ISSN 0001-4346, Mathematical Notes, 2014, Vol. 96, No. 6, pp. 957–964. © Pleiades Publishing, Ltd., 2014.
Published in Russian in Matematicheskie Zametki, 2014, Vol. 96, No. 6, pp. 896–904.

Interpolation of Nonlinear Maps*

T. Kappeler[1], A. M. Savchuk[2]***, A. A. Shkalikov[2]****, and P. Topalov[3]*****

[1]*Institut für Mathematik, Universität Zürich, Zürich, Switzerland*
[2]*Moscow State University, Moscow, Russia*
[3]*Northeastern University, Boston, MA, USA*

Received September 9, 2014

Abstract—Let (X_0, X_1) and (Y_0, Y_1) be complex Banach couples and assume that $X_1 \subseteq X_0$ with norms satisfying $\|x\|_{X_0} \le c\|x\|_{X_1}$ for some $c > 0$. For any $0 < \theta < 1$, denote by $X_\theta = [X_0, X_1]_\theta$ and $Y_\theta = [Y_0, Y_1]_\theta$ the complex interpolation spaces and by $B(r, X_\theta)$, $0 \le \theta \le 1$, the open ball of radius $r > 0$ in X_θ centered at zero. Then, for any analytic map $\Phi\colon B(r, X_0) \to Y_0 + Y_1$ such that $\Phi\colon B(r, X_0) \to Y_0$ and $\Phi\colon B(c^{-1}r, X_1) \to Y_1$ are continuous and bounded by constants M_0 and M_1, respectively, the restriction of Φ to $B(c^{-\theta}r, X_\theta)$, $0 < \theta < 1$, is shown to be a map with values in Y_θ which is analytic and bounded by $M_0^{1-\theta} M_1^\theta$.

DOI: 10.1134/S0001434614110339

Keywords: *interpolation, nonlinear maps, Banach couple, Hausdorff space.*

1. INTRODUCTION

Let us first recall some basic notations and definitions of the interpolation theory for Banach spaces. Following [1], we say that two complex Banach spaces X_0 and X_1 are a *complex Banach couple* or *Banach couple* for short, (X_0, X_1), if they are both linearly and continuously embedded into a linear complex Hausdorff space $\mathcal{X}$, i.e., $X_0 \subseteq \mathcal{X}$ and $X_1 \subseteq \mathcal{X}$. The spaces $X_\cap := X_0 \cap X_1$ and

$$X_+ := X_0 + X_1 = \{x \in \mathcal{X} \mid x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}$$

are also Banach spaces when endowed with the norms

$$\|x\|_\cap := J(1, x), \qquad \text{respectively,} \qquad \|x\|_+ := K(1, x),$$

where, for any $t \ge 0$,

$$J(t, x) := \max\{\|x\|_{X_0}, t\|x\|_{X_1}\} \qquad \forall\, x \in X_\cap$$

and for any $x \in X_+$

$$K(t, x) := \inf\bigl\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} \mid x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\bigr\}. \tag{1}$$

Without loss of generality, we will assume in what follows that $\mathcal{X} = X_+$. We say that (X_0, X_1) is a *regular Banach couple* if X_1 is continuously embedded into X_0, i.e., $X_1 \subseteq X_0$ and there exists $c > 0$ so that

$$\|x\|_{X_0} \le c\|x\|_{X_1} \qquad \forall\, x \in X_1.$$

There are several methods of constructing, for any given Banach couple (X_0, X_1), interpolation spaces X, $X_\cap \subseteq X \subseteq X_+$, that satisfy the interpolation property discussed below. The most familiar

*The text was submitted by the authors in English.

**E-mail: thomas.kappeler@math.uzh.ch
***E-mail: artem_savchuk@mail.ru
****E-mail: ashkaliko@yandex.ru
*****E-mail: p.topalov@neu.edu

interpolation methods are the real and the complex ones. The real method comprises among others the K-method, the L-method, the J-method, the mean-methods, and the trace method (see, e.g., [2], [1]). Up to equivalent norms, these methods all lead to the same interpolation spaces $X_{\theta,p}$, where $0 < \theta < 1$ and $1 \le p \le \infty$. In particular, for any fixed $1 \le p < \infty$ and $0 < \theta < 1$, the K-method defines the interpolation spaces $X_{\theta,p} \equiv (X_0, X_1)_{\theta,p}$ as follows

$$X_{\theta,p} := \left\{ x \in X_+ \;\middle|\; \|x\|_{\theta,p} = \left(\int_0^\infty (t^{-\theta} K(t,x))^p \frac{dt}{t} \right)^{1/p} < \infty \right\}, \tag{2}$$

where $K(t,x)$ is defined by (1). If $p = \infty$, the space $(X_0, X_1)_{\theta,\infty}$ consists of the elements $x \in X_+$ with $\|x\|_{\theta,\infty} = \sup_{0<t<\infty} t^{-\theta} K(t,x) < \infty$.

Now let us turn to the complex method of interpolation. For the notion of an analytic map between complex Banach spaces, we refer the reader, e.g., to [3] or [4, Appendix A]. Following [1], denote by S the vertical strip in the complex plane given by

$$S := \{ z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1 \},$$

and by $\overline{S}$ its closure. For any given Banach couple (X_0, X_1), we then introduce the complex vector space $\mathcal{H}(X_0, X_1)$ of maps $f\colon \overline{S} \to X_+$ with the following properties:

($\mathcal{H}0$) $f\colon \overline{S} \to X_+$ is continuous and bounded;

($\mathcal{H}1$) $f|_S\colon S \to X_+$ is analytic;

($\mathcal{H}2$) for any $t \in \mathbb{R}$ $f(it) \in X_0$, $f(1+it) \in X_1$, and the maps $\mathbb{R} \to X_0$, $t \mapsto f(it)$, and $\mathbb{R} \to X_1$, $t \mapsto f(1+it)$, are bounded and continuous.

Then

$$\|f\|_{\mathcal{H}} := \max\left\{ \sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1} \right\}$$

defines the norm on $\mathcal{H}(X_0, X_1)$. For any $0 < \theta < 1$, the complex interpolation space $X_\theta := [X_0, X_1]_\theta$ is the space

$$X_\theta := \{ x \in X_+ \mid \exists\, f(z) \in \mathcal{H}(X_0, X_1) \text{ with } f(\theta) = x \}$$

endowed with the norm

$$\|x\|_{X_\theta} := \inf_{f\in\mathcal{H}} \{ \|f\|_{\mathcal{H}} \mid f(\theta) = x \}. \tag{3}$$

It is well known that the spaces X_θ (and $X_{\theta,p}$) have the *interpolation property*: given any two Banach couples (X_0, X_1) and (Y_0, Y_1) and any bounded linear operator $T\colon X_+ \to Y_+$ such that

$$\|Tx\|_{Y_0} \le M_0 \|x\|_{X_0} \quad \forall\, x \in X_0 \qquad \text{and} \qquad \|Tx\|_{Y_1} \le M_1 \|x\|_{X_1} \quad \forall\, x \in X_1$$

for some $M_0 > 0$ and $M_1 > 0$, it follows that, for any $0 < \theta < 1$, T maps X_θ into Y_θ and

$$\|Tx\|_{Y_\theta} \le M_0^{1-\theta} M_1^\theta \|x\|_{X_\theta} \qquad \forall\, x \in X_\theta.$$

An important problem is to identify classes of nonlinear maps for which (a version of) the above interpolation property holds. It was investigated by many authors, see, e.g., [5]–[12], as well as the books [10], [1] and references therein for results on nonlinear maps defined on the entire spaces X_0 and X_1. In applications, such an assumption is often too restrictive. The first results for nonlinear maps which are defined only locally were obtained by Tartar [13, Theorem 1]. Using the real interpolation method (constructed with the help of the K-functional (see (2)) and a setup where $X_0 = X_+$ and $Y_0 = Y_+$, Tartar proved the interpolation property for a class of nonlinear maps $\Phi\colon U \to Y_0$ defined on an open, nonempty subset $U \subset X_0$ with the following properties:

(T1) Φ is locally α-Hölder continuous for some $0 < \alpha \le 1$;

(T2) for any $x \in U$, there exist a neighborhood $V \subset U$ of x and a constant $c > 0$ such that

$$\forall\, y \in V \cap X_1 \qquad \|\Phi(y)\|_{Y_1} \le c(\|y\|_{X_1}^{\beta} + 1) \quad \text{for some} \quad \beta > 0.$$

Note that the set $V \cap X_1$ may be of infinite diameter in X_1 even if V is a (small) ball in X_0. Hence in applications, property (T2) is often not satisfied or difficult to verify. It is this fact that motivated our study of nonlinear interpolation in [14], [15], and [16], where applications to inverse problems of spectral theory were considered. Within the setup of the complex method of interpolation, in [14] and [15], we established the interpolation property for a class of nonlinear maps, defined on balls with center at the origin, satisfying assumptions that are rather easy to prove in many applications. In this paper, our aim is to extend the results of [14] and [16] to a more general setup. Results in this direction have been obtained previously by Böhm [17] and Cwikel [18]. For a discussion of these results, we refer the reader to Remark 1 below. To the best of our knowledge, the type of result obtained in Theorem 2 below, is new.

2. MAIN RESULTS

To state our results, we first need to introduce some additional notation and establish some auxiliary results. For any given complex Banach space X, denote by $B(r, X)$ the open ball in X of radius r centered at the origin. Throughout this section, let (X_0, X_1) be a complex Banach couple with $X_1 \subseteq X_0$, and let $X_\theta = [X_0, X_1]_\theta$, $0 < \theta < 1$, be the complex Banach spaces constructed by the complex method of interpolation. We have the following auxiliary lemma.

Lemma 1. *Assume that the norms of the Banach spaces $X_1 \subseteq X_0$ satisfy*

$$\|x\|_{X_0} \le c\|x\|_{X_1} \qquad \forall\, x \in X_1 \tag{4}$$

for some positive constant $c > 0$. Then, for any $0 < \theta < 1$,

$$\begin{aligned} &\text{(i)} \quad \|x\|_{X_0} \le c^{\theta}\|x\|_{X_\theta} \qquad \forall\, x \in X_\theta, \\ &\text{(ii)} \quad \|x\|_{X_\theta} \le c^{1-\theta}\|x\|_{X_1} \qquad \forall\, x \in X_1. \end{aligned} \tag{5}$$

In particular,

$$B(c^{-1}r, X_1) \subseteq B(c^{-\theta}r, X_\theta) \subseteq B(r, X_0). \tag{6}$$

Proof. Inequality (i) in (5) follows from the interpolation property of linear operators. Indeed, by (4) the identity operators

$$I\colon X_1 \to X_0 \qquad \text{and} \qquad I\colon X_0 \to X_0$$

are bounded by c and 1, respectively. Hence,

$$I\colon X_\theta = [X_0, X_1]_\theta \to [X_0, X_0]_\theta = X_0$$

is bounded by c^θ. To prove inequality (ii) in (5), consider for any given $x \in X_1$, the analytic function $f\colon \mathbb{C} \to X_1$, $f(z) := c^{z-\theta}x$. Clearly, $f \in \mathcal{H}(X_0, X_1)$. Furthermore, in view of the definition of the norms $\|\cdot\|_{X_\theta}$, $\|f\|_{\mathcal{H}}$ and of (4), we have

$$\|x\|_{X_\theta} \le \|f\|_{\mathcal{H}} = \max\{c^{-\theta}\|x\|_{X_0}, c^{1-\theta}\|x\|_{X_1}\} \le c^{1-\theta}\|x\|_{X_1}. \qquad \square$$

The first main result of this paper is the following.

Theorem 1. *Assume that the norms of the Banach spaces $X_1 \subseteq X_0$ satisfy*

$$\|x\|_{X_0} \le c\|x\|_{X_1} \qquad \forall\, x \in X_1 \tag{7}$$

for some positive constant $c > 0$. Let (Y_0, Y_1) be an arbitrary Banach couple and, for some $r > 0$, let

$$\Phi\colon B(r, X_0) \to Y_+ \tag{8}$$

be an analytic map with $\Phi(B(r, X_0)) \subseteq Y_0$ *and* $\Phi(B(c^{-1}r, X_1)) \subseteq Y_1$*, so that*

$$\Phi\colon B(r, X_0) \to Y_0 \tag{9}$$

and

$$\Phi|_{B(c^{-1}r,X_1)}\colon B(c^{-1}r, X_1) \to Y_1 \tag{10}$$

are continuous and bounded by the constants M_0 *and* M_1*, respectively,*

$$\sup_{x\in B(r,X_0)} \|\Phi(x)\|_{Y_0} \le M_0 \qquad \text{and} \qquad \sup_{x\in B(c^{-1}r,X_1)} \|\Phi(x)\|_{Y_1} \le M_1. \tag{11}$$

Then, for any $0 < \theta < 1$*, the inclusions* $B(c^{-\theta}r, X_\theta) \subseteq B(r, X_0)$ *and* $\Phi(B(c^{-\theta}r, X_\theta)) \subseteq Y_\theta$ *hold. Furthermore,*

$$\Phi|_{B(c^{-\theta}r,X_\theta)}\colon B(c^{-\theta}r, X_\theta) \to Y_\theta \tag{12}$$

is bounded. More precisely, for any $x \in B(c^{-\theta}r, X_\theta)$

$$\|\Phi(x)\|_{Y_\theta} \le M_0^{1-\theta} {M_1}^{\theta}. \tag{13}$$

Remark 1. Theorem 1 is related to [18, Theorem 1.1], where sufficient conditions are provided for nonlinear maps defined on balls to have the interpolation property. Our proof of Theorem 1 is self-contained, making no reference to Theorem 1.1 in [18]. At the same time, it implicitly verifies some conditions which appear in [18]. Theorem 1 is also related to [17, Theorem 1] stating a result comparable to ours. However, condition (P2), assumed in Theorem 1 in [17], is too strong for our purposes: expressed in the setup of our paper, this condition says that the map Φ continuously maps $B(r, X_0) \cap X_1$ into the space Y_1. As already pointed out in our discussion of Tartar's result of [13], $B(r, X_0) \cap X_1$ may be of infinite diameter in X_1. In comparison, in Theorem 1, it is only assumed that Φ continuously maps $B(c^{-1}r, X_1)$ into Y_1.

Proof of Theorem 1. By Lemma 1, for any $0 < \theta < 1$,

$$B(c^{-\theta}r, X_\theta) \subseteq B(r, X_0).$$

Hence the map Φ is well defined on $B(c^{-\theta}r, X_\theta)$. It remains to prove that $\Phi(B(c^{-\theta}r, X_\theta)) \subseteq Y_\theta$ and that (13) holds. Take an arbitrary $x \in B(c^{-\theta}r, X_\theta)$. By the definition of the norm $\|x\|_\theta$, there exists a function $f \in \mathcal{H}(X_0, X_1)$ such that

$$f(\theta) = x \qquad \text{and} \qquad \|f\|_{\mathcal{H}} < c^{-\theta}r.$$

In particular, in view of the definition of the norm $\|f\|_{\mathcal{H}}$, one has, for any $t \in \mathbb{R}$,

$$\|f(it)\|_{X_0} < c^{-\theta}r \qquad \text{and} \qquad \|f(1 + it)\|_{X_1} < c^{-\theta}r. \tag{14}$$

Consider the function

$$g\colon \overline{S} \to X_+, \qquad z \mapsto g(z) := c^{\theta-z}f(z).$$

Clearly, $g \in \mathcal{H}(X_0, X_1)$ and $x = g(\theta)$. Since X_0 and X_+ coincide and the norms of X_0 and X_+ are equivalent, it follows that $g\colon \overline{S} \to X_0$ is continuous and bounded, and $f|_S\colon S \to X_0$ is analytic. Moreover, in view of (7) and (14), for any $t \in \mathbb{R}$, we have

$$\|g(1 + it)\|_{X_0} = c^{\theta-1}\|f(1 + it)\|_{X_0} \le c^\theta\|f(1 + it)\|_{X_1} < r \tag{15}$$

$$\|g(it)\|_{X_0} \le c^\theta\|f(it)\|_{X_0} < r. \tag{16}$$

Applying Hadamard's three line theorem (see, e.g., [2, Lemma 1.1.2]), then yields

$$\|g(z)\|_{X_0} < r \qquad \forall\, z \in \overline{S}. \tag{17}$$

Further, it follows from (14) that

$$\|g(1 + it)\|_{X_1} = c^{\theta-1}\|f(1 + it)\|_{X_1} < c^{-1}r. \tag{18}$$

Using inequality (17), we can define $F\colon \overline{S} \to Y_+$ by setting

$$F(z) := M_0^{z-1} M_1^{-z} \Phi(g(z)).$$

Since $g\colon \overline{S} \to B(r, X_0) \subseteq X_0$ is continuous and, by assumption, $\Phi\colon B(r, X_0) \to Y_0$ is bounded and continuous, we conclude from the continuity of the embedding $Y_0 \subseteq Y_+$ that $F\colon \overline{S} \to Y_+$ is also continuous and bounded. Similarly, since $g|_S\colon S \to B(r, X_0) \subseteq X_0$ and, by our assumption, $\Phi\colon B(r, X_0) \to Y_+$ are analytic, the function $F|_S\colon S \to Y_+$ is analytic as well. The inequalities (16) and (18), together with the bounds (11), as well as the continuity and the boundedness of the maps $\Phi\colon B(r, X_0) \to Y_0$ and $\Phi|_{B(c^{-1}r, X_1)}\colon B(c^{-1}r, X_1) \to Y_1$, then imply that $F \in \mathcal{H}(Y_0, Y_1)$ and $\|F\|_{\mathcal{H}} \le 1$. Further, in view of the equality $F(\theta) = M_0^{\theta-1} M_1^{-\theta} \Phi(x)$, we conclude that

$$\|\Phi(x)\|_{Y_\theta} = M_0^{1-\theta} M_1^{\theta} \|F(\theta)\|_{Y_\theta} \le M_0^{1-\theta} M_1^{\theta} \|F(z)\|_{\mathcal{H}} \le M_0^{1-\theta} M_1^{\theta}.$$

This completes the proof of Theorem 1. □

If the nonlinear maps $\Phi\colon B(r, X_0) \to Y_0$ and $\Phi|_{B(c^{-1}r, X_1)}\colon B(c^{-1}r, X_1) \to Y_1$ in Theorem 1 admit polynomial bounds of order n for some $n \ge 1$, then it follows from the next statement that the same is true for the maps $\Phi|_{B(c^{-\theta}r, X_\theta)}\colon B(c^{-\theta}r, X_\theta) \to Y_\theta$ with arbitrary $0 < \theta < 1$. Actually, the following statement generalizes the corresponding result of [16].

Corollary. *Assume that all assumptions of Theorem* 1 *are satisfied except that the boundedness condition* (11) *is replaced for some integer* $n \ge 0$ *by*

$$\|\Phi(x)\|_{Y_0} \le M_0 \|x\|_{X_0}^n \qquad \forall\, x \in B(r, X_0), \tag{19}$$

$$\|\Phi(x)\|_{Y_1} \le M_1 \|x\|_{X_1}^n \qquad \forall\, x \in B(c^{-1}r, X_1). \tag{20}$$

Then, for any $0 \le \theta \le 1$ *and any* $x \in B(c^{-\theta}r, X_\theta)$,

$$\|\Phi(x)\|_{Y_\theta} \le M_0^{1-\theta} M_1^{\theta} \|x\|_{X_\theta}^n. \tag{21}$$

Proof of the corollary. Let $x \in B(c^{-\theta}r, X_\theta)$, and set $\rho := \|x\|_{X_\theta}$. Choose $\varepsilon > 0$ so that

$$r_1 := c^{\theta}(\rho + \varepsilon) < r.$$

Then, by (19),

$$\|\Phi(y)\|_{Y_0} \le M_0 r_1^n \qquad \forall\, y \in B(r_1, X_0)$$

and, by (20),

$$\|\Phi(y)\|_{Y_1} \le M_1 (c^{-1} r_1)^n \qquad \forall\, y \in B(c^{-1} r_1, X_1).$$

Theorem 1 then implies that, for all $y \in B(c^{-\theta} r_1, X_\theta)$,

$$\|\Phi(y)\|_{Y_\theta} \le M_0^{1-\theta} M_1^{\theta} (r_1)^{n(1-\theta)} (c^{-1} r_1)^{n\theta} = M_0^{1-\theta} M_1^{\theta} (\rho + \varepsilon)^n.$$

In particular, the above inequality holds for $y = x$. Since $\varepsilon > 0$ can be chosen arbitrarily small, the corollary follows. □

Our second main result says that the map

$$\Phi|_{B(c^{-\theta}r, X_\theta)}\colon B(c^{-\theta}r, X_\theta) \to Y_\theta,$$

in Theorem 1 is analytic.

Theorem 2. *The map* $\Phi|_{B(c^{-\theta}r, X_\theta)}\colon B(c^{-\theta}r, X_\theta) \to Y_\theta$, *in Theorem* 1 *is analytic.*

Proof. Inspired by arguments used in [16], we prove the claim by showing that $\Phi|_{B(c^{-\theta}r,X_\theta)}$ is represented by a series of analytic maps and the series converges absolutely and uniformly in Y_θ on any ball $B(c^{-\theta}\rho, X_\theta)$ with $0<\rho<r$ (see [4, Theorem 2, Appendix A]. First, note that Y_0 and Y_1 are continuously embedded in Y_+, so, by Lemma 2 in the Appendix, the maps

$$\Phi\colon B(r,X_0)\to Y_0 \qquad \text{and} \qquad \Phi|_{B(c^{-1}r,X_1)}\colon B(c^{-1}r,X_1)\to Y_1$$

of (9) and (10) are analytic. Being analytic, the map $\Phi\colon B(r,X_0)\to Y_0$ is represented by its Taylor's series at 0 with values in Y_0,

$$\Phi(h)=\Phi(0)+\sum_{n=1}^{\infty}\Phi_n(h), \qquad h\in B(r,X_0). \tag{22}$$

Here

$$\Phi(0)\in Y_1 \qquad \text{and} \qquad \Phi_n(h)=\frac{1}{n!}\,d_0^n\Phi(h,\dots,h), \quad n\ge 1,$$

with $d_0^n\Phi$ denoting the nth derivative of Φ at 0 (cf. (A4) in Appendix). We remark that, for any $n\ge 1$, $\Phi_n(h)$ is a bounded homogeneous polynomial of degree n in $h\in X_0$ with values in Y_0 and hence is analytic, and that the series in (22) converges absolutely and uniformly in h on any ball $B(\rho,X_0)$ with $0<\rho<r$. Moreover, in view of Cauchy's formula (cf. the Appendix), for any $h\in X_0$ with $h\ne 0$,

$$\Phi_n(h)=\frac{1}{2\pi i}\oint_{|z|=\rho}\frac{\Phi(zh)}{z^{n+1}}\,dz, \tag{23}$$

where ρ is chosen arbitrarily so that $0<\rho<r/\|h\|_{X_0}$. Using (23) and the first inequality in (11), we see that, for any $h\in X_0$ with $h\ne 0$,

$$\|\Phi_n(h)\|_{Y_0}\le\frac{1}{2\pi}\int_0^{2\pi}\frac{\|\Phi(\rho e^{it}h)\|_{X_0}}{|\rho e^{it}|^{n+1}}|\rho\, ie^{it}|\,dt\le\frac{M_0}{\rho^n}.$$

Since this inequality holds for any $0<\rho<r/\|h\|_{X_0}$, it follows that, for any $h\in X_0$ with $h\ne 0$,

$$\|\Phi_n(h)\|_{Y_0}\le\frac{M_0}{r^n}\|h\|_{X_0}^n. \tag{24}$$

Note that the latter estimate holds trivially for $h=0$. Applying the arguments above to the analytic map $\Phi|_{B(c^{-1}r,X_1)}\colon B(c^{-1}r,X_1)\to Y_1$, we see that, for any $h\in B(c^{-1}r,X_1)$, $\Phi_n(h)\in Y_1$ and the series (22) converges in Y_1. Further, by (23), for any $h\in X_1$,

$$\|\Phi_n(h)\|_{Y_1}\le\frac{M_1}{(c^{-1}r)^n}\,\|h\|_{X_1}^n\,. \tag{25}$$

Applying the corollary to the the map $\Phi_n\colon X_0\to Y_0\subseteq Y_+$, we conclude from (24) and (25) that, for any $h\in X_\theta$, $\Phi_n(h)\in Y_\theta$ and

$$\|\Phi_n(h)\|_{Y_\theta}\le\left(\frac{M_0}{r^n}\right)^{1-\theta}\left(\frac{M_1}{(c^{-1}r)^n}\right)^{\theta}\|h\|_{X_\theta}^n\le M_0^{1-\theta}M_1^{\theta}\left(\frac{\|h\|_{X_\theta}}{c^{-\theta}r}\right)^n.$$

This inequality shows that, for any $h\in B(c^{-\theta}r,X_\theta)$, the series in (22) converges in Y_θ and that it converges absolutely and uniformly on any ball $B(c^{-\theta}\rho,X_\theta)$ with $0<\rho<r$. □

Finally, we remark that Theorem 1, combined with Theorem 2, generalizes Theorem 1.1 in [14] in the context of the setup chosen in this paper.

3. APPENDIX

In this appendix, we review the notion of an analytic map between complex Banach spaces and discuss properties of such maps used in Sec. 2. For more details we refer the reader, e.g., to [3] or [4, Appendix A].

Let X and Y be complex Banach spaces and let $U \subseteq X$ be an open set in X. A map $F\colon U \to Y$ is said to be *analytic* if it is Fréchet differentiable over $\mathbb{C}$ at any point $x \in U$. The map $F\colon U \to Y$ is said to be *weakly analytic* if for any $x \in U$, $h \in X$, and $f \in Y^*$, the complex-valued function $f(F(x+zh))$ is holomorphic in a small disk in $\mathbb{C}$ centered at zero.

Let us recall the following analyticity criteria (see, e.g., [4, Theorem 1.1, Appendix A]). Assume that X and Y are complex Banach spaces and that $\Phi\colon U \to Y$ is a map defined on an open subset U of X, $U \subseteq X$. Then the following statements are equivalent:

(A1) $\Phi\colon U \to Y$ *is analytic*;

(A2) $\Phi\colon U \to Y$ *is weakly analytic and locally bounded*;

(A3) $\Phi\colon U \to Y$ *is continuous and for any* $x \in U$ *and* $h \in X$*, there exists a disk* $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ *such that, for any* $0 < \rho < r$*, Cauchy's formula holds*

$$F(z) = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{\Phi(\xi)}{\xi - z}\, d\xi \qquad \forall\, |z| < \rho; \tag{26}$$

(A4) $\Phi\colon U \to Y$ *is infinitely differentiable on* U *and is represented by its Taylor series in a neighborhood of each point of* U*, i.e., for any* $x \in U$,

$$\Phi(x+h) = \Phi(x) + \sum_{n=1}^{\infty} d_x^n \Phi(h, \dots, h),$$

where $d_x^n \Phi$ *denotes the nth derivative of* Φ *at* x *and the series converges absolutely and uniformly for any* $h \in B(\rho, X)$*, with* $0 < \rho < r$ *so that* $B(r, X) \subset U$.

Now let us prove the following lemma.

Lemma 2. *Let* X*,* Y*, and* Y_+ *be Banach spaces such that* $Y \subset Y_+$*. Let* $U \subseteq X$ *be an open set in* X *and let* $\Phi\colon U \to Y_+$ *be analytic,* $\Phi(U) \subseteq Y$*, and* $\Phi\colon U \to Y$ *be continuous. Then* $\Phi\colon U \to Y$ *is analytic.*

Proof. For any $x \in U$ and $h \in X$, consider the map

$$F\colon D_r \to Y_+, \qquad F(z) := \Phi(x+zh),$$

where the disk $D_r \subseteq \mathbb{C}$ is centered at zero and chosen so that $x + zh \in U$ for any $z \in D_r$. Since, by assumption, $F\colon D_r \to Y_+$ is analytic in the Y_+-norm, it follows that Cauchy's formula (26) holds in Y_+ for any given $z \in D$. Furthermore, it follows from the assumptions of the lemma that $F\colon D_r \to Y$ is continuous with respect to the Y-norm; then the integral on the right-hand side of (26) defines a continuous map $G\colon D_r \to Y$. In view of the embedding $Y \subseteq Y_+$, one has $G = F$. □

ACKNOWLEDGMENTS

The work of the first author was supported by the Swiss National Science Foundation. The work of the second and third authors was supported by the Russian Science Foundation (RSF) (grant no. 14-11-00754). The work of the fourth author was supported by the National Science Foundation (NSF) (grant no. DMS-0901443).

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