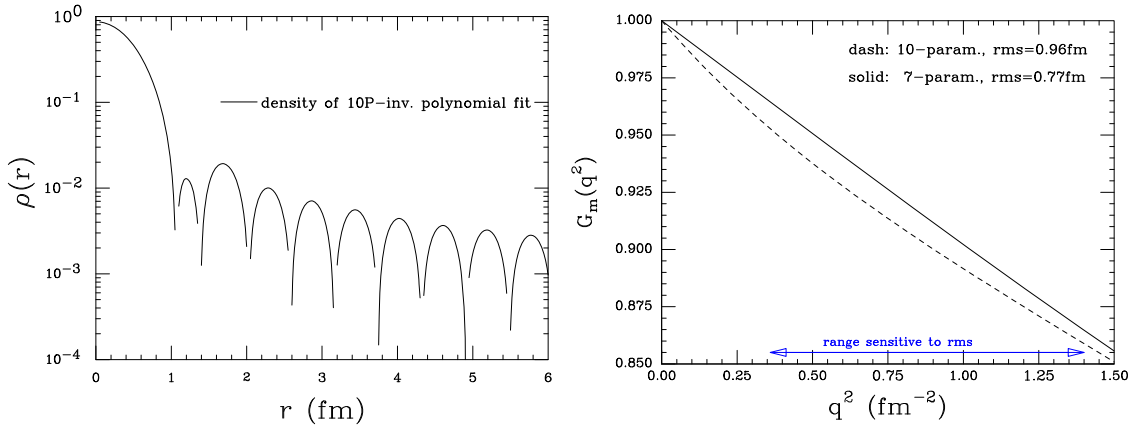

Fig. 2 Density corresponding to the 10P inverse polynomial fit (*left*) and form factors $G(q)$ at low q (*right*). Note that the data of [2] (not shown) have a floating normalization, so both fits give excellent χ^2

the $\chi^2 = 1.06$ per degree of freedom is excellent (as low as a spline fit) and $G(q)$ has no poles, the *rms*-radius R is found to be 1.49 fm! Again the problem results from an uncontrolled behavior of $G(q)$ at $q > q_{max}$ which in the density happens to imply a tail to extremely large r and a resulting falsification of the *rms*-radius.

4 Origin of Problem

In order to better understand the problems discussed above, it is useful to confront the approaches used for a determination of R for nuclei $A \leq 2$ and $A > 2$. For $A \leq 2$ the form factor is parameterized, the parameters are fitted to the data, and the $q = 0$ slope is used to determine R. For $A > 2$ the density is parameterized, the corresponding cross sections are fitted to the data, and R is obtained from the integral over the density or the $q = 0$ slope of the corresponding form factor.

These two approaches are *not equivalent*. When using parameterizations such as Woods-Saxon, Fourier-Bessel or Sum Of Gaussians (SOG) the density is automatically confined to a given range in r. This reflects the basic property that the density at large r must fall as a Whittaker function depending on the separation energy of the least-bound charged constituent and is, for practical purposes, zero outside some radius. This *physical condition* is absent when parameterizing $G(q)$; depending on the parameterization chosen, the density corresponding to $G(q)$ can have an unphysical tail to large r, which in turn affects the *rms*-radius and the behavior of $G(q)$ at extremely low q. This generic problem must be avoided.

This problem unfortunately concerns many of the fits made to e–p scattering data. Least affected are parameterizations that have been used to fit the data up to the maximal momentum transfer q_{max} where data are available ($q \sim 12\,\mathrm{fm}^{-1}$ for the charge form factor). In this case the data fix the shape of the density, and with it the large-r behavior, pretty well; in addition, $G(q > q_{max})$ is constrained by the small values of $G(q \sim q_{max})$ if the parameterization insures that $G(q)$ falls faster than q^{-4} as required to get a regular $\rho(0)$.

5 Solution

One could imagine to always check, during a fit of $G(q)$ to the e–p data, that the corresponding density falls in a sensible way at large r. This is difficult as the above examples show, and does not work at all for parameterizations that do not have a Fourier transform (such as the popular expansion of $G(q)$ in terms of powers of q^2).

One also could imagine to parameterize $\rho(r)$ with expressions that have an a-priori sensible large-r fall off, and fit the parameters after Fourier transform to the data. This complicates life, but not too much. The tricky point is the definition of "sensible".

The only effective solution we have found involves:

– Parameterization of $\rho(r)$ in a basis with analytic Fourier transform such as SOG, Hermite or Laguerre polynomials, such that quantities in q- and r-space can be fitted simultaneously.

– Adding in the fit of the (e, e) data information on the large-r behavior of $\rho(r)$ obtained from a physical model of the proton. While at small radii the proton has a complicated quark/gluon structure, at large radii where the density has fallen to, say, $\rho(r) < 0.01\rho(0)$, the density is given by the Fock component with the smallest separation energy; in the case of the proton this is the $\pi^+ + n$ configuration. Hereby one *explicitly* adds physics knowledge to constrain a low-q property that is not adequately fixed by the data. While the *shape* of the tail can be reliably calculated, the overall normalization of the tail density is hard to predict and therefore treated as a free parameter.
– Fitting the data up to the largest q where experimental information is available. This approach is feasible with the above parameterizations and helps to constrain the large-r tail of the density by also using the full information content of the electron scattering data.

When studying *quantitatively* the properties of the proton charge density in the tail [7] one needs to account for relativistic effects, two-photon exchange etc. Excellent agreement has been found between the *shape* of the density as computed from the $n + \pi^+$ and $\Delta + \pi^+$ components and the result of an analysis of the world data up to the largest $q's$ available using a [3,5] Pade parameterization.

6 Conclusions

We have found that many of the parameterizations of $G(q)$ fitted to e–p scattering data have problems. These can only be avoided if the large-r behavior of the corresponding density is studied and is verified to correspond to the behavior we expect from our physics understanding of the proton. This, unfortunately, is almost never done.

Fits respecting the solution presented in the previous section have been performed in [7]. These fits give stable radii free of the problems discussed above. When analyzing the *world* data on e–p scattering that include or exclude the recent Mainz data, with both fixed or floating normalizations of the world cross sections, one finds a charge-*rms* radius of the proton of 0.886 ± 0.008 fm.

This, unfortunately, does not help in solving the discrepancy with the radius from muonic hydrogen discussed in the introduction.

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