

Coisotropic Submanifolds and Dual Pairs

ALBERTO S. CATTANEO

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zurich, Switzerland. e-mail: alberto.cattaneo@math.uzh.ch

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Abstract. The Poisson sigma model is a widely studied two-dimensional topological field theory. This note shows that boundary conditions for the Poisson sigma model are related to coisotropic submanifolds (a result announced in [math.QA/0309180]) and that the corresponding reduced phase space is a (possibly singular) dual pair between the reduced spaces of the given two coisotropic submanifolds. In addition the generalization to a more general tensor field is considered and it is shown that the theory produces Lagrangian evolution relations if and only if the tensor field is Poisson.

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1. Introduction

Let M be a finite-dimensional manifold and I the unit interval $[0, 1]$. We denote by $PM := C^1(I, M)$ the space of differentiable paths on M and by T^*PM the space of bundle maps $TI \rightarrow T^*M$ with continuously differentiable base map and continuous fiber map. We consider PM as a Banach manifold and T^*PM as a Banach, weak symplectic manifold (a weak symplectic form is a closed 2-form that induces an injective map from the tangent to the cotangent bundle). The canonical symplectic form Ω is the differential of the canonical 1-form Θ . If we denote an element of T^*PM by (X, η) where X is the differentiable base map and the fiber map η is regarded as a continuous section of $T^*I \otimes X^*T^*M$, we have

$$\Theta(X, \eta)(\widehat{\xi}) = \int_I \langle \eta, \xi \rangle,$$

where $\widehat{\xi}$ is a tangent vector at (X, η) , ξ is its projection to $T_X PM = \Gamma(X^*TM)$, and $\langle \cdot, \cdot \rangle$ is the canonical pairing between the cotangent and the tangent bundles to M .

Let now π be a section of $TM \otimes TM$. We denote by π^\sharp the induced bundle map $T^*M \rightarrow TM$ satisfying

$$\langle \pi^\sharp(x)\sigma, \tau \rangle = \pi(x)(\sigma, \tau), \quad \forall x \in M, \quad \forall \sigma, \tau \in T_x^*M.$$

Given two submanifolds C_0 and C_1 of M , we denote by $\mathcal{C}_\pi(M; C_0, C_1)$ the space of “ π -compatible paths” from C_0 to C_1 ; i.e.,

$$\mathcal{C}_\pi(M; C_0, C_1) := \{(X, \eta) \in T^*PM : dX + \pi^\sharp(X)\eta = 0, X(0) \in C_0, X(1) \in C_1\},$$

where the differential dX of the base map is regarded as a section of $T^*I \otimes X^*TM$. By using the implicit function theorem, one can easily prove [6] that $\mathcal{C}_\pi(M; M, M)$ is a Banach submanifold of T^*PM .

In general, for other submanifolds C_0 and C_1 , $\mathcal{C}_\pi(M; C_0, C_1)$ is not a Banach submanifold.¹

Anyway, even if $\mathcal{C}_\pi(M; C_0, C_1)$ is not a submanifold, we may define its Zarisky tangent space at each point. Namely, we first define the submanifold $T^*PM(C_0, C_1)$ of T^*PM consisting of bundle maps whose base maps connect C_0 to C_1 . Then the tangent space at $(X, \eta) \in \mathcal{C}_\pi(M; C_0, C_1)$ is defined as the subspace of vectors in $T_{(X, \eta)}T^*PM(C_0, C_1)$ satisfying the linearized equation.

In general, we will call subvariety (of a smooth manifold) the common zero set of a family of smooth functions and, by abuse of notation, we will call tangent bundle the union of the Zarisky tangent spaces to a subvariety. Given a symplectic manifold (\mathcal{M}, ω) and a subvariety \mathcal{C} , we define, again by abuse of notation, the symplectic orthogonal bundle $T^\perp\mathcal{C}$ to \mathcal{C} as the union $T_x^\perp\mathcal{C}$, $x \in \mathcal{C}$, with

$$T_x^\perp\mathcal{C} := \{v \in T_x\mathcal{M} : \omega_x(v, w) = 0 \quad \forall w \in T_x\mathcal{C}\}.$$

The subvariety \mathcal{C} is said to be coisotropic if $T^\perp\mathcal{C} \subset T\mathcal{C}$.

1.1. THE MAIN RESULTS

In Section 2 we study the tangent bundle and the symplectic orthogonal bundle to $\mathcal{C}_\pi(M; C_0, C_1)$. Relying on the results of Section 2, we prove in Section 3.1 (viz., Propositions 3.1 and 3.2) the following

THEOREM 1.1. *$\mathcal{C}_\pi(M; M, M)$ is coisotropic in T^*PM iff π is a Poisson bivector field.*

Recall that a Poisson bivector field π is a skew-symmetric 2-tensor field satisfying $[\pi, \pi] = 0$, where $[\ , \]$ is the Schouten–Nijenhuis bracket. A proof of the

¹The simplest example is when π is identically zero. In this case $\mathcal{C}_\pi(M; C_0, C_1)$ is a fibration over $B := C_0 \cap C_1$ with fiber at x given by $\Omega^1(I, T_x^*M)$. If the basis B is not a manifold, so neither is $\mathcal{C}_\pi(M; C_0, C_1)$.

if-part of the Theorem is contained in [6], but the result was already known [11, 13] in the case when one considers loops instead of paths.

If \mathcal{C} is a coisotropic submanifold of a weak symplectic Banach manifold, $T^\perp\mathcal{C}$ is an integrable distribution and the leaf space $\underline{\mathcal{C}}$, also called the reduced phase space, inherits a weak symplectic structure if it is a smooth manifold. The reduced phase space $\underline{\mathcal{C}}_\pi(M; M, M)$ has been shown in [6] to also have the structure of a topological groupoid; if it is smooth, it is a symplectic groupoid integrating the Poisson manifold M .

Now notice that not all possible boundary conditions allow solutions to the constraint equation. In other words, the maps

$$p_i: \mathcal{C}_\pi(M; C_0, C_1) \rightarrow C_i, \quad i=0, 1, \quad (1.1)$$

associating $X(i)$ to a solution (X, η) may not be surjective. With this notation we may give the precise formulation, and a proof (see Section 3.2, in particular Propositions 3.5 and 3.6), of a result announced in [7] as Theorem 3.1 (implicitly assuming there that the C_i s were chosen so that the p_i s were surjective):

THEOREM 1.2. *If π is a Poisson bivector field, then $\mathcal{C}_\pi(M; C_0, C_1)$ is coisotropic in T^*PM iff $\text{Im } p_0$ and $\text{Im } p_1$ are coisotropic in M relative to C_0 and C_1 respectively.*

Recall that a submanifold C of a finite-dimensional manifold M endowed with a 2-tensor field π is called coisotropic if $\pi^\sharp(N^*C) \subset TC$, where N^*C denotes the conormal bundle to C :

$$N_x^*C := \{\alpha \in T_x^*M : \langle \alpha, v \rangle = 0 \ \forall v \in T_xC\}, \quad \forall x \in C.$$

If S is a subset of a submanifold C of a Poisson manifold M , we say that S is coisotropic in M relative to C if $\pi^\sharp(N_x^*C) \subset T_xC$ for all $x \in S$. We consider the empty set as a coisotropic submanifold of any symplectic or Poisson manifold.

Observe that having $\mathcal{C}_\pi(M; C_0, C_1)$ coisotropic may not be enough to conclude that π is Poisson as it is possible that the base paths contained in $\mathcal{C}_\pi(M; C_0, C_1)$ do not explore the whole of M .²

By definition we may also reformulate the if-part of Theorem 1.2 in the following (slightly weaker) form:

COROLLARY 1.3. *If π is Poisson and C_0 and C_1 are coisotropic, then $\mathcal{C}_\pi(M; C_0, C_1)$ is coisotropic.*

In this case, as recalled above, $T^\perp\mathcal{C}_\pi(M; C_0, C_1)$ is an integrable distribution on $\mathcal{C}_\pi(M; C_0, C_1)$, which we describe in details in Section 3.2.1.

²For example, assume that π has a zero at a point x . Then $\mathcal{C}_\pi(M; \{x\}, M)$ consists of pairs (X, η) where X is the constant path at x and there are no conditions on η . It is then clear that $\mathcal{C}_\pi(M; \{x\}, M)$ is coisotropic (actually Lagrangian) whatever the tensor π is.

In Section 4 we concentrate on the case when C_0 and C_1 are coisotropic submanifolds of a Poisson manifold M and discuss how the reduction of $\mathcal{C}_\pi(M; C_0, C_1)$ may be understood as a (singular) dual pair.

Finally in Section 5 we show that this reduction may also be recovered by Poisson reduction of the coisotropic submanifold of the symplectic groupoid of M determined by the intersection of the preimages of C_0 and C_1 under the source and target maps, respectively.

Remark 1.4. One may consider the weaker condition that $\mathcal{C}_\pi(M; C_0, C_1)$ be presymplectic (i.e., that the kernel of the restriction of the symplectic form is a subbundle of the tangent bundle of $\mathcal{C}_\pi(M; C_0, C_1)$). As shown in [1], a sufficient condition for this to happen is that C_0 and C_1 are pre-Poisson submanifolds of M (according to the definition in [8]). We will not elaborate on this in this paper.

1.2. THE CASE OF THE CIRCLE

In this paper we mainly work on the path space PM as it is interesting to have boundary components (to be associated to the submanifolds C_0 and C_1). A very similar, actually a bit easier, story works in the case of the loop space $LM := C^1(S^1, M)$. In this case, we can analogously define the weak symplectic Banach manifold T^*LM as the space of bundle maps $TS^1 \rightarrow T^*M$ with continuously differentiable base map and continuous fiber map. The weak symplectic form Ω is again the differential of the canonical 1-form Θ :

$$\Theta(X, \eta)(\widehat{\xi}) = \int_{S^1} \langle \eta, \xi \rangle,$$

where we use the same notations as above. To the tensor π we now associate the Banach submanifold

$$\mathcal{C}_\pi(M) := \{(X, \eta) \in T^*LM : dX + \pi^\sharp(X)\eta = 0\}.$$

A very similar proof to Theorem 1.1, see Section 3.3, yields the following

THEOREM 1.5. *$\mathcal{C}_\pi(M)$ is coisotropic in T^*LM iff π is a Poisson bivector field.*

In this case, as recalled above, $T^\perp\mathcal{C}_\pi(M)$ is an integrable distribution on $\mathcal{C}_\pi(M)$, which we describe in details in Section 3.3.1.

1.3. LAGRANGIAN FIELD THEORIES WITH BOUNDARY

In [4] and [5] the general notion of Lagrangian field theories on manifolds with boundary is studied. The symplectic manifolds T^*PM and T^*LM described above arise as spaces of boundary fields of a two-dimensional Lagrangian field theory

with $\mathcal{C}_\pi(M; C_0, C_1)$ and $\mathcal{C}_\pi(M)$ as its “spaces of Cauchy data.” The requirement of π being Poisson turns out to be equivalent to the requirement that the theory is “good” in the sense that the evolution relations determined by solutions to the Euler–Lagrange equations are (immersed) Lagrangian submanifolds. We discuss this in more details in Section 6.

2. Tangent and Orthogonal Bundles

By choosing a linear connection on M , which for simplicity we assume to be torsion free, we may identify the tangent bundle to T^*M with the vector bundle $E := T^*M \oplus TM \oplus T^*M$. Explicitly this is done as follows. First observe that both TT^*M and E can be regarded as vector bundles over T^*M with fiber at point (x, p) given by the vector space $T_xM \oplus T_x^*M$; the transition functions in the two vector bundles are however different. Choosing on M local coordinates $\{x^i\}_{i=1, \dots, m}$, $m = \dim M$, and the corresponding dual coordinates $\{p_i\}$ on $\widehat{T_x^*M}$, we consider the fiber isomorphism

$$\begin{aligned} \Phi_{(x,p)}: T_{(x,p)}T^*M = T_xM \oplus T_x^*M &\rightarrow E_{(x,p)} = T_xM \oplus T_x^*M \\ (\dot{x}^i, \dot{p}_i) &\mapsto (\dot{x}^i, \dot{p}_i - \Gamma_{si}^r(x) p_r \dot{x}^s) \end{aligned}$$

where the Γ s are the Christoffel symbols of the given connection, and we use Einstein’s convention that a sum over upper and lower repeated indices is understood. Then $\Phi: TT^*M \rightarrow E$ is a vector bundle isomorphism.

By this we may also identify TT^*PM with $T^*PM \oplus TPM \oplus T^*PM$, regarded as vector bundles over T^*PM . Recall that $T_X PM = \Gamma(X^*TM)$ and $T_X^* PM := \Gamma(T^*I \otimes X^*T^*M)$.³ To describe this isomorphism explicitly, we observe that, given a continuous path X , we may subdivide the interval I into finitely many subintervals I_α such that $X(I_\alpha)$ is contained in a coordinate patch $\forall \alpha$. For a given I_α , we denote by X^i and η_i , $i = 1, \dots, m := \dim M$, the components in local coordinates of the restrictions to I_α of X and η . The restriction to I_α of a tangent vector $\widehat{\xi}$ can then be split correspondingly into its components ξ^i in the X^i -direction and ζ_i in the η_i -direction. We then define

$$e_i = \zeta_i - \Gamma_{si}^r(X) \eta_r \xi^s. \tag{2.1}$$

The map $(X^i, \eta_i, \xi^i, \zeta_i) \mapsto (X^i, \eta_i, \xi^i, e_i)$ is well-defined globally and yields the required vector bundle isomorphism $TT^*PM \rightarrow T^*PM \oplus TPM \oplus T^*PM$. If we now impose boundary conditions, by the same map we may finally identify the fiber at $(X, \eta) \in T^*PM(C_0, C_1)$ with

$$\{\xi \oplus e \in \Gamma(X^*TM) \oplus \Gamma(T^*I \otimes X^*T^*M) : \xi(0) \in T_0, \xi(1) \in T_1\},$$

³To avoid cumbersome notations, from now on we will avoid indicating which maps or forms are continuous or differentiable.

with

$$T_0 := T_{X(0)}C_0 \quad \text{and} \quad T_1 := T_{X(1)}C_1.$$

The linear connection on M also induces a connection on the vector bundle X^*TM . We denote by $\partial: \Gamma(X^*TM) \rightarrow \Gamma(T^*I \otimes X^*TM)$ the corresponding covariant exterior derivative. In local coordinates, its action on a section σ of X^*TM is given by

$$(\partial\sigma)^i = d\sigma^i + \Gamma_{rs}^i dX^r \sigma^s. \tag{2.2}$$

It is convenient to modify this connection by using

$$A := \nabla\pi^\sharp(X)\eta \in \Gamma(T^*I \otimes X^*\text{End}(TM)), \tag{2.3}$$

where ∇ denotes the covariant derivative. In local coordinates we have

$$(\nabla\pi)_k^{ij} = \partial_k\pi^{ij} + \Gamma_{kr}^i\pi^{rj} + \Gamma_{kr}^j\pi^{ir} \tag{2.4}$$

and

$$A_k^i = \eta_j(\nabla\pi)_k^{ji}(X). \tag{2.5}$$

We will denote by D the covariant exterior derivative $\partial + A$.

2.1. THE TANGENT SPACES TO COMPATIBLE PATHS

Let (X, η) be a point in $C_\pi(M; C_0, C_1)$.

PROPOSITION 2.1. *After choosing a connection on M and using the above notations, we have*

$$T_{(X,\eta)}C_\pi(M; C_0, C_1) = \{\xi \in \Gamma(X^*TM), e \in \Gamma(T^*I \otimes X^*T^*M): \\ (D\xi) + \pi^\sharp(X)e = 0, \xi(0) \in T_0, \xi(1) \in T_1\}.$$

Proof. Let us restrict our attention to a subinterval I_α such that $X(I_\alpha)$ is contained in a coordinate patch. The restriction to I_α of the equation satisfied by X and η then reads

$$dX^i + \eta_j \pi^{ji}(X) = 0 \tag{2.6}$$

in local coordinates. Let ξ^i and ζ_i denote the local-coordinate expression of a tangent vector. They then satisfy the equation

$$d\xi^i + \eta_j \partial_l \pi^{ji}(X) \xi^l + \zeta_j \pi^{ji}(X) = 0,$$

or equivalently by (2.4)

$$(d\xi^i - \xi^l \Gamma_{lr}^i(X) \eta_j \pi^{jr}) + \eta_j (\nabla \pi)_l^{ji}(X) \xi^l + (\zeta_j - \Gamma_{lj}^s(X) \eta_s \xi^l) \pi^{ji}(X) = 0.$$

Since the connection is torsion free, by (2.6), (2.1), (2.2), (2.3) and (2.5) we conclude the proof. \square

We then consider the parallel transport $U \in \Gamma(\text{Iso}(X^*TM, T_{X(0)}M))$ of the connection D , viz., the solution to the Cauchy problem

$$\begin{cases} d \circ U = U \circ D, \\ U(0) = Id. \end{cases} \tag{2.7}$$

In local coordinates, we may also write

$$\begin{cases} dU_j^i = U_l^i (A_j^l + \Gamma_{sj}^l dX^s), \\ U(0)_j^i = \delta_j^i. \end{cases} \tag{2.8}$$

We may then simplify the equation satisfied by the tangent vector (ξ, e) into

$$d\lambda + P^\sharp \phi = 0, \tag{2.9}$$

with

$$\lambda := U\xi \in \Omega^0(I, T_{X(0)}M), \tag{2.10a}$$

$$\phi := (U^l)^{-1}e \in \Omega^1(I, T_{X(0)}^*M), \tag{2.10b}$$

$$P^\sharp := U\pi^\sharp U^t \in \Omega^0(I, \text{Hom}(T_{X(0)}^*M, T_{X(0)}M)). \tag{2.10c}$$

So we get

$$\begin{aligned} T_{(X,\eta)}\mathcal{C}_\pi(M; C_0, C_1) &\cong T_{(X,\eta)}\mathcal{C}_\pi(M; C_0, C_1)^{\text{twisted}} \\ &:= \{ \lambda \in \Omega^0(I, T_{X(0)}M), \phi \in \Omega^1(I, T_{X(0)}^*M) : d\lambda + P^\sharp \phi = 0, \\ &\quad \lambda(0) \in T_0, U(1)^{-1}\lambda(1) \in T_1 \}. \end{aligned} \tag{2.11}$$

Equation (2.9) may be easily solved for any ϕ just assigning the initial condition $\lambda(0) = \lambda_0$:

$$\lambda(u) = \lambda_0 - \int_0^u P^\sharp \phi. \tag{2.12}$$

Se we get the alternative description

$$\begin{aligned} T_{(X,\eta)}\mathcal{C}_\pi(M; C_0, C_1) &\cong T_{(X,\eta)}\mathcal{C}_\pi(M; C_0, C_1)_0 \\ &:= \left\{ \lambda_0 \in T_0, \phi \in \Omega^1(I, T_{X(0)}^*M) : U(1)^{-1} \left(\lambda_0 - \int_I P^\sharp \phi \right) \in T_1 \right\}. \end{aligned}$$

2.1.1. *Properties of P*

The tensor π has been replaced by P in (2.10c). Just by differentiating, it is not difficult to see that P^\sharp is the solution to the Cauchy problem

$$\begin{cases} dP^\sharp = UT^\sharp U^t, \\ P^\sharp(0) = \pi_0^\sharp, \end{cases}$$

with $\pi_0 := \pi(X(0))$ and $T^\sharp := D\pi^\sharp$. Using (2.6), (2.5) and (2.8), we obtain in local coordinates

$$T^{ls} = \eta_k (\pi^{rs}(X) (\nabla\pi)_r^{kl}(X) - \pi^{kr}(X) (\nabla\pi)_r^{ls}(X) + \pi^{lr}(X) (\nabla\pi)_r^{ks}(X)).$$

Recall that, in local coordinates, the vanishing of the Schouten–Nijenhuis bracket of a bivector field π may also be written, by using any connection, as

$$\pi^{sr} (\nabla\pi)_r^{lk} + \pi^{kr} (\nabla\pi)_r^{sl} + \pi^{lr} (\nabla\pi)_r^{ks} = 0.$$

This immediately implies the following

LEMMA 2.2. *If π is a Poisson bivector field, then $P = \pi_0$.*

Observe that P depends on the chosen (X, η) . We also have the following

LEMMA 2.3. *If P is skew-symmetric and constant for all $(X, \eta) \in \mathcal{C}_\pi(M; M, M)$, then π is a Poisson bivector field.*

Proof. If P is skew symmetric, then so is π_0 , that is, π at any possible starting point of a path X . Thus, π is a bivector field. Moreover, for any $x \in M$, we can choose a solution to (2.6) with X passing through x for some $u_0 \in I$ and, in a neighborhood of u_0 , $\eta_j = \mathbf{e}_j du$, with \mathbf{e}_j a basis element of $(\mathbb{R}^m)^*$. This implies that T vanishes at u_0 and hence that

$$\pi^{rs}(x) (\nabla\pi)_r^{kl}(x) - \pi^{kr}(x) (\nabla\pi)_r^{ls}(x) + \pi^{lr}(x) (\nabla\pi)_r^{ks}(x) = 0.$$

Since this holds for all $x \in M$, it follows that π is Poisson. □

2.2. THE SYMPLECTIC ORTHOGONAL SPACES TO COMPATIBLE PATHS

Assuming that the chosen connection is torsion-free, the symplectic form Ω evaluated at tangent vectors (ξ, ζ) and $(\tilde{\xi}, \tilde{\zeta})$ to T^*PM at a point (X, η) reads

$$\Omega_{(X,\eta)}((\xi, \zeta), (\tilde{\xi}, \tilde{\zeta})) = \int_I \langle e, \tilde{\xi} \rangle - \langle \tilde{e}, \xi \rangle,$$

where $(\xi \oplus e)$ and $(\tilde{\xi} \oplus \tilde{e})$ are the corresponding elements of $T_X PM \oplus T_X^* PM$. By the transformation (2.10a) and (2.10b), and the analogous ones $\tilde{\lambda} = U\xi$, $\tilde{\phi} = (U^t)^{-1}\tilde{e}$, we get

$$\Omega_{(X,\eta)}((\xi, \zeta), (\tilde{\xi}, \tilde{\zeta})) = \int_I \langle \phi, \tilde{\lambda} \rangle - \langle \tilde{\phi}, \lambda \rangle. \quad (2.13)$$

Assume now that (ξ, e) is tangent to $\mathcal{C}_\pi(M; C_0, C_1)$. Then, by (2.12),

$$\begin{aligned} \int_I \langle \tilde{\phi}, \lambda \rangle &= \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle - \int_I \left\langle \tilde{\phi}, \int_0^{\bullet} P^\sharp \phi \right\rangle \\ &= \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle - \int_I \left\langle \phi, (P^\sharp)^t \int_0^1 \tilde{\phi} \right\rangle. \end{aligned}$$

We thus obtain

$$\begin{aligned} T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1) &\cong T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{implicit}} \\ &:= \left\{ \tilde{\lambda} \in \Omega^0(I, T_{X(0)}M), \tilde{\phi} \in \Omega^1(I, T_{X(0)}^*M) : \right. \\ &\quad \left. \int_I \left\langle \phi, \tilde{\lambda} + (P^\sharp)^t \int_0^1 \tilde{\phi} \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0, \quad \forall (\lambda_0, \phi) \in T_{(X,\eta)} \mathcal{C}_\pi(M; C_0, C_1)_0 \right\}. \end{aligned}$$

3. Proofs to the Main Theorems

Using the results and notations of Section 2, we are now going to prove the main Theorems 1.1 and 1.2 and to draw further consequences.

3.1. PROOF OF THEOREM 1.1

In the case $C_0 = C_1 = M$, we have

$$T_{(X,\eta)} \mathcal{C}_\pi(M; M, M)_0 := \{ \lambda_0 \in T_{X(0)}M, \phi \in \Omega^1(I, T_{X(0)}^*M) \}.$$

Thus, $(\tilde{\lambda}, \tilde{\phi})$ belongs to $T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; M, M)^{\text{implicit}}$ iff

$$\int_I \left\langle \phi, \tilde{\lambda} + (P^\sharp)^t \int_0^1 \tilde{\phi} \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0$$

for all $\lambda_0 \in T_{X(0)}M$ and $\phi \in \Omega^1(I, T_{X(0)}^*M)$. This implies that $(\tilde{\lambda}, \tilde{\phi})$ belongs to $T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; M, M)^{\text{implicit}}$ iff

$$\tilde{\lambda}(u) + (P^\sharp)^t \int_u^1 \tilde{\phi} = 0 \tag{3.1}$$

and

$$\int_I \tilde{\phi} = 0. \tag{3.2}$$

Now we have

PROPOSITION 3.1. *If π is a Poisson bivector field, then $\mathcal{C}_\pi(M; M, M)$ is coisotropic.*

Proof. By Lemma 2.2 we have $P = \pi_0$. So (3.1) implies that $(\tilde{\lambda}, \tilde{\phi})$ belongs to $T_{(X,\eta)}\mathcal{C}_\pi(M; M, M)^{\text{twisted}}$. Thus, $\mathcal{C}_\pi(M; M, M)$ is coisotropic. \square

PROPOSITION 3.2. *If $\mathcal{C}_\pi(M; M, M)$ is coisotropic, then π is a Poisson bivector field.*

Proof. Since $T_{(X,\eta)}^\perp\mathcal{C}_\pi(M; M, M) \subset T_{(X,\eta)}\mathcal{C}_\pi(M; M, M)$, any pair $(\tilde{\lambda}, \tilde{\phi})$ satisfying (3.1) and (3.2) also belongs to $T_{(X,\eta)}\mathcal{C}_\pi(M; M, M)^{\text{twisted}}$; i.e., it satisfies

$$d\tilde{\lambda} + P^\sharp\tilde{\phi} = 0.$$

On the other hand, differentiating (3.1) yields

$$d\tilde{\lambda} + d(P^\sharp)^t \int_u^1 \tilde{\phi} - (P^\sharp)^t \tilde{\phi} = 0.$$

So we get,

$$d(P^\sharp)^t \int_u^1 \tilde{\phi} - ((P^\sharp)^t + P^\sharp)\tilde{\phi} = 0 \tag{3.3}$$

for any $\tilde{\phi}$ satisfying (3.2). Now let $u_0 < u_1$ be points on I . Let U_0 and U_1 be disjoint neighborhoods of u_0 and u_1 with $U_0 < U_1$. We then choose $\tilde{\phi}$ to vanish outside $U_0 \cup U_1$. For $U_0 < u < U_1$, (3.3) yields

$$d(P^\sharp)^t(u)\sigma = 0$$

with $\sigma := \int_u^1 \tilde{\phi}$. Since this holds for all $\sigma \in T_{X(0)}^*M$, we see that P must be constant. So now (3.3) reads

$$((P^\sharp)^t + P^\sharp)\tilde{\phi} = 0.$$

Again this must hold for all $\tilde{\phi}$ satisfying (3.2). From this we conclude that P must be skew-symmetric. Since these conclusions must hold for any solution (X, η) , Lemma 2.3 completes the proof. \square

This concludes the proof of Theorem 1.1.

3.2. PROOF OF THEOREM 1.2

Assuming that π is a Poisson bivector field, we know by Lemma 2.2 that P is constant and equal to $\pi_0 = \pi(X(0))$. So $(\tilde{\lambda}, \tilde{\phi})$ belongs to $T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{implicit}}$ iff

$$\int_I \left\langle \phi, \tilde{\lambda} - \pi_0^\sharp \int_I \tilde{\phi} \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0 \tag{3.4}$$

for all $\lambda_0 \in T_0$ and $\phi \in \Omega^1(I, T_{X(0)}^* M)$ such that

$$U(1)^{-1} \left(\lambda_0 - \pi_0^\sharp \int_I \phi \right) \in T_1. \tag{3.5}$$

PROPOSITION 3.3. *Let $N_i^* = N_{X(i)}^* C_i$, $i = 1, 2$. If π is Poisson, then*

$$\begin{aligned} T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{implicit}} &= T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{explicit}} \\ &:= \{ \tilde{\lambda} \in \Omega^0(I, T_{X(0)} M), \tilde{\phi} \in \Omega^1(I, T_{X(0)}^* M) : d\tilde{\lambda} + \pi_0^\sharp \tilde{\phi} = 0, \\ &\quad \tilde{\lambda}(0) \in \pi_0^\sharp(N_0^*), U(1)^{-1} \tilde{\lambda}(1) \in \pi_1^\sharp(N_1^*) \}. \end{aligned}$$

Proof. We may first consider $\lambda_0 = 0$ and ϕ such that $\int_I \phi = 0$. Since (3.4) must hold in particular for all (λ_0, ϕ) of this kind, we obtain that there must be a constant $\tilde{\lambda}_1 \in T_{X(0)} M$ such that

$$\tilde{\lambda}(u) - \pi_0^\sharp \int_u^1 \tilde{\phi} = \tilde{\lambda}_1, \quad \forall u \in I. \tag{3.6}$$

So (3.4) simplifies to

$$\left\langle \int_I \phi, \tilde{\lambda}_1 \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0. \tag{3.7}$$

Observe now that $\langle \int_I \phi, \tilde{\lambda}_1 \rangle = \langle U(1)^t \int_I \phi, U(1)^{-1} \tilde{\lambda}_1 \rangle$. Set $\pi_1 = \pi(X(1))$. Since

$$U(1) \pi_1^\sharp U(1)^t = P(1)^\sharp = \pi_0^\sharp, \tag{3.8}$$

we have that $U(1)^{-1}\pi_0^\sharp \int_I \phi = \pi_1^\sharp U(1)^t \int_I \phi$. By choosing again $\lambda_0 = 0$, we get the condition

$$\left\langle U(1)^t \int_I \phi, U(1)^{-1}\tilde{\lambda}_1 \right\rangle = 0,$$

$$\forall \phi \in \Omega^1(I, T_{X(0)}^*M) \text{ such that } \pi_1^\sharp U(1)^t \int_I \phi \in T_1.$$

Thus,

$$\left\langle \alpha, U(1)^{-1}\tilde{\lambda}_1 \right\rangle = 0, \quad \forall \alpha \in T_{X(1)}^*M \text{ such that } \pi_1^\sharp \alpha \in T_1. \tag{3.9}$$

We use now the following simple fact from linear algebra:

LEMMA 3.4. *Let V and W be vector spaces. Let F be a linear map $V \rightarrow W$ and T a linear subspace of W . Then*

$$\text{Ann}(F^{-1}(T)) = F^t(\text{Ann}(T)),$$

where Ann denotes the annihilator of a subspace (e.g., $\text{Ann}(T) = \{\tau \in W^* : \tau(t) = 0 \ \forall t \in T\}$).

Proof. It is obvious that $F^t(\text{Ann}(T)) \subset \text{Ann}(F^{-1}(T))$. We now prove the other inclusion. Let V' be a complement of $F^{-1}(T)$ in V and W' a complement of $T \oplus F(V')$ in W . Since the restriction $F|_{V'}$ of F to V' establishes an isomorphism between V' and $F(V')$, for any $\psi \in \text{Ann}(F^{-1}(T)) \subset V^*$ there is a unique $\phi \in F(V')^*$ with $\psi = F|_{V'}^t(\phi)$. Now let $\varphi \in W^*$ be equal to ϕ when evaluated on elements of $F(V')$ and zero when evaluated on elements of T or W' . So $\varphi \in \text{Ann}(T)$. Since $\psi = F^t(\varphi)$, this concludes the proof. \square

We apply the Lemma to (3.9) with $V = T_{X(1)}^*M$, $W = T_{X(1)}M$, $T = T_1$ and $F = \pi_1^\sharp$. Since π_1 is skew-symmetric, $F^t = -\pi_1^\sharp$. So we get that necessarily

$$U(1)^{-1}\tilde{\lambda}_1 \in \pi_1^\sharp N_1^*,$$

where $N_1^* = N_{X(1)}^*C$ is the annihilator of T_1 . So there exists $\theta \in N_1^*$ such that $\tilde{\lambda}_1 = U(1)\pi_1^\sharp \theta$, and we may rewrite (3.7) as

$$\left\langle \int_I \phi, U(1)\pi_1^\sharp \theta \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0,$$

or equivalently, using again (3.8) and the skew-symmetry of π ,

$$\left\langle \theta, U(1)^{-1}\pi_0^\sharp \int_I \phi \right\rangle + \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0.$$

This equation has to be satisfied for all $\lambda_0 \in T_0$ and $\phi \in \Omega^1(I, T_{X(0)}^*M)$ satisfying (3.5). This is equivalent to imposing

$$\left\langle (U(1)^t)^{-1}\theta + \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0 \quad (3.10)$$

for all $\lambda_0 \in T_0$. That is,

$$(U(1)^t)^{-1}\theta + \int_I \tilde{\phi} \in N_0^*, \quad (3.11)$$

where $N_0^* = N_{X(0)}^*C_0$ is the annihilator of T_0 . Recalling (3.6),

$$\tilde{\lambda}(u) - \pi_0^\sharp \int_u^1 \tilde{\phi} = \tilde{\lambda}_1 = U(1)\pi_1^\sharp\theta = \pi_0^\sharp(U^t)^{-1}\theta, \quad \forall u \in I, \quad (3.12)$$

we see that a pair $(\tilde{\lambda}, \tilde{\phi})$ belongs to $T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{implicit}}$ iff there exists $\theta \in N_1^*$ such that (3.11) and

$$\tilde{\lambda}(u) = \pi_0^\sharp \left((U^t)^{-1}\theta + \int_u^1 \tilde{\phi} \right), \quad \forall u \in I, \quad (3.13)$$

are satisfied. By differentiating, in order to get rid of θ , we finally obtain

$$T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{implicit}} \subset T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{explicit}}.$$

To prove the other inclusion, consider a pair $(\tilde{\lambda}, \tilde{\phi}) \in T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{explicit}}$. Since $U(1)^{-1}\tilde{\lambda}(1) \in \pi_1^\sharp(N_1^*)$, there exists $\theta \in N_1^*$ such that $\tilde{\lambda}_1 = U(1)\pi_1^\sharp\theta$. Then the solution to the equation has the form in (3.13) and satisfies (3.11). \square

PROPOSITION 3.5. *Assume π to be Poisson. If C_0 and C_1 are coisotropic, then so is $\mathcal{C}_\pi(M; C_0, C_1)$.*

Proof. In this case, by (2.11), we immediately have

$$\begin{aligned} T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{explicit}} &\subset T_{(X,\eta)} \mathcal{C}_\pi(M; C_0, C_1)^{\text{twisted}} \\ &= \{\lambda \in \Omega^0(I, T_{X(0)}M), \phi \in \Omega^1(I, T_{X(0)}^*M) : d\lambda + \pi_0^\sharp\phi = 0, \\ &\quad \lambda(0) \in T_0, U(1)^{-1}\lambda(1) \in T_1\}, \end{aligned}$$

for all $(X, \eta) \in \mathcal{C}_\pi(M; C_0, C_1)$. So $\mathcal{C}_\pi(M; C_0, C_1)$ is coisotropic. \square

PROPOSITION 3.6. *Assume π to be Poisson. If $\mathcal{C}_\pi(M; C_0, C_1)$ is coisotropic, then so are $\text{Im } p_0$ and $\text{Im } p_1$.*

Proof. If $\mathcal{C}_\pi(M; C_0, C_1)$ is coisotropic, then any pair $(\tilde{\lambda}, \tilde{\phi})$ that belongs to $T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{explicit}}$ must also belong to $T_{(X,\eta)} \mathcal{C}_\pi(M; C_0, C_1)^{\text{twisted}} \forall (X, \eta) \in \mathcal{C}_\pi(M; C_0, C_1)$. Thus, in particular, we must have $\tilde{\lambda}(0) \in T_0$ and $U(1)^{-1} \tilde{\lambda}(1) \in T_1$.

We may arbitrarily choose the end condition $\tilde{\lambda}(1)$ such that $U(1)^{-1} \tilde{\lambda}(1) \in \pi_1^\sharp(N_1^*)$ since condition (3.11) will always be satisfied by an appropriate choice of $\tilde{\phi}$ (e.g., such that $\int_I \tilde{\phi} = -(U(1)^t)^{-1} \theta$). So we see that $\pi_1^\sharp(N_1^*) \subset T_1$. Since this must hold for all $(X, \eta) \in \mathcal{C}_\pi(M; C_0, C_1)$, we obtain that $N_x^* C_1 \subset T_x C_1 \forall x \in \text{Im } p_1$.

Similarly, we may arbitrarily choose $\tilde{\lambda}(0) \in \pi_0^\sharp(N_0^*)$ since the condition $U(1)^{-1} \tilde{\lambda}(1) \in \pi_1^\sharp(N_1^*)$ will always be satisfied by an appropriate choice of $\tilde{\phi}$ (e.g., such that $\int_I \tilde{\phi} = \tau$ if $\tilde{\lambda}(0) = \pi_0^\sharp \tau$). Then we see that $\pi_0^\sharp(N_0^*) \subset T_0$ and, since this must hold for all $(X, \eta) \in \mathcal{C}_\pi(M; C_0, C_1)$, we obtain that $N_x^* C_0 \subset T_x C_0 \forall x \in \text{Im } p_0$. \square

This concludes the proof of Theorem 1.2.

3.2.1. Symmetries

From now on we assume that π is Poisson and that C_0 and C_1 are coisotropic. By defining

$$b := (U(1)^t)^{-1} \theta + \int_{\bullet}^1 \tilde{\phi} \in \Omega^0(I, T_{X(0)}^* M),$$

we finally obtain

$$\begin{aligned} T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1) &= T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)^{\text{explicit}} \\ &:= \{ \tilde{\lambda} \in \Omega^0(I, T_{X(0)} M), \tilde{\phi} \in \Omega^1(I, T_{X(0)}^* M) : \\ &\quad \exists b \in \Omega^0(I, T_{X(0)}^* M), b(0) \in N_0^*, U(1)^t b(1) \in N_1^*, \tilde{\lambda} = \pi_0^\sharp b, \tilde{\phi} = -db \}. \end{aligned}$$

Observe that from this description of $T_{(X,\eta)}^\perp \mathcal{C}_\pi(M; C_0, C_1)$ it follows immediately that

$$T_{(X,\eta)}^{\perp\perp} \mathcal{C}_\pi(M; C_0, C_1) = T_{(X,\eta)} \mathcal{C}_\pi(M; C_0, C_1). \tag{3.14}$$

If we now invert the transformations (2.10a) and (2.10b) to go back to tangent vectors at (X, η) ,

$$\begin{aligned} \tilde{\xi} &= U^{-1} \tilde{\lambda} \in \Gamma(X^* T M), \\ \tilde{e} &= U^t \tilde{\phi} \in \Gamma(T^* I \otimes X^* T^* M), \end{aligned}$$

and introduce

$$\beta = U^t b \in \Gamma(X^* T^* M),$$

we obtain that the characteristic distribution of the coisotropic submanifold $\mathcal{C}_\pi(M; C_0, C_1)$ is given, at the point (X, η) , by the family of vectors $(\tilde{\xi}, \tilde{e}) \in T_{(X,\eta)} \mathcal{C}_\pi(M; C_0, C_1)$ defined by

$$\tilde{\xi} = \pi^\sharp(X)\beta, \quad (3.15a)$$

$$\tilde{e} = -D\beta, \quad (3.15b)$$

for $\beta \in \Gamma(X^*T^*M)$ with $\beta(0) \in N_{X(0)}^*C_0$ and $\beta(1) \in N_{X(1)}^*C_1$. In local coordinates the above formulae read,

$$\tilde{\xi}^i = -\pi^{ij}(X)\beta_j,$$

$$\tilde{e}_i = -d\beta_i + \Gamma_{ri}^k(X) dX^r \beta_k + (\nabla\pi)_i^{jk}(X) \eta_j \beta_k.$$

These are the symmetries of the Poisson sigma model as presented in [6,11,13]. The boundary conditions for β in case of coisotropic boundary conditions has been introduced in [7].

3.2.2. “Equivariant Momentum Map”

Denote by ι_C the inclusion map of a submanifold C into a manifold M and define

$$\Omega_C^1(M) = \{\alpha \in \Omega^1(M) : \iota_C^* \alpha = 0\}.$$

LEMMA 3.7. *If M is a Poisson manifold and C is a coisotropic submanifold, then $\Omega_C^1(M)$ is a Lie subalgebra of $\Omega^1(M)$.*

We leave the proof of this simple fact (which directly follows from N^*C being a Lie subalgebroid of T^*M) to the reader.

Now, given two coisotropic submanifolds C_0 and C_1 , we define the Lie algebra

$$P_{C_0, C_1} \Omega^1(M) = \{B : I \rightarrow \Omega^1(M) : B(i) \in \Omega_{C_i}^1(M), i = 0, 1\},$$

where the Lie bracket is defined pointwise. The map B is assumed to be continuously differentiable. Given a path X on M and an element B of this Lie algebra, we define $B_X \in \Gamma(X^*T^*M)$ by

$$B_X(u) = B(u)(X(u)).$$

If $X(i) \in C_i$, $i = 0, 1$, then $B_X(i) \in N_{X(i)}^*M$. It is not difficult to check that, replacing β by B_X in (3.15), one may define an infinitesimal action of $P_{C_0, C_1} \Omega^1(M)$ on $C_\pi(M; C_0, C_1)$ whose induced foliation is the canonical foliation. On $T^*PM(C_0, C_1)$ one may also define Hamiltonian functions for this action; viz.,

$$\mu_B(X, \eta) := \int_I \langle B_X, dX + \pi^\sharp(X)\eta \rangle. \quad (3.16)$$

3.3. PROOF OF THEOREM 1.5

We proceed as in Section 2.1. Proposition 2.1 still holds, so we have

$$T_{(X,\eta)}\mathcal{C}_\pi(M) = \{\xi \in \Gamma(X^*TM), e \in \Gamma(T^*S^1 \otimes X^*T^*M) : (D\xi) + \pi^\sharp(X)e = 0\}.$$

We now regard S^1 as the interval $I = [0, 1]$ with identified end points. The fields are then regarded as periodic sections on it. We then continue likewise up to (2.10) getting

$$\begin{aligned} T_{(X,\eta)}\mathcal{C}_\pi(M) &\cong T_{(X,\eta)}\mathcal{C}_\pi(M)^{\text{twisted}} \\ &:= \{\lambda \in \Omega^0(I, T_{X(0)}M), \phi \in \Omega^1(I, T_{X(0)}^*M) : d\lambda + P^\sharp\phi = 0, \\ &\quad \lambda(1) = U(1)\lambda(0), \phi(0) = U(1)^t\phi(1)\} \end{aligned}$$

and

$$\begin{aligned} T_{(X,\eta)}\mathcal{C}_\pi(M) \cong T_{(X,\eta)}\mathcal{C}_\pi(M)_0 := &\left\{ \lambda_0 \in T_{X(0)}M, \phi \in \Omega^1(I, T_{X(0)}^*M) : \right. \\ &\left. U(1)\lambda_0 = \lambda_0 - \int_I P^\sharp\phi, \phi(0) = U(1)^t\phi(1) \right\}. \end{aligned}$$

We then proceed as in Section 2.2 getting

$$\begin{aligned} T_{(X,\eta)}^\perp\mathcal{C}_\pi(M) &\cong T_{(X,\eta)}^\perp\mathcal{C}_\pi(M)^{\text{implicit}} \\ &:= \left\{ \tilde{\lambda} \in \Omega^0(I, T_{X(0)}M), \tilde{\phi} \in \Omega^1(I, T_{X(0)}^*M) : \tilde{\lambda}(1) = U(1)\tilde{\lambda}(0), \tilde{\phi}(0) = U(1)^t\tilde{\phi}(1), \right. \\ &\quad \left. \int_I \left\langle \phi, \tilde{\lambda} + (P^\sharp)^t \int_{\bullet}^1 \tilde{\phi} \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0, \forall (\lambda_0, \phi) \in T_{(X,\eta)}\mathcal{C}_\pi(M)_0 \right\}. \end{aligned}$$

PROPOSITION 3.8. *If π is a Poisson bivector field, then $\mathcal{C}_\pi(M)$ is coisotropic.*

Proof. By Lemma 2.2 we have $P = \pi_0$, so the condition on λ_0 becomes $U(1)\lambda_0 = \lambda_0 - \pi_0^\sharp \int_I \phi$, which is in particular satisfied by $\lambda_0 = 0$ assuming $\int_I \phi = 0$. In particular, we get

$$\int_I \left\langle \phi, \tilde{\lambda} - \pi_0^\sharp \int_{\bullet}^1 \tilde{\phi} \right\rangle = 0,$$

for all $\phi \in \Omega^1(I, T_{X(0)}^*M)$ satisfying $\phi(0) = U(1)^t\phi(1)$ and $\int_I \phi = 0$. This implies $d(\tilde{\lambda} - \pi_0^\sharp \int_{\bullet}^1 \tilde{\phi}) = 0$, so $(\tilde{\lambda}, \tilde{\phi}) \in T_{(X,\eta)}\mathcal{C}_\pi(M)_0$. □

PROPOSITION 3.9. *If $\mathcal{C}_\pi(M)$ is coisotropic, then π is a Poisson bivector field.*

Proof. Notice that any $(\tilde{\lambda}, \tilde{\phi})$ with $\tilde{\phi}(0) = U(1)^t\tilde{\phi}(1)$ satisfying (3.1) and (3.2) belongs to $T_{(X,\eta)}^\perp\mathcal{C}_\pi(M)^{\text{implicit}}$ and hence, since $\mathcal{C}_\pi(M)$ is coisotropic, to $T_{(X,\eta)}\mathcal{C}_\pi(M)^{\text{twisted}}$. The proof then proceeds exactly as in the proof to Proposition 3.2. □

This concludes the proof of Theorem 1.5.

3.3.1. *Symmetries*

Assume π is Poisson. In the proof to Proposition 3.8 we have only considered $\lambda_0 = 0$ and $\int_I \phi = 0$, which yields only some necessary condition to be satisfied by $(\tilde{\lambda}, \tilde{\phi})$. We now want to characterize $T^\perp \mathcal{C}_\pi(M)$ completely.

PROPOSITION 3.10. *The characteristic distribution of $\mathcal{C}_\pi(M)$ at (X, η) is given by the family of vectors $(\tilde{\xi}, \tilde{e}) \in T_{(X, \eta)} \mathcal{C}_\pi(M)$ defined by*

$$\tilde{\xi} = \pi^\sharp(X)\beta, \quad (3.17a)$$

$$\tilde{e} = -D\beta, \quad (3.17b)$$

with $\beta \in \Gamma(X^*T^*M)$.

Proof. We have already obtained that if $(\tilde{\lambda}, \tilde{\phi})$ belongs to $T^\perp \mathcal{C}_\pi(M)$ implicit then, in addition to $\tilde{\lambda}(1) = U(1)\tilde{\lambda}(0)$ and $\tilde{\phi}(0) = U(1)^t \tilde{\phi}(1)$, it satisfies $d(\tilde{\lambda} - \pi_0^\sharp \int_I \tilde{\phi}) = 0$. This implies that $(\tilde{\lambda}, \tilde{\phi})$ belongs to $T^\perp \mathcal{C}_\pi(M)$ implicit iff in addition

$$\left\langle \int_I \phi, \tilde{\lambda}(1) \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = 0 \quad (3.18)$$

for all $(\lambda_0, \phi) \in T_{(X, \eta)} \mathcal{C}_\pi(M)_0$.

First consider $\lambda_0 = 0$ and $\int_I \phi \in \ker \pi_0^\sharp$. This yields, by Lemma 3.4, that there is a $\theta \in T_{X(0)}^* M$ with $\tilde{\lambda}(1) = \pi_0^\sharp(\theta)$; hence

$$\tilde{\lambda}(u) = \pi_0^\sharp \left(\theta + \int_u^1 \tilde{\phi} \right). \quad (3.19)$$

As a consequence $\langle \int_I \phi, \tilde{\lambda}(1) \rangle = 0$, so we are left with the condition $\langle \int_I \tilde{\phi}, \lambda_0 \rangle = 0$ for all λ_0 such that there is a ϕ with $(\lambda_0, \phi) \in T_{(X, \eta)} \mathcal{C}_\pi(M)_0$. In particular, we may take ϕ such that $\int_I \phi$ is in the kernel of π_0^\sharp ; since $\pi_0^\sharp \int_I \phi = \lambda_0 - U(1)\lambda_0$, this yields that λ_0 must now lie in the kernel of the operator $G := U(1) - \text{id}$. Since $\langle \int_I \tilde{\phi}, \lambda_0 \rangle$ must vanish for all λ_0 satisfying this condition, we get that $\int_I \tilde{\phi}$ must be in the image of G^t . Hence there is a $\gamma \in T_{X(0)}^* M$ with

$$\int_I \tilde{\phi} = U(1)^t \gamma - \gamma. \quad (3.20)$$

Using again $\pi_0^\sharp \int_I \phi = \lambda_0 - U(1)\lambda_0$, we now get $\langle \int_I \tilde{\phi}, \lambda_0 \rangle = \langle \int_I \phi, \pi_0^\sharp \gamma \rangle$; so condition (3.18) finally reads $\langle \int_I \phi, \tilde{\lambda}(1) - \pi_0^\sharp \gamma \rangle = 0$ for all ϕ such that there is a λ_0 with $(\lambda_0, \phi) \in T_{(X, \eta)} \mathcal{C}_\pi(M)_0$. Using (3.19), the condition becomes $\langle \theta - \gamma, \pi_0^\sharp \int_I \phi \rangle = 0$ or, using $\pi_0^\sharp \int_I \phi = -G\lambda_0$, $\langle \theta - \gamma, G\lambda_0 \rangle = 0$. Hence we have that $\theta - \gamma$ must be in the annihilator of $\text{im } \pi_0^\sharp \cap \text{im } G$ which is $\ker \pi_0^\sharp + \ker G^t$. We hence have $\mu, \nu \in T_{X(0)}^* M$ with $\pi_0^\sharp \mu = 0$ and $U(1)^t \nu = \nu$ such that $\theta - \gamma = \mu + \nu$. Finally, define

$$b(u) := \theta - \mu + \int_u^1 \tilde{\phi} = \nu + \gamma + \int_u^1 \tilde{\phi}.$$

Since $U(1)^t \nu = \nu$, we have thanks to (3.20) that $b(0) = U(1)^t b(1)$. This shows that

$$\beta := U^t b$$

is a periodic section of X^*T^*M . Since $\pi_0^\# \mu = 0$, we have from (3.19) that $\tilde{\lambda} = \pi_0^\# b$. Moreover, since θ and μ are constant, we have $\tilde{\phi} = -db$.

If we now invert the transformations (2.10a) and (2.10b) to go back to tangent vectors at (X, η) ,

$$\begin{aligned} \tilde{\xi} &= U^{-1} \tilde{\lambda} \in \Gamma(X^*TM), \\ \tilde{e} &= U^t \tilde{\phi} \in \Gamma(T^*S^1 \otimes X^*T^*M), \end{aligned}$$

we obtain that a pair $(\tilde{\xi}, \tilde{e})$ belongs to $T^\perp \mathcal{C}_\pi(M)$ iff equations (3.17a) and (3.17b) are satisfied. □

Remark 3.11. One may define functions μ_B on T^*LM as in (3.16) with B now a map from S^1 to $\Omega^1(M)$. Notice that the functions μ_B generate the vanishing ideal of $\mathcal{C}_\pi(M)$ and that their Hamiltonian vector fields generate the distribution defined by (3.17a) and (3.17b). This remark however does not replace the proof above as in the infinite dimensional case it is not automatic that the Hamiltonian vector fields of functions in the vanishing ideal span the whole characteristic distribution.

4. Dual Pairs

In this section we assume that C_0 and C_1 are coisotropic submanifolds of a Poisson manifold M . In this case $\mathcal{C}_\pi(M; C_0, C_1)$ is a coisotropic submanifold of T^*PM and its leaf space $\underline{\mathcal{C}}_\pi(M; C_0, C_1)$ is endowed with a symplectic structure. On the other hand, the leaf spaces \underline{C}_0 and \underline{C}_1 are endowed with a Poisson structure. By (3.15a) and the conditions on β , the maps p_i of (1.1) descend to the quotients

$$p_i : \underline{\mathcal{C}}_\pi(M; C_0, C_1) \rightarrow \underline{C}_i, \quad i = 0, 1.$$

Proceeding as in the proof of Theorem 4.6 in [6], one may prove that \underline{p}_0 and \underline{p}_1 are a Poisson and an anti-Poisson map respectively. We will prove the following

LEMMA 4.1. *ker d \underline{p}_0 and ker d \underline{p}_1 are symplectically orthogonal at any point of $\underline{\mathcal{C}}_\pi(M; C_0, C_1)$, i.e.,*

$$(\ker d\underline{p}_0(x))^\perp = \ker d\underline{p}_1(x), \quad \forall x \in \underline{\mathcal{C}}_\pi(M; C_0, C_1).$$

In other words, $\underline{C}_0 \xleftarrow{\underline{p}_0} \underline{\mathcal{C}}_\pi(M; C_0, C_1) \xrightarrow{\underline{p}_1} \underline{C}_1$ is a Lie–Weinstein dual pair [14]. Observe that the maps \underline{p}_i may fail to be surjective submersions, so this dual pair is in general not full.

Remark 4.2. Since the quotient $\underline{\mathcal{C}_\pi(M; C_0, C_1)}$ is finite dimensional, the above condition is equivalent to

$$(\ker d\underline{p}_1(x))^\perp = \ker d\underline{p}_0(x), \quad \forall x \in \underline{\mathcal{C}_\pi(M; C_0, C_1)}.$$

Notice that the maps \underline{p}_i s are defined, as continuous maps, even if the leaf spaces are not smooth. Lemma 4.1 makes sense also in the nonsmooth case if we define $T_{[x]}\underline{\mathcal{C}_\pi(M; C_0, C_1)}$ as $T_x\underline{\mathcal{C}_\pi(M; C_0, C_1)}$, $x \in [x] \in \underline{\mathcal{C}_\pi(M; C_0, C_1)}$, and $T_{[x]}\underline{C}_i$ as $T_x\underline{C}_i$, $x \in [x] \in \underline{C}_i$. The linear maps $\ker d\underline{p}_0$ and $\ker d\underline{p}_1$ are also well defined. Thus, we may think of $\underline{C}_0 \xleftarrow{\underline{p}_0} \underline{\mathcal{C}_\pi(M; C_0, C_1)} \xrightarrow{\underline{p}_1} \underline{C}_1$ as of a singular Lie–Weinstein dual pair.

Proof of Lemma 4.1. Let (X, η) be a representative of $x \in \underline{\mathcal{C}_\pi(M; C_0, C_1)}$. We introduce the following notations:

$$\begin{aligned} V &:= T_{(X, \eta)}\underline{\mathcal{C}_\pi(M; C_0, C_1)}, \\ Z_i &:= T_{X(i)}\underline{C}_i, \\ Z_i^\perp &:= \pi^\sharp(X(i))(N_{X(i)}^*\underline{C}_i), \\ \varpi_i &:= dp_i(X, \eta), \end{aligned}$$

for $i=0, 1$. So we have the following commutative diagram of vector spaces:

$$\begin{array}{ccc} V & \xrightarrow{\varpi_i} & Z_i \\ \downarrow & & \downarrow \\ V/V^\perp & \xrightarrow{d\underline{p}_i(x)} & Z_i/Z_i^\perp \end{array}$$

We then have

$$\begin{aligned} \ker d\underline{p}_i(x) &= \varpi_i^{-1}(Z_i^\perp)/V^\perp, \\ (\ker d\underline{p}_i(x))^\perp &= (\varpi_i^{-1}(Z_i^\perp))^\perp/V^\perp, \end{aligned}$$

and

$$\begin{aligned} \varpi_0^{-1}(Z_0^\perp) &= \{\lambda \in \Omega^0(I, T_{X(0)}M), \phi \in \Omega^1(I, T_{X(0)}^*M) : d\lambda + \pi_0^\sharp\phi = 0, \\ &\quad \lambda(0) \in \pi_0^*(N_0^*), U(1)^{-1}\lambda(1) \in T_1\}, \\ \varpi_1^{-1}(Z_1^\perp) &= \{\lambda \in \Omega^0(I, T_{X(0)}M), \phi \in \Omega^1(I, T_{X(0)}^*M) : d\lambda + \pi_0^\sharp\phi = 0, \\ &\quad \lambda(0) \in T_0, U(1)^{-1}\lambda(1) \in \pi_1^*(N_1^*)\}. \end{aligned}$$

Step 1: $(\ker d\underline{p}_1(x)) \subset \ker d\underline{p}_0(x)^\perp$.

It is enough to show that

$$\Omega_{(X, \eta)}((\xi, \zeta), (\tilde{\xi}, \tilde{\zeta})) = 0, \quad \forall (\xi, \zeta) \in \varpi_0^{-1}(Z_0^\perp), (\tilde{\xi}, \tilde{\zeta}) \in \varpi_1^{-1}(Z_1^\perp),$$

since this implies $\varpi_1^{-1}(Z_1^\perp)^\perp \subset \varpi_0^{-1}(Z_0^\perp)^\perp$ which in turn implies the desired result on the quotient. Using (2.13) with

$$\begin{aligned} \lambda &= \lambda_0 - \pi_0^\sharp \int_0^\bullet \phi, \\ \tilde{\lambda} &= \tilde{\lambda}_1 + \pi_0^\sharp \int_0^1 \tilde{\phi}, \end{aligned}$$

we get

$$\Omega_{(x,\eta)}((\xi, \zeta), (\tilde{\xi}, \tilde{\zeta})) = \left\langle \int_I \phi, \tilde{\lambda}_1 \right\rangle - \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle.$$

If we now write

$$\begin{aligned} \lambda_0 &= \pi_0^\sharp \alpha, \\ \tilde{\lambda}_1 &= U(1)\pi_1^\sharp \beta = \pi_0^\sharp (U^t)^{-1} \beta, \end{aligned}$$

with $\alpha \in N_0^*$ and $\beta \in N_1^*$, we get

$$\left\langle \int_I \phi, \tilde{\lambda}_1 \right\rangle = \left\langle \int_I \tilde{\phi}, \lambda_0 \right\rangle = - \left\langle U(1)^{-1} \pi_0^\sharp \alpha, \beta \right\rangle,$$

which completes Step 1.

Step 2: $(\ker d\underline{p}_0(x))^\perp \subset \ker d\underline{p}_1(x)$.

Let $(\tilde{\lambda}, \tilde{\phi})$ be an element of $(\varpi_0^{-1}(Z_0^\perp))^\perp$. Then proceeding exactly as in the proof of Proposition 3.3, we see that this element must satisfy (3.6) with $\tilde{\lambda}_1 = U(1)\pi_1^\sharp \theta$ for some $\theta \in N_1^*$. We still have condition (3.10) but now for all $\lambda_0 \in \pi_0^\sharp(N_0^*)$. This implies that

$$(U^t)^{-1} \theta + \int_I \tilde{\phi} \in \text{Ann}(\pi_0^\sharp N_0^*).$$

In finite dimensions, Lemma 3.4 on page 12 implies $F^{-1}(T) = \text{Ann}(F^t(\text{Ann}(T)))$. Taking $T = T_0$ and $F = \pi_0^\sharp$, we then get

$$\text{Ann}(\pi_0^\sharp N_0^*) = \text{Ann}(\pi_0^\sharp \text{Ann}(T_0)) = (\pi_0^\sharp)^{-1}(T_0).$$

Thus, (3.13) implies

$$\tilde{\lambda}(0) = \pi_0^\sharp \left((U^t)^{-1} \theta + \int_I \tilde{\phi} \right) \in T_0.$$

Hence $(\tilde{\lambda}, \tilde{\phi})$ is an element of $\varpi_1^{-1}(Z_1^\perp)$, and its class modulo V^\perp is an element of $\ker d\underline{p}_1(x)$. □

4.1. COMPOSITION

Under certain technical conditions (see [12] and references therein), dual pairs can be composed by symplectic reduction. Namely, let P_0, P_1 and P_2 be Poisson manifolds, S_0 and S_1 symplectic manifolds, together with Poisson maps I_0, J_0 , and anti-Poisson maps I_1 and J_1 as in the following diagram:

$$P_0 \xleftarrow{I_0} S_0 \xrightarrow{I_1} P_1 \xleftarrow{J_0} S_1 \xrightarrow{J_1} P_2,$$

then $S_0 \times_{P_1} S_1$ is a coisotropic submanifold of $S_0 \times \overline{S_1}$ ($\overline{S_1}$ denotes S_1 with opposite symplectic structure), and the maps I_0, J_1 descend to the symplectic quotients, so that

$$P_0 \xleftarrow{I_0} \underline{S_0 \times_{P_1} S_1} \xrightarrow{J_1} P_2$$

is a new dual pair which we will denote by

$$S_0 \star S_1.$$

Of course, without the appropriate assumptions, this might be quite singular; even if we started with smooth manifolds, already the fibered product $S_0 \times_{P_1} S_1$ might not be a manifold, unless I_1 and J_0 are surjective submersions. For the reduced space to be smooth as well, one need some more assumptions, see [12].

In our case, we allow all sorts of singularity. Given coisotropic submanifolds C_0, C_1 and C_2 , we can construct singular dual pairs by $\underline{C_\pi(M; C_0, C_1)}$ and $\underline{C_\pi(M; C_1, C_2)}$. A natural question is whether $\underline{C_\pi(M; C_0, C_2)} = \underline{C_\pi(M; C_0, C_1)} \star \underline{C_\pi(M; C_1, C_2)}$. Roughly speaking the composition of these dual pairs arises by joining paths at a fiber of $C_1 \rightarrow C_1$. So we may expect the above identity to hold only if every path from C_0 to C_2 is equivalent to a path that passes through C_1 . Otherwise in the composition we will select only paths with this property, so we may expect that, in general, only the following inclusion relation holds:

$$\underline{C_\pi(M; C_0, C_2)} \supset \underline{C_\pi(M; C_0, C_1)} \star \underline{C_\pi(M; C_1, C_2)}.$$

In the next section, through another description of our singular dual pairs, this will be more clear. Another way out is the extension to this case of the construction in [3,9], where we might speak of relational dual pairs.

5. Reduced Spaces

Let (M, π) be a Poisson manifold, then $\mathbf{G}(M) = \underline{C_\pi(M; M, M)}$ is the (possibly singular) source-simply-connected symplectic groupoid of M [6]. In this case we will denote by s and t (instead of $\underline{p_0}$ and $\underline{p_1}$) the Poisson and anti-Poisson maps to M . Given two submanifolds C_0 and C_1 of M , we define

$$C_\pi^{M; C_0, C_1} = s^{-1}(C_0) \cap t^{-1}(C_1).$$

If C_0 and C_1 are coisotropic, then so are $s^{-1}(C_0), t^{-1}(C_1)$ and (because of the symplectic orthogonality of the s - and t -fibers) also $C_\pi^{M; C_0, C_1}$. We may then consider its reduction $\underline{C_\pi^{M; C_0, C_1}}$. We have the following

THEOREM 5.1. $\underline{\mathcal{C}_\pi^{M;C_0,C_1}} = \underline{\mathcal{C}_\pi(M; C_0, C_1)}$.

This Theorem is a consequence of the following

LEMMA 5.2 (Reduction in stages). *Let S be a (possibly infinite-dimensional, weak) symplectic space. Let V be a subspace of S and W a subspace of V . If W is coisotropic in S , then:*

- (1) $V^\perp \subset W^\perp \subset W \subset V$, and in particular V is also coisotropic.
- (2) W/V^\perp is coisotropic in V/V^\perp .
- (3) $W/W^\perp = (W/V^\perp)/(W/V^\perp)^\perp$.

The proof is a simple exercise in linear algebra.

Proof of Theorem 5.1. Let x be a point in $\mathcal{C}_\pi^{M;C_0,C_1} \subset \mathbf{G}(M)$ and (X, η) a representative of x in $\mathcal{C}_\pi(M; C_0, C_1) \subset \mathcal{C}_\pi(M; M, M)$. Then we apply the reduction in stages to the following spaces

$$\begin{aligned} S &:= T_{(X,\eta)} T^* P M, \\ V &:= T_{(X,\eta)} \mathcal{C}_\pi(M; M, M), \\ W &:= T_{(X,\eta)} \mathcal{C}_\pi(M; C_0, C_1), \end{aligned}$$

observing that

$$W/V^\perp = T_x \mathcal{C}_\pi^{M;C_0,C_1}.$$

□

Thanks to Theorem 5.1 we may now easily discuss a few examples.

EXAMPLE 5.3 (*Trivial Poisson structure*). For $\pi = 0$ we have $\mathbf{G}(M) = T^*M$ with canonical symplectic structure and with $s = t = \text{projection } T^*M \rightarrow M$. Any submanifold of M is automatically coisotropic with trivial foliation. Then we have

$$\mathcal{C}_0(M; C_0, C_1) = T_{C_0 \cap C_1}^* M$$

and

$$\underline{\mathcal{C}_0(M; C_0, C_1)} = T^*(C_0 \cap C_1)$$

which is a manifold iff $C_0 \cap C_1$ is so. Moreover, a simple computation shows that

$$\underline{\mathcal{C}_0(M; C_0, C_1)} \star \underline{\mathcal{C}_0(M; C_1, C_2)} = T^*(C_0 \cap C_1 \cap C_2) \subset \underline{\mathcal{C}_0(M; C_0, C_2)}.$$

EXAMPLE 5.4 (*Symplectic case*). Let M be a symplectic manifold, and π the corresponding Poisson structure. For simplicity we assume M to be simply connected. Then $\mathbf{G}(M) = M \times \overline{M}$, where \overline{M} denotes M with opposite symplectic structure. The maps s and t are the projections to the factors. Thus,

$$\mathcal{C}_\pi(M; C_0, C_1) = C_0 \times C_1,$$

and, in case C_0 and C_1 are coisotropic,

$$\underline{\mathcal{C}}_\pi(M; C_0, C_1) = \underline{C}_0 \times \underline{C}_1.$$

In this case,

$$\underline{\mathcal{C}}_\pi(M; C_0, C_2) = \underline{\mathcal{C}}_\pi(M; C_0, C_1) \star \underline{\mathcal{C}}_\pi(M; C_1, C_2),$$

for any three coisotropic submanifolds C_0, C_1 and C_2 .

Observe that the map to M from $\mathcal{C}_\pi^{M;C_0,M}$ (resp., $\mathcal{C}_\pi^{M;M,C_1}$) is a surjective submersion if C_0 (resp., C_1) has a clean intersection with every symplectic leaf of M . Under this condition, the dual pairs behave well and the composition is well defined (in the world of differentiable stacks). Thus, if we define a coisotropic submanifold to be nice when it has a clean intersection with every symplectic leaf, we have the following

THEOREM 5.5. *To every Poisson manifold (M, π) , we may associate a category $\mathbf{C}(M)$ where the objects are Poisson reductions (as differentiable stacks) of nice coisotropic submanifolds of M and the morphisms from the object \underline{C}_0 to the object \underline{C}_1 are elements of the form $\underline{\mathcal{C}}_\pi^{M;C_0,C_1}$ (as a differentiable stack) where C_0 and C_1 are nice coisotropic submanifolds of M which reduce to \underline{C}_0 and \underline{C}_1 .*

5.1. GROUPOID QUOTIENTS

As shown in [2], there is a one-to-one correspondence between coisotropic submanifolds of a given Poisson manifold M and (possibly singular) Lagrangian subgroupoids of the (possibly singular) source-simply-connected symplectic groupoid $\mathbf{G}(M)$ of M . It turns out that $\underline{\mathcal{C}}_\pi^{M;C_0,C_1}$ may be understood as a quotient of groupoids. Namely, given a groupoid $\mathbf{G} \rightrightarrows M$ and a subgroupoid $\mathbf{L} \rightrightarrows C \subset M$, we have a left action of \mathbf{L} on $s^{-1}(C)$ and a right action on $t^{-1}(C)$. We will write $\mathbf{L} \backslash \mathbf{G}$ (resp., \mathbf{G} / \mathbf{L}) as a shorthand notation for $\mathbf{L} \backslash s^{-1}(C)$ (resp., $t^{-1}(C) / \mathbf{L}$). If $\mathbf{L}(C)$ is the Lagrangian subgroupoid of $\mathbf{G}(M)$ corresponding to the coisotropic submanifold C , we have

$$\begin{aligned} \underline{\mathcal{C}}_\pi(M; C, M) &= \mathbf{L}(C) \backslash \mathbf{G}(M), \\ \underline{\mathcal{C}}_\pi(M; M, C) &= \mathbf{G}(M) / \mathbf{L}(C). \end{aligned}$$

For two given coisotropic submanifolds C_0 and C_1 , we have instead

$$\begin{aligned} \underline{\mathcal{C}_\pi(M; C_0, C_1)} &= \mathbf{L}(C_0) \backslash \mathbf{G}(M) / \mathbf{L}(C_1) \\ &= \mathbf{L}(C_0) \backslash (s^{-1}(C_0) \cap t^{-1}(C_1)) / \mathbf{L}(C_1). \end{aligned}$$

This can be verified by recalling the construction of [2] and comparing it with the one in this paper. Namely, the results of Section 3.2.1, may be rephrased as follows:

$$\begin{aligned} \mathcal{C}_\pi(M; C_0, C_1) &= \{ \text{Lie algebroid morphisms } T I \rightarrow T^* M \\ &\quad \text{with base maps connecting } C_0 \text{ to } C_1 \}. \end{aligned}$$

Two elements γ_0 and γ_1 of $\mathcal{C}_\pi(M; C_0, C_1)$ are defined to be equivalent if there exists a Lie algebroid morphism $\Gamma : T(I \times J) \rightarrow T^*M$, with J an interval, such that

- (1) the restriction of Γ to the boundaries of J are the two given morphisms γ_0 and γ_1 , and
- (2) the restriction of Γ to the boundaries $\{0\}$ and $\{1\}$ of I are Lie algebroid morphism $TJ \rightarrow N^*C_i, i=0, 1$.

Then $\underline{\mathcal{C}_\pi(M; C_0, C_1)}$ may be regarded as the quotient of $\mathcal{C}_\pi(M; C_0, C_1)$ by this equivalence relation.

On the other hand, the Lagrangian subgroupoid corresponding to a coisotropic submanifold C is shown in [2] to be the source-simply-connected groupoid whose Lie algebroid is N^*C , and this is exactly [10] the quotient of the space of Lie algebroid morphisms $TJ \rightarrow N^*C$ by Lie algebroid morphisms $T(J \times K) \rightarrow N^*C$ (where K is another interval) which are trivial on the boundary of J . Finally observe that equivalent Lie algebroid morphisms $TJ \rightarrow N^*C_i$ act the same way on $\mathcal{C}_\pi(M; C_0, C_1)$.

Remark 5.6. This construction may be generalized to any Lie groupoid.

6. Lagrangian Field Theories with Boundary

As mentioned in Section 1.3, the constructions in this paper are related to the general ones for Lagrangian field theories on manifolds with boundary as in [4] and [5].

We start recalling a few facts. In general to a compact oriented manifold N (possibly with boundary), of fixed dimension d , the theory associates a space of fields \mathcal{F}_N and a function S_N on \mathcal{F}_N called the action functional. The case of this paper, fixing a manifold M and a tensor π on it, corresponds to $d=2$, \mathcal{F}_N the space of bundle maps $TN \rightarrow T^*M$ and

$$S_N(X, \eta) = \int_N \langle \eta, dX \rangle + \frac{1}{2} \langle \eta, \pi^\# \eta \rangle$$

where X is a map $N \rightarrow M$, η a section of $T^*N \otimes X^*T^*M$ and again $\langle \cdot, \cdot \rangle$ is the canonical pairing between the cotangent and the tangent bundles to M . Notice that the action functional does not see the symmetric part of π , so it may be convenient to assume that π is a bivector field. When π is Poisson, this theory is called the Poisson sigma model [11,13]. We will call the general case the bivector sigma model (BSM).

Under certain assumptions (in particular locality), to a compact oriented $(d - 1)$ -manifold Σ the theory also associates an exact weak symplectic manifold $(\mathcal{F}_\Sigma^\partial, \Omega_\Sigma = d\Theta_\Sigma)$ such that whenever $\Sigma = \partial M$ for a compact d -manifold M there is a surjective submersion $\pi_M: \mathcal{F}_M \rightarrow \mathcal{F}_{\partial M}^\partial$. Denoting by EL_M the zero set⁴ of the 1-form $EL_M := dS_M - \pi_M^* \Theta_{\partial M}$, one obtains that the “evolution relation” $L_M := \pi_M(EL_M)$ is isotropic in $\mathcal{F}_{\partial M}^\partial$. In “good” theories the L_M s should be Lagrangian. Observe that L_M is here defined just as a subset; a good additional condition is that it should be a (possibly immersed) submanifold. One also has $\mathcal{F}_{\Sigma \sqcup \Sigma'}^\partial = \mathcal{F}_\Sigma^\partial \times \mathcal{F}_{\Sigma'}^\partial$ as a product of exact weak symplectic manifolds and $\mathcal{F}_{\Sigma^{\text{op}}}^\partial = \overline{\mathcal{F}_\Sigma^\partial}$, where Σ^{op} denotes Σ with opposite orientation and bar denotes the same manifold with opposite one-form. Finally one defines the “space of Cauchy data” \mathcal{C}_Σ as the space of points in $\mathcal{F}_\Sigma^\partial$ that can be completed to a pair of points in $L_{\Sigma \times [0, \epsilon]} \subset \mathcal{F}_\Sigma^\partial \times \mathcal{F}_\Sigma^\partial$ for some $\epsilon > 0$. Under some mild assumptions, one can show that, if $L_{\Sigma \times [0, \epsilon]}$ is Lagrangian for all ϵ , then \mathcal{C}_Σ is coisotropic.⁵ In the BSM, we get $(\mathcal{F}_{S^1}^\partial, \Theta_{S^1}) = (T^*LM, \Theta)$ and $\mathcal{C}_{S^1} = \mathcal{C}_\pi(M)$ with the notations of Section 1.2. We have the following

THEOREM 6.1. *$L_{S^1 \times [0, \epsilon]}$ is an immersed Lagrangian submanifold $\forall \epsilon > 0$ iff π is Poisson.*

Proof. If $L_{S^1 \times [0, \epsilon]}$ is Lagrangian $\forall \epsilon > 0$, then $\mathcal{C}_{S^1} = \mathcal{C}_\pi(M)$ is coisotropic by the general theory. By Theorem 1.5, π is then Poisson.

⁴This is the set of solutions to Euler–Lagrange equations where one “ignores” the boundary.

⁵The first assumption is that the theory behaves well under diffeomorphisms: viz., a diffeomorphism of the bulk manifolds induces a diffeomorphism of the corresponding EL spaces. This implies that each $L_{\Sigma \times [0, \epsilon]}$ is symmetric: $(x, y) \in L_{\Sigma \times [0, \epsilon]}$ iff $(y, x) \in L_{\Sigma \times [0, \epsilon]}$. The second assumption is locality which in particular implies that we can restrict solutions; so, if we know that x lies in \mathcal{C}_Σ because there is a y with $(x, y) \in L_{\Sigma \times [0, \epsilon]}$, then we also know that for all $\epsilon' < \epsilon$ there is a y' with $(x, y') \in L_{\Sigma \times [0, \epsilon']}$. (In the case of topological field theories, like the BSM, this part of the argument is much easier since $L_{\Sigma \times [0, \epsilon]} = L_{\Sigma \times [0, \epsilon']}$ for all ϵ, ϵ' .)

Suppose now that x lies in \mathcal{C}_Σ . Pick a compact neighborhood U_x of x in \mathcal{C}_Σ . For each z in U_x there is then an $\epsilon_z > 0$ and a y such that $(z, y) \in L_{\Sigma \times [0, \epsilon_z]}$. Let ϵ be the maximum ϵ_z for z in U_x . Thanks to the locality assumption, we then have the simplified statement that for all $z \in U_x$ there is a y_z such $(z, y_z) \in L_{\Sigma \times [0, \epsilon]}$. This in particular shows that $T_x \mathcal{C}_\Sigma$ consists of all $v \in T_x \mathcal{F}_\Sigma^\partial$ such that there is a $w \in T_y \mathcal{F}_\Sigma^\partial$, with $y := y_x$, such that $(v, w) \in T_{(x, y)} L_{\Sigma \times [0, \epsilon]}$. Thanks to the symmetry property, we also see that $T_{(x, y)} L_{\Sigma \times [0, \epsilon]}$ is contained in $T_x \mathcal{C}_\Sigma \oplus T_y \mathcal{C}_\Sigma$. The orthogonal space of the latter, in $T_x \mathcal{F}_\Sigma^\partial \oplus T_y \mathcal{F}_\Sigma^\partial$, is readily seen to be $(T_x \mathcal{C}_\Sigma)^\perp \oplus (T_y \mathcal{C}_\Sigma)^\perp$. Since $L_{\Sigma \times [0, \epsilon]}$ is Lagrangian by assumption, we conclude that $(T_x \mathcal{C}_\Sigma)^\perp \oplus (T_y \mathcal{C}_\Sigma)^\perp$ is contained in $T_x \mathcal{C}_\Sigma \oplus T_y \mathcal{C}_\Sigma$, so $T_x \mathcal{C}_\Sigma$ is coisotropic (and so is $T_y \mathcal{C}_\Sigma$). Since this can be shown for all $x \in \mathcal{C}_\Sigma$, we have that \mathcal{C}_Σ is coisotropic.

On the other hand, if π is Poisson, then $C_{S^1} = C_\pi(M)$ is coisotropic by Theorem 1.5. In addition, $\forall \epsilon > 0$, $L_{S^1 \times [0, \epsilon]}$ consists of all pairs $(x, x') \in C_\pi(M) \times C_\pi(M)$ such that x and x' are on the same characteristic leaf.⁶ This implies that $L_{S^1 \times [0, \epsilon]}$ is an immersed Lagrangian submanifold.⁷ \square

This story extends to the case when Σ is only part of ∂M . If $\partial M = \Sigma \sqcup \Sigma'$, then π_M is the product of two surjective submersions $\pi_{M, \Sigma}$ and $\pi_{M, \Sigma'}$ to $\mathcal{F}_\Sigma^\partial$ and $\mathcal{F}_{\Sigma'}^\partial$, respectively. Upon picking a Lagrangian submanifold L' of $\mathcal{F}_{\Sigma'}^\partial$, on which $\Theta_{\Sigma'}$ vanishes, one sets $\mathcal{F}_M^{L'} := \pi_{M, \Sigma'}^{-1}(L')$. Denoting by $S_M^{L'}$, $EL_M^{L'}$ and $\pi_{M, \Sigma}^{L'}$ the restrictions of S_M , EL_M and $\pi_{M, \Sigma}$ to $\mathcal{F}_M^{L'}$, one then has $EL_M^{L'} = dS_M^{L'} - (\pi_{M, \Sigma}^{L'})^* \Theta_\Sigma$ and therefore $L_M^{L'} := \pi_{M, \Sigma}^{L'}(EL_M^{L'})$ is isotropic, where $EL_M^{L'}$ denotes the zero set of $EL_M^{L'}$.

A further extension occurs when M is a compact manifold with corners and the codimension-one boundary stratum of M , which we denote by ∂M , is the union of compact manifolds with boundary Σ and Σ' joined along their common boundary, the codimension-two boundary stratum of M (we assume that there are no further lower dimensional boundary strata). The story described in the previous paragraph extends verbatim to this case.

⁶The critical points of S_N are solutions to

$$\begin{aligned} dX &= -\pi^\sharp(X)\eta, \\ d\eta_i &= \frac{1}{2} \partial_i \pi^{jk}(X) \eta_j \eta_k. \end{aligned}$$

For simplicity of notations we work in local coordinates (the rest of the computation may be done in covariant form since, using the first equation, the second can be written as $\partial \eta_i = \frac{1}{2} (\nabla \pi)_i^{jk}(X) \eta_j \eta_k$). We write $\eta = \eta_\parallel - \beta dt$, where η_\parallel is a 1-form in the S^1 direction, β a function and t the coordinate on $[0, \epsilon]$. Writing $d = d_\parallel + dt \delta$, with d_\parallel the differential in the S^1 direction and δ the partial derivative with respect to t , we may rewrite the equations as

$$\begin{aligned} d_\parallel X &= -\pi^\sharp(X)\eta_\parallel, \\ \delta X &= \pi^\sharp(X)\beta, \\ \delta \eta_{\parallel i} &= -d_\parallel \beta_i + \partial_i \pi^{jk}(X) \beta_j \eta_{\parallel k}. \end{aligned}$$

The first equation says that the restriction of (X, η) to $S^1 \times \{t\}$ for each t yields an element of $C_\pi(M)$. The two other equations say that (X, η) evolves in the t direction along the characteristic distribution of $C_\pi(M)$, cf. (3.17).

⁷We follow the analogue proof in [9, Section 3.5.2]). Since $L_{S^1 \times [0, \epsilon]}$ consists of pair of points on the same leaf of the characteristic distribution of $C_\pi(M)$, it follows that it is an immersed submanifold. We now prove that it is Lagrangian. Let (x, y) be in $L_{S^1 \times [0, \epsilon]}$. Then, in addition to knowing that $T_{(x, y)} L_{S^1 \times [0, \epsilon]}$ is a subspace of $T_x C_\pi(M) \oplus T_y C_\pi(M)$, which in turn is a coisotropic subspace of $T_x(T^*LM) \oplus T_y(T^*LM)$, we now also know that it contains $(T_x C_\pi(M))^\perp \oplus (T_y C_\pi(M))^\perp = (T_x C_\pi(M) \oplus T_y C_\pi(M))^\perp$. Moreover, $T_{[x]} C_\pi(M) := T_x C_\pi(M) / (T_x C_\pi(M))^\perp$ gets canonically identified with $T_y C_\pi(M) / (T_y C_\pi(M))^\perp$. Finally, $T_{(x, y)} L_{S^1 \times [0, \epsilon]} / (T_x C_\pi(M) \oplus T_y C_\pi(M))^\perp$ is the diagonal in $T_{[x]} C_\pi(M) \oplus T_{[y]} C_\pi(M)$, which is Lagrangian. This proves that $T_{(x, y)} L_{S^1 \times [0, \epsilon]}$ itself is Lagrangian, see [4, Proposition A.1(3)].

Finally, one can define the “space of Cauchy data” $\mathcal{C}_\Sigma^{L'}$ as the space of points in $\mathcal{F}_\Sigma^\partial$ that can be completed to a pair of points in $L_{\Sigma \times [0, \epsilon]}^{L'} \subset \mathcal{F}_\Sigma^\partial \times \overline{\mathcal{F}_\Sigma^\partial}$ for some $\epsilon > 0$, where L' is a fixed Lagrangian submanifold of $\mathcal{F}_{\partial\Sigma \times [0, \epsilon]}$.⁸ Again, one shows that, if $L_{\Sigma \times [0, \epsilon]}^{L'}$ is Lagrangian for all ϵ , then $\mathcal{C}_\Sigma^{L'}$ is coisotropic.

In the BSM, we get $(\mathcal{F}_I^\partial, \Theta_I) = (T^*PM, \Theta)$ with the notations of the Introduction. Moreover, for a submanifold C of M , the space L_C of bundle maps $T[0, \epsilon] \rightarrow N^*C$ is a Lagrangian submanifold of $\mathcal{F}_{[0, \epsilon]}^\partial$. Finally, one has $C_I^{L_{C_0} \times L_{C_1}} = C_\pi(M; C_0, C_1)$ and

THEOREM 6.2. $L_{I \times [0, \epsilon]}^{L_M \times L_M}$ is an immersed Lagrangian submanifold $\forall \epsilon > 0$ iff π is Poisson. If π is Poisson and C_0 and C_1 are coisotropic, then $L_{I \times [0, \epsilon]}^{L_{C_0} \times L_{C_1}}$ is Lagrangian $\forall \epsilon > 0$.

The proof is similar to the case of S^1 but now uses Theorem 1.1, Corollary 1.3 and the results of Section 3.2.1.

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⁸Since a diffeomorphism $[0, \epsilon] \rightarrow [0, \epsilon']$ induces a symplectomorphism $\mathcal{F}_{\gamma \times [0, \epsilon]}^\partial \rightarrow \mathcal{F}_{\gamma \times [0, \epsilon']}^\partial$, it is enough to select L' in $\mathcal{F}_{\gamma \times [0, \epsilon]}$ for a fixed ϵ .

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