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Conflict-Free Chromatic Art Gallery Coverage

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Abstract We consider a *chromatic* variant of the art gallery problem, where each guard is assigned one of k distinct colors. A placement of such colored guards is *conflict-free* if each point of the polygon is seen by some guard whose color appears exactly once among the guards visible to that point. What is the smallest number $k(n)$ of colors that ensure a conflict-free covering of all n-vertex polygons? We call this the *conflict-free chromatic art gallery problem*. Our main result shows that $k(n)$ is $O(\log n)$ for orthogonal and for monotone polygons, and $O(\log^2 n)$ for arbitrary simple polygons. By contrast, if *all* guards visible from each point must have distinct colors, then $k(n)$ is $\Omega(n)$ for arbitrary simple polygons, as shown by Erickson and LaValle (Robotics: Science and Systems, vol. VII, pp. 81–88, 2012). The problem is motivated by applications in distributed robotics and wireless sensor networks but is also of interest from a theoretical point of view.

Keywords Art gallery problem · Conflict-free coloring · Visibility · Polygon partitioning

1 Introduction

The Art Gallery Theorem is a classical result in computational geometry, first posed by Klee and proved by Chvátal [2], which says that $\lfloor n/3 \rfloor$ (point) guards are always

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sufficient, and sometimes necessary, to cover a simply-connected n-vertex polygon. In the last 30 years, many extensions, variations, and generalizations involving different types of guards, polygons, and visibility constraints have been investigated. (See [4] and [6], for instance.)

Besides their mathematical elegance and appeal, the interest in art gallery problems is also spurred by applications in distributed surveillance, robotics, and monitoring. In many of these applications, the "guards" are "landmarks" deployed in an environment to help provide navigation and localization service to mobile robots. The mobile device communicates with these landmarks through wireless, or other "line-of-sight" signaling mechanisms. In order for the signaling mechanism to work correctly, the different landmarks visible to the robot at any position must operate on different frequency—the robot is unable to receive the signal if multiple landmarks in its range are transmitting at the same frequency. This motivates a "chromatic" version of the art gallery theorem, where the goal is not to optimize the *number of guards*, but rather the *number of distinct colors* needed to distinguish the guards.

Problem Motivation and the Results Radio transceivers are cheap but tuning them to many different frequencies requires costly hardware. If the polygons can be covered by guards of very few *distinct colors* (frequencies), then it would enable inexpensive robot localization and navigation. This was the motivation behind the work of Erickson and LaValle who sought to guard the polygon so that each point of the polygon is seen by guards of distinct colors only—that is, the robot located anywhere in the polygon is able to communicate without interference with *any* of the guards in its line-of-sight. Surprisingly, Erickson and LaValle discovered that this *strong chromatic* condition does not lead to much savings in the number of colors: there are simple polygons that require $\Omega(n)$ colors, and even monotone orthogonal polygons require $\Omega(\sqrt{n})$ colors [3].

Motivated by this negative result, we consider a weaker chromatic condition, which is sufficient for the original robotics application of interference-free communication with a guard at all locations. Specifically, we call a placement of colored guards *conflict-free* if each point of the polygon is seen by some guard whose color appears exactly once among the guards visible to that point. Thus, for any placement of the robot in the polygon, there is at least one guard that can communicate with the robot without interference. We want to determine the smallest number $k(n)$ of colors that ensure a conflict-free coloring of some guard set in all n-vertex polygons. We call this the *conflict-free chromatic art gallery problem*.

The main result of our paper is to prove that $k(n)$ is $O(\log n)$ for orthogonal and for monotone polygons, and $k(n) = O(\log^2 n)$ for arbitrary simple polygons. Thus, not only does the conflict-free coloring yield significantly smaller bounds for distinct colors, it also fulfills the hopeful vision of robotics application that a few colors suffice. Furthermore we introduce a new method to partition a simple polygon into monotone polygons that might be of independent interest by itself.

Related Work and Hypergraph Coloring The chromatic art gallery problem is related to hypergraph coloring, where one must assign colors to the vertices of a hypergraph $H = (V, \mathcal{E})$, so that its edges, which are subsets of vertices, are appropriately colored. In the most basic form, called the proper coloring, every edge e with at least two vertices must be non-monochromatic; that is, there must be two vertices $x, y \in e$ whose colors are distinct. In the conflict-free coloring of H, every edge e must have a vertex that is uniquely colored among the vertices in e. Smorodinsky [7, 9] considers several simple geometric hypergraphs, such as those induced by disks or rectangles. For instance, the rectangle hypergraph has a finite set of axis-aligned rectangles, and each maximal subset of rectangles with a common intersection forms a hyperedge. For these hypergraphs, it is known that the conflict-free chromatic number is $\Theta(\log n)$, where the upper bound of $O(\log n)$ was shown by Smorodinsky in [8], and the matching lower bound of $\Omega(\log n)$ has been shown by Pach and Tardos in [5].

To see the connection between chromatic art gallery and the hypergraph coloring, consider a guard set S, and let $\mathcal{R}$ be the set of the guards' visibility regions in the polygon. Then we have a hypergraph $H = (V, \mathcal{E})$, whose vertices correspond to S and in which a subset $S_e \subseteq S$ corresponds to an edge if there is a point p_e in the polygon contained exactly in the visibility regions of the guards in S_e and no others. A *conflict-free hypergraph coloring* of H is easily seen to be also a conflict-free coloring of the guard set S. Of course, in the chromatic art gallery, we need to simultaneously choose the guard set and color it, so it does not quite reduce to the hypergraph coloring. Even if we were to consider a fixed guard set, the visibility regions are not as well-behaved as disks or rectangles, and no non-trivial bound is known for their conflict-free chromatic number.

The previous result that is most directly relevant to our work is the mentioned version of the chromatic art gallery, with a stronger chromatic condition on the guard's coloring. This original version relates to a *strong hypergraph coloring* of the corresponding hypergraph H.

Organization In Sect. 2, we introduce the basic definitions and concepts used throughout the paper. In Sect. 3, we prove the $O(\log n)$ upper bound for the conflict-free coloring of orthogonal polygons. This section also introduces a new scheme for partitioning a polygon into monotone pieces that may have other applications as well. In Sect. 4, we prove the $O(\log n)$ bound for non-orthogonal monotone polygons, which is the key to establishing the $O(\log^2 n)$ upper bound for general simple polygons in Sect. 5. We conclude with a few remarks and open questions in Sect. 6.

2 Conflict-Free Chromatic Art Gallery Problems

Let P be a simple polygon, whose boundary we denote as $\partial P \subset P$. We say that two points $p, q \in P$ are *visible* to each other if the line segment $\overline{pq}$ is a subset of P. We would like to remark that visibility is always defined with respect to the polygon P, i.e. in a subpolygon $P_s \subseteq P$, two points $p, q \in P_s$ are visible to each other if the line segment $\overline{pq}$ is a subset of P (and not of P_s). The *visibility region* of a point p is defined as $V(p) := \{q \in P \mid q \text{ is visible from } p\}$. A finite point set $S \subset P$ is called a *guard set* if $\bigcup_{p \in S} V(p) = P$ and we call the points in S *guards*. A coloring $c\colon S \to \{1, \dots, k\}$ of the guards with k colors is called *conflict-free* if each point $p \in P$ is seen by a guard whose color appears exactly once among all guards that see p.

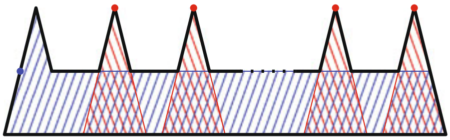

Fig. 1 A polygon P that requires $\lfloor n/3 \rfloor$ guards and has a conflict-free chromatic guard number of $\chi(P) = 2$

Let $k_{cf}(S)$ be the minimum number of colors required to color a guard set S conflict-free and let $\mathcal{S}(P)$ be the set of all guard sets of P. Then the *conflict-free chromatic guard number* of a polygon P is defined as $\chi(P) := \min_{S \in \mathcal{S}(P)} k_{cf}(S)$. We want to determine the smallest number $k(n)$ such that for all n-vertex polygons P_n we have $\chi(P_n) \leq k(n)$. The classical art gallery theorem says that $\lfloor n/3 \rfloor$ guards are both necessary and sufficient for covering a n-vertex polygon, but the number of colors needed to ensure a conflict-free covering may be significantly smaller. For instance, the classical comb construction that forces $\lfloor n/3 \rfloor$ guards (Fig. 1) has a conflict-free chromatic guard number of 2.

A polygon is called *orthogonal* if its edges meet at right angles. A polygon P is called *monotone with respect to a line* ℓ if every line orthogonal to ℓ intersects the boundary of P at most twice. P is called x*-monotone* (y-monotone) if P is monotone with respect to the x-axis (respectively the y-axis).

The following concept of *independence* is central to our proofs, and forms a basis for a conflict-free covering by partitioning into independent subpolygons.

Definition 1 (Independence) Let P be a polygon. We call two subpolygons P_1 and P_2 of P *independent* if there are no two points $p_1 \in P_1$ and $p_2 \in P_2$ that are mutually visible.

Lemma 1 *Let* $\{A_1, \dots, A_m\}$ *be a partition of the polygon* P *into* m *families of pairwise independent subpolygons. That is, each* $A_i = \{P_{i1}, \dots, P_{ik_i}\}$ *is a collection of subpolygons that are pairwise independent and all the subpolygons in the* m *families form a partition of* P. *Then we have* $\chi(P) \leq \sum_{i=1}^{m} \max_{P_{ij} \in A_i} \{\chi(P_{ij})\}$.

Proof Let $\{C_1, \dots, C_m\}$ be m disjoint color sets, where in each set the number of colors is $|C_i| = \max_{P_{ij} \in A_i} \{\chi(P_{ij})\}$. Then we can guard every subpolygon $P_{ij} \in A_i$ conflict-free in itself with guards that get colors from C_i, giving a total number of $|C_1| + \cdots + |C_m|$ colors. We claim that this coloring ensures that every point $p \in P$ sees a guard of unique color among all guards that see p. To prove this claim, without loss of generality, suppose that p is contained in a subpolygon P_{ij_1} of A_i and s_1 is its guard of unique color in P_{ij_1}. Any other guard s_2 in $P \setminus P_{ij_1}$ that has the same color as s_1 must lie in a subpolygon $P_{ij_2} \neq P_{ij_1}$, which is contained in A_i and hence independent of P_{ij_1}. Thus s_2 does not see p, and s_1 is not only a guard of unique color among all guards in P_{ij_1}, but among all guards in P. Thus, we have found a conflict-free covering with $|C_1| + \cdots + |C_m|$ colors, which completes the proof. □

Lemma 1 naturally suggests a divide-and-conquer strategy: we partition the polygon into four sets of subpolygons and then conquer each set by recursively splitting the regions into sets of independent subpolygons and applying Lemma 1.

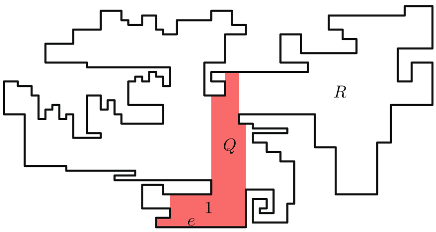

Fig. 2 The first step of the partitioning process

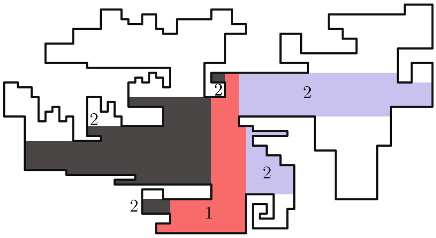

Fig. 3 The second step of the partitioning process

Remark 1 We only require the interiors of subpolygons P_1 and P_2 to be independent, and allow mutual visibility among their boundary points as long as these points *also belong* to the boundary of another subpolygon that is responsible for their conflict-free covering. In particular, for a line segment e contained in two boundaries ∂P_1 and ∂P_2, we will explicitly mention whether P_1 or P_2 is "responsible" for guarding e.

3 Orthogonal Polygons

Our basic strategy is to partition the orthogonal polygon P into four types of monotone orthogonal subpolygons. These subpolygons have a boundary consisting of a single base edge and another subchain that is either x-monotone or y-monotone. The chain can be either above the base edge or below in the former case, and to the left or to the right in the latter case. We use mnemonic identifiers U (up), D (down), L (left) and R (right) to refer to these four types. When we show all or parts of the partition, we display these types with the colors red, green, black and blue, always using the following consistent mapping U $\rightarrow$ red, D $\rightarrow$ green, L $\rightarrow$ black and R $\rightarrow$ blue. In a gray-scale version of this paper, the two types we will mostly focus on—type L and U—will have colors black and dark gray.

The Partitioning Process Given a polygon P we construct a partition by iteratively adding monotone subpolygons. In each *odd-numbered* step we add subpolygons of Type U and D, and in each *even-numbered* step we add subpolygons of Type L and R. Figures 2 and 3 illustrate the construction.

Step 1 Let e be the lowest horizontal edge of P's boundary. Let Q be the set of all points $q \in P$ which are vertically visible from e and lie on or above e. Q is the first subpolygon in our partitioning, and it is of type U. Because P is a simply-connected region, with no holes, it is easy to see that Q splits it in parts that lie entirely to its

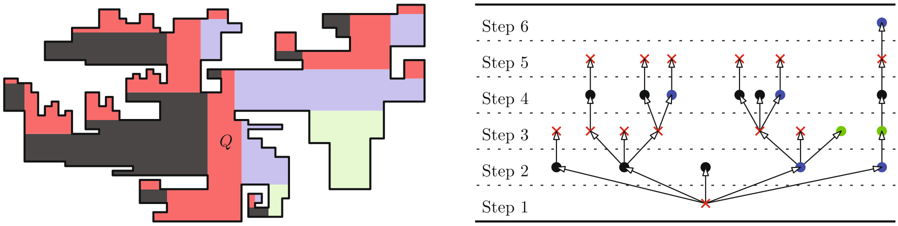

Fig. 4 The complete partition and the corresponding schematic tree. Vertices of T corresponding to type U subpolygons are marked with an x

left or entirely to its right, and each part R shares exactly one edge with Q, which is a vertical line segment.

Step 2 The line segments on the boundary of Q become the base edges for new subpolygons of Type L and R, which are defined analogously as the first subpolygon, with vertical visibility replaced by horizontal visibility. We note that the remaining regions lie entirely above or below a subpolygon of type L or R and share exactly one horizontal line segment with these subpolygons, but not with the first subpolygon Q.

Step 3 The horizontal line segments from Step 2 in turn generate subpolygons of Type U and D.

We repeat steps 2 and 3 until we have a complete partition. In each odd-numbered step we construct U and D subpolygons and in each even-numbered step L and R subpolygons.

Lemma 2 *The partitioning process terminates within $n + 2$ steps.*

Proof In each step at least one subpolygon is added to the partition. Such a subpolygon touches at least one edge $e = \{u, v\}$ previously not touched. In at most two additional steps, both the endpoints of e, u and v, become completely surrounded by subpolygons of the partition. The polygon is completely covered if all vertices are surrounded, hence the partitioning process ends after at most $n + 2$ steps. □

The Schematic Tree The recursive partitioning generates four *families* of polygons: up-polygons A_U, down-polygons A_D, left-polygons A_L, and right-polygons A_R. Ideally, we would like to invoke Lemma 1 on this partition partitioned into families $\{A_U, A_D, A_L, A_R\}$. Unfortunately the subpolygons in each family are *not independent*, see Fig. 4 for an example. We, therefore, introduce a condition that allows us to subdivide the group A_U into sets of independent subpolygons. In the following, we focus exclusively on the type U subpolygon group; the other three groups are handled in the same way.

We first introduce a schematic tree that is a convenient graphical representation of the polygon partition we have. This graph is a 4-partite directed graph T, where the four independent vertex sets of $V(T) = U \cup D \cup L \cup R$ correspond to the four families $\{A_U, A_D, A_L, A_R\}$, i.e. each vertex in U represents a type U subpolygon etc. There exists a directed edge from a subpolygon P_i to a subpolygon P_j if and only if P_j has been constructed over a line segment e that is part of P_i's boundary. As mentioned earlier, we consider e to be part of P_i but not of P_j, i.e. P_i is responsible for guarding e conflict-free. Since P has no holes, T contains no cycle and is a tree. The first constructed subpolygon Q has no incoming edge, and it represents the root of our tree. (The base edge of Q is considered to be a part of this subpolygon.) Since all other vertices have indegree 1, T is a rooted directed tree and any subpolygon constructed in Step k has depth $k-1$ in T. Hence all vertices in $U \cup D$ have even height and all vertices in $L \cup R$ have odd height. Therefore every directed path in T alternates between vertices in U or D and vertices in L or R.

Remark 2 Let $p_i \in P_i$ and $p_j \in P_j$ be two points of two subpolygons of the partition. Then the shortest path between p_i and p_j in P goes through a subpolygon P_k if and only if P_k lies on the shortest path between P_i and P_j in T.

Lemma 3 *Let P be a polygon with the given partition and the schematic tree T. Let P_i and P_j be two arbitrary subpolygons of type U. Then*, *either* (*i*) *P_i and P_j are independent*, *or* (*ii*) *there must exist a U-L-alternating* (*or a U-R-alternating*) *directed path in T between P_i and P_j.*

Proof Suppose P_i and P_j are not independent, then there exist points $p_i \in P_i$ and $p_j \in P_j$ that are mutually visible. The shortest path in P between p_i and p_j, therefore, must be a line segment. The way we included the base edges to be part of just one subpolygon excludes the possibility of the line segment being horizontal or vertical. Without loss of generality, let us assume that the line segment is directed up and to the left, with p_i at the bottom-right, and p_j at the top-left. Since P_i is a U polygon, the visibility ray $\overrightarrow{p_i p_j}$ can only leave it through its left boundary, and therefore it must enter a type L subpolygon. Next, by the upward direction of $\overrightarrow{p_i p_j}$, it can leave this L subpolygon only through a top boundary edge, which forces it to enter a U subpolygon. This process repeats until we reach P_j, showing that the sequence of polygons traversed by the shortest path from p_i to p_j is an alternating U-L sequence, which corresponds to a U-L-alternating path in T. □

Conquering U-L-alternating Trees: Staircase and Recursion Deriving a bound on the conflict-free chromatic guard number for the family A_U directly seems difficult, because of inter-dependence of the subpolygons within the family. Instead, we use the property of Lemma 3 to look at that portion of A_U that is contained in a *U-L-alternating tree*. That is, consider the union of the subpolygons that corresponds to a U-L-alternating tree in T. Suppose P_n is such a n-vertex orthogonal polygon, $P_n \subseteq A_U \cup A_L$. We will cover a part of P_n with a *staircase polygon* in such a way that all other relevant parts (containing type U subpolygons) are independent. Then we proceed recursively for all of the independent parts.

Recall that a vertex of a polygon is called a *reflex vertex* (respectively a *convex vertex*) if the interior angle between its adjacent edges is greater than 180° (respectively $< 180°$). A *staircase (orthogonal) polygon* is an orthogonal polygon whose boundary can be split into two subchains with alternating convex and reflex interior vertices, with the two endpoints being convex. A staircase polygon in which one of the subchains has only one interior vertex is called a *convex fan*. Convex fans are star-shaped and can clearly be guarded with one guard (and one color).

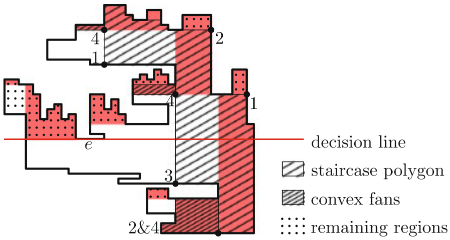

Fig. 5 The first inserted staircase subpolygon and the convex fans to its left

Lemma 4 *The conflict-free chromatic guard number for a staircase polygon P is at most* 3.

Proof Consider the following placement of colored guards in a staircase polygon: Starting from the top, we place a guard s_1 on the first convex vertex of the lower subchain. Then, alternating between the two subchains, we iteratively place a guard s_{i+1} on the lowest convex vertex visible from s_i until the staircase polygon is covered. To each guard s_i we assign the color in $\{1, 2, 3\}$ with the same residue class as i modulo 3. One can check that this coloring is conflict-free, and a complete proof can be found in [3]. □

Let $f(n)$ denote the smallest number of colors that ensure a conflict-free covering of all type U subpolygons in any orthogonal P_n corresponding to a U-L-alternating tree. In other words, for every P_n there is a guard set $S \subset P_n$ that can be colored with $f(n)$ colors such that each point of a type U subpolygon is seen by some guard whose color appears exactly once among the guards visible to that point. In the following we give a placement of colored guards, which shows that $f(n)$ is at most $4 \log n$.

Since P_n consists of type U and type L subpolygons, it "grows to the left". Therefore we will cover P_n with staircases ascending to the left in a natural way: Let e be a horizontal edge with two reflex vertices. We call the horizontal line through e a *decision line*, see Fig. 5. A decision line splits P_n in a lower part and two or more independent upper parts, of which at most one upper part contains more than $\lfloor n/2 \rfloor$ vertices. Starting from the lowest and rightmost vertex of P_n we construct a staircase ascending to the left, which at every decision line follows the upper part with the most vertices. We guard this staircase with colors $\{1, 2, 3\}$.

Furthermore at every intersection of the staircase's lower subchain with a base edge of a type U subpolygon, we insert a convex fan that is oriented to the left and to the top. These convex fans are bounded from the right by the staircase polygon and hence independent. We guard every convex fan with a guard of color 4 placed on the intersection.

The still remaining type U subpolygons (and subpolygon parts) build smaller U-L-alternating subpolygon subtrees. For these regions we iteratively add staircases together with convex fans as shown in Fig. 6. Doing this we can prove an upper bound on $f(n)$:

Lemma 5 *Suppose P_n is an orthogonal polygon with a partition that has a U-L-alternating schematic tree. Then a conflict-free coloring of all the type U subpolygons of P_n needs at most $4 \log n$ colors. The same bound also holds for a U-R-alternating schematic tree.*

Fig. 6 An iterative covering with staircases and convex fans

Proof We cover a part of the type U subpolygons with a staircase and convex fans as described, using 4 colors. The remaining regions of the type U subpolygons are parts of smaller U-L-alternating trees. These smaller trees are all bounded from below by a decision line and from above and from the side by P_n's boundary, hence they are independent. Furthermore all of the smaller trees contain at most $\lfloor n/2 \rfloor$ of P_n's vertices because during the construction we choose at every decision line the upper part with the most remaining vertices.

Thus, the chromatic number follows the recurrence $f(n) \leq f(n/2) + 4$, which yields $f(n) \leq 4 \log n$. By symmetry, the same holds for the U-R-alternating trees. □

A Logarithmic Upper Bound for Orthogonal Polygons We will show how one can cover *all* type U subpolygons in an arbitrary orthogonal polygon P_n with $O(\log n)$ colors. Let T be the schematic tree of the partition and let A and B be two disjoint color sets of size $f(n)$. We use the sets A and B to iteratively cover U-L-alternating and U-R-alternating subtrees of T. In each step we must ensure that the subtrees of the same type are independent so that we can use the same colors for all of the subtrees:

Step 1 Take a not yet covered subpolygon P_s corresponding to a vertex v_s of minimal depth in T. Let T_s denote the inclusion-maximum U-L-alternating subtree rooted at v_s. By Lemma 5 we can guard all type U subpolygons corresponding to vertices in $U \cap V(T_s)$ with A.

Step 2 For every type U subpolygon in T_s (which now are all guarded) check whether there are vertices of U with distance 2 from T_s, which are not yet guarded. Such vertices must be connected to T_s through a vertex of R. These “grandchildren nodes” are pairwise independent by Lemma 3, hence for each grandchild v it is possible to cover the inclusion-maximum U-R-alternating subtree rooted at v with guards colored with colors in B conflict-free by Lemma 5. Furthermore we have no conflicts with the type U subpolygons covered before since A and B are disjoint.

Step 3 As in Step 2, cover the independent inclusion-maximum U-L-alternating subtrees rooted at not yet covered grandchildren of type U subpolygons in one of the U-R-alternating subtrees. We use the color set A, which gives no conflicts with the guards in the U-R-alternating subtrees, since they have colors from B. Furthermore we have also no conflicts with the guards in a previous U-L-alternating subtree by Lemma 3, since the shortest path must go through the root of a U-R-alternating subtree and hence through both a type L and a type R subpolygon.

Step 4 Repeat Step 2 and Step 3 as long as there are grandchildren. Otherwise we either have covered all type U subpolygons, or there remain type U subpolygons connected through a vertex of D, which are thus independent by Lemma 3. In that case we start over with Step 1.

In this way, we get a conflict-free covering of all type U subpolygons in P_n with at most $|A|+|B|=2f(n)=O(\log n)$ colors. By symmetry, we can apply the same procedure to type D, L and R subpolygons in alternating trees. For each type we use two new color sets of size $f(n)$, which yields a conflict-free coloring of all subpolygons of the partition of an orthogonal polygon, where we use at most $8f(n)=O(\log n)$ colors in total. We have established the main result of this section:

Theorem 1 *The conflict-free chromatic guard number for orthogonal polygons on n vertices is $k(n)=O(\log n)$.*

Remark 3 Although we only care about the number of colors, we still want to mention that the number of guards given by our algorithm is in $O(n)$. This is easy to see, as each guard is placed in one of three possible locations: (i) a vertex of the polygon, (ii) an edge of the polygon or (iii) the base edge of a subpolygon of the partition.

In the covering of the type U subpolygons, on any vertex and on any edge of the polygon at most one guard is placed, and at most 2 guards are placed on the base edge of a subpolygon. Any vertex of the polygon is adjacent to at most three subpolygons of the partition. Therefore we have a linear number of subpolygons and the algorithm needs at most $O(n)$ guards.

4 Monotone Polygons: A Step Towards Simple Polygons

The recursive partitioning technique of the previous section will form the basis for our proof of the general (non-orthogonal) polygons as well. However, the more complex visibility structure of non-orthogonal polygons forces us to first establish an intermediate result for *monotone* polygons. Specifically, our proof structure works by partitioning the polygon into families of simpler *staircase-shaped* subpolygons. In the orthogonal case, staircase polygons are easily covered using 3 colors (Lemma 4), but non-orthogonal staircases appear to be more complicated. In non-orthogonal polygons a staircase subpolygon will remain to be both x-monotone and y-monotone, but we lose the property of having alternating convex and reflex right angles. To obtain our main result on arbitrary simple polygons we first show that these basic building blocks (more specific: non-orthogonal x-monotone polygons) admit a conflict-free coloring with $O(\log n)$ colors.

A second (albeit minor) is that a naive recursive partitioning using x-aligned and y-aligned visibility may not even terminate in general polygons, and so we appropriately modify the partitioning in the next section to ensure finite termination. We then use these results to show that arbitrary simple polygons have conflict-free chromatic guard number $O(\log^2 n)$.

In the following, we assume without loss of generality that our polygon is x-monotone.

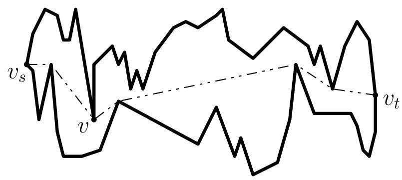

Fig. 7 Partition of a monotone polygon

Monotone Polygons Even monotone polygons require a careful analysis, and we are forced to consider special subcases, which are x-monotone polygons in which one of the two x-monotone chains is either completely convex or completely non-convex (concave). First we show that any monotone polygon can be split into two sets of mutually independent subpolygons that lie above or under a concave subchain. Then we will explain how a conflict-free covering of such subpolygons can be achieved with a composition mainly consisting of subpolygons over a convex subchain.

Lemma 6 *The conflict-free chromatic guard number for monotone polygons is at most twice the conflict-free chromatic guard number for monotone polygons over a concave chain.*

The monotone polygons are easily reduced to a collection of *independent* monotone polygons with a specialized structure, where one of the chains is either a line segment or a completely non-convex (concave) chain.

Specifically, given an x-monotone polygon, consider the shortest path between the leftmost vertex v_s and the rightmost vertex v_t, see Fig. 7. We can cut the polygon with line segments along this shortest path. Let A be the set of the subpolygons above the path from v_s to v_t and let B be the set of the subpolygons below the path. We prove Lemma 6 with the following claim:

Claim *The subpolygons in A (respectively B) are monotone polygons over (respectively under) a single edge or over (respectively under) a concave subchain. Furthermore all subpolygons in the same set are pairwise independent.*

(We would like to point out that two neighboring subpolygons of A always intersect in exactly one point v. This point is a vertex of the polygon. We can clearly see that v therefore also belongs to exactly *one* subpolygon of B, which will hence be responsible to guard v with a guard of unique color.)

Proof The subpolygons in A are clearly monotone because the original polygon P is monotone. Look at the lower subchain of a subpolygon in A. If it is a line segment then we are done, otherwise it is a subchain with at least one interior vertex. Assume for the sake of contradiction that one of the interior vertices is a convex vertex. The subchain is also a part of the shortest path from v_s to v_t. But since it contains a convex vertex one can take a shortcut by walking horizontally above that vertex, which is a contradiction to making cuts along the *shortest* path. Hence the subchain must be a concave subchain.

The subpolygons are also independent: Assume that there are two subpolygons P_1, P_2 with points $p_1 \in P_1$ and $p_2 \in P_2$ that are mutually visible. Then the line segment $\overline{p_1 p_2}$ doesn't intersect with the polygon P's boundary, hence it must intersect with both of the lower subchains of P_1 and P_2. But then we could take a shortcut along the line segment $\overline{p_1 p_2}$ instead of following the path from v_s to v_t, which is a contradiction to making cuts along the *shortest* path. Therefore the subpolygons in A must be independent.

Hence by Lemma 1 the conflict-free chromatic guard number of an arbitrary monotone polygon is at most twice the conflict-free chromatic guard number for monotone polygons over a concave chain. □

In the following, we show that monotone polygons over a concave chain have conflict-free chromatic guard number $O(\log n)$. The basic units of interest, however, turn out to be monotone polygons over an edge or over a *convex chain*.

Monotone Polygons over a Convex Subchain Let P_n be a monotone polygon over a single horizontal edge. Let $g(n)$ denote the smallest number of colors that ensure a conflict-free covering for any such P_n. Similar to our method for constructing staircases in orthogonal polygons, we consider decision lines through either horizontal edges with adjacent reflex vertices or through a vertex for which both of its neighbors have a higher y-coordinate. A decision line splits P_n in a lower part and two or more independent upper parts, of which at most one part contains more than $\lfloor n/2 \rfloor$ vertices. Then we construct a subpolygon that is x-monotone and y-monotone such that it contains the base edge of P_n and at each decision line follows the part with the most remaining vertices, see the left picture in Fig. 8. This subpolygon is *star-shaped* and can thus be guarded with a single guard. The remaining regions are mutually independent, x-monotone over a horizontal edge and contain at most $\lfloor n/2 \rfloor$ of P_n's vertices. We get the recurrence $g(n) \leq g(n/2) + 1$, which yields $g(n) \leq \log n$.

Now let's look at a monotone polygon over a single non-horizontal edge, without loss of generality ascending to the right. We show in the middle picture of Fig. 8 that P_n can be partitioned into a set of independent monotone polygons over a horizontal edge and a tilted monotone polygon over a horizontal edge. Hence by Lemma 1 for any such polygon we have $\chi(P_n) \leq 2g(n) \leq 2\log n$.

Monotone polygons P_n over a convex subchain are easily covered with $O(\log n)$ colors as well. The shortest path in P_n from the leftmost vertex to the rightmost vertex cuts off independent monotone polygons over a single edge. The remaining subpolygon is bounded by a concave chain on top and the convex chain at the bottom, see the right picture in Fig. 8. A polygon whose boundary can be separated at two convex vertices v_s and v_t into a convex subchain and a concave subchain is called a *spiral* polygon. Note that no new reflex vertex in a subpolygon can ever be created

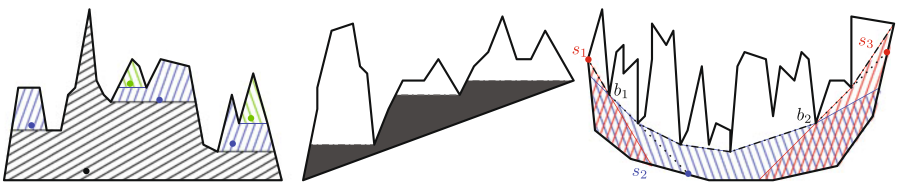

Fig. 8 x-monotone polygons over a single horizontal edge, a sloped edge and a convex chain

by our partitioning (since we use the shortest path to cut the polygon into pieces). Hence the spiral polygon has no reflex vertices that are not also reflex vertices of the original polygon and we can use the following result:

Lemma 7 *The conflict-free chromatic guard number for a spiral polygon P is at most* 2.

Proof Consider the following placement of colored guards in a spiral polygon: Starting at v_s, we place a guard s_1 on the first vertex of the convex subchain. Until we have covered the whole polygon we do the following: Let b_i be the last vertex on the concave chain visible from s_i. Draw a ray from b_i through the next vertex of the concave chain. Then we place s_{i+1} at the intersection of the ray with the convex subchain. To each guard s_i with odd i we assign color 1 and to each guard s_j with even j we assign color 2. This is indeed a conflict-free covering of P, see also [3]. □

Therefore by Lemma 1, for any monotone polygon P_n over a convex chain we have $\chi(P_n) \leq 2g(n) + \chi(P_{\text{spiral}}) \leq 2\log n + 2$.

Monotone Polygons over a Concave Subchain For monotone polygons P_n over a concave chain, we cut off independent monotone subpolygons over a horizontal edge as we did before in the case of a non-horizontal base edge.

This results in two additional independent subpolygons whose boundary consists of a lower subchain which is concave and strictly increasing (respectively strictly decreasing) and an upper subchain which is monotonically increasing (respectively monotonically decreasing).

Using at most $\log n$ colors, we place colored guards on the vertices of both of these concave subchains by the following recursive process: place a guard of color $i = 1$ at the middle vertex of the concave chain; increment the color to $i = 2$, place guards of color 2 at the middle vertex of the two halves of the subchain, and so on. Clearly this requires at most $\log n$ colors. We show the partition and the guard placement and coloring in Fig. 9.

Let P be the subpolygon over the strictly increasing concave subchain. If a point p in P is guarded by a guard on the concave subchain, it has a guard of unique color among all other guards on the concave subchain that see p: Let $l(p)$ be the list of all guard colors p can see. Between any two guards on the concave subchain that have the same color there must lie a guard of lower color between them. Hence the minimal color in $l(p)$ is a unique color among all guards that contain p in their visibility region. However, there may be regions in P not guarded by the

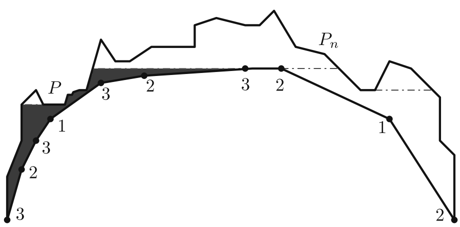

Fig. 9 Guard placement for a monotone polygon over a concave subchain

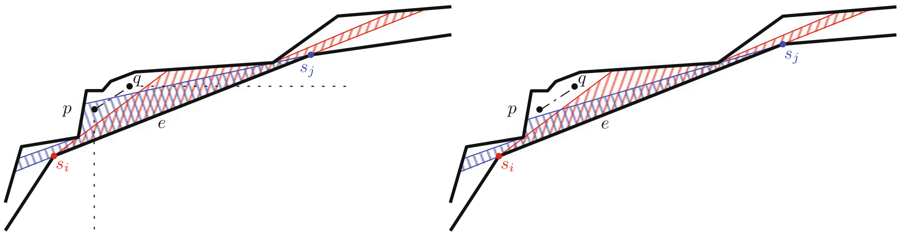

Fig. 10 Proof that the remaining subpolygons are independent subpolygons over convex chains

guards on the concave subchain. For these regions we have the following technical lemma:

Lemma 8 *If a point $p \in P$ is not visible from any of the guards on the concave subchain, then p lies in a not yet guarded simply connected region, which has the shape of a monotone subpolygon over a convex chain. Furthermore all such regions are independent.*

Proof To prove that (i) p lies in an unguarded region that has the shape of a monotone subpolygon over a convex subchain and that (ii) all such unguarded regions are independent, we show that any unguarded point q seen by p must lie in the same unguarded region as p and that this region has the claimed shape.

Hence assume that q is another point in the polygon that is not seen by any of the guards but is seen by p. This means that both p and q can't see any vertex of the lower subchain of the polygon and that the closed line segment $\overline{pq}$ does not intersect the polygon's boundary. Let e be the edge of the lower concave subchain that is closest to $\overline{pq}$ and let s_i and s_j be the guards placed on the vertices of e. Since p and q are not visible from s_i and s_j (by assumption) and the upper subchain of P is monotonically increasing, without loss of generality s_i must lie to the left of both p and q while s_j must lie above both p and q, see the left picture in Fig. 10.

Assume for the sake of contradiction that p and q are not in the same unguarded region. This means that the visibility region of some guard on the concave subchain intersects the line segment $\overline{pq}$. But then also at least one the visibility regions of s_i, s_j must intersect the line segment $\overline{pq}$. Assume without loss of generality that it is $V(s_j)$. Since the upper subchain is monotonically increasing and doesn't intersect $\overline{pq}$ and since s_i lies to the left of both p and q, there is no edge of the polygon that would "block" s_j from seeing all of the lower half of the line segment $\overline{pq}$ and hence the lower of the points p, q as well. But this contradicts the assumption that both p and q are unguarded.

Since the lower subchain is concave and the upper subchain is monotonically increasing, the boundary of the visibility regions $V(s_i)$ and $V(s_j)$ bound the region from below and the unguarded region is a monotone polygon over a convex subchain (see the right picture in Fig. 10). □

Thus, we have a partition into monotone polygons over a single horizontal edge (where we need at most $\log n$ colors), monotone polygons over a convex chain (at most $2\log n + 2$ colors) and the two independent subpolygons guarded by the guards on the concave chain (at most $\log n$ colors). By Lemma 1 we have that for any monotone polygon P_n over a concave chain, $\chi(P_n) \leq 4\log n + 2$. In view of Lemma 6, we now have the main result of this section.

Theorem 2 *The conflict-free chromatic guard number for monotone polygons on n vertices is* $k(n) = O(\log n)$.

5 Arbitrary Simple Polygons

Our proof structure for orthogonal polygons had the following form: We first partitioned the polygon into four different types of subpolygons and showed that the process terminates after a finite number of steps (Lemma 2). We then derived a necessary condition for two subpolygons of the same type not to be independent (Lemma 3). We then found a conflict-free covering using three colors for the basic building blocks, the staircase polygons (Lemma 4). We used this to get an upper bound of $4\log n$ for polygons corresponding to U-L-alternating subtrees (Lemma 5). Finally we put all subtrees together to achieve an $O(\log n)$ upper bound on the conflict-free chromatic guard number $k(n)$ for orthogonal polygons.

Our proof for non-orthogonal simple polygons follows the same outline, with appropriate differences spelled out. Specifically, given a n-vertex polygon P_n, we construct a partition $\{A_U, A_D, A_L, A_R\}$, where A_U, A_D, A_L, A_R, respectively, is the collection of up-polygons, down-polygons, left-polygons and right-polygons. We rotate P_n in such a way that we can start with a horizontal line segment which gives rise to a first subpolygon of type U. Now just applying our partitioning method from Sect. 3 as is has one potential problem: The partitioning might not stop.

Recall that the proof of Lemma 2 depended on the following fact: After an edge gets touched by a subpolygon of the partition for the first time, its vertices are at latest touched in the next step and completely surrounded by subpolygons in another step.

In the non-orthogonal case a vertex v can run through at most six states (Fig. 11): It can be not yet touched, or its interior angle can be covered by less than 90°,

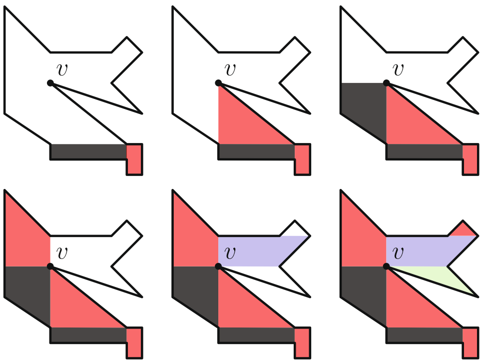

Fig. 11 The six possible vertex states

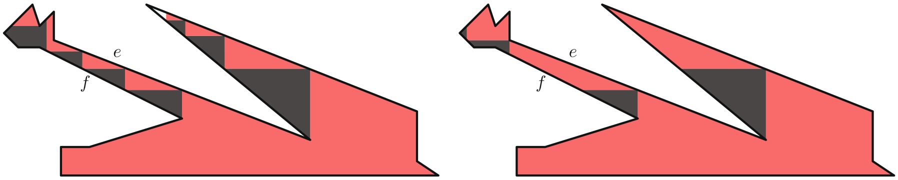

Fig. 12 Replacing a U-L-alternating path with an augmented U subpolygon

by less than 180°, by less than 270°, by less than 360°, or the vertex is completely surrounded.

Assume that in every step of the partitioning process a vertex of the polygon changes its state. Then after at most $5n$ steps every vertex of the polygon is completely surrounded by subpolygons of the partition, and hence we have a complete polygon partition. Furthermore if we can show that each subpolygon is adjacent to a vertex, then the number of subpolygons is in $O(n)$ (since a vertex is adjacent to at most 5 subpolygons, corresponding to its non-initial states during the process). However, we will have to adapt the partitioning process to achieve these properties.

The difference to the partitioning of orthogonal polygons is the following: Since the polygon's edges are no longer axis parallel, the partitioning process can be trapped between two edges e and f that ascend (respectively descend) to the same direction. This gives rise to a long and possibly infinite alternating path. Assume that the chain of subpolygons is without loss of generality ascending to the left, see the left picture in Fig. 12. In order to deal with this difficulty, we replace such an alternating path, i.e. *a path trapped between two edges ascending to the left*, with an augmented subpolygon as follows: When an edge e of P gets touched by a subpolygon P_s during the partitioning process and *no vertex changes its state during this step*, we extend the subpolygon P_s by a horizontal sweep to the left until it touches a vertex of e or f, see the right picture in Fig. 12. Note that we used a horizontal sweep since the depicted P_s is of type U; in case of a type L subpolygon one would use a vertical sweep going upwards.

All the other cases of edges ascending/descending to the left/right can be treated in a similar fashion. Hence the modification of the partitioning process with the described augmenting procedure allows us to prove a result on the number of steps of the partitioning algorithm:

Lemma 9 *The revised partitioning process gives a complete partition after a finite number of* (*at most* $5n$) *steps.*

Proof To prove the lemma we show that in every step of the revised partitioning process a vertex of the polygon changes its state. Assume for the sake of contradiction that there exists a *Step k* where for the first time in the process an edge $e = \{u, v\}$ is touched by a subpolygon and no vertex changes its state during this step. Without loss of generality e is ascending to the left and gets touched by a subpolygon P_{e1} of type U.

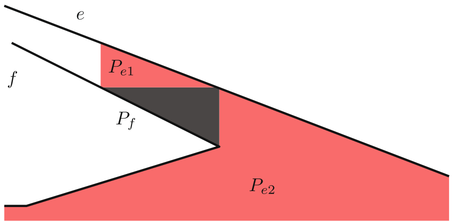

Fig. 13 Step k in the proof of Lemma 9

But this means that e must have been touched before, since otherwise (i) at least one of the vertices u, v would have changed its state or (ii) there exists a vertex w on one of the vertical line segments of P_{e1}'s boundary that would thus have changed its state, both cases being a contradiction to our assumption. Hence there exists a type U subpolygon P_{e2} that has touched e in Step $k-2$ and is connected to P_{e1} through a subpolygon P_f. Since e is ascending to the left, P_f is of type L, see Fig. 13.

The subpolygon P_f gives rise to the base edge of P_{e1} and this base edge can't contain a polygon vertex (otherwise this vertex would change its state in Step k because of P_{e1}, which contradicts our assumption). Consider the polygon edge f that contains the left vertex of P_{e1}'s base edge. Assume f ascends to the right. Then its top vertex would get touched by P_{e1} in Step k and would change its state, contradiction. Hence f ascends to the left, as does e. But then the revised partitioning process extends P_{e1} to the left until it touches a vertex of e or f and thus that vertex' state changes.

This again contradicts the assumption and proves that in every step of the revised partitioning process a vertex changes its state. Thus we get a complete partition after at most $5n$ steps. □

The replacement of alternating paths with a single polygon slightly changes the definitions of the subpolygon types used in the partitioning, but it does not change (i) the property of the building blocks being still x-monotone and y-monotone and (ii) the relations between subpolygons of the same type when it comes to visibility—we simply replaced an alternating path with a *shorter* alternating path. This means that the schematic tree of the revised partitioning process has the same properties as the original partitioning process in orthogonal polygons, in particular we get as a corollary from Lemma 3:

Lemma 10 *Let P be a polygon with the given revised partition and the schematic tree T. Let P_i and P_j be two arbitrary subpolygons of type U. Then, either* (*i*) *P_i and P_j are independent, or* (*ii*) *there must exist a U-L-alternating* (*or a U-R-alternating*) *directed path in T between P_i and P_j.*

This allows us to invoke the same coloring strategy as used in orthogonal polygons. We first focus on polygon regions corresponding to U-L-alternating trees. A polygon P_n corresponding to a U-L-alternating tree consists of type U and type L subpolygons; it "grows to the left". In place of Lemma 4, which states a constant conflict-free chromatic guard number for staircase polygons, we have Theorem 2, which gives an $O(\log n)$ upper bound for monotone polygons. We cover a part of

P_n with a polygon that is both x- and y-monotone: Starting from the lowest and rightmost vertex of P_n, at every decision line we follow the upper part with the most vertices. We need $O(\log n)$ colors to do this plus an additional color to cover the convex fans to its left as before. We are left with independent subtrees, all of size $\leq \lfloor n/2 \rfloor$. We recursively deal with each of them and cover all type U subpolygons of P_n in at most $\log n$ rounds. This leads to the following result.

Lemma 11 *Suppose P_n is a simple polygon with a partition that corresponds to a U-L-alternating schematic tree. Then a conflict-free coloring of all the type U subpolygons of P_n needs at most $O(\log^2 n)$ colors. The same bound also holds for a U-R-alternating schematic tree.*

The composition of U-L-alternating trees and U-R-alternating trees that we described earlier depended only on the condition of Lemma 3, which we preserved in the revised partition of arbitrary polygons, see Lemma 10. Considering this, we can put subtrees together as we did in the case of orthogonal polygons. Thus we finally get an upper bound for simple polygons.

Theorem 3 *The conflict-free chromatic guard number for non-orthogonal simple polygons on n vertices is $k(n) = O(\log^2 n)$.*

Remark 4 Similarly to the case of orthogonal polygons, a careful analysis of the guards placement in the "building blocks" of the covering of a U-L-alternating tree shows that the number of guards given by our algorithm is in $O(n)$. With respect to the existence of polygons where *any* guard set is of size $\Omega(n)$, it seems unlikely to come up with a conflict-free covering that needs much fewer guards.

6 Conclusions

The art gallery problems provide a conceptually clean and mathematically elegant framework to study many applied questions related to surveillance, monitoring and covering of a physical environment. In this paper, we studied a *chromatic* variant of the art gallery, where the primary concern is to minimize the number of distinct *colors* assigned to guards. Our two main results are that (i) every n-vertex simple polygon has a conflict-free chromatic art gallery coverage with $O(\log^2 n)$ colors, and (ii) if the polygon is orthogonal, then the number of colors is only $O(\log n)$. A stronger form of coloring, which requires all guards visible to a point to be distinct in colors, needs $\Omega(n)$ colors for simple polygons and $\Omega(\sqrt{n})$ for orthogonal polygons [3], showing that the weaker conflict-free condition gives a significant improvement in the number of colors.

Our work suggests several directions for future research. Perhaps the most natural question is to investigate the lower bounds on the number of colors needed. Currently, we have none. What is the tight bound for the simple non-orthogonal polygons? Finally, the line-of-sight visibility model is a crude model for wireless communication. Recently, Aichholzer et al. [1] have investigated an art gallery problem that allows the signal to penetrate k walls. One could consider our chromatic art gallery in a similar setting.

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