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Spans of special cycles of codimension less than 5

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Abstract. We show that the span of special cycles in the r-th Chow group of a Shimura variety of orthogonal type is finite dimensional, if $r < 5$. As our main tool, we develop the theory of Jacobi forms with rational index $M \in \mathrm{Mat}_N(\mathbb{Q})$.

1. Introduction

In 1956, Chow introduced cohomology groups attached to any variety X (see [6]), which, by now, are called Chow groups and are denoted by $\mathrm{CH}^r(X)$. They are constructed as formal sums up to rational equivalence of subvarieties of constant codimension r. The first Chow group is isomorphic to the Picard group, which classifies holomorphic line bundles up to isomorphisms. While Picard groups are difficult to study, our knowledge about Chow groups is even more restricted. Even their rank stays mysterious in most cases.

Special cycles generalize CM points and Hirzebruch–Zagier cycles, which historically, were the first instances whose class in cohomology were studied [9, 10]. In the 1980s, Kudla and Milson published a series of papers, in which they studied intersection numbers of special cycles, that is cohomology classes of special cycles. They found that the generating functions of intersection numbers of special cycles of codimension r are Siegel modular forms of genus r. Kudla [11] conjectured that these modularity results follow from a much strong modular property of special cycles considered as elements of the Chow group.

Conjecture 1.1. Given a Shimura variety X_Γ of orthogonal type, the generating function of codimension r cycles on X_Γ is a Siegel modular form of degree r taking values in the r-th Chow group $\mathrm{CH}^r(X_\Gamma)_\mathbb{C}$ of X_Γ.

In particular, the span of special cycles in $\mathrm{CH}^r(X_\Gamma)_\mathbb{C}$ has finite dimension.

In parallel, to Kudla's work, Borcherds finished his studies of product expansions of automorphic forms [2]. As a result, he was able to resolve completely Kulda's conjecture in the case of $r = 1$.

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Theorem 1.2 (Borcherds [3,4]). *For $r = 1$, Kudla's conjecture is true.*

Zhang dedicated his thesis to the above conjecture, and he was able to obtain partial results.

Theorem 1.3 (Zhang [21]). *If $r = 2$, the span of special cycles in $\mathrm{CH}^r(X_\Gamma)$ has finite dimension.*

In this paper, we extend the range for which this statement holds.

Theorem 1.4. *If $r < 5$, the span of special cycles in $\mathrm{CH}^r(X_\Gamma)$ has finite dimension.*

A more detailed statement can be found in Theorem 4.1.

Zhang used an ad hoc method to prove his result, and he stated that it "does not generalize to higher codimension cycles" ([21, p. 35]). We show that similar ideas allow to cover all cases $r < 5$. Zhang employed induction and standard vanishing results for elliptic modular forms of higher level. In his proof, these elliptic modular forms arose form Jacobi forms, which he did not consider directly. The key to our result is not to pass to elliptic modular forms, but to study Jacobi forms directly. This leads us to develop a theory of vector valued Jacobi forms whose indices are matrices with rational entries.

Jacobi forms show up in many important places, including the proof of Saito–Kurokawa's conjecture [1, 12–14, 20] and modern string theory [5, 7, 8]. They were originally defined as functions on $\mathbb{H} \times \mathbb{C}$ with an index $m \in \mathbb{Z}$ attached to them. Later, this notion was extended to functions on $\mathbb{H} \times \mathbb{C}^N$ with matrix index $M \in \mathrm{Mat}_N(\mathbb{Z})$ (see [19,22]). A further generalization shows up naturally while studying theta lifts or special cycles. We extend the theory of Jacobi forms to indices $M \in \mathrm{Mat}_N(\mathbb{Q})$. We show that there is an analog of the theta decomposition, which links Jacobi forms to vector valued elliptic modular forms. Theta decompositions of usual Jacobi forms in the most general case is due to Shimura and Ziegler [18, 22]. We will write $\mathrm{J}_{k,M}(\rho)$ for the space of Jacobi form of weight $k \in \frac{1}{2}\mathbb{Z}$, index $M \in \mathrm{Mat}_N(\mathbb{Q})$ and type ρ, where M is positive definite and symmetric, and ρ is a finite-dimensional unitary representation of the metaplectic Jacobi group (see Definition 3.1).

Theorem 1.5. *For given M and ρ there are functions $\theta_{M,\rho;\nu}$ such that for every k and $\phi \in \mathrm{J}_{k,M}(\rho)$, we have*

$$\phi(\tau, z) = \sum_\nu h_\nu(\tau)\theta_{M,\rho;\nu}(\tau, z).$$

The components h_ν form a vector valued elliptic modular form of weight $k - \frac{N}{2}$.

A more detailed statement and exact expressions for $\theta_{M,\rho;\nu}$ are provided in Theorem 3.17 below. The vanishing statement that is crucial to our argument, is given in Section 3.5.

The study of Jacobi forms requires classification of those representations of the full Jacobi group that occur for non-trivial Jacobi forms. We build standard representations of the metaplectic Jacobi group, that, as we show, occur as tensor factors of every irreducible representation. As a next step, we construct theta functions attached to each of these standard representations. This finally allows us to decompose any Jacobi forms.

A brief word on notation. This paper addresses people working on Shimura varieties of orthogonal type and working on Jacobi forms. Notation in these areas varies slightly, and we have decided to accommodate both as much as possible. In Section 3, which is dedicated to Jacobi forms, we use notation common in this area. In Section 4, we adopt the notation used by Zhang [21].

Section 2 contains preliminaries on modular forms and a vanishing result. In Section 3, we build up the theory of Jacobi forms. This includes theta decomposition, a vanishing theorem, and a proposition on the geometry of lattices. In Section 4, we combine these results in order to prove the main theorem.

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2. Vector valued modular forms

We start by defining vector valued elliptic modular forms, that we will later relate to Jacobi forms. The Poincaré upper half plane is $\mathbb{H} = \{\tau = u + iv : v > 0\} \subset \mathbb{C}$. We write q for $\exp(2\pi i \cdot \tau) = e(\tau)$. The metaplectic cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ is the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_2(\mathbb{R})$, the connected double cover of $\mathrm{SL}_2(\mathbb{R})$. Write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for a typical element of $\mathrm{SL}_2(\mathbb{R})$. Note that the elements of $\mathrm{Mp}_2(\mathbb{R})$ can be written as $(\gamma, \tau \mapsto \sqrt{c\tau + d})$, where the first component is an element of $\mathrm{SL}_2(\mathbb{R})$ and the second is a holomorphic root on $\mathbb{H}$, that we usually write as $\omega : \mathbb{H} \to \mathbb{C}$. Since there are two branches of the square root, this yields indeed a double cover of $\mathrm{SL}_2(\mathbb{R})$. The product in $\mathrm{Mp}_2(\mathbb{Z})$ is defined as

$$(\gamma_1, \omega_1)(\gamma_2, \omega_2) = (\gamma_1\gamma_2, \omega_1 \circ \gamma_2 \cdot \omega_2).$$

The elements

$$T := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \tau \mapsto 1\right) \quad \text{and} \quad S := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \tau \mapsto \sqrt{\tau}\right)$$

generate $\mathrm{Mp}_2(\mathbb{Z})$. The root in the definition of the element S is the principal branch, which maps i to $\exp(\frac{1}{4}\pi i)$.

An action of $\mathrm{SL}_2(\mathbb{R})$ (and thus of $\mathrm{Mp}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z})$) on $\mathbb{H}$ is given by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

Given a representation ρ of $\mathrm{Mp}_2(\mathbb{Z})$ with representation space $V(\rho)$, $k \in \frac{1}{2}\mathbb{Z}$, and a map $f : \mathbb{H} \to V(\rho)$, we define

$$(f|_{k,\rho}(\gamma, \omega))(\tau) := \omega(\tau)^{-2k}\rho((\gamma, \omega))^{-1} f(\gamma\tau)$$

for all $(\gamma, \omega) \in \mathrm{Mp}_2(\mathbb{R})$. In this paper, our interest lies in Weil representations, and restrictions of them. Their crucial property is that their kernel has finite index.

Definition 2.1. Let $(\rho, V(\rho))$ be a representation of $\mathrm{Mp}_2(\mathbb{Z})$ with finite index kernel. A vector valued modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ and type ρ is a holomorphic function $f : \mathbb{H} \to V(\rho)$ such that the following conditions are satisfied:

(i) For all $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ we have $f|_{k,\rho}\gamma = f$.

(ii) We have $\|f(\tau)\| = O(1)$ as $y \to \infty$, where $\|\cdot\|$ is some norm on $V(\rho)$.

We write $\mathrm{M}_k(\rho)$ for the space of modular forms of weight k and type ρ.

Representations ρ with finite index kernel are automatically unitary with respect to a suitable scalar product, $\rho(T)$ is diagonalizable, and $f \in \mathrm{M}_k(\rho)$ has Fourier expansion

$$f(\tau) = \sum_{0 \le m \in \mathbb{Q}} c(\phi; m) q^m$$

with $c(\phi; m) \in V(\rho)$.

Our chief interest in vector valued modular forms originates in the following vanishing result.

Proposition 2.2. *Suppose that $\phi \in M_k(\rho)$ and $c(\phi; m) = 0$ for all $m < \frac{k}{12} + 1$. Then we have $\phi = 0$.*

Proof. Let $l = \lfloor \frac{k}{12} \rfloor + 1 > \frac{k}{12}$. Given ϕ as in the statement, we have $\Delta^{-l}\phi \in M_{k-12l}(\rho)$, where Δ is the unique cusp form of weight 12 and trivial type. If $\phi \neq 0$, then we obtain a non-zero modular form of negative weight, which cannot be. □

3. Jacobi forms

We now prepare for defining vector valued Jacobi forms. Given $0 < N \in \mathbb{Z}$, let

$$\mathbb{H}^{\mathrm{J}(N)} := \mathbb{H} \times \mathbb{C}^N$$

be the Jacobi upper half space. The metaplectic cover of the centrally extended Jacobi group is

$$\widetilde{\Gamma}^{\mathrm{J}(N)} := \mathrm{Mp}_2(\mathbb{Z}) \ltimes \mathrm{Mat}_{N,2}(\mathbb{Z}) \widetilde{\times} \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Z}),$$

where $\widetilde{\times}$ denotes the central extension as specified by the equation below. We denote the group of symmetric matrices by $\mathrm{Mat}_N^{\mathrm{t}}(\,\cdot\,)$, and accordingly we write ${}^{\mathrm{t}}(\,\cdot\,)$ for the transpose of a vector or matrix.

Typical elements of the second component are written (λ, μ), $\lambda, \mu \in \mathbb{Z}^N$. The product in $\widetilde{\Gamma}^{\mathrm{J}(N)}$ is given by

$$\begin{aligned}&((\gamma_1, \omega_1), (\lambda_1, \mu_1), \kappa_1)((\gamma_2, \omega_2), (\lambda_2, \mu_2), \kappa_2)\\ &\qquad = ((\gamma_1, \omega_1)(\gamma_2, \omega_2), (\lambda_1, \mu_1)\gamma_2 + (\lambda_2, \mu_2), \kappa_1 + \kappa_2 + \lambda_1\,{}^{\mathrm{t}}\mu_2 - \mu_1\,{}^{\mathrm{t}}\lambda_2).\end{aligned}$$

In particular, the semidirect product is defined via the natural right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{Mat}_{N,2}(\mathbb{Z})$. As before, we usually write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for elements of $\mathrm{SL}_2(\mathbb{Z})$. We frequently use the notation

$$[\lambda, \mu, \kappa]^{\mathrm{J}} = ((I_2, \tau \mapsto 1), (\lambda, \mu), \kappa)$$

for elements of the Heisenberg-like group

$$\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)}) := \mathrm{Mat}_{N,2}(\mathbb{Z}) \widetilde{\times} \mathrm{Mat}_N(\mathbb{Z}) \subset \widetilde{\Gamma}^{\mathrm{J}(N)}.$$

In most cases, elements $\gamma^{\mathrm{J}} \in \widetilde{\Gamma}^{\mathrm{J}(N)}$ are given as tuples $((\gamma, \omega), (\lambda, \mu), \kappa)$. In order to lighten notation, we occasionally switch the position of ω, writing

$$(\gamma^{\mathrm{J}}, \omega) = ((\gamma, (\lambda, \mu), \kappa), \omega)$$

for $((\gamma, \omega), (\lambda, \mu), \kappa) \in \widetilde{\Gamma}^{\mathrm{J}(N)}$.

There is an action $\widetilde{\Gamma}^{\mathrm{J}(N)} \times \mathbb{H}^{\mathrm{J}(N)} \to \mathbb{H}^{\mathrm{J}(N)}$ given by

$$(\gamma^{\mathrm{J}}, \omega)(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right). \tag{3.1}$$

We next give a family of actions on functions on $\mathbb{H}^{\mathrm{J}(N)}$. Let $\mathrm{tr}(\cdot)$ denote the trace of a matrix. For convenience, we set

$$\alpha_M\big((\gamma, (\lambda, \mu), \kappa); \tau, z\big) = e\left(\frac{-cM[z + \lambda\tau + \mu]}{c\tau + d} - M[\lambda]\tau - 2{}^{\mathrm{t}}\lambda M z - \tfrac{1}{2}\mathrm{tr}(M\kappa)\right)$$

for any $M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$. Let ρ be a finite-dimensional representation of $\widetilde{\Gamma}^{\mathrm{J}(N)}$ with representation space $V(\rho)$. Fix a weight $k \in \frac{1}{2}\mathbb{Z}$ and a (Jacobi) index $M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Z})$ which is positive definite and has even diagonal entries. Note that in the case $N = 1$, it is more common to use indices $m \in \mathbb{Q}$. These notions are related by $M = (2m)$. We abbreviate ${}^{\mathrm{t}}zMz$ ($z \in \mathbb{C}^N$) by $M[z]$. Recall that $e(x) = \exp(2\pi i \cdot x)$. Given $\phi : \mathbb{H}^{\mathrm{J}(N)} \to V(\rho)$, we define

$$(\phi|_{k,M,\rho}(\gamma^{\mathrm{J}}, \omega))(\tau, z) = \omega(c\tau + d)^{-2k}\alpha_M(\gamma^{\mathrm{J}}; \tau, z)^{-1}\rho((\gamma^{\mathrm{J}}, \omega))^{-1}\phi(\gamma^{\mathrm{J}}(\tau, z)).$$

The next definition generalizes a definition made in [22] to vector valued Jacobi forms which have non-integral indices.

Definition 3.1. Suppose that $k \in \frac{1}{2}\mathbb{Z}$, $M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$, and $(\rho, V(\rho))$ is a representation of $\widetilde{\Gamma}^{\mathrm{J}(N)}$ whose kernel has finite index. A holomorphic function $\phi : \mathbb{H}^{\mathrm{J}(N)} \to V(\rho)$ is called a Jacobi form of weight k, index M, and type ρ if

(i) we have $\phi|_{k,M,\rho}(\gamma^{\mathrm{J}}, \omega) = \phi$ for all $(\gamma^{\mathrm{J}}, \omega) \in \widetilde{\Gamma}^{\mathrm{J}(N)}$,

(ii) we have $\|\phi(\tau, \alpha\tau + \beta)\| = O(1)$ for all $\alpha, \beta \in \mathbb{Q}^N$ and some norm $\|\cdot\|$ on $V(\rho)$.

We denote the space of Jacobi forms of weight k, index M, and type ρ by $\mathrm{J}_{k,M}(\rho)$.

Throughout this section, we assume that k, M, and ρ satisfy the assumptions in the previous definition.

We write ζ^r for $e({}^{\mathrm{t}}rz)$. If ρ is unitary, the second condition in Definition 3.1 is equivalent to

$$\phi(\tau, z) = \sum_{m \in \mathbb{Q},\, r \in \mathbb{Q}^N} c(\phi; m, r) q^m \zeta^r,$$

for Fourier coefficients $c(\phi; m, r)$ that vanish if $2\det(M)n - M^{\#}[r] < 0$ or $m < 0$. The adjunct of M is denoted by $M^{\#}$. Because Jacobi forms are invariant under the action of $[\lambda, \mu, 0]^{\mathrm{J}}$ ($\lambda, \mu \in \mathbb{Z}^N$), the Fourier coefficients $c(\phi; m, r)$ satisfy the following relations, which are connected to the statement of Theorem 3.17:

$$c(\phi; m, r) = c(\phi; m + \tfrac{1}{2}M[\lambda] + {}^{\mathrm{t}}r\lambda, r + M\lambda), \tag{3.2}$$

$$c(\phi; m, -r) = \omega(-1)^{2k}\rho((-I_2, \omega))\, c(\phi; m, r). \tag{3.3}$$

Note that we have $\mathrm{J}_{k,M}(\rho_1 \oplus \rho_2) = \mathrm{J}_{k,M}(\rho_1) \oplus \mathrm{J}_{k,M}(\rho_2)$. This allows us to reduce most considerations to the case of irreducible ρ.

Among experts it is well known that on can restrict to positive definite M. The next proposition, however, is not in the literature. We do not need it, but it seems appropriate to justify our future assumption $M > 0$.

Proposition 3.2. *Suppose that M is not positive semidefinite. Then*

$$\mathrm{J}_{k,M}(\rho) = \{0\}.$$

If M is positive semidefinite, but not positive definite, then there is an $s \in \mathrm{Mat}^{\mathrm{t}}_{N,N-1}(\mathbb{Z})$ such that

$$\mathrm{J}_{k,M}(\rho) \cong \mathrm{J}_{k,M[s]}(\rho) \quad \textit{via} \quad \phi(\tau, z) \mapsto \phi(\tau, sz'),$$

where $z' \in \mathbb{C}^{N-1}$.

Proof. Assume that $\mathrm{J}_{k,M}(\rho) \neq \{0\}$. We first show that M is positive semidefinite. Suppose there was $\lambda \in \mathbb{Z}^N$ such that $M[\lambda] < 0$. Fix some $0 \neq \phi \in \mathrm{J}_{k,M}(\rho)$. By (3.2), for any $m \in \mathbb{Q}$, $r \in \mathbb{Q}^N$, and $l \in \mathbb{Z}$, we have

$$c(\phi; m, r) = c(\phi; m + \tfrac{l^2}{2} M[\lambda] + l\,{}^{\mathrm{t}}r\lambda, r + lM\lambda).$$

By choosing l large enough we can force the second argument on the right hand side to be negative, which shows that $c(\phi; m, r) = 0$. Since m and r were arbitrary, this shows that $\phi = 0$, contradicting our choice.

We are reduced to the case of positive semidefinite M. If M is not positive definite, there is a primitive vector $\lambda \in \mathbb{Z}^N$ with $M[\lambda] = 0$. Fix $g \in \mathrm{GL}_N(\mathbb{Z})$ with last column λ. Then

$$\mathrm{J}_{k,M} \cong \mathrm{J}_{k,M[g]} \quad \text{via} \quad \phi(\tau, z) \mapsto \phi(\tau, gz).$$

Writing $M_{i,j}$ for the (i, j)-th entry of M, it thus suffices to treat the case of $M_{N,N} = 0$. Since M is positive semidefinite, we have $M_{N,i} = 0$ for all $1 \leq i \leq N$.

Fix τ and $z_1, \ldots, z_{N-1}$, and let z_N vary. Applying the transformation law for Jacobi forms, we see that

$$\phi(\tau, (z_1, \ldots, z_{N-1}, z_N + \lambda\tau + \mu)) = \phi(\tau, (z_1, \ldots, z_N))$$

for all $\lambda, \mu \in \mathbb{Z}$. That is

$$z_N \mapsto \phi(\tau, (z_1, \ldots, z_N))$$

is holomorphic and bounded. Hence it is constant. This shows that we can choose

$$s = (\delta_{i,j})_{1 \leq i \leq N,\, 1 \leq j \leq N-1},$$

where δ is the Kronecker delta. □

3.1. Representations of the Heisenberg-like part of $\Gamma^{\mathrm{J}(N)}$. Representations of the Heisenberg-like group $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)}) \subset \widetilde{\Gamma}^{\mathrm{J}(N)}$ that occur for non-zero Jacobi forms can be classified with little effort. Here and throughout Section 3.1, we assume that π is an irreducible representation of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$ with finite index kernel.

Since $\mathrm{Mat}_N(\mathbb{Z})$ it central in $\mathrm{H}(\widetilde{\Gamma}^{(\mathrm{J}(N)})$, there is $M_\pi \in \mathrm{Mat}_N(\mathbb{Q})$ such that $\pi([0, 0, -\kappa]^{\mathrm{J}})$ acts as $e(\mathrm{tr}(\kappa M_\pi))$. Note that M_π is only well-defined up to elements in $\mathrm{Mat}_N(\mathbb{Z})$. Throughout the paper we assume that M_π is symmetric. We justify this assumption in Proposition 3.16. Since M_π can be freely modified by elements in $\mathrm{Mat}_N(\mathbb{Z})$, we can replace M_π by $M_\pi + l\,\mathrm{id}_N$ for sufficiently large $l \in \mathbb{Z}$, and thus focus on positive definite M_π. We say that π has central character $e(\mathrm{tr}(M_\pi \cdot))$.

Given a matrix $0 < M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$, fix a Smith normal form $M = UDV$. Decompose D as $D_{\mathbb{Z}} D^{\mathbb{Z}}$, where $D_{\mathbb{Z}}$ is the diagonal matrix whose entries are the numerators of the entries of D. Set

$$M_{\mathbb{Z}} = UD_{\mathbb{Z}}V \quad \text{and} \quad M^{\mathbb{Z}} = UD^{\mathbb{Z}}V. \tag{3.4}$$

We can view $M_{\mathbb{Z}}$ and $M^{\mathbb{Z}}$ as the integral and fractional part of M, respectively.

Lemma 3.3. *We have*

$$M_{\mathbb{Z}}\mathbb{Z}^N = M\mathbb{Z}^N \cap \mathbb{Z}^N, \qquad M_{\mathbb{Z}}^{-1}\mathbb{Z}^N = M^{-1}\mathbb{Z}^N + \mathbb{Z}^N,$$
$$M^{\mathbb{Z}}\mathbb{Z}^N = M\mathbb{Z}^N + \mathbb{Z}^N, \quad (M^{\mathbb{Z}})^{-1}\mathbb{Z}^N = M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N.$$

Proof. Given the Smith normal form of $M = UDV$, write $M_{\mathbb{Z}} = UD_{\mathbb{Z}}V$ as in (3.4). We have

$$M_{\mathbb{Z}}\mathbb{Z}^N = UD_{\mathbb{Z}}\mathbb{Z}^N = U(D\mathbb{Z}^N \cap \mathbb{Z}^N) = M\mathbb{Z}^N \cap \mathbb{Z}^N$$

and

$$M_{\mathbb{Z}}^{-1}\mathbb{Z}^N = V^{-1}D_{\mathbb{Z}}^{-1}\mathbb{Z}^N = V^{-1}(D^{-1}\mathbb{Z}^N + \mathbb{Z}^N) = M^{-1}\mathbb{Z}^N + \mathbb{Z}^N.$$

This proves the first and second equality. The remaining assertions can be proved in the same way. □

Lemma 3.4. *For $\lambda \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$, we have $M[\lambda] \in \mathbb{Z}$.*

Proof. We have $\lambda \in (M^{\mathbb{Z}})^{-1}\mathbb{Z}^N$, and hence the claim follows from

$$M[(M^{\mathbb{Z}})^{-1}] = (UDV)[V^{-1}(D^{\mathbb{Z}})^{-1}U^{-1}] = {}^{\mathrm{t}}U^{-1}(D^{\mathbb{Z}})^{-1}\,{}^{\mathrm{t}}V^{-1}U(D(D^{\mathbb{Z}})^{-1})U^{-1},$$

where U and V were used to define $M^{\mathbb{Z}}$ in (3.4). The fact $D(D^{\mathbb{Z}})^{-1} \in \mathrm{Mat}_N(\mathbb{Z})$ shows that

$$M[(M^{\mathbb{Z}})^{-1}\mathbb{Z}^N] = M[(M^{\mathbb{Z}})^{-1}][\mathbb{Z}^N] \subseteq \mathbb{Z}.$$ □

For $\alpha, \beta \in \mathbb{Q}^N$, define representations $\pi_{M,\alpha,\beta}$ of $\mathrm{H}(\widetilde{\Gamma}^{(\mathrm{J}(N)})$ as follows. The central character of $\pi_{M,\alpha,\beta}$ is $e(\frac{1}{2}\mathrm{tr}(M\,\cdot))$. Its representation space is

$$V_{M,\alpha,\beta} = \mathbb{C}[(M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,\mathbb{Z}^N] = \mathrm{span}(\mathrm{e}_r : r \in (M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,\mathbb{Z}^N). \tag{3.5}$$

The action on e_r is given by

$$\pi([\lambda, 0, 0]^{\mathrm{J}})\mathrm{e}_r = e({}^{\mathrm{t}}\alpha\lambda)\mathrm{e}_{r+M^{\mathbb{Z}}\lambda},$$
$$\pi([0, \mu, 0]^{\mathrm{J}})\mathrm{e}_r = e({}^{\mathrm{t}}(\beta + r)\mu)\mathrm{e}_r.$$

Remark 3.5. The referee pointed out that Propositions 3.6 and 3.7 can be considered as a special case of the Stone–von-Neumann Theorem, and that the representations that we study are expressible in terms of Schrödinger representations. Indeed, since we are interested in representations whose kernel has finite index, both propositions are covered by Mumford's generalization of the Stone–von-Neumann Theorem to group schemes ([15, Proposition 3.7], see [17] for a complete proof).

Proposition 3.6. *For* $0 < M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$ *and* $\alpha, \beta \in \mathbb{Q}^N$, $\pi_{M,\alpha,\beta}$ *is an irreducible representation of* $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$. *Two representations* $\pi_{M_1,\alpha_1,\beta_1}$ *and* $\pi_{M_2,\alpha_2,\beta_2}$ *are isomorphic if and only if*

$$M_1 - M_2 \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Z}), \quad \alpha_1 - \alpha_2, \beta_1 - \beta_2 \in M\mathbb{Z}^N + \mathbb{Z}^N.$$

Proof. Clearly, the action of both $[\lambda, 0, 0]^{\mathrm{J}}$ and $[0, \mu, 0]^{\mathrm{J}}$ on e_r only depends on r modulo $\mathbb{Z}^N$. We have to check that

$$\pi([\lambda, 0, 0]^{\mathrm{J}})\pi([0, \mu, 0, 0]^{\mathrm{J}})\pi([0, 0, 2\lambda\,{}^{\mathrm{t}}\mu]^{\mathrm{J}})\mathrm{e}_r = e({}^{\mathrm{t}}\mu M\lambda)e({}^{\mathrm{t}}\alpha\lambda)e({}^{\mathrm{t}}(\beta + r)\mu)\mathrm{e}_{r+M^{\mathbb{Z}}\lambda}$$

equals

$$\pi([0, \mu, 0]^{\mathrm{J}})\pi([\lambda, 0, 0]^{\mathrm{J}})\mathrm{e}_r = e({}^{\mathrm{t}}\alpha\lambda)e({}^{\mathrm{t}}(\beta + r + M\lambda)\mu)\mathrm{e}_{r+M^{\mathbb{Z}}\lambda},$$

which is obvious. This show that $\pi_{M,\alpha,\beta}$ is a representation.

Observe that the set $\{[0, \mu, 0]^{\mathrm{J}} : \mu \in \mathbb{Z}^N\}$ forms a commutative subgroup of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$. The system of eigenvalues of e_r under the action of $[0, \mu, 0]^{\mathrm{J}}$ is different for distinct elements $r \in (M\mathbb{Z}^N + \mathbb{Z}^N) / \mathbb{Z}^N$. Further, the corresponding eigenspaces are permuted transitively by the action of $[\lambda, 0, 0]^{\mathrm{J}}$ ($\lambda \in \mathbb{Z}^N$). Thus we see that $\pi_{M,\alpha,\beta}$ is irreducible.

In the remainder of this proof, we use Lemma 3.3 without further reference. Let $\pi_{M_1,\alpha_1,\beta_1}$ and $\pi_{M_2,\alpha_2,\beta_2}$ by isomorphic representations. By checking central characters, we find that $M_1 - M_2 \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Z})$. The subgroup of all $[0, \mu, 0]^{\mathrm{J}}$ ($\mu \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$) acts by $e({}^{\mathrm{t}}\beta_1\mu)$ and $e({}^{\mathrm{t}}\beta_2\mu)$ on V_{M_1,α_1,β_2} and V_{M_2,α_2,β_2}, respectively. Consequently, we have

$${}^{\mathrm{t}}\mu(\beta_1 - \beta_2) \in \mathbb{Z} \quad \text{for all } \mu \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N.$$

We conclude that $\beta_1 - \beta_2 \in M\mathbb{Z}^N + \mathbb{Z}^N$. Finally, $[\lambda, 0, 0]^{\mathrm{J}}$ ($\lambda \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$) acts by scalars and the attached eigenvalues are $e({}^{\mathrm{t}}\alpha_1\lambda)$ and $e({}^{\mathrm{t}}\alpha_2\lambda)$. In analogy to the previous considerations, we find that ${}^{\mathrm{t}}\lambda(\alpha_1 - \alpha_2) \in \mathbb{Z}$ for all $\lambda \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$. Hence $\alpha_1 - \alpha_2 \in M\mathbb{Z}^N + \mathbb{Z}^N$.

Conversely, suppose that

$$M_1 - M_2 \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Z}), \quad \alpha_1 - \alpha_2,\ \beta_1 - \beta_2 \in M\mathbb{Z}^N + \mathbb{Z}^N.$$

We have to show that $\pi_{M_1,\alpha_1,\beta_1} \cong \pi_{M_2,\alpha_2,\beta_2}$. First of all, the central characters coincide. Denote the previously chosen basis elements of V_{M_1,α_1,β_1} and V_{M_2,α_2,β_2} by $\mathrm{e}_{r,1}$ and $\mathrm{e}_{r,2}$. Let $\alpha = (M^{\mathbb{Z}})^{-1}(\alpha_2 - \alpha_1) \in \mathbb{Z}^N$ and $\beta = \beta_1 - \beta_2 \in M\mathbb{Z}^N + \mathbb{Z}^N$. We claim that the map

$$\iota : \mathrm{e}_{r,1} \mapsto e({}^{\mathrm{t}}\alpha r)\mathrm{e}_{r+\beta,2}$$

intertwines $\pi_{M_1,\alpha_1,\beta_1}$ and $\pi_{M_2,\alpha_2,\beta_2}$. We have

$$\begin{aligned}\iota(\pi_{M_1,\alpha_1,\beta_1}([0, \mu, 0]^{\mathrm{J}})\mathrm{e}_{r,1}) &= e({}^{\mathrm{t}}\alpha r)e({}^{\mathrm{t}}(\beta_1 + r)\mu)\mathrm{e}_{r+\beta,2}\\ &= e((\beta_2 + r + \beta)\,{}^{\mathrm{t}}\mu)e({}^{\mathrm{t}}\alpha r)\mathrm{e}_{r+\beta,2}\\ &= \pi_{M_2,\alpha_2,\beta_2}([0, \mu, 0]^{\mathrm{J}})\iota(\mathrm{e}_{r,1})\end{aligned}$$

and

$$\begin{aligned}\iota(\pi_{M_1,\alpha_1,\beta_1}([\lambda, 0, 0]^{\mathrm{J}})\mathrm{e}_{r,1}) &= e({}^{\mathrm{t}}\alpha_1\lambda)e({}^{\mathrm{t}}\alpha(r + M^{\mathbb{Z}}\lambda))\mathrm{e}_{r+M^{\mathbb{Z}}\lambda+\beta,2}\\ &= e({}^{\mathrm{t}}\alpha_2\lambda)e({}^{\mathrm{t}}\alpha r)\mathrm{e}_{r+M^{\mathbb{Z}}\lambda+\beta,2}\\ &= \pi_{M_2,\alpha_2,\beta_2}([\lambda, 0, 0]^{\mathrm{J}})\iota(\mathrm{e}_{r,1}).\end{aligned}$$

This completes the proof of Proposition 3.6. □

Proposition 3.7. *Fix $0 < M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$. Any irreducible, finite-dimensional, unitary representation of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$ with central character $e(\frac{1}{2}\mathrm{tr}(M\,\cdot))$ is isomorphic to $\pi_{M,\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{Q}^N$.*

Proof. Fix a representation π as in the assumption. Since the set $\{[0,\mu,0]^{\mathrm{J}} : \mu \in \mathbb{Z}^N\}$ is a commutative subgroup of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$, we can find a basis of eigenvectors v_i, $1 \le i \le \dim \pi$. Denote the eigenvalue of an eigenvector v under $[0,\mu,0]^{\mathrm{J}}$ by $e({}^{\mathrm{t}}\beta(v)\mu)$. A calculation that is analogous to the computations in the proof of Proposition 3.6 shows that $\pi([\lambda,0,0]^{\mathrm{J}})\,v$ is an eigenvector under the action of $[0,\mu,0]^{\mathrm{J}}$ if so is v. We have $\beta(\pi([\lambda,0,0]^{\mathrm{J}})\,v) = \beta(v) + M^{\mathbb{Z}}\lambda$.

Fix any eigenvector v. Since the representation π is unitary and finite dimensional, the orbit $\pi(v) = \{[\lambda,0,0]^{\mathrm{J}} \cdot \mathrm{span}_{\mathbb{C}}(v) : \lambda \in \mathbb{Z}^N\}$ is finite. By inspection of $\beta(\pi([\lambda,0,0]^{\mathrm{J}})\,v,\mu)$, we find that the cardinality $\#\pi(v)$ is a multiple of $\#(M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,\mathbb{Z}^N$. If it is equal, let $e({}^{\mathrm{t}}\alpha\lambda)$ be the eigenvalue of v under $[\lambda,0,0]^{\mathrm{J}}$ ($\lambda \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$). It is straightforward to verify that π is isomorphic to $\pi_{M,\alpha,\beta(v)}$ via the map

$$[\lambda,0,0]^{\mathrm{J}} v \mapsto e({}^{\mathrm{t}}\alpha\lambda)\,\mathrm{e}_{M^{\mathbb{Z}}\lambda}.$$

To complete the proof, we show that π is reducible if $\#\pi(v) \neq \#(M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,\mathbb{Z}^N$. Define

$$B = \mathbb{Z}^N / \{\lambda \in \mathbb{Z}^N : [\lambda,0,0]^{\mathrm{J}}\ \mathrm{span}_{\mathbb{C}}(v) = \mathrm{span}_{\mathbb{C}}(v)\}.$$

Set $\tilde{v} = \sum_{\lambda \in B} [\lambda,0,0]^{\mathrm{J}}\,v$, where we fix one choice of representatives of B. By the construction of B, we see that $\tilde{v}$ is an eigenvector under all $[0,\mu,0]^{\mathrm{J}}$ ($\mu \in \mathbb{Z}^N$). Straightforward computation of the corresponding orbit yields $\#\pi(\tilde{v}) = \#(M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,\mathbb{Z}^N$, and so $\tilde{v}$ generates a subrepresentation of π. This finishes the proof. □

3.2. Representations of $\Gamma^{\mathrm{J}(N)}$. We have classified representations of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$, and the next step is to study in which manner they occur as building blocks of representations of $\widetilde{\Gamma}^{\mathrm{J}(N)}$. We henceforth assume that ρ is an irreducible representation of $\widetilde{\Gamma}^{\mathrm{J}(N)}$ with finite index kernel and central character $e(\frac{1}{2}\mathrm{tr}(M\,\cdot))$. The restriction of ρ to $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$ is denoted by $\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})}$.

Lemma 3.8. *Suppose that π is a subrepresentation of $\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})}$. If π is isomorphic to $\pi_{M,\alpha,\beta}$, then for $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$, $\gamma\pi$ is also a subrepresentation of $\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})}$. It is isomorphic to $\pi_{M,\alpha',\beta'}$, where $(\alpha',\beta') = (\alpha,\beta)\,{}^{\mathrm{t}}\gamma$ and γ acts via the projection $\mathrm{Mp}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z})$.*

Furthermore, the image of e_r given in (3.5) *under γ is uniquely determined by relations in $\widetilde{\Gamma}^{\mathrm{J}(N)}$.*

Corollary 3.9. *Let $\tilde{\pi}_{M,\alpha,\beta}$ denote the isotypical components of $\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})}$. Then we have*

$$\gamma\tilde{\pi}_{M,\alpha,\beta} = \tilde{\pi}_{M,\alpha',\beta'},$$

where $(\alpha',\beta') = (\alpha,\beta)\gamma$.

Proof of Lemma 3.8. Any irreducible representation of the group $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$ with central character $e(\frac{1}{2}\mathrm{tr}(M\,\cdot))$ has dimension $(M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,\mathbb{Z}^N$. By the commutation rules in $\widetilde{\Gamma}^{\mathrm{J}(N)}$, $\gamma\tilde{\pi}_{M,\alpha,\beta}$ is hence irreducible. Proposition 3.7 shows that it is isomorphic to $\pi_{M,\alpha',\beta'}$ for some $\alpha',\beta' \in \mathbb{Q}^N$. In order to compute the vectors α' and β', it suffices to consider the action of $[\lambda,0,0]^{\mathrm{J}}$ ($\lambda \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$) and $[0,\mu,0]^{\mathrm{J}}$ ($\lambda \in M^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$) on $\mathrm{e}_0 \in V_{M,\alpha,\beta}$.

We find that

$$[\lambda,0,0]^{\mathrm{J}}\gamma e_0 = \gamma[a\lambda,b\lambda,0]^{\mathrm{J}}e_0 = e(-abM[\lambda])e({}^{\mathrm{t}}\beta b\lambda)e({}^{\mathrm{t}}\alpha a\lambda)e_0,$$
$$[0,\mu,0]^{\mathrm{J}}\gamma e_0 = \gamma[c\mu,d\mu,0]^{\mathrm{J}}e_0 = e(-cdM[\mu])e({}^{\mathrm{t}}\beta d\mu)e({}^{\mathrm{t}}\alpha c\mu)e_0.$$

Since, by Lemma 3.4, $M[\lambda]$ and $M[\mu]$ are integral, the proof of the first half of Lemma 3.8 is complete.

Since π is unitary, the second half of Lemma 3.8 is a consequence of the simple observation that the system eigenvalues under $[0,\mu,0]^{\mathrm{J}}$ ($\mu\in\mathbb{Z}$) of eigenvectors in $V_{M,\alpha',\beta'}$ are pairwise different. □

The preceding lemma enables us to define minimal representations $\rho_{M,\alpha,\beta}$ of $\widetilde{\Gamma}^{\mathrm{J}(N)}$. As a representation of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$, let

$$\rho_{M,\alpha,\beta}|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})} = \bigoplus_{(\alpha',\beta')\in(\alpha,\beta)\mathrm{SL}_2(\mathbb{Z})/\sim} \pi_{M,\alpha',\beta'},$$

where

$$(\alpha_1',\beta_1')\sim(\alpha_2',\beta_2') \iff (\alpha_1'-\alpha_2',\beta_1'-\beta_2')\in(M^{\mathbb{Z}}\mathbb{Z}^N)^2.$$

Note that each $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$-isomorphism class occurs at most once. In accordance with Lemma 3.8, we let $\gamma\in\mathrm{Mp}_2(\mathbb{Z})$ permute components $\pi_{M,\alpha',\beta'}$ by the right action of ${}^{\mathrm{t}}\gamma$ on (α',β'). By the same lemma, we can define the image of $v\in V_{M,\alpha,\beta}$ under γ by the commutation relations in $\widetilde{\Gamma}^{\mathrm{J}(N)}$:

$$[\lambda,\mu,0]^{\mathrm{J}}\gamma v = \gamma[a\lambda+c\mu,b\lambda+d\mu,0]^{\mathrm{J}}v.$$

Proposition 3.10. *Let ρ be an irreducible representation of $\widetilde{\Gamma}^{\mathrm{J}(N)}$. Suppose that $\pi_{M,\alpha,\beta}$ occurs in $\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})}$. Then $\rho\cong\rho'\otimes\rho_{M,\alpha,\beta}$ for a unitary representation ρ' of $\mathrm{Mp}_2(\mathbb{Z})$.*

Proof. First consider the $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$-isotypical decomposition of ρ:

$$\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})} \cong \sum_{(\alpha',\beta')} \tilde{\pi}_{M,\alpha',\beta'}.$$

Because ρ is irreducible and $\pi_{M,\alpha,\beta}$ occurs in $\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})}$, Lemma 3.8 implies that (α',β') runs through $(\alpha,\beta)\mathrm{SL}_2(\mathbb{Z})/\sim$. Since, for the same reason, $\mathrm{SL}_2(\mathbb{Z})$ permutes the isotypical components transitively, we find that

$$\rho|_{\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})} \cong \pi_{\mathbb{1}}\otimes\rho_{M,\alpha,\beta},$$

where $\pi_{\mathbb{1}}$ a multiple of the trivial representation of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$. By the construction of $\rho_{M,\alpha,\beta}$, we have

$$\rho(\gamma)(\pi_{\mathbb{1}}\otimes\rho_{M,\alpha,\beta}^{-1}(\gamma))\rho([\lambda,\mu,\kappa]^{\mathrm{J}}) = \rho([\lambda,\mu,\kappa]^{\mathrm{J}})\rho(\gamma)(\pi_{\mathbb{1}}\otimes\rho_{M,\alpha,\beta}^{-1}(\gamma))$$

for all $\lambda,\mu\in\mathbb{Z}^N$, and $\kappa\in\mathrm{Mat}_N(\mathbb{Z})$. Consequently, $\rho\cong\rho'\otimes\rho_{M,\alpha,\beta}$ is a tensor product. □

3.3. Theta functions. In this subsection, we set up the basic tool, the theta functions, to deduce the theta decomposition for Jacobi forms of arbitrary index. Theta functions for indices in $\mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$ come with non-trivial representations of $\mathrm{H}(\widetilde{\Gamma}^{\mathrm{J}(N)})$ attached to them.

We denote Fourier transforms by $\mathcal{FT}$. Recall that the Fourier transform of a function $f : \mathbb{R}^N \to \mathbb{C}$, by definition, is

$$\mathcal{FT}(f)(\xi) = \int_{-\infty}^{\infty} f(\xi') \exp(2\pi i \cdot {}^t\xi\xi')\, d\xi'.$$

Lemma 3.11. *Let K be a symmetric, positive definite matrix. The Fourier transforms with respect to ξ' of* $\exp(-\pi K[\xi' + h])$ *equals*

$$(\det K)^{-\frac{1}{2}} \exp(2\pi i \cdot {}^th\xi) \exp(-\pi K^{-1}[\xi]).$$

Proof. This is standard and a well-organized proof can be found, e.g., in [2]. □

Corollary 3.12. *Given $v \in \mathbb{Q}^N$, the Fourier transform of*

$$\exp(\pi i \tau M^{-1}[M_{\mathbb{Z}}\xi + v] + 2\pi i\ {}^tz(M_{\mathbb{Z}}\xi + v))$$

equals

$$\sqrt{\det M^{-1}[M_{\mathbb{Z}}]}^{-1} \exp(2\pi i \cdot \xi M_{\mathbb{Z}}^{-1} v)\left(\frac{i}{\tau}\right)^{\frac{N}{2}} \exp\left(\pi i \cdot \frac{-M[z]}{\tau}\right)$$
$$\cdot \exp\left(\pi i \cdot \frac{-1}{\tau} M^{-1}[M\,{}^tM_{\mathbb{Z}}^{-1}\xi]\right) \exp\left(2\pi i \cdot {}^t(M\,{}^tM_{\mathbb{Z}}^{-1}\xi)\frac{z}{\tau}\right).$$

Proof. For $\tau = iy$ and $z = iv$ the result follows Lemma 3.11 with $K = yM^{-1}[M_{\mathbb{Z}}]$ and $h = M_{\mathbb{Z}}^{-1}(v + \frac{1}{y}Mv)$. A straightforward calculation gives

$$\sqrt{\det M^{-1}[M_{\mathbb{Z}}]}^{-1} \exp(2\pi i \cdot \xi M_{\mathbb{Z}}^{-1} v)$$
$$\cdot y^{-\frac{N}{2}} \exp\left(\pi \frac{M[v]}{y}\right) \exp(-\pi y^{-1} M[{}^tM_{\mathbb{Z}}^{-1}\xi]) \exp\left(2\pi i \cdot {}^t(M\,{}^tM_{\mathbb{Z}}^{-1}\xi)\frac{v}{y}\right).$$

Since the first and second expression in the corollary's statement represent holomorphic functions, this proves the corollary. □

Lemma 3.13. *We have*

$$M\,{}^tM_{\mathbb{Z}}^{-1}\mathbb{Z}^N = M\mathbb{Z}^N + \mathbb{Z}^N.$$

Proof. We proceed as in the proof of Lemma 3.3. Recall that we have $M = UDV$ and $M_{\mathbb{Z}} = UD_{\mathbb{Z}}V$. In order to simplify computations, we use the fact that if $M' = M[g]$ for $g \in \mathrm{GL}_N(\mathbb{Z})$, then

$$M'({}^tM'_{\mathbb{Z}})^{-1} = {}^tgM\,{}^tM_{\mathbb{Z}}^{-1}\,{}^tg^{-1}.$$

That is, we may assume that V is the identity matrix. Because M is symmetric, we have $UD = D\,{}^tU$. Now, the computation

$$M\,{}^tM_{\mathbb{Z}}^{-1}\mathbb{Z}^N = UD\,{}^tU^{-1}D_{\mathbb{Z}}^{-1}\mathbb{Z}^N = DD_{\mathbb{Z}}^{-1}\mathbb{Z}^N = M\mathbb{Z}^N + \mathbb{Z}^N$$

proves the lemma. □

We call $\operatorname{disc} M = (M\mathbb{Z}^N + \mathbb{Z}^N)/(M\mathbb{Z}^N \cap \mathbb{Z}^N)$ the (generalized) discriminant form associated to M. Note that is not a discriminant form in the sense of [16], since the associated quadratic form $q_M(\lambda) = M^{-1}[\lambda]$ does not necessarily take integral values on $M\mathbb{Z}^N \cap \mathbb{Z}^N$. Let $\mathbb{C}[\operatorname{disc} M]$ be the group algebra of $\operatorname{disc} M$, which has basis e_ν ($\nu \in \operatorname{disc} M$).

For $\nu \in \operatorname{disc} M$, define component functions

$$\theta_{M,\nu}(\tau, z) = \sum_{\xi \equiv \nu \pmod{M\mathbb{Z}^N \cap \mathbb{Z}^N}} q^{\frac{1}{2}M^{-1}[\xi]} \zeta^\xi, \tag{3.6}$$

where ξ runs through $M\mathbb{Z}^N \cap \mathbb{Z}^N$. We find that

$$\theta_M(\tau, z) = \sum_{\nu \in (M\mathbb{Z}^N + \mathbb{Z}^N)/(M\mathbb{Z}^N \cap \mathbb{Z}^N)} \theta_{M,\nu}(\tau, z) e_\nu,$$

as a function $\mathbb{H}^{\mathrm{J}(N)} \to \mathbb{C}[\operatorname{disc} M]$, transforms as a vector valued modular form under $\mathrm{Mp}_2(\mathbb{Z})$.

Let ρ_M be the representation of $\widetilde{\Gamma}^{\mathrm{J}(N)}$ on $\mathrm{GL}(\mathbb{C}[\operatorname{disc} M])$, whose central character is $e(\frac{1}{2}\operatorname{tr}(\cdot M))$ and which is defined as

$$\begin{aligned}
\rho_M(S)e_\nu &= \sqrt{\#\operatorname{disc} M}^{-1} \sum_{\nu' \in \operatorname{disc} M} \exp(2\pi i \cdot {}^{\mathrm{t}}\nu M^{-1} \nu') e_{\nu'}, \\
\rho_M(T)e_\nu &= \exp(\pi i \cdot M^{-1}[\nu]) e_\nu, \\
\rho_M([\lambda, 0, 0]^{\mathrm{J}})e_\nu &= e_{\nu + M\lambda}, \\
\rho_M([0, \mu, 0]^{\mathrm{J}})e_\nu &= \exp(2\pi i \cdot {}^{\mathrm{t}}\nu\mu) e_\nu,
\end{aligned}$$

where we identify S and T with the corresponding elements in $\widetilde{\Gamma}^{\mathrm{J}(N)}$.

Theorem 3.14. *For any positive definite matrix $M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$, the theta function θ_M is a vector valued Jacobi form of weight $\frac{N}{2}$, index M, and type ρ_M.*

Proof. We have to prove that

$$\theta_M|_{\frac{N}{2},M}\gamma^{\mathrm{J}} = \rho_M(\gamma^{\mathrm{J}})\theta_M \quad \text{for all } \gamma^{\mathrm{J}} \in \{T, S, [\lambda, 0, 0]^{\mathrm{J}}, [0, \mu, 0]^{\mathrm{J}}, [0, 0, \kappa]^{\mathrm{J}}\}.$$

All cases but $\gamma^{\mathrm{J}} = S$ are clear. In order to check the case $\gamma^{\mathrm{J}} = S$, we apply Poisson summation and Corollary 3.12. Since we have $M_{\mathbb{Z}}\mathbb{Z}^N = M\mathbb{Z}^N \cap \mathbb{Z}^N$ by Lemma 3.3, we find

$$\begin{aligned}
\theta_{M,\nu}(\tau, z) &= \sum_{\xi \in \mathbb{Z}^N} \exp\big(\pi i \cdot \tau M^{-1}[M_{\mathbb{Z}}\xi + \nu] + 2\pi i \cdot {}^{\mathrm{t}}\nu(M_{\mathbb{Z}}\xi + \nu)\big) \\
&= \sqrt{\det M^{-1}[M_{\mathbb{Z}}]}^{-1} \sum_{\xi' \in \mathbb{Z}^N} \exp(2\pi i \cdot \xi' M_{\mathbb{Z}}^{-1}\nu) \left(\frac{i}{\tau}\right)^{\frac{N}{2}} \exp\left(\pi i \cdot \frac{-M[z]}{\tau}\right) \\
&\qquad \cdot \exp\left(\pi i \cdot \frac{-1}{\tau} M^{-1}[M\,{}^{\mathrm{t}}M_{\mathbb{Z}}^{-1}\xi'] + 2\pi i \cdot {}^{\mathrm{t}}(M\,{}^{\mathrm{t}}M_{\mathbb{Z}}^{-1}\xi')\frac{z}{\tau}\right).
\end{aligned}$$

By Lemma 3.13, $M\,{}^{\mathrm{t}}M_{\mathbb{Z}}^{-1}\mathbb{Z}^N = M\mathbb{Z}^N + \mathbb{Z}^N$. This allows us to split the sum over ξ' with respect to congruence classes in $(M\mathbb{Z}^N + \mathbb{Z}^N)/(M\mathbb{Z}^N \cap \mathbb{Z}^N)$. We then set $\xi = M\,{}^{\mathrm{t}}M_{\mathbb{Z}}^{-1}\xi'$, and obtain the result. □

By Proposition 3.10, we have

$$\rho_M \cong \rho'_M \otimes \rho_{M,0,0} \tag{3.7}$$

for a suitable representation ρ'_M of $\mathrm{Mp}_2(\mathbb{Z})$. We shift the elliptic variable z, to obtain representations $\rho'_M \otimes \rho_{M,\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{Q}^N$.

Given $\alpha, \beta \in \mathbb{Q}^N$ and $\nu \in (M\mathbb{Z}^N + \mathbb{Z}^N)/(M\mathbb{Z}^N \cap \mathbb{Z}^N)$, define

$$\theta^{(\alpha,\beta)}_{M,\nu} = \theta_{M,\nu}(\tau, z)|_{k,M}[M^{-1}\alpha, 0, 0]^{\mathrm{J}}[0, M^{-1}\beta, 0]^{\mathrm{J}}$$

and

$$\theta^{(\alpha,\beta)}_M : \mathbb{H}^{\mathrm{J}(N)} \to \mathbb{C}[\mathrm{disc}\, M] \otimes \mathbb{C}[(\alpha,\beta)\mathrm{SL}_2(\mathbb{Z})/\sim], \tag{3.8}$$

$$\theta^{(\alpha,\beta)}_M(\tau, z) = \sum_{\substack{(\alpha',\beta')\in(\alpha,\beta)\mathrm{SL}_2(\mathbb{Z})/\sim \\ \nu\in(M\mathbb{Z}^N+\mathbb{Z}^N)/(M\mathbb{Z}^N\cap\mathbb{Z}^N)}} \theta^{(\alpha',\beta')}_{M,\nu} e_\nu \otimes e_{\alpha',\beta'},$$

where the sum runs over representatives modulo $(M^{\mathbb{Z}}\mathbb{Z}^N)^2$, and $e_{\alpha',\beta'}$ is a basis element of $\mathbb{C}[(\alpha,\beta)\mathrm{SL}_2(\mathbb{Z})/\sim]$.

Proposition 3.15. *For any vectors $\alpha, \beta \in \mathbb{Q}^N$, $\theta^{(\alpha,\beta)}_M$ is a vector valued Jacobi form of weight $\frac{N}{2}$, index M, and type $\rho'_M \otimes \rho_{M,\alpha,\beta}$.*

Proof. In conjunction with Proposition 3.10 and Theorem 3.14, it suffices to check the action of $\mathrm{Mp}_2(\mathbb{Z})$ and $[\lambda, \mu, 0]^{\mathrm{J}}$ ($\lambda, \mu \in M\mathbb{Z}^N \cap \mathbb{Z}^N$) on

$$\theta_M(\tau, z)|_{k,M}[M^{-1}\alpha', 0, 0]^{\mathrm{J}}[0, M^{-1}\beta', 0]^{\mathrm{J}}.$$

A straightforward computation yields the claim. □

3.4. Theta decomposition. The aim of this section to deduce theta decomposition for all Jacobi forms of index $0 < M \in \mathrm{Mat}^{\mathrm{t}}_N(\mathbb{Q})$. We start by limiting the possible action of the center $\mathrm{Mat}_N(\mathbb{Z})$ of $\widetilde{\Gamma}^{\mathrm{J}(N)}$.

Proposition 3.16. *Suppose that ρ is irreducible. If $\rho([0,0,\kappa]^{\mathrm{J}}) \neq e(\frac{1}{2}\mathrm{tr}(M\kappa))$, then*

$$\mathrm{J}_{k,M}(\rho) = \{0\}.$$

Proof. Any Jacobi form satisfies

$$\phi = \phi|_{k,M,\rho}[0,0,\kappa]^{\mathrm{J}} = e(\tfrac{1}{2}\mathrm{tr}(M\kappa))\rho([0,0,-\kappa]^{\mathrm{J}})\phi.$$

This implies the statement straightforwardly. □

Recall that any Jacobi form ϕ has Fourier expansion

$$\phi(\tau, z) = \sum_{m\in\mathbb{Q},\, r\in\mathbb{Q}^N} c(\phi; m, r) q^m \zeta^r, \quad q = e(\tau), \quad \zeta^r = e({}^{\mathrm{t}}rz). \tag{3.9}$$

The Fourier coefficients $c(\phi; m, r)$ vanish if $2\det(M)m - M^{\#}[r] < 0$.

We write $\check{\rho}$ for the dual of any representation ρ.

Theorem 3.17. *Let ϕ be a Jacobi form of irreducible type ρ. Then there are $\alpha, \beta \in \mathbb{Q}^N$ such that:*

(i) *There is a unitary, irreducible representation ρ' of $\mathrm{Mp}_2(\mathbb{Z})$ with $\rho \cong \rho' \otimes \rho_{M,\alpha,\beta}$.*

(ii) *We have*

$$\phi(\tau, z) = \sum_{\nu \in (M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,M\mathbb{Z}^N} h_\nu(\tau) \cdot \sum_{\substack{(\alpha',\beta') \in (\alpha,\beta)\mathrm{SL}_2(\mathbb{Z}) \\ \nu' \in M\mathbb{Z}^N\,/\,(M\mathbb{Z}^N \cap \mathbb{Z}^N)}} \theta^{(\alpha',\beta')}_{M,\nu+\nu'}(\tau, z) e_{\nu+\nu'} \otimes e_{\alpha',\beta'} \tag{3.10}$$

for some $(h_\nu) \in \mathrm{M}_{k-\frac{N}{2}}(\rho' \otimes \check{\rho}'_M)$. Conversely, given a vector valued modular form of weight $k - \frac{N}{2}$ and type $\rho \otimes \check{\rho}'_M$, equation (3.10) *defines an element of $\mathrm{J}_{k,M}(\rho \otimes \rho_{M,\alpha,\beta})$.*

Remark 3.18. If ρ arises as an irreducible component of the restriction of the Weil representation for the metaplectic cover $\widetilde{\mathrm{Sp}}_r(\mathbb{Z})$ of $\mathrm{Sp}_r(\mathbb{Z})$, as it will be the case in Section 4, then ρ' is the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. An explicit decomposition required lengthy arguments, and will be given at another occasion.

Proof of Theorem 3.17. Note that the sum $\nu + \nu'$, which occurs in the subscript, is well-defined, since ν' runs through all representatives modulo $M\mathbb{Z}^N \cap \mathbb{Z}^N$.

By Proposition 3.10, there is a representation ρ' such that $\rho \cong \rho' \otimes \rho_{M,\alpha,\beta}$ for some vectors $\alpha, \beta \in \mathbb{Q}^N$. Since ρ is irreducible, ρ' must be irreducible, too. We can assume that $\rho = \rho' \otimes \rho_{M,\alpha,\beta}$, thus identifying representation spaces.

The Fourier expansion of ϕ is uniquely determined by $c(\phi; n, r)$, where r runs through a system of representatives of $(M\mathbb{Z}^N + \mathbb{Z}^N)\,/\,M\mathbb{Z}^N$. Fix some lift r of ν to $M\mathbb{Z}^N + \mathbb{Z}^N$ and one pair (α, β). It is straightforward to check corresponding Fourier coefficients $c(\,\cdot\,; n, r)$ of the left and right hand side. This proves the first part of the theorem.

The second part follows directly, when checking transformation properties. □

We denote the induced bijection between $\mathrm{J}_{k,M}(\rho)$ and $\mathrm{M}_{k-\frac{N}{2}}(\check{\rho}'_M \otimes \rho')$ by Θ_M, suppressing the weight.

3.5. Vanishing of Jacobi forms. We deduce vanishing statements for Jacobi forms based on their connection to vector valued elliptic modular forms. The theta decomposition allows us to do this.

Let

$$\mathrm{rd}(M) = \max_{\xi \in \mathbb{R}^N} \min_{\xi' \in \mathbb{Z}^N} M[\pm\xi + \xi'].$$

Note that $\mathrm{rd}(M)$ is related to the radius of a Voronoi cell of a lattice with Gram matrix M.

Proposition 3.19. *If $\phi \in \mathrm{J}_{k,M}(\rho)$ and $c(\phi; m, r) = 0$ for all $0 \le m < \frac{k}{12} + 1 + \frac{1}{2}\,\mathrm{rd}(M)$ and all $r \in M\mathbb{Z}^N + \mathbb{Z}^N$, then $\phi = 0$.*

Proof. We can assume that ρ is irreducible. According to Theorem 3.17, we have

$$\rho \cong \rho' \otimes \rho_{M,\alpha,\beta}$$

for some $\alpha, \beta \in \mathbb{Q}^N$.

We apply Proposition 2.2 to $\Theta_M(\phi)$. Given $\nu \in (M\mathbb{Z}^N + \mathbb{Z}^N) \,/\, M\mathbb{Z}^N$ and a lift $r \in \mathbb{Q}^N$ of ν, we have

$$c(\Theta_M(\phi)_\nu; m) = \big(c(\phi; m + \tfrac{1}{2}M^{-1}[r+\alpha], r+\alpha)\big)_{r;\alpha,\beta},$$

where the subscripts r and α, β refer to the corresponding coordinates in $\mathbb{C}[\operatorname{disc} M]$ and $\mathbb{C}[(\alpha,\beta)\mathrm{SL}_2(\mathbb{Z}) \,/\, \sim]$, respectively.

Combining this with relation (3.3), we see that it suffices to prove that for every vector $\nu \in (M\mathbb{Z}^N + \mathbb{Z}^N) \,/\, M\mathbb{Z}^N$, there is $\xi \in \pm\nu \subset \mathbb{Q}^N$ with $M^{-1}[\xi] \le \operatorname{rd}(M)$. In order to connect ν to the definition of $\operatorname{rd}(M)$, we consider $M^{-1}\nu \in (M^{-1}\mathbb{Z}^N + \mathbb{Z}^N) \,/\, \mathbb{Z}^N$ instead of ν. We have to find $M^{-1}\xi \in \pm M^{-1}\nu$ with $M^{-1}[\xi] = M[M^{-1}\xi] \le \operatorname{rd}(M)$. By the definition of $\operatorname{rd}(M)$, this is possible, since $M^{-1}\xi$ is defined up to elements in $\mathbb{Z}^N$. This finishes the proof. □

Given $0 < M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$, define

$$\operatorname{md}(M) = \min_{g \in \mathrm{GL}_N(\mathbb{Z})} \max_{1 \le i \le N} (M[g]_{i,i}).$$

This is the length of the longest vector within a basis of shortest vectors.

Proposition 3.20. *Given* $0 < M \in \mathrm{Mat}_N^{\mathrm{t}}(\mathbb{Q})$*, we have* $\operatorname{rd}(M) \le \frac{N(N+1)}{8} \operatorname{md}(M)$.

Proof. Since the proposition holds for $M[g]$, $g \in \mathrm{GL}_N(\mathbb{Z})$ if it holds for M, we can assume that $M_{i,i} \le M_{i+1,i+1}$ for $1 \le i < N$. Further, we can assume that for all $1 \le i < N$, we have

$$M_{i,i} = \min\{M[g]_{i,i} : M[g]_{j,j} = M_{j,j} \text{ for } 1 \le j < i,\ g \in \mathrm{GL}_N(\mathbb{Z})\}.$$

In particular, we have $\operatorname{md}(M) = M_{N,N}$. Moreover, we find that $|M_{i,j}| \le \frac{1}{2}\min\{M_i, M_j\}$. Indeed, suppose that this was not the case for some $i < j$. Then we could replace the basis vector e_j by $e_i \pm e_j$. We would then have

$$M[e_i \pm e_j] = M_i + M_j \pm 2M_{i,j} < M_j$$

for the right choice of sign. This would contradict the minimality of diagonal elements of M.

Given any $\xi \in \mathbb{R}^N$, we can find $w = \pm\xi + \xi'$ for some choice of sign and $\xi' \in \mathbb{Z}^N$ such that $|\pm\xi_i + \xi_i'| \le \frac{1}{2}$ for $1 \le i \le N$. We then have

$$\begin{aligned}
\operatorname{rd}(M) \le M[w] &\le \sum_{1 \le i,j \le N} |M_{i,j}||w_i||w_j| \\
&\le \frac{1}{4} \sum_{1 \le i,j \le N} |M_{i,j}| \\
&\le \frac{1}{4} \sum_{1 \le i \le j \le N} |M_{N,N}| = \frac{N(N+1)}{8} \operatorname{md}(M).
\end{aligned}$$

□

Remark 3.21. In the next section, we use $\operatorname{rd}(M) < (2-\epsilon)\operatorname{md}(M)$ for $N < 4$. In the case $N = 4$, the diagonal Gram matrix $M = \operatorname{diag}(2,2,2,2)$ has $\operatorname{md}(M) = 2$ and $\operatorname{rd}(M) = 1$.

4. Special cycles

As announced in the introduction, we switch notation and adopt Zhang's.

Let $\mathcal{L}$ be an integral, even lattice of signature $(n,2)$, and write $\mathcal{L}^\#$ for its dual. The Grassmannian of 2-dimensional negative subspaces of $\mathcal{L}\otimes\mathbb{R}$ is denoted by $\mathrm{Gr}^-(\mathcal{L}\otimes\mathbb{R})$. Fix a subgroup Γ of $\mathrm{O}(\mathcal{L})$ that acts trivially on $\operatorname{disc}\mathcal{L} := \mathcal{L}^\#/\mathcal{L}$. The quotient

$$X_\Gamma = \Gamma\backslash\mathrm{Gr}^-(\mathcal{L}\otimes\mathbb{R})$$

is a Shimura variety of orthogonal type, on which there are rational quadratic cycles. Given a tuple of vectors $v=(v_1,\dots,v_r)\in(\mathcal{L}\otimes\mathbb{Q})^r$, let

$$Z(v) = \{W\in\mathrm{Gr}^-(\mathcal{L}\otimes\mathbb{R}) : \operatorname{span}(v)\perp W\}.$$

If $\operatorname{span}(v)\subseteq\mathcal{L}\otimes\mathbb{Q}$ is positive, this cycle is non-trivial.

Kudla [11] built so-called special cycles from the $Z(v)$. To each v, we attach a moment matrix $q_{\mathcal{L}}(v)=\frac{1}{2}(\langle v_i,v_j\rangle_{\mathcal{L}})_{1\le i,j\le r}$, where $\langle v,w\rangle_{\mathcal{L}} = q_{\mathcal{L}}(v+w)-q_{\mathcal{L}}(v)-q_{\mathcal{L}}(w)$. Given a positive semidefinite matrix $0\le T\in\mathrm{Mat}_r^{\mathrm{t}}(\mathbb{Q})$ and $\lambda\in(\mathcal{L}^\#/\mathcal{L})^r$, set

$$\Omega(T,\mu) = \{v\in\mu+\mathcal{L}^r : T = q_{\mathcal{L}}(v)\}.$$

The symmetries Γ act on $\Omega(T,\mu)$, and there are finitely many orbits. Kudla defines special cycles as

$$Z(T,\mu) = \sum_{v\in\Gamma\backslash\Omega(T,\mu)} Z(v).$$

All cycles of this form descend to cycles on X_Γ, which we also denote by $Z(T,\mu)$. Writing $\mathrm{rk}(T)$ for the rank of T, we find that $Z(T,\mu)$ is a cycle of codimension $\mathrm{rk}(T)$, whose class in $\mathrm{CH}^{\mathrm{rk}(T)}(X_\Gamma)_{\mathbb{C}}$ is denoted by $\{Z(T,\mu)\}$.

The aim of this section is to prove the next theorem.

Theorem 4.1. *Suppose that $\Gamma\subset\mathrm{O}(\mathcal{L})$ acts trivially on $\operatorname{disc}\mathcal{L}$. Then for $r<5$, the space*

$$\operatorname{span}(\{Z(T,\mu)\} : 0<T\in\mathrm{Mat}_r^{\mathrm{t}}(\mathbb{Q}),\ \mu\in\operatorname{disc}^r\mathcal{L}) \subseteq \mathrm{CH}^r(X_\Gamma)_{\mathbb{C}}$$

is finite dimensional.

Given $\mu\in\operatorname{disc}\mathcal{L}$, $\mu'\in\operatorname{disc}^{r-1}\mathcal{L}$ and $M\in\mathrm{Mat}_{r-1}^{\mathrm{t}}(\mathbb{Q})$, Zhang formed the partial generating series

$$\theta_{\mu,\mu',M} = \sum_{\substack{0\le n\in\mathbb{Z}\\ p\in\mathbb{Z}^{r-1}}} \{Z(T,(\mu,\mu'))\}\, q^n\zeta^p, \quad T=\begin{pmatrix} n & \frac{1}{2}{}^{\mathrm{t}}p\\ \frac{1}{2}p & M\end{pmatrix}. \tag{4.1}$$

Based on results in [3,4], Zhang proved in his thesis [21] that $\theta_{\mu,\mu',M}$ is a vector valued Jacobi form of weight $1+\frac{n}{2}$. In order to state his result, we make a connection to the notation in Section 3. We write $\rho_{\mathcal{L},r}$ for the Weil representation on the metaplectic double cover $\widetilde{\mathrm{Sp}}_r(\mathbb{Z})$ of $\mathrm{Sp}_r(\mathbb{Z})$ associated to $\mathcal{L}$. Precise formulas are given by Zhang [21, Section 2].

Using notation from Section 3, we have an embedding $\widetilde{\Gamma}^{\mathrm{J}(r-1)}$ into $\widetilde{\mathrm{Sp}}_r(\mathbb{Q})$ via

$$\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), [\lambda, \mu, \kappa]\right) \mapsto \begin{pmatrix} a & & b & a\,{}^{\mathrm{t}}\mu - b\,{}^{\mathrm{t}}\lambda \\ \lambda & \mathrm{id}_{r-1} & \mu & \frac{1}{2}\kappa \\ c & & d & c\,{}^{\mathrm{t}}\mu - d\,{}^{\mathrm{t}}\lambda \\ & & & \mathrm{id}_{r-1} \end{pmatrix}.$$

Beware that the notation μ for a vector in $\mathbb{Z}^{r-1}$, used in the previous section, conflicts with the use of μ in the present section. We consider $\rho_{\mathcal{L},r}$ as a representation of $\widetilde{\Gamma}^{\mathrm{J}(r-1)}$ by means of this embedding.

Proposition 4.2 ([21, Proposition 2.6 and (2.10)]). *The right hand side of* (4.1) *is absolutely convergent, and we have*

$$\sum_{\substack{\mu \in \mathrm{disc}\,\mathcal{L} \\ \mu' \in \mathrm{disc}^{r-1}\,\mathcal{L}}} \theta_{\mu,\mu',M}\, e_{\mu,\mu'} \in \mathrm{J}_{1+\frac{n}{2},M}(\rho_{\mathcal{L},r}).$$

Proposition 4.3. *Given any* $0 < B \in \mathbb{R}$*, the space of cycles on* X_Γ

$$\mathrm{span}\Big(Z(T,(\mu,\mu')) : T = \left(\begin{smallmatrix} m & \frac{1}{2}\,{}^{\mathrm{t}}p \\ \frac{1}{2}p & M \end{smallmatrix}\right) \in \mathrm{Mat}_r^{\mathrm{t}}(\mathbb{Q}),\ \mu \in \mathrm{disc}\,\mathcal{L}, \mu' \in \mathrm{disc}^{r-1}\,\mathcal{L},\ m, \mathrm{md}(M) < B + \tfrac{1}{2}\,\mathrm{rd}(M)\Big)$$

is finite dimensional, if $r < 5$*.*

Proof. Because of the $\mathrm{GL}_r(\mathbb{Z})$ invariance of $Z(T,(\mu,\mu'))$ and because the denominator of $q_{\mathcal{L}}(\lambda)$, $\lambda \in \mu$ is bounded, it is sufficient to consider cycles associated to $\mathrm{GL}_{r-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{r-1}$ classes

$$\left\{\left(\begin{smallmatrix} m & {}^{\mathrm{t}}({}^{\mathrm{t}}gp + \lambda M[g]) \\ {}^{\mathrm{t}}gp + \lambda M[g] & M[g] \end{smallmatrix}\right) : g \in \mathrm{GL}_{r-1}(\mathbb{Z}),\ \lambda \in \mathbb{Z}^{r-1}\right\}$$

of positive matrices with bounded denominator.

First of all there are only finitely many $\mathrm{GL}_{r-1}(\mathbb{Z})$ classes of matrices M with bounded denominator that satisfy $\mathrm{md}(M) < B + \frac{1}{2}\,\mathrm{rd}(M)$. This follows from Proposition 3.20, and the fact that, for any B', there are only finitely many $\mathrm{GL}_{r-1}(\mathbb{Z})$ classes with bounded denominator and $\mathrm{md}(M) < B'$.

Next, we can assume that p is reduced with respect to the column space of M, so that there are only finitely many p that can occur for any given M. Only finitely many m occur for given M and p, since $0 < m < B + \frac{1}{2}\,\mathrm{rd}(M)$. This completes the proof. □

Theorem 4.4. *We have*

$$\begin{aligned} &\mathrm{span}\big(\{Z(T,\mu)\} : 0 < T \in \mathrm{Mat}_r^{\mathrm{t}}(\mathbb{Q}),\ \mu \in \mathrm{disc}^r\,\mathcal{L}\big) \\ &\quad = \mathrm{span}\Big(\{Z(T,\mu)\} : 0 < T = \left(\begin{smallmatrix} m & \frac{1}{2}\,{}^{\mathrm{t}}p \\ {}^{\mathrm{t}}p & M \end{smallmatrix}\right) \in \mathrm{Mat}_r^{\mathrm{t}}(\mathbb{Q}),\ \mu \in \mathrm{disc}^r\,\mathcal{L}, \\ &\qquad\qquad m, \mathrm{md}(M) < 1 + \tfrac{2+2n}{24} + \tfrac{1}{2}\,\mathrm{rd}(M)\Big). \end{aligned}$$

Proof. Assume that the above is not true. Then there is a functional f on $\mathrm{CH}^r(X_\Gamma)_{\mathbb{C}}$ that vanishes on the right hand side but not on the left hand side. Write

$$f(\theta_{\mu,\mu',M}) = \sum_{\substack{0\le m\in\mathbb{Q}\\ p\in\mathbb{Q}^{r-1}}} f(Z(T,(\mu,\mu')))q^m\zeta^p.$$

By the choice of f, we have $c(f(\theta_{\mu,\mu',M});m,p)\neq 0$ for some m, p, and M. We show that this cannot be.

Set $B = 1+\frac{2+2m}{24}$. We first consider M with $\mathrm{md}(M)\le B+\frac{1}{2}\mathrm{rd}(M)$. Then

$$c(f(\theta_{\mu,\mu',M});m,p)=0$$

by assumption for all $m < 1+\frac{2+2n}{24}+\mathrm{rd}(M)$. By Proposition 3.19 and 4.2, we find that

$$f(\theta_{\mu,\mu',M})=0.$$

Next, we use induction on $(M_{i,i})_{1\le i\le N}\in\mathbb{Z}^{r-1}$ to show that $f(\theta_{\mu,\mu',M})=0$ for all M. We impose the componentwise comparison on $\mathbb{Z}^{r-1}$. In other words, for fixed M with the property that $\mathrm{md}(M) > B+\frac{1}{2}\mathrm{rd}(M)$, we suppose that $f(\theta_{\mu_2,\mu_2',M_2})=0$ whenever $(M_2)_{i,i}\le M_{i,i}$ for all $1\le i\le r-1$, and there is one $1\le i\le r-1$ such that $(M_2)_{i,i} < M_{i,i}$.

We can make the same assumptions as in the proof of Proposition 3.20. That is, we can assume that $M_{i,i} < M_{i+1,i+1}$ for $1\le i < r-1$, and for all $1\le i\le r-1$

$$M_{i,i} = \min\{M[g]_{i,i} : M[g]_{j,j} = M_{j,j} \text{ for } 1\le j < i,\ g\in \mathrm{GL}_{r-1}(\mathbb{Z})\}.$$

We want to show that $f(\theta_{\mu,\mu',M})$ vanishes using Proposition 3.19. We must argue that $c(f(\theta_{\mu,\mu',M});m,p)=0$ for all $m < B+\frac{1}{2}\mathrm{rd}(M) < \mathrm{md}(M)$. For any such m we have

$$c(f(\theta_{\mu,\mu',M});m,p) = c(f(\theta_{\mu'_{r-1},(\mu'_1,\dots,\mu'_{r-2},\mu),M'});M_{r-1,r-1},p'),$$

where

$$\begin{pmatrix} m & \frac{1}{2}{}^t p\\ \frac{1}{2}p & M\end{pmatrix} = \begin{pmatrix} M_{r-1,r-1} & \frac{1}{2}{}^t p'\\ \frac{1}{2}p' & M'\end{pmatrix}\left[\begin{pmatrix} 0 & 0 & 1\\ 0 & \mathrm{id}_{r-2} & 0\\ 1 & 0 & 0\end{pmatrix}\right].$$

We have $M'_{i,i} = M_{i,i}$ if $i\neq r-1$, and $M'_{r-1,r-1} = m < \mathrm{md}(M) = M_{r-1,r-1}$. Thus we can employ the induction hypothesis to prove that the right hand side is zero. This way, we establish the hypothesis of Proposition 3.19, and finish the induction.

We have shown $f(\theta_{\mu,\mu',M})=0$ for all M. This contradicts the choice of f, and finishes the proof. □

Proof of Theorem 4.1. Combine Theorem 4.4 and Proposition 4.3. □

References

[1] *A. Andrianov*, Modular descent and the Saito–Kurokawa conjecture, Invent. Math. **53** (1979), no. 3, 267–280.
[2] *R. Borcherds*, Automorphic forms with singularities on Grassmannians, Invent. Math. **132** (1998), no. 3, 491–562.
[3] *R. Borcherds*, The Gross–Kohnen–Zagier theorem in higher dimensions, Duke Math. J. **97** (1999), no. 2, 219–233.
[4] *R. Borcherds*, Correction to "The Gross–Kohnen–Zagier theorem in higher dimensions" [Duke Math. J. 97 (1999), no. 2, 219–233], Duke Math. J. **105** (2000), no. 1, 183–184..

[5] *M. Cheng* and *J. Duncan*, The largest Mathieu group and (mock) automorphic forms, in: String-Math 2011, Proc. Sympos. Pure Math. **85**, American Mathematical Society, Providence (2012), 53–82.
[6] *W.-L. Chow*, On equivalence classes of cycles in an algebraic variety, Ann. of Math. (2) **64** (1956), 450–479.
[7] *A. Dabholkar* and *D. Gaiotto*, Spectrum of CHL dyons from genus-two partition function, J. High Energy Phys. **2007** (2007), no. 12, Paper No. 87.
[8] *A. Dabholkar*, *S. Murthy* and *D. Zagier*, Quantum black holes, wall crossing, and mock modular forms, preprint 2012, `http://arxiv.org/abs/1208.4074`.
[9] *E. Hecke*, Neue Herleitung der Klassenzahlrelationen von Hurwitz und Kronecker, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. **1926** (1926), 244–249.
[10] *F. Hirzebruch* and *D. B. Zagier*, Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, Invent. Math. **36** (1976), 57–113.
[11] *S. Kudla*, Algebraic cycles on Shimura varieties of orthogonal type, Duke Math. J. **86** (1997), no. 1, 39–78.
[12] *H. Maass*, Über eine Spezialschar von Modulformen zweiten Grades, Invent. Math. **52** (1979), no. 1, 95–104.
[13] *H. Maass*, Über eine Spezialschar von Modulformen zweiten Grades. II, Invent. Math. **53** (1979), no. 3, 249–253.
[14] *H. Maass*, Über eine Spezialschar von Modulformen zweiten Grades. III, Invent. Math. **53** (1979), no. 3, 255–265.
[15] *D. Mumford*, Tata lectures on theta. III, Progr. Math. **97**, Birkhäuser-Verlag, Boston 1991.
[16] *V. Nikulin*, Integer symmetric bilinear forms and some of their geometric applications, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238.
[17] *R. Sekiguchi*, On projective normality of Abelian varieties. II, J. Math. Soc. Japan **29** (1977), no. 4, 709–727.
[18] *G. Shimura*, On certain reciprocity-laws for theta functions and modular forms, Acta Math. **141** (1978), no. 1–2, 35–71.
[19] *N. Skoruppa*, Jacobi forms of critical weight and Weil representations, in: Modular forms on Schiermonnikoog (Schiermonnikoog 2006), Cambridge University Press, Cambridge (2008), 239–266.
[20] *D. Zagier*, Sur la conjecture de Saito-Kurokawa (d'après H. Maass), in: Théorie des nombres. Séminaire Delange-Pisot-Poitou (Paris 1979–80), Progr. Math. **12**, Birkhäuser-Verlag, Boston (1981), 371–394.
[21] *W. Zhang*, Modularity of generating functions of special cycles on Shimura varieties, Ph.D. thesis, Columbia University, New York 2009.
[22] *C. Ziegler*, Jacobi forms of higher degree, Abh. Math. Semin. Univ. Hambg. **59** (1989), 191–224.

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