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An invariance principle for isotropic diffusions in random environment

Alain-Sol Sznitman[1], **Ofer Zeitouni**[2,3,⋆]

[1] Departement Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland
[2] Department of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455, USA
[3] Departments of Electrical Engineering and of Mathematics, Technion, Israel Institute of Technology, Haifa 32000, Israel

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Abstract. We investigate in this work the asymptotic behavior of isotropic diffusions in random environment that are small perturbations of Brownian motion. When the space dimension is three or more, we prove an invariance principle as well as transience. Our methods also apply to questions of homogenization in random media.

0. Introduction

The mathematical investigation of transport in disordered media has been an active field of research over the last thirty years, rich in surprising effects and mathematical challenges. In a number of cases the method of the environment viewed from the particle has proven a powerful tool, cf. De Masi et al. [7], Kipnis-Varadhan [12], Kozlov [13], Molchanov [17], Olla [18], [19], Papanicolaou-Varadhan [21], [22]. However basic models such as random walk in random environment or Brownian motion perturbed by an environment-dependent drift, when typically the random drift is neither the gradient of a stationary function nor incompressible, have in essence not been amenable to this approach and remain to this day mathematical challenges. An intensive effort to understand these models has been launched in the last five years. Progress has been made, especially in the case of ballistic behavior, i.e. when the particle has a non-degenerate velocity, see for instance [27], [32] and the references therein. As for diffusive behavior, there has been some progress, cf. [4], but overall the topic has

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been little touched. The present work is precisely concerned with diffusive behavior, and investigates isotropic diffusions in random environment that are small perturbations of Brownian motion. When the space dimension is three or more, we prove transience and an invariance principle. The model we analyze is a continuous counterpart of the model studied by Bricmont-Kupiainen [5]. However our strategy of proof is different and we believe more transparent.

Let us first describe the setting in more details. The local characteristics, i.e. covariance and drift, of the diffusion in random environment are bounded stationary functions $a(x,\omega)$, $b(x,\omega)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, with respective values in the non-negative d-matrices and $\mathbb{R}^d$, $d \geq 3$; the set Ω is endowed with a group $(t_x)_{x\in\mathbb{R}^d}$ of jointly measurable transformations preserving the probability $\mathbb{P}$ on Ω. We assume that for $\omega \in \Omega$, $a(\cdot,\omega)$ is uniformly elliptic, see (1.5), and that

$$\begin{aligned} &a(\cdot,\omega) \text{ and } b(\cdot,\omega) \text{ satisfy a Lipschitz condition} \\ &\text{with constant } K, \text{ cf. (1.4)}\,. \end{aligned} \tag{0.1}$$

We denote with $P_{x,\omega}$ the law of the diffusion in the environment ω, starting from x, i.e. the unique probability on $C(\mathbb{R}_+,\mathbb{R}^d)$ solution of the martingale problem attached to x and

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(y,\omega)\,\partial^2_{ij} + \sum_{i=1}^{d} b_i(y,\omega)\,\partial_i\,, \tag{0.2}$$

cf. [26]. We let $(X_t)_{t\geq 0}$ stand for the canonical process on $C(\mathbb{R}_+,\mathbb{R}^d)$.

The random characteristics of the diffusion are assumed to have finite range dependence, namely for some $R > 0$, under $\mathbb{P}$,

$$\begin{aligned} &\sigma\big(a(x,\cdot), b(x,\cdot), x \in A\big) \text{ and } \sigma\big(a(y,\cdot), b(y,\cdot), y \in B\big) \\ &\text{are independent when } A, B \subseteq \mathbb{R}^d \text{ have mutual distance at least } R\,. \end{aligned} \tag{0.3}$$

Further they also fulfill a restricted isotropy condition, namely for any rotation matrix r preserving the union of coordinate axes of $\mathbb{R}^d$,

$$\begin{aligned} &\big(a(rx,\omega), b(rx,\omega)\big)_{x\in\mathbb{R}^d} \text{ has same law under } \mathbb{P} \text{ as} \\ &\big(ra(x,\omega)r^T, rb(x,\omega)\big)_{x\in\mathbb{R}^d}\,, \end{aligned} \tag{0.4}$$

we refer to Sect. 1 for details.

The main result of this article, cf. Theorem 6.3, states that

Theorem. $(d \geq 3)$
There is an $\eta_0(d,K,R) > 0$*, such that if*

$$|a(x,\omega) - I| \leq \eta_0,\ \ |b(x,\omega)| \leq \eta_0, \textit{for all } x \in \mathbb{R}^d, \omega \in \Omega\,, \tag{0.5}$$

then for $\mathbb{P}$*-a.e.* ω,

$$\frac{1}{\sqrt{t}}\, X_{\cdot t} \text{ under } P_{0,\omega} \text{ converges in law to Brownian motion on } \mathbb{R}^d \text{ with deterministic variance } \sigma^2 > 0, \text{ as } t \to \infty\,, \tag{0.6}$$

and

$$\text{for all } x \in \mathbb{R}^d,\ P_{x,\omega}\text{-a.s.},\ \lim_{t\to\infty} |X_t| = \infty\,. \tag{0.7}$$

In other words for diffusions in random environment that are small perturbations of Brownian motion and satisfy the restricted isotropy condition (0.4), we prove transience and diffusive behavior. Our results also apply to questions of homogenization in random media, cf. Theorem 6.4, and show that

Theorem. $(d \geq 3)$
One can choose $\eta_0(d, K, R) > 0$*, so that when* (0.5) *holds, on a set of full* $\mathbb{P}$*-probability, for any bounded functions* f, g *on* $\mathbb{R}^d$*, respectively continuous and Hölder continuous, the solution of the Cauchy problem:*

$$\begin{cases} \partial_t\, u_\epsilon = L_\epsilon\, u_\epsilon + g, \ \text{in } (0,\infty) \times \mathbb{R}^d\,, \\ u_\epsilon|_{t=0} = f\,, \end{cases} \tag{0.8}$$

where for $\epsilon > 0$,

$$L_\epsilon = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\Big(\frac{x}{\epsilon}, \omega\Big)\, \partial^2_{ij} + \sum_{i=1}^{d} \frac{1}{\epsilon}\, b_i\Big(\frac{x}{\epsilon}, \omega\Big)\, \partial_i\,, \tag{0.9}$$

converges uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{R}^d$*, as* $\epsilon \to 0$*, to the solution of the Cauchy problem*

$$\begin{cases} \partial_t\, u_0 = \dfrac{\sigma^2}{2}\, \Delta\, u_0 + g, \ \text{in } (0,\infty) \times \mathbb{R}^d\,, \\ u_0|_{t=0} = f\,. \end{cases} \tag{0.10}$$

When $b(\cdot, \omega) \equiv 0$, cf. [22], [31], or when L is in divergence form, cf. [7], [13], [19], [20], [21], the method of the environment viewed from the particle applies successfully, and there is an extensive literature on invariance principles describing diffusive behavior and applications to homogenization. There is also ample literature on analogous discrete situations, cf. [2], [3], [12], [13], [14], [15]. On the other hand the case of general diffusions in random environment of type (0.2) remains poorly understood, reflecting the genuine non self-adjoint character of the problem and the absence of invariant measure at hand. We do not know of any work proving diffusive behavior, and in the context of random walks in random environment only of [4], [5]. The restricted isotropy condition (0.4) provides us with a convenient way to rule out the presence of a non-degenerate limiting velocity (i.e. so-called ballistic behavior). This is a somewhat delicate matter because there is no explicit formula in dimension bigger than one expressing what the limiting velocity of the particle ought to be. Examples have for instance been provided in [4], showing that in the discrete context of random walks in random environment, the assumption of mean zero drift does not rule out ballistic behavior. So in this work (0.4) grants a convenient centering condition for the diffusion in random environment.

We will now give some description of the proof of our results. We construct a sequence of measures coupling on increasing space and time scales the diffusion in random environment to a sequence of Brownian motions with respective variances α_n, cf. (0.12) below, that converge to σ^2 in (0.6). These couplings yield efficient approximations of the diffusion in random environment, cf. Proposition 6.2, from which the claims (0.6), (0.7), (0.10) follow straightforwardly. The construction of this sequence of couplings involves an induction (or renormalization) scheme propagating controls from one scale to the next. In this scheme a sequence of Hölder-norms plays a central role via estimates in operator norm of the difference of (a truncation of) the transition kernel of the diffusion in random environment with that of Brownian motion with variance α_n. These Hölder-norm controls are used in at least three ways. First, together with the Kantorovich-Rubinstein theorem, cf. [8], they provide estimates on Vasserstein distances and enable to construct good couplings, cf. Proposition 3.1. A second use stems from the fact that when the medium behaves nicely in a given scale, these Hölder-norm controls have good contraction properties, when moving to the next scale, at least when the dimension d is three or more, cf. Remark 4.7. Finally, in the induction scheme we have to face the occurrence of certain deviations from "nice behavior". Some of these deviations arise from defects in the medium that have no real trapping power, but where nevertheless the Hölder-norm controls pertinent to "nice behavior" in a given scale, are violated. Here comes a third role of Hölder norm controls. Namely they enable to smooth out, when looking at a higher scale, the presence of a (few) defects on a lower scale, with no trapping power, cf. Proposition 5.1. In addition to the above mentioned defects that can be handled through the use of Hölder norms, one also has to handle the potential appearance of traps, i.e. pockets in the medium that may emprison the particle for a long time, and thus destroy its diffusive character. As part of the induction scheme, we show that traps are rare, by constructing suitable escape strategies for the diffusion, that prove that it is very unlikely for the medium to entrap the particle, cf. Proposition 3.3.

We will now discuss the renormalization scheme in a somewhat more precise fashion. The main point appears in Theorem 1.1. It states an induction step concerning the behavior of the diffusion in random environment along a sequence of length scales $L_n \simeq L_0^{(1+a)^n}$ and time scales L_n^2, where a is a small positive number and L_0 in a large enough number, cf. (1.14), (1.15). Several assumptions are propagated from level n to level $n+1$. A first assumption, cf. (1.47), states that up to a $\mathbb{P}$-probability decaying like a large negative power of L_n, the following holds. On the one hand, for starting points x with distance const L_n from the origin, the displacements of the path of the diffusion in the environment ω slightly beyond distances of order L_n satisfy under $P_{x,\omega}$ a certain exponential control, cf. (1.39), and on the other hand the transition kernel at time L_n^2 of the diffusion:

$$R_n(x, dy) = P_{x,\omega}[X_{L_n^2} \in dy] \tag{0.11}$$

is in a sense that we explain below "close" to the Gaussian kernel

$$\begin{aligned} &R_n^0(x, dy) = \left(2\pi\alpha_n L_n^2\right)^{-d/2} \exp\left\{ - \frac{|y-x|^2}{2\alpha_n L_n^2} \right\} dy, \quad \text{with} \\ &\alpha_n \approx \mathbb{E}E_{0,\omega}\left[|X_{L_n^2}|^2\right] / \left(dL_n^2\right), \end{aligned} \tag{0.12}$$

(cf. (1.22) for the precise definition), after localization of x in a box of size *const* L_n around the origin. The way in which "close" is defined plays a pivotal role in this work. It refers to the operator norm $\|\cdot\|_n$, for linear transformations on the space of bounded Hölder continuous functions of order β (some fixed number in $(0, \frac{1}{2}]$, cf. (1.13)), endowed with the norm $|\cdot|_{(n)}$, cf. (1.28), adapted to functions "living in scale L_n":

$$|f|_{(n)} = \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\left|\frac{x-y}{L_n}\right|^\beta}. \tag{0.13}$$

In essence "close" means $\|\chi_{n,0}(R_n - R_n^0)\|_n \leq \text{const } L_n^{-\delta}$, where $\chi_{n,0}$ is a cut-off function localizing x in (0.11), (0.12), within distance const L_n of the origin, cf. (1.38), and $\delta > 0$ is a fraction of β, cf. (1.40).

A second assumption being propagated, cf. (1.48), states quantitatively the rarity of traps by describing the domination of the tails under $\mathbb{P}$ of certain variables measuring the strength of traps in boxes of size L_n, cf. (1.44), by the corresponding tails of i.i.d. variables equal to 0 with overwhelming probability.

The third and last assumption entering the induction step, cf. (1.49), controls the behavior of α_n.

Once Theorem 1.1 is proved, we show in Sect. 6 that when the local characteristics of the diffusion satisfy (0.5), we can start the induction stated in Theorem 1.1. So the induction assumptions propagate to all levels n, and with Borel-Cantelli's lemma we see that all boxes L_n within distance const L_{n+3}^2 of the origin "behave well". With the Kantorovich-Rubinstein Theorem, cf. [8], the Hölder-norm estimates and the controls on displacements of the diffusion, cf. (1.47), enable to construct "good couplings" between the diffusion in random environment and Brownian motion with variance α_n, cf. Proposition 6.2. Since α_n converges to a positive limit, namely σ^2 of (0.6), the invariance principle easily follows. The transience of the diffusion, cf. (0.7), and the homogenization result (0.8), (0.10), also come as easy consequences of these coupling measures.

Let us explain how the article is organized and briefly comment on each section. Section 1 presents the setting and states Theorem 1.1. The proof of Theorem 1.1 occupies Sects. 2 to 5 of the article.

Section 2 propagates from level n to level $n+1$ the controls on the displacement of the path, cf. Proposition 2.2.

Section 3 propagates the controls on traps, cf. (1.48) and Proposition 3.3. Traps are a serious matter in our problem because a pocket of size L has the potential, depending on the realization of the medium, to entrap the particle for times of exponential order in L. Hence pockets of relatively modest size may distort the diffusive behavior of the particle on many time scales L_n^2. This feature naturally affects the distribution of the variables in (1.44) that measure the strength of traps. We are in fact mainly interested in a small portion of the information contained in (1.48), namely ensuring that the variables in (1.44) vanish with "overwhelming probability", cf. (5.2), (5.3). But the inductive proof requires a control on the tails of the variables in (1.44). To carry the tail domination control (1.48) from level n to level $n+1$, in essence we exhibit exit strategies for the particle from boxes of size L_{n+1} before time L_{n+1}^2, which show that it is costly for the medium to produce a trap at level $n+1$ of a given strength. Depending on the strength in question, the exit strategy that is employed varies, and we distinguish four distinct regimes, (three regimes suffice when $d \geq 4$), cf. (3.20).

Sections 4 and 5 are devoted to the propagation from level n to level $n+1$ of the Hölder-norm controls contained in (1.47).

In Sect. 4, we perform "surgery" in a large box of size const L_{n+1}^2 around the origin, which contains the relevant portion of the medium for our purpose. We investigate at a finite depth $n-m_0-1$, with m_0 a fixed number, cf. (1.17), this large box, remove all boxes of size L_{n-m_0-1} where bad behavior in the sense of (1.47) occurs, and in essence replace them with good boxes. In this new artificial environment "after surgery", we analyze the diffusion at all the levels n' between $n-m_0-1$ and $n+1$. We show that with overwhelming $\mathbb{P}$-probability this environment not only does not develop in these intermediate levels bad Hölder-norm behavior with distance L_{n+1}^2 from the origin, but produces a decay of the relevant $\|\cdot\|_{n'}$-norms faster than $L_{n'}^{-\delta}$, cf. Proposition 4.11. Wavelets, cf. [6], [16], turn out to provide a powerful tool in the control of the $\|\cdot\|_{n'}$-norms of certain random linear operators, cf. Lemma 4.5 and 4.6. Isotropy also provides crucial centerings, cf. (4.78), (4.79). Collecting Lemmas 4.2 to 4.6, one can read that the relevant $\|\cdot\|_{n'}$-norms mentioned above "contract like $L_{n'}^{-\beta/3\wedge(1-\beta)\wedge(d/2-1)}$", see also Remark 4.7.

In Sect. 5, we compare at level $n+1$ the true environment with the environment after surgery constructed in Sect. 4. The difference between them resides in a few defects of size L_{n-m_0-1}. Thanks to the controls on traps in (1.48), we can assume that these defects have no trapping power. Then using a strategy close in spirit to Sect. 2 of [25], we show that the Hölder regularity of the kernels of the diffusion in the environment after surgery performed in Sect. 4, tends to repair the small defects of the true environment, cf. Proposition 5.1. One can then recover with large $\mathbb{P}$-probability the bound $\|\chi_{n+1,0}(R_{n+1} - R^0_{n+1})\|_{n+1} \le \text{const } L^{-\delta}_{n+1}$, required to prove (1.47) at level $n+1$, and the discrepancy $|\alpha_{n+1} - \alpha_n|$ is controlled in Proposition 5.7.

Section 6 as indicated previously applies Theorem 1.1 to the proof of the main Theorem 6.3, cf. also (0.6), (0.7), and to the derivation of an homogenization result, cf. Theorem 6.4 and (0.8), (0.10).

The Appendix collects some useful results on the norms $|\cdot|_{(n)}$ on the space of β-Hölder continuous functions, cf. (0.13), and on the control of the corresponding operator norms $\|\cdot\|_n$ with wavelets, cf. Proposition A.2.

The work by Bricmont-Kupiainen [5] was certainly a source of inspiration for the present work even if we had difficulty to follow some of their arguments. Our proof albeit using renormalization follows a different track. It may be helpful to highlight some of the differences beyond the fact that in [5] the setting is discrete and here it is continuous. In this article we introduce a family of Hölder-norms that play an important role both for their contraction properties and the couplings they enable to construct. They also motivate the use of wavelets. Further we directly compare the quenched transition kernels of the diffusion, cf. (0.11) to certain Gaussian kernels, cf. (0.12), and not to the $\mathbb{P}$-average of the kernels in (0.11). This simplifies the proof. Our bounds on traps are conducted in a different fashion, that is more in line with [29]. We do not carry in our induction a decomposition of the kernels into "small field" and "large field". The scales along which we perform renormalization here grow faster than geometrically, and we perform surgery at a finite depth, and compare what happens in true and "after surgery" environments. Our proof also enables to have, unlike [5], a concise induction step stated in Theorem 1.1. We believe this is a source of clarity.

Finally let us say a few words concerning the decision to work in a continuous rather than discrete setting. It entails some simplifications because a number of scaling arguments become natural and straightforward. But it also bears some technical intricacies related to regularity questions at small scales. Decisive was perhaps the fact that some of the calculations involving wavelets are more transparent and standard when one uses wavelets on $\mathbb{R}^d$, rather than wavelets on $\mathbb{Z}^d$, cf. [16], §7.3.3.

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1. Setting and main induction step

In this section we introduce notation for the main objects of interest and collect some of their elementary properties. We also present in Theorem 1.1 the induction assumption that will be propagated. The proof of Theorem 1.1 occupies the next four sections.

We let $(e_i)_{1\le i\le d}$ stand for the canonical basis of $\mathbb{R}^d$, and $d \ge 3$ throughout the article. We respectively denote with $|\cdot|$ and $|\cdot|_\infty$ the Euclidean and supremum distances on $\mathbb{R}^d$. We let $B(x,r)$ and $\overline{B}(x,r)$ stand for the open and closed Euclidean balls with center $x \in \mathbb{R}^d$ and radius $r > 0$, and write $B_\infty(x,r)$, $\overline{B}_\infty(x,r)$ for the corresponding $|\cdot|_\infty$-balls. For A, B subsets of $\mathbb{R}^d$ we denote with

$$d(A,B) = \inf\{|x-y|;\ x \in A, y \in B\}, \tag{1.1}$$

their mutual $|\cdot|$-distance, and with $d_\infty(A,B)$ their analogously defined mutual $|\cdot|_\infty$-distance. When $\mathcal{U}$ is a finite subset, we write $|\mathcal{U}|$ for the cardinality of $\mathcal{U}$.

The *random environment* is described by $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space endowed with $(t_x)_{x\in\mathbb{R}^d}$ a bi-measurable group of $\mathbb{P}$-preserving transformations. The diffusion matrix and the drift of the diffusion in random environment are stationary functions $a(x,\omega)$, $b(x,\omega)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, with respective values in the space M_d^+ of non-negative d-matrices and $\mathbb{R}^d$:

$$\begin{aligned} &a(x, t_y\,\omega) = a(x+y,\omega)\,,\\ &b(x, t_y\,\omega) = b(x+y,\omega), \ \text{ for } x, y \in \mathbb{R}^d,\ \omega \in \Omega\,. \end{aligned} \tag{1.2}$$

We assume that these functions are bounded and uniformly Lipschitz, i.e. there is $K > 1$, such that for $x, y \in \mathbb{R}^d$, $\omega \in \Omega$,

$$|b(x,\omega)| + |a(x,\omega)| \le K\,, \tag{1.3}$$

$$|b(x,\omega) - b(y,\omega)| + |a(x,\omega) - a(y,\omega)| \le K|x-y|\,. \tag{1.4}$$

Further we assume that the diffusion matrix is uniformly elliptic, i.e. there is a $\nu > 1$, such that for $x \in \mathbb{R}^d$, $\omega \in \Omega$:

$$\frac{1}{\nu}\, I \le a(x,\omega) \le \nu\, I\,. \tag{1.5}$$

As mentioned in (0.3) the local characteristics of the diffusion satisfy a condition of finite range dependence. Namely for $A \subseteq \mathbb{R}^d$, we define

$$\mathcal{G}_A = \sigma\big(a(x,\cdot),\ b(x,\cdot);\ x \in A\big)\,, \tag{1.6}$$

and assume that for some $R > 0$,

$$\mathcal{G}_A \text{ and } \mathcal{G}_B \text{ are independent under } \mathbb{P} \text{ whenever } d(A,B) \ge R\,. \tag{1.7}$$

Finally we assume that the local characteristics of the diffusion satisfy the restricted isotropy condition stated in (0.4).

We recall that $(X_t)_{t\ge0}$ denotes the canonical process on $C(\mathbb{R}_+,\mathbb{R}^d)$. We write $(\mathcal{F}_t)_{t\ge0}$ and $(\theta_t)_{t\ge0}$ for the respective canonical right-continuous filtration and canonical shift on $C(\mathbb{R}_+,\mathbb{R}^d)$. We also write H_B and T_U for the respective entrance time of X in the closed set $B \subseteq \mathbb{R}^d$ and exit time of X from the open set $U \subseteq \mathbb{R}^d$:

$$H_B = \inf\{u \geq 0,\ X_u \in B\},\ T_U = \inf\{u \geq 0,\ X_u \notin U\}\,. \tag{1.8}$$

In view of (1.2)–(1.5), for any $\omega \in \Omega$, $x \in \mathbb{R}^d$, the martingale problem attached to $(a(\cdot,\omega), b(\cdot,\omega), x)$, (or alternatively to L in (0.2), and x) is well-posed, cf. [26]. The corresponding law $P_{x,\omega}$ on $C(\mathbb{R}_+, \mathbb{R}^d)$, unique solution of the above martingale problem, describes the *diffusion in the environment* ω and starting from x. We write $E_{x,\omega}$ for the expectation under $P_{x,\omega}$. Under $P_{x,\omega}$, $(X_.)$ satisfies the stochastic differential equation

$$\begin{cases} dX_t = \sigma(X_t,\omega)\, d\beta_t + b(X_t,\omega)dt\,, \\ X_0 = x,\ \ P_{x,\omega}\text{-a.s.}\,, \end{cases} \tag{1.9}$$

where $\sigma(\cdot,\omega) = a(\cdot,\omega)^{\frac{1}{2}}$ and $\beta_.$ is some d-dimensional $(\mathcal{F}_t)$-Brownian motion under $P_{x,\omega}$.

The laws $P_{x,\omega}$ are sometimes called "quenched laws" of the diffusion in random environment. We also need the "annealed laws", P_x, $x \in \mathbb{R}^d$, that are the semi-direct products on $\Omega \times C(\mathbb{R}_+, \mathbb{R}^d)$:

$$P_x = \mathbb{P} \times P_{x,\omega}\,. \tag{1.10}$$

We denote with E_x the corresponding expectations. These laws typically destroy the Markovian property of $(X_.)$ but restore translation invariance and isotropy:

$$\begin{aligned} &\text{the law of } (X_. + y) \text{ under } P_x \text{ equals that of } (X_.) \\ &\text{under } P_{x+y}, \text{ for } x, y \in \mathbb{R}^d\,, \end{aligned} \tag{1.11}$$

and for r a rotation matrix preserving the union of coordinate axes of $\mathbb{R}^d$, and $x \in \mathbb{R}^d$,

$$\text{the law of } (rX_.) \text{ under } P_x \text{ equals that of } (X_.) \text{ under } P_{rx}\,. \tag{1.12}$$

We now turn to the description of *spatial scales*. We first choose

$$\beta \in \Big(0, \frac{1}{2}\Big]\,, \tag{1.13}$$

that will later appear as an exponent of Hölder-continuous functions, as well as

$$a \in \Big(0, \frac{\beta}{1000d}\Big], \text{ and } c_0 > 1, \text{ with } 2\,c_0 \log\Big(1 + \frac{a}{2}\Big) > 1\,. \tag{1.14}$$

Then for $L_0 \geq 10^{a^{-1}}$, an integer multiple of 5, we define $L_n, n \geq 0$, by induction via:

$$L_{n+1} = \ell_n\, L_n \text{ with } \ell_n = 5\big[L_n^a/5\big],\ n \geq 0\,, \tag{1.15}$$

and by convention we set $L_{-1} = 1$. We also need the auxiliary scales

$$D_n = L_n \, \exp\left\{c_0 (\log\log L_n)^2\right\}, \quad \widetilde{D}_n = L_n \, \exp\left\{2c_0 (\log\log L_n)^2\right\}, \; n \geq 0. \tag{1.16}$$

The proof of Theorem 1.1, when deriving controls on certain Hölder-norms at scale L_{n+1}, requires one to work at depth $m_0 + 2$ in scale L_{n-m_0-1}, see Sects. 4 and 5, with $m_0 \geq 2$ determined by

$$(1+a)^{m_0-2} \leq 100 < (1+a)^{m_0-1}. \tag{1.17}$$

We can now introduce the *probability kernels* that enter the renormalization scheme. To this end we first define

$$X^*_u = \sup_{s \leq u} |X_s - X_0|, \; u \geq 0, \tag{1.18}$$

as well as the $(\mathcal{F}_t)$-stopping times describing the first time $X_.$ travels a distance $\widetilde{D}_n$ from its starting point:

$$T_n = \inf\{u \geq 0, \; X^*_u \geq \widetilde{D}_n\}, \; n \geq 0. \tag{1.19}$$

We can then consider $n \geq 0$, $\omega \in \Omega$, the probability kernels on $\mathbb{R}^d$

$$R_n(x, dy) = P_{x,\omega}[X_{L_n^2} \in dy], \; \widetilde{R}_n(x, dy) = P_{x,\omega}[X_{L_n^2 \wedge T_n} \in dy]. \tag{1.20}$$

In the renormalization scheme we compare R_n and $\widetilde{R}_n$ to a Gaussian probability kernel R^0_n that we now define. To this end we denote with W_x the d-dimensional Wiener measure starting from $x \in \mathbb{R}^d$. Then for $n \geq 0$, we set

$$R^0_n(x, dy) = W_x[X_{\alpha_n L_n^2} \in dy], \quad \widetilde{R}^0_n(x, dy) = W_x[X_{(\alpha_n L_n^2) \wedge T_n} \in dy], \tag{1.21}$$

where $\widetilde{R}^0_n$ is not used until (4.7), and the positive constant α_n is such that:

$$E_0\left[|X_{L_n^2 \wedge T_n}|^2\right] = E^{W_0}\left[|X_{\alpha_n L_n^2}|^2\right] = \alpha_n \, d L_n^2, \; n \geq 0. \tag{1.22}$$

To compare R_n and $\widetilde{R}_n$ to R^0_n, we will use the kernels

$$S_n = R_n - R^0_n, \; \widetilde{S}_n = \widetilde{R}_n - R^0_n, \; n \geq 0, \omega \in \Omega. \tag{1.23}$$

The *local drift* and the *compensated second moments* at level n at site x in the environment ω are defined via:

$$\widetilde{d}_n(x, \omega) = \int (y - x) \, \widetilde{R}_n(x, dy) = \int (y - x) \, \widetilde{S}_n(x, dy), \quad \widetilde{\gamma}^{i,j}_n(x, \omega) = \int (y - x)_i \, (y - x)_j \, \widetilde{S}_n(x, dy), \; 1 \leq i, j \leq d. \tag{1.24}$$

In view of the translation invariance and isotropy of $X_.$ under the annealed measure, cf. (1.11), (1.12), and of (1.22), we see that

$$\mathbb{E}[\widetilde{d}_n(x,\omega)] = 0, \ \mathbb{E}[\widetilde{\gamma}_n(x,\omega)] = 0, \ \text{ for } n \geq 0, \ x \in \mathbb{R}^d . \tag{1.25}$$

Note also that for $x \in \mathbb{R}^d$, $n \geq 0$,

$$\widetilde{S}_n(x, dy) \text{ depends in a } \mathcal{G}_{\overline{B}(x,\widetilde{D}_n)}\text{-fashion on } \omega , \tag{1.26}$$

(see (1.6) for the notation), and in particular

$$\widetilde{d}_n(x,\omega), \widetilde{\gamma}_n(x,\omega) \text{ are } \mathcal{G}_{\overline{B}(x,\widetilde{D}_n)}\text{-measurable} . \tag{1.27}$$

The finite range dependence property (1.7), together with stationarity and (1.25) yields the fact that $\big(\widetilde{d}_n(x,\omega), \widetilde{\gamma}_n(x,\omega)\big)_{x\in\mathcal{V}}$ are i.i.d. centered variables under $\mathbb{P}$, whenever $\mathcal{V}$ is a collection of points of $\mathbb{R}^d$ with mutual distance at least $2\widetilde{D}_n + R$. This will be especially useful in Sect. 4.

In what follows we will use various *norms*. For $p \in [1,\infty]$, we denote with $|f|_p$ the L^p-norm of a measurable scalar function f on $\mathbb{R}^d$. We also consider as already mentioned in (0.13) the Hölder-norm of order β, cf. (1.13), in scale L_n:

$$|f|_{(n)} = \sup_{x\in\mathbb{R}^d} |f(x)| + L_n^\beta \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta}, \ n \geq 0 . \tag{1.28}$$

Note that for f, g scalar functions on $\mathbb{R}^d$:

$$|fg|_{(n)} \leq |f|_{(n)}\,|g|_{(n)}, \ n \geq 0 . \tag{1.29}$$

The operator norm corresponding to $|\cdot|_{(n)}$ is denoted with $\|\cdot\|_n$:

$$\|A\|_n = \sup_{|f|_{(n)}=1} |Af|_{(n)} , \tag{1.30}$$

for A a linear operator mapping the space of Hölder-continuous functions of order β into itself.

In Sect. 4 we need to compute in an efficient way the $\|\cdot\|_{n+1}$-norm of certain operators entering the linearization of S_{n+1} expressed in terms of n, for $n_0 - m_0 - 1 \leq n \leq n_0$, cf. Theorem 1.1 for the notation. This is done with the help of *wavelets*. Namely we choose a scaling function φ and a mother wavelet ψ, which are compactly supported on $\mathbb{R}$, of class C^4, cf. Daubechies [6, Chaps. 5, 6], Mallat [16, Chap. 7]. In particular φ, ψ have unit L^2-norms and $\int_{\mathbb{R}} \psi(t)dt = 0$, cf. [6, p. 153], (intuitively one can think of the Haar wavelets $\varphi(t) = 1_{[0,1)}(t)$, $\psi(t) = 1_{[0,\frac{1}{2})}(t) - 1_{[\frac{1}{2},1)}(t)$, which of course do not fulfill the smoothness assumption we require). Attached to this choice we have a multiresolution approximation of $L^2(\mathbb{R})$, namely a decreasing sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R})$:

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots , \tag{1.31}$$

with dyadic scaling sending one space into the next, $V_{-\infty} = L^2(\mathbb{R})$, $V_\infty = \{0\}$, and $\varphi(\cdot - k)$, $k \in \mathbb{Z}$, an orthonormal basis of V_0, $\psi(\cdot - k)$, $k \in \mathbb{Z}$, an orthonormal basis of the complement of V_0 in V_{-1}. Since we are interested in functions on $\mathbb{R}^d$, we write

$$\theta_0 = \varphi,\ \theta_1 = \psi\,, \tag{1.32}$$

and for $\alpha \in \{0,1\}^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define:

$$\theta_\alpha(x) = \theta_{\alpha_1}(x_1) \dots \theta_{\alpha_d}(x_d)\,, \tag{1.33}$$

as well as for $\ell \in \mathbb{Z}$, $p \in \mathbb{Z}^d$:

$$\theta_{\alpha,\ell,p}(x) = \theta_\alpha\Big(\frac{x}{2^\ell} - p\Big)\,. \tag{1.34}$$

In this way given any "top scale" 2^{j_0}, we have an orthogonal basis of $L^2(\mathbb{R}^d)$ made of $\theta_{\alpha,\ell,p}$, $\ell \le j_0$, $p \in \mathbb{Z}^d$, with $\alpha \neq 0$ if $\ell < j_0$, and any $f \in L^2(\mathbb{R}^d)$ can be expanded as

$$f(x) = \sum_{\substack{\ell \le j_0,\ p\in\mathbb{Z}^d \\ \alpha\neq 0, \text{ for } \ell < j_0}} c^{j_0}_{\alpha,\ell,p}\, \theta_\alpha\Big(\frac{x}{2^\ell} - p\Big)\,. \tag{1.35}$$

For our purpose the interest of this expansion stems from the fact that with an adequate choice of j_0 (i.e. $2^{j_0} \approx L_n$) the norm $|f|_{(n)}$ is comparable to $\sup\{|c^{j_0}_{\alpha,\ell,p}| 2^{\beta(j_0-\ell)};\ \ell \le j_0,\ p \in \mathbb{Z}^d,\ \alpha \neq 0 \text{ for } \ell < j_0\}$. This leads to effective estimates on $\|\cdot\|_n$, cf. Proposition A.2 from the Appendix. These controls will be very useful in the proof of Lemmas 4.5 and 4.6.

To formulate the Hölder-norm controls that enters the induction assumption of Theorem 1.1 we need certain *cut-off functions* which we now describe. We consider the $[0,1]$-valued radial function:

$$\chi(x) = 1 \wedge (2 - |x|)_+,\ x \in \mathbb{R}^d\,, \tag{1.36}$$

so that $\chi = 1$ on $\overline{B}(0,1)$, $\chi = 0$ on $B(0,2)^c$. For $u \ge 1$, $x \in \mathbb{R}^d$, $n \ge 0$, we also consider

$$\chi_u(\cdot) = \chi\Big(\frac{\cdot}{u}\Big),\ \text{as well as} \tag{1.37}$$

$$\chi_{n,x}(\cdot) = \chi_{10\sqrt{d}\,L_n}(\cdot - x) = \chi\Big(\frac{\cdot - x}{10\sqrt{d}\,L_n}\Big)\,. \tag{1.38}$$

Of special importance for us will be the control of the norm $\|\chi_{n,x}\,\widetilde{S}_n\|_n$ to measure the closeness of $\widetilde{R}_n$ to R^0_n, for starting points in a neighborhood of size const L_n of x, (we incidentally mention that $\|\chi_{n,x}\,\widetilde{S}_n\|_n$ is finite, cf. Remark 2.6.2)).

We are now ready to describe the induction assumption we will propagate. Part of the induction assumption, cf. (1.47), expresses the fact that with "high probability", $\|\chi_{n,0}\,\widetilde{S}_n\|_n$ is "small" and for starting points

$|y| \le 30\sqrt{d}\, L_n$, the tail of $X^*_{L_n^2}$ under $P_{y,\omega}$ has exponential decay. More precisely we introduce for $\omega \in \Omega$, $n \ge 0$, the set

$$\begin{aligned}\mathcal{B}_n(\omega) = \big\{x \in L_n\,\mathbb{Z}^d;\ \text{for } |y-x| \le 30\sqrt{d}\, L_n,& \\ P_{y,\omega}[X^*_{L_n^2} \ge v] \le e^{-\frac{v}{D_n}},\ \text{for } v \ge D_n,& \\ \text{and } \|\chi_{n,x}\, \widetilde{S}_n\|_n \le L_n^{-\delta}\big\}\,,&\end{aligned} \tag{1.39}$$

with δ a number slightly larger than $\frac{\beta}{8}$, specifically:

$$\delta = \frac{5}{32}\,\beta\,. \tag{1.40}$$

We will in particular propagate an upper bound on $\mathbb{P}[0 \notin \mathcal{B}_n(\omega)]$, cf. (1.47).

Another part of the induction assumption involves the control of traps in the medium. For $n \ge 0$, $x \in L_n\,\mathbb{Z}^d$, we write

$$C_n(x) = x + L_n[0,1]^d,\ \ C_n'(x) = x + L_n\Big(-\frac{1}{4}, \frac{5}{4}\Big)^d\,. \tag{1.41}$$

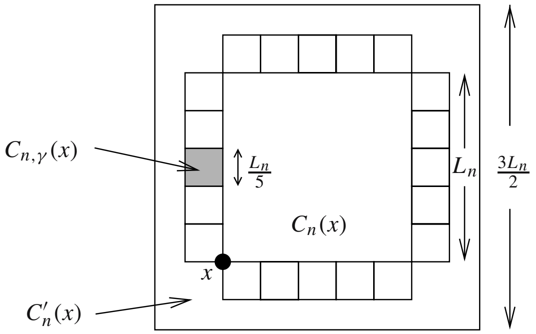

Fig. 1. The boxes $C_n(x)$, $C_n'(x)$, $C_{n,\gamma}(x)$

We then chop each of the $2d$ faces of $\partial C_n(x)$ into $5^{(d-1)}$ closed $(d-1)$-dimensional cubes of side-length $L_n/5$, see (1.15), and denote with $C_{n,\gamma}(x)$, $1 \le \gamma \le 2d\,5^{(d-1)}$, the resulting closed d-dimensional cubes obtained by "expanding" in the outwards normal direction to $\partial C_n(x)$ the above mentioned $(d-1)$-dimensional cubes, (with some specific labelling of the collection of cubes expressed by the index γ). We clearly have

$$C_{n,\gamma}(x) \subseteq C_n'(x),\ \ \text{for } 1 \le \gamma \le 2d\,5^{(d-1)},\ n \ge 0,\ x \in L_n\,\mathbb{Z}^d\,. \tag{1.42}$$

To measure the possible presence of traps in $C_n(x)$, we want to control how well the diffusion starting in the smaller box $C_n(x)$ travels to the boundary boxes $C_{n,\gamma}(x)$ without leaving the larger box $C'_n(x)$, within time L_n^2. To this end we pick a number

$$\zeta \in (0,2), \quad \text{with } \zeta^{-1} \geq \frac{1}{2} + d\,3^{d+1}\,, \tag{1.43}$$

see also (3.85), and introduce for $n \geq 0$, $x \in L_n\,\mathbb{Z}^d$, $A \subseteq C_n(x)$, $1 \leq \gamma \leq 2d\,5^{(d-1)}$, the random variables measuring the presence and strength of traps:

(1.44)
$$J_{n,x,A,\gamma}(\omega) = \inf\big\{u \geq 0;\ \inf_{y\in A} P_{y,\omega}\big[H_{C_{n,\gamma}(x)} \leq L_n^2 \wedge T_{C'_n(x)}\big] \geq c_1\,L_n^{-\zeta u}\big\}\,,$$

where $c_1 \in (0,1)$ is the constant depending on d and ν, see also above (3.67):

$$\begin{aligned} c_1 = \frac{1}{4}\ \inf\Big\{ & W_x\big[X_u \in B,\ u < T_{(-\frac{9}{40},\frac{49}{40})^d}\big]; \\ & u \in \Big[\frac{1}{40\nu}, \frac{4\nu}{10}\Big],\ x \in \Big[-\frac{1}{10}, \frac{11}{10}\Big]^d, \\ & \text{and } B \text{ is a closed cube with side-length } \tfrac{1}{10}, \\ & \text{contained in } \Big[-\frac{1}{5}, \frac{6}{5}\Big]^d\Big\} > 0\,. \end{aligned}$$

We call *n-admissible family*, for $n \geq 0$, an arbitrary collection

$$\begin{aligned} & (u_x, A_x, \gamma_x)_{x\in\mathcal{A}}, \text{ where } \mathcal{A} \text{ is a finite subset of } L_n\,\mathbb{Z}^d, \\ & \text{and for } x \in \mathcal{A},\ u_x > 0,\ \gamma_x \in \{1,\dots,2d\,5^{(d-1)}\}, \\ & \text{and } A_x \subseteq C_n(x) \text{ is a union of boxes } C_{n-1}(z) \\ & \text{(with the convention } L_{-1} = 1, \text{ when } n = 0, \text{ cf. below (1.15)),} \\ & \text{such that } d_\infty(A_x, A_{x'}) \geq 10d\,L_{n-1}, \text{ when } x \neq x'\,. \end{aligned} \tag{1.45}$$

In the induction step we will propagate an upper bound on $\mathbb{P}$[for $x \in \mathcal{A}$, $J_{n,x,A_x,\gamma_x} \geq u_x$] for n-admissible families that will show that with overwhelming probability the variables in (1.44) vanish. We are now almost ready to state the main Theorem 1.1. We just need to introduce two numbers M_0 and M that will respectively govern the estimates on $\mathbb{P}[0 \notin \mathcal{B}_n(\omega)]$ and on the tail of the variables in (1.44).

$$M_0 \geq 100d(1+a)^{m_0+2},\ \ M \geq 1000M_0\,. \tag{1.46}$$

Throughout this article we denote with c a positive constant varying from place to place that solely depends on $d, K, \nu, R, \beta, a, c_0, \varphi, \psi, \zeta, M_0, M$, cf. (1.3), (1.4), (1.5), (1.13), (1.7), (1.14), (1.32), (1.43), (1.46). Any additional dependence of the constant will appear in the notation. So for instance if μ is a parameter, $c(\mu)$ denotes a positive constant depending solely on $\mu, d, K, \nu, R, \beta, a, c_0, \varphi, \psi, \zeta, M_0, M$.

Theorem 1.1. *(Main induction step)*
There are positive constants c_2, c, *such that for* $L_0 \ge c$, *for* $n_0 \ge m_0 + 1$, *(cf.* (1.17)*), if for all* $0 \le n \le n_0$,

$$\mathbb{P}[0 \notin \mathcal{B}_n(\omega)] \le L_n^{-M_0}, \tag{1.47}$$

and for all n-admissible families $(u_x, A_x, \gamma_x)_{x \in \mathcal{A}}$,

$$\begin{aligned} &\mathbb{P}[\text{for all } x \in \mathcal{A},\ J_{n,x,A_x,\gamma_x} \ge u_x] \le L_n^{-\overline{M}_n \sum_{x \in \mathcal{A}} (u_x+1)}, \\ &\text{with } \overline{M}_n = M \prod_{0 \le j < n} \Big(1 - \frac{c_2}{\log L_j}\Big), \end{aligned} \tag{1.48}$$

and if, with δ *as in* (1.40),

$$\begin{aligned} &\text{i)} \quad \frac{1}{2\nu} \le \alpha_n \le 2\nu,\ \ 0 \le n \le n_0, \\ &\text{ii)} \quad |\alpha_{n+1} - \alpha_n| \le L_n^{-(1+\frac{9}{10})\delta},\ \ 0 \le n < n_0, \end{aligned} \tag{1.49}$$

then the estimates (1.47), (1.48) *hold with* $n_0 + 1$ *in place of* n_0, *and*

$$|\alpha_{n_0+1} - \alpha_{n_0}| \le L_{n_0}^{-(1+\frac{9}{10})\delta}. \tag{1.50}$$

The proof of Theorem 1.1 is the scope of the next four sections. The crucial control is (1.47). In Sect. 2 we propagate the localization estimate contained in (1.47), that pertains to the tail behavior of $X^*_{L_n^2}$. In Sect. 3 we propagate the control on traps that appears in (1.48). It is in fact used in a rather special case, at the beginning of Sect. 5, cf. (5.3). As mentioned in the Introduction, the more detailed (1.48) enables the induction proof to function. In Sect. 4 we perform surgery on the environment at scale $L_{n_0'}$, with $n_0' = n_0 - m_0 - 1$ and m_0 from (1.17), and remove possible defects within distance const $L^2_{n_0+1}$ from the origin, which (in essence) belong to $L_{n_0'} \mathbb{Z}^d \backslash \mathcal{B}_{n_0'}(\omega)$, and show that with high probability this modified environment behaves very well up to scale L_{n_0+1}. In Sect. 5 we compare the true and modified environment, and show with the help of the smoothness estimates of Sect. 4, and the control on traps from (1.48) and Sect. 3, that one can repair the defects possibly present in the true environment. Later on in Sect. 6, cf. Proposition 6.2, we choose η_0, cf. (0.5), small enough, i.e. we consider small perturbations of Brownian motion, in order to initiate the induction.

We have already discussed our convention concerning positive constants above Theorem 1.1. We will use in the sequel the expression "for large L_0" in place of "when $L_0 \ge c$". We will recurrently use the shorthand notation

$$\kappa_n = \exp\big\{c\,(\log\log L_n)^2\big\},\ \ n \ge 0. \tag{1.51}$$

From now on we assume $L_0 \ge 10$, large enough so that

$$L_n < D_n < \widetilde{D}_n < L_{n+1},\ \ \text{for } n \ge 0. \tag{1.52}$$

We close this section with some bounds on the Brownian semigroup and on the semigroup of diffusion in random environment. We write $(P_t)_{t\geq 0}$ for the *Brownian semigroup* and $p_t(x, y)$ for its transition density so that

(1.53) $$p_t(x, y) = (2\pi t)^{-\frac{d}{2}} \exp\Big\{ - \frac{|y-x|^2}{2t}\Big\}, \; t > 0, x, y \in \mathbb{R}^d, \text{ and}$$

(1.54) $$\begin{aligned} P_t f(x) &= \int p_t(x, y) f(y) dy, \; t > 0\,, \\ &= f(x), \; t = 0, \text{ with } x \in \mathbb{R}^d, f \text{ bounded measurable}\,. \end{aligned}$$

Note that P_t, $t \geq 0$, contracts the $|\cdot|_{(n)}$-norm and

(1.55) $$\|P_t\|_n = 1, \text{ for } t \geq 0\,.$$

Also for $\gamma = (\gamma_1, \dots, \gamma_d)$ a multi-index (i.e. $\gamma_i \geq 0$, integer), f bounded measurable, $x \in \mathbb{R}^d$, $t > 0$, one has

(1.56) $$|D^\gamma (P_t f)(x)| \leq \frac{c(\gamma)}{t^{\frac{|\gamma|}{2}}} \exp\Big\{ - \frac{d(x, \operatorname{Supp} f)^2}{4t}\Big\} \Big[\Big(\frac{|f|_1}{t^{\frac{d}{2}}}\Big) \wedge |f|_\infty\Big],$$

with $|\gamma| = \gamma_1 + \cdots + \gamma_d$, (the estimate readily follows from the identity: $D_x^\gamma\, p_t(x, y) = (-1)^{|\gamma|}\, t^{-\frac{d+|\gamma|}{2}}\, D^\gamma q\big(\frac{y-x}{\sqrt{t}}\big)$, with $q(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{|z|^2}{2}}$).

The *semigroup of the diffusion in the environment* ω

(1.57) $$(P_{t,\omega} f)(x) = E_{x,\omega}[f(X_t)], \; t \geq 0, x \in \mathbb{R}^d, f \text{ as in (1.54)}\,,$$

thanks to (1.3)–(1.5), is known to admit a density $p_{t,\omega}(x, y)$, cf. Friedman [9], p. 24, which satisfies for $0 < t \leq 1$, $x, y \in \mathbb{R}^d$:

(1.58) $$p_{t,\omega}(x, y) \leq \frac{c}{t^{\frac{d}{2}}} \exp\Big\{ - \frac{c\,|y-x|^2}{t}\Big\}\,,$$

(1.59) $$|D_x\, p_{t,\omega}(x, y)| \leq \frac{c}{t^{\frac{d+1}{2}}} \exp\Big\{ - \frac{c\,|y-x|^2}{t}\Big\}\,.$$

As a consequence we can bound the norm $\|P_t\|_{L^\infty \to (n)}$ of P_t between $L^\infty(\mathbb{R}^d)$ and the space of β-Hölder-continuous functions endowed with the norm $|\cdot|_{(n)}$.

Lemma 1.2.

(1.60) $$\|P_{t,\omega}\|_{L^\infty \to (n)} \leq c\, L_n^\beta, \;\; \textit{for} \;\; t \geq 1, n \geq 0, \omega \in \Omega\,.$$

Proof. First note that for $s \geq 0$, $\omega \in \Omega$,

(1.61) $$|P_{s,\omega} f|_\infty \leq |f|_\infty\,.$$

Then with (1.59) and the above we see that

$$
\begin{aligned}
|P_{1,\omega} f(x) - P_{1,\omega} f(y)| &\le c(|x-y| \wedge 1)\, |f|_\infty \\
&\le c\, L_n^\beta \Big(\Big| \frac{x-y}{L_n} \Big|^\beta \wedge 1 \Big)\, |f|_\infty .
\end{aligned} \tag{1.62}
$$

We thus find

$$
|P_{1,\omega} f|_{(n)} \le c\, L_n^\beta\, |f|_\infty , \tag{1.63}
$$

and writing for $t \ge 1$, $P_{t,\omega} = P_{1,\omega}\, P_{t-1,\omega}$, the claim (1.60) now follows from (1.61), (1.63). □

2. Localization estimates

We keep the notation of the previous section and in particular of Theorem 1.1. We begin here the proof of Theorem 1.1, the principal aim of this section is to propagate to level $n_0 + 1$ the tail estimates on X^* implicit in (1.47), see also (1.39). This is achieved in Proposition 2.5. We also derive controls in Proposition 2.5 which in particular imply that S_{n_0+1} and $\widetilde{S}_{n_0+1}$ are typically close in $\| \ \|_{n_0+1}$-norm. We begin with some additional notation. With K from (1.3), (1.4), and $n \ge 0$, we define:

$$
\mathcal{T}_n = \big(-2K\, L_n^2,\ 2K\, L_n^2 \big)^d , \tag{2.1}
$$

and also introduce for $\omega \in \Omega$, the modification of $\mathcal{B}_n(\omega)$ in (1.39), see (1.16) for notation:

$$
\begin{aligned}
\widetilde{\mathcal{B}}_n(\omega) = \big\{ x \in L_n\, \mathbb{Z}^d;\ & \text{for } |y-x| \le 30\sqrt{d}\, L_n, \\
& P_{y,\omega}\big[X^*_{L_n^2} \ge v\big] \le \exp\Big\{ -\frac{v}{D_n} \Big\},\ \text{for } D_n \le v \le \widetilde{D}_n, \\
& \text{and } \|\chi_{n,x}\, \widetilde{S}_n\|_n \le L_n^{-\delta} \big\} .
\end{aligned} \tag{2.2}
$$

Note that for $n \ge 0$, $x \in L_n\, \mathbb{Z}^d$, the event $\{x \in \widetilde{\mathcal{B}}_n(\omega)\}$ unlike $\{x \in \mathcal{B}_n(\omega)\}$ has a local dependence:

$$
\{x \in \widetilde{\mathcal{B}}_n(\omega)\} \in \mathcal{G}_{\overline{B}(x,\widetilde{D}_n + 30\sqrt{d}\, L_n)},\ \text{(see (1.6) for the notation)} . \tag{2.3}
$$

In the terminology introduced above (1.51), and the notation of (1.22), (1.24), one has

Lemma 2.1. *There is a constant $\overline{c} > 0$, such that for large L_0, for any $\omega \in \Omega$, $n \ge 0$, with $\alpha_n \le 2\nu$, $x \in L_n\, \mathbb{Z}^d$ with $\|\chi_{n,x} \widetilde{S}_n\|_n \le L_n^{-\delta}$, and $|y-x| \le 10\sqrt{d}\, L_n$:*

$$
\begin{aligned}
& |\widetilde{d}_n(y,\omega)| \le \overline{\kappa}_n\, L_n^{1-\delta},\ \ |\widetilde{\gamma}_n(y,\omega)| \le \overline{\kappa}_n\, L_n^{2-\delta}, \\
& \textit{with}\ \ \overline{\kappa}_n = \exp\big\{ \overline{c} (\log\log L_n)^2 \big\} .
\end{aligned} \tag{2.4}
$$

Proof. For y as above and $1 \le i,\, j \le d$, we define, cf. (1.37),

$$f_i(z) = \chi_{\widetilde{D}_n}(z-y)\,\frac{(z-y)_i}{L_n}\,,\ \text{ and} \tag{2.5}$$

$$f_{i,j}(z) = f_i(z)\, f_j(z)\,. \tag{2.6}$$

Observe that

$$|f_i|_{(n)} \le \kappa_n \text{ and } |f_{i,j}|_{(n)} \overset{(1.29)}{\le} \kappa_n\,. \tag{2.7}$$

Further using that $f_i(z) = (\frac{z-y}{L_n})_i$ for $|z-y| \le \widetilde{D}_n$, and Gaussian estimates, see (1.53), (here the control on α_n comes in play), one finds that

$$\Big|\frac{\widetilde{d}_n(y,\omega)_i}{L_n} - (\widetilde{S}_n\, f_i)(y)\Big| \le e^{-\kappa_n},\quad \Big|\frac{\widetilde{\gamma}_n^{i,j}(y,\omega)}{L_n^2} - (\widetilde{S}_n\, f_{i,j})(y)\Big| \le e^{-\kappa_n}\,. \tag{2.8}$$

Since $\chi_{n,x}(y) = 1$, cf. (1.38), and $\|\chi_{n,x}\,\widetilde{S}_n\|_n \le L_n^{-\delta}$, cf. (2.2), the claim now follows (L_0 is large). $\square$

We now turn to the localization estimates.

Proposition 2.2. *For large L_0, if for $n \ge 0$,* (1.47) *and $\frac{1}{2\nu} \le \alpha_n \le 2\nu$ hold, then*

$$\mathbb{P}\begin{bmatrix} \text{for } |y| \le 30\sqrt{d}\, L_{n+1},\ \text{ and } v \ge D_{n+1}, \\ P_{y,\omega}[X^*_{L^2_{n+1}} \ge v] \le \exp\Big\{-f\,\dfrac{v}{D_{n+1}}\Big\} \end{bmatrix} \ge 1 - \frac{1}{10}\,L_{n+1}^{-M_0}\,. \tag{2.9}$$

Proof. Using the exponential inequality for martingales, cf. Revuz-Yor [23], p. 145, for large L_0, $n \ge 0$, $\omega \in \Omega$, $v \ge 2K\,L^2_{n+1}$, and arbitrary y we find

$$\begin{aligned} P_{y,\omega}\big[X^*_{L^2_{n+1}} \ge v\big] &\le c\,\exp\Big\{-\frac{c\,v^2}{L^2_{n+1}}\Big\} \\ &\le c\,\exp\{-cv\} \le \exp\Big\{-\frac{v}{D_{n+1}}\Big\}\,. \end{aligned} \tag{2.10}$$

Hence for proving (2.9) we can restrict v to

$$D_{n+1} \le v < 2K\,L^2_{n+1}\,. \tag{2.11}$$

For such v and $\omega \in \Omega$, we define

$$\begin{aligned} \mathcal{B}_{n,v}(\omega) = \Big\{&x \in L_n\,\mathbb{Z}^d,\ P_{y,\omega}[X^*_{L^2_n} \ge u] \le \exp\Big\{-\frac{u}{D_n}\Big\}, \\ &\text{for } D_n \le u \le \frac{v}{100}\ \text{ and} \\ &|\widetilde{d}_n(y,\omega)| \le \overline{\kappa}_n\,L_n^{1-\delta},\ \text{for } |y-x| \le 10\sqrt{d}\,L_n\Big\}, \end{aligned} \tag{2.12}$$

where $\overline{\kappa}_n$ appears in (2.4). As in (2.3) the local dependence of the event $\{x \in \mathcal{B}_{n,v}(\omega)\}$, for $x \in L_n\,\mathbb{Z}^d$, is expressed by

$$\{x \in \mathcal{B}_{n,v}(\omega)\} \in \mathcal{G}_{\overline{B}(x,(\frac{v}{100} \vee \widetilde{D}_n)+10\sqrt{d}\,L_n)}\,. \tag{2.13}$$

In particular with (1.7) and (2.11), we see that when L_0 is large,

$$\begin{aligned}&\text{for } x, x' \in L_n\,\mathbb{Z}^d, \text{ with } |x-x'| \geq \tfrac{v}{40},\ \{x \in \mathcal{B}_{n,v}(\omega)\} \text{ and}\\ &\{x' \in \mathcal{B}_{n,v}(\omega)\} \text{ are independent}\,.\end{aligned} \tag{2.14}$$

We then introduce, see (2.1):

$$\begin{aligned}\Omega_{n,v} = \Big\{\omega \in \Omega,\ \mathcal{T}_{n+1} \cap L_n\,\mathbb{Z}^d \cap \mathcal{B}^c_{n,v}(\omega) \subset B\Big(x_0, \frac{v}{70}\Big),\\ \text{for some } x_0 \in L_n\,\mathbb{Z}^d\Big\}\,.\end{aligned} \tag{2.15}$$

Observe that when L_0 is large, $n \geq 0$, v as in (2.11),

(2.16)

$$\begin{aligned}\mathbb{P}[\Omega^c_{n,v}] \leq \mathbb{P}\Big[\mathcal{T}_{n+1} \cap L_n\,\mathbb{Z}^d \cap \mathcal{B}^c_{n,v}(\omega) \text{ has diameter} \geq \frac{2v}{70} - \sqrt{d}\,L_n\Big] \leq\\ \mathbb{P}\begin{bmatrix}\text{for some } x, x' \in \mathcal{T}_{n+1} \cap L_n\,\mathbb{Z}^d,\\ \text{with } |x-x'| \geq \dfrac{v}{40}, x \text{ and } x' \notin \mathcal{B}_{n,v}(\omega)\end{bmatrix} \leq\\ \big(c\,L^2_{n+1}/L_n\big)^{2d}\,L_n^{-2M_0} \leq L^{4d}_{n+1}\,L_n^{-2M_0}\,,\end{aligned}$$

where we have used $\mathcal{B}_n(\omega) \subset \widetilde{\mathcal{B}}_n(\omega)$, and hence with (2.2), (2.4), $\mathcal{B}_n(\omega) \subset \mathcal{B}_{n,v}(\omega)$, as well as (1.47) and (2.14) in the last step. We now pick some $\omega \in \Omega_{n,v}$. We can find some $x_0(\omega) \in \mathcal{T}_{n+1} \cap L_n\,\mathbb{Z}^d$, such that

$$\mathcal{T}_{n+1} \cap L_n\,\mathbb{Z}^d \cap \mathcal{B}^c_{n,v}(\omega) \subseteq B\Big(x_0(\omega), \frac{v}{70}\Big)\,. \tag{2.17}$$

We introduce the successive entrance times H_i and exit times V_i of $X_{\boldsymbol{.}}$ in $\overline{B}(x_0, \frac{v}{50})$ and out of $B(x_0, \frac{v}{40})$, (see (1.8) for the notation):

$$\begin{aligned}H_1 = H_{\overline{B}(x_0,\frac{v}{50})},\ V_1 = T_{B(x_0,\frac{v}{40})} \circ \theta_{H_1} + H_1,\ \text{ and for } i \geq 1\,,\\ H_{i+1} = H_1 \circ \theta_{V_i} + V_i,\ V_{i+1} = V_1 \circ \theta_{V_i} + V_i\,,\end{aligned} \tag{2.18}$$

so that

$$H_1 \leq V_1 \leq H_2 \leq \cdots \leq \infty\,. \tag{2.19}$$

We first discuss the more complicated case where

$$|x_0(\omega)| \leq \frac{v}{2}\,. \tag{2.20}$$

Then for $|y| \le 30\sqrt{d}\, L_{n+1}$, we write for large L_0,

$$\begin{aligned} &P_{y,\omega}[X^*_{L^2_{n+1}} \ge v] \le \\ &P_{y,\omega}\big[X^*_{L^2_{n+1}} \ge v,\, H_1 \le L^2_{n+1}\big] + P_{y,\omega}\big[T_{B(0,\frac{3}{4}\,v)} < H_1 \wedge L^2_{n+1}\big]\,, \end{aligned} \tag{2.21}$$

where we have used that $P_{y,\omega}$-a.s., $T_{B(0,\frac{3}{4}\,v)} < L^2_{n+1}$, on $\{X^*_{L^2_{n+1}} \ge v\}$. To bound the first term on the right-hand side of (2.21), we consider on the event $\{X^*_{L^2_{n+1}} \ge v,\, H_1 \le L^2_{n+1}\}$ the last exit time of $B(x_0, \frac{v}{40})$ before $T_{B(0,\frac{3}{4}\,v)}$ ($< L^2_{n+1}$, $P_{y,\omega}$-a.s. on this event), and the integer part of this time. We then find:

$$\begin{aligned} &P_{y,\omega}\big[X^*_{L^2_{n+1}} \ge v,\ H_1 \le L^2_{n+1}\big] \le \\ &P_{y,\omega}\Big[\text{for some } k \le L^2_{n+1},\ \sup_{u\in[k,k+1]} |X_u - X_k| \ge \frac{v}{100}\Big] + \\ &P_{y,\omega}\Big[\bigcup_{m \le L^2_{n+1}} \big(\{X_m \in K(x_0)\} \cap \theta_m^{-1}\big\{T_{B(0,\frac{3}{4}\,v)} < H_1 \wedge L^2_{n+1}\big\}\big)\Big], \\ &\text{with } m \text{ integer and } K(x_0) = \partial B\big(x_0, \frac{v}{40}\big) + \overline{B}\big(0, \frac{v}{100}\big)\,. \end{aligned} \tag{2.22}$$

Using an exponential inequality as in (2.10) to bound the first term on the right-hand side of (2.22), we find:

$$\begin{aligned} &P_{y,\omega}\big[X^*_{L^2_{n+1}} \ge v,\ H_1 \le L^2_{n+1}\big] \le \\ &c\, L^2_{n+1}\big(\exp\{-cv^2\} + \sup_{z\in K(x_0)} P_{z,\omega}\big[T_{B(0,\frac{3}{4}\,v)} < H_1 \wedge L^2_{n+1}\big]\big)\,. \end{aligned} \tag{2.23}$$

For convenience we write $\mathcal{K}_n = \{k \ge 0;\, kL^2_n < H_1 \wedge L^2_{n+1} \wedge T_{\mathcal{T}_{n+1}}\}$. Keeping in mind the last term of (2.21), we write for $|z| \le 30\sqrt{d}\, L_{n+1}$, or $z \in K(x_0)$:

$$\begin{aligned} &P_{z,\omega}\big[T_{B(0,\frac{3}{4}\,v)} < H_1 \wedge L^2_{n+1}\big] \le \\ &P_{z,\omega}\Big[\text{for some } k \in \mathcal{K}_n,\ \sup_{u\in[kL^2_n,(k+1)L^2_n]} |X_u - X_{kL^2_n}| \ge \frac{v}{100}\Big] + \\ &P_{z,\omega}\Big[\text{for each } k \in \mathcal{K}_n,\ \sup_{u\in[kL^2_n,(k+1)L^2_n]} |X_u - X_{kL^2_n}| < \frac{v}{100}, \text{ and} \\ &\qquad T_{B(0,\frac{3}{4}\,v)} < H_1 \wedge L^2_{n+1} \wedge T_{\mathcal{T}_{n+1}}\Big] \overset{(2.12),(2.17)}{\le} c\,\ell^2_n \exp\Big\{-\frac{v}{100 D_n}\Big\} + \\ &P_{z,\omega}\Big[\text{for each } k \in \mathcal{K}_n,\ \sup_{u\in[kL^2_n,(k+1)L^2_n]} |X_u - X_{kL^2_n}| < \frac{v}{100}, \\ &\qquad \text{and } X^*_{H_1 \wedge L^2_{n+1} \wedge T_{\mathcal{T}_{n+1}}} > \frac{v}{5}\Big] \le c\,\ell^2_n\, \exp\Big\{-\frac{v}{100 D_n}\Big\} + \\ &P_{z,\omega}\Big[\text{for each } k \in \mathcal{K}_n,\ \sup_{u\in[kL^2_n,(k+1)L^2_n]} |X_u - X_{kL^2_n}| < \frac{v}{100}, \\ &\qquad \text{and for some } m \in \mathcal{K}_n,\ |X_{mL^2_n} - z| > \frac{v}{10}\Big]\,. \end{aligned} \tag{2.24}$$

We now have to bound the last term of (2.24). To this end we will use an exponential estimate. But we first need the following

Lemma 2.3. *If Z is a random variable on some probability space such that*

$$E[e^{Z}] \le 2,\ \ E[e^{-Z}] \le 2 \text{ and} \tag{2.25}$$

$$E[Z] = 0\,, \tag{2.26}$$

then for $L \ge 1$,

$$E\Big[\exp\Big\{\sqrt{\frac{\log 2}{2}}\ \frac{Z}{L}\Big\}\Big] \le 2^{1/L^2}\,. \tag{2.27}$$

Proof. For $\alpha \in (0,1]$ and $u \in \mathbb{R}$, one has the inequality

$$\alpha^{-2}(e^{\alpha u} - 1 - \alpha u) \le e^{u} + e^{-u} - 2\,, \tag{2.28}$$

that can be verified by expanding both sides in Taylor series and using that $\sum_{k\ge 2,\text{even}} \frac{u^k}{k!} \ge \sum_{k\ge 2,\text{odd}} \frac{u^k}{k!}$. Hence we find

$$e^{\alpha Z} \le 1 + \alpha Z + \alpha^2[e^{Z} + e^{-Z} - 2]\,. \tag{2.29}$$

Substituting $\alpha = \sqrt{\frac{\log 2}{2}}\, L^{-1}$, and taking expectations we find with (2.25), (2.26), that the left-hand side of (2.27) is smaller than

$$1 + \frac{\log 2}{L^2} \le \exp\Big\{\frac{\log 2}{L^2}\Big\} \le 2^{1/L^2}\,.$$

This proves (2.27). □

The desired exponential estimate comes in the next lemma where y' plays the role of $X_{mL_n^2}$ in the last term of (2.24). For $u \ge 0$, we write

$$\psi_u(\cdot) = [-u \vee \cdot\,] \wedge u\,. \tag{2.30}$$

Lemma 2.4. *There is a constant c such that for L_0 large, if $x \in \mathcal{B}_{n,v}(\omega)$ and $|y' - x| \le 10\sqrt{d}\, L_n$, then for any $e \in \mathbb{Z}^d$, with $|e| = 1$,*

$$\begin{aligned} E_{y',\omega}\Big[\exp\Big\{\frac{c}{\ell_n\, D_n}\,\big[\psi_{\frac{v}{100}}\big((X_{L_n^2} - y')\cdot e\big) \\ - E_{y',\omega}\big[\psi_{\frac{v}{100}}\big((X_{L_n^2} - y')\cdot e\big)\big]\big]\Big\}\Big] \le 2^{\ell_n^{-2}}\,. \end{aligned} \tag{2.31}$$

Proof. In view of Lemma 2.3, we only need to prove that for some c and all e as above:

$$\begin{aligned} E_{y',\omega}\Big[\exp\Big\{\frac{c}{D_n}\,\big[\psi_{\frac{v}{100}}\big((X_{L_n^2} - y')\ \cdot e\big) \\ - E_{y',\omega}\big[\psi_{\frac{v}{100}}\big((X_{L_n^2} - y')\cdot e\big)\big]\big]\Big\}\Big] \le 2\,. \end{aligned} \tag{2.32}$$

To this end note that with a small enough c one has

$$\begin{aligned}
&E_{y',\omega}\Big[\exp\Big\{\frac{c}{D_n}\,\psi_{\frac{v}{100}}\big((X_{L_n^2}-y')\cdot e\big)\Big\}\Big] \le \\
&1+E_{y',\omega}\Big[\big(X_{L_n^2}-y'\big)\cdot e>0\,, \\
&\int_0^{\frac{v}{100}\wedge(X_{L_n^2}-y')\cdot e}\frac{c}{D_n}\,\exp\big\{\frac{c}{D_n}\,u\big\}du\Big] \overset{(2.12)}{\le} \\
&1+\int_0^{\frac{v}{100}}\frac{c}{D_n}\,\exp\big\{(c-1)\,\frac{u}{D_n}+1\big\}\,du \le \\
&1+\frac{c}{1-c}\,e\le\sqrt{2}\,.
\end{aligned} \tag{2.33}$$

Then observe that when L_0 is large:

$$\begin{aligned}
&\big|E_{y',\omega}\big[\psi_{\frac{v}{100}}\big((X_{L_n^2}-y')\cdot e\big)\big]-\widetilde{d}_n(y',\omega)\cdot e\big| \overset{(1.24)}{=} \\
&\big|E_{y',\omega}\big[\psi_{\frac{v}{100}}\big((X_{L_n^2}-y')\cdot e\big)-(X_{L_n^2\wedge T_n}-y')\cdot e\big]\big|
\end{aligned} \tag{2.34}$$

and since the integrand vanishes when $T_n > L_n^2$, (because $\frac{v}{100} > \widetilde{D}_n$),

$$\begin{aligned}
&\le \frac{2v}{100}\,P_{y',\omega}[T_n\le L_n^2] \overset{(2.11),(2.12)}{\le} c\,L_{n+1}^2\,\exp\Big\{-\frac{\widetilde{D}_n}{D_n}\Big\} \\
&\overset{(1.15),(1.16)}{\le} c\,\exp\big\{2(1+a)\log L_n-\exp\big\{c_0(\log\log L_n)^2\big\}\big\}\,.
\end{aligned}$$

Moreover with (2.12) we find:

$$|\widetilde{d}_n(y',\omega)\cdot e|\le\overline{\kappa}_n\,L_n^{1-\delta}\,. \tag{2.35}$$

Hence where L_0 is large, combining (2.33)–(2.35), we obtain (2.32). This concludes the proof of Lemma 2.4. □

With the same c as in (2.31), introducing for $e\in\mathbb{Z}^d$, with $|e|=1$, and $m\ge 0$, the notation

$$\begin{aligned}
\mathcal{E}_{e,m}=\exp\Big\{\frac{c}{\ell_n D_n}\sum_{0\le j<m}&\big(\psi_{\frac{v}{100}}\big((X_{(j+1)L_n^2}-X_{jL_n^2})\cdot e\big) \\
&-E_{X_{jL_n^2},\omega}\big[\psi_{\frac{v}{100}}\big((X_{L_n^2}-X_0)\cdot e\big)\big]\big)\Big\}\,,
\end{aligned} \tag{2.36}$$

we see as an application of (2.31) and the Markov property that for $m<\ell_n^2$, $|z|\le 30\sqrt{d}\,L_{n+1}$ or $z\in K(x_0)$, e as above:

$$E_{z,\omega}\big[m\,L_n^2<H_1\wedge T_{\mathcal{T}_{n+1}},\,\mathcal{E}_{e,m}\big]\le 2\,. \tag{2.37}$$

Note also that for large L_0, for $0 \le m < \ell_n^2$, $P_{z,\omega}$-a.s. on the event $\{|X_{m L_n^2} - z| > \frac{v}{10}, m L_n^2 < H_1 \wedge T_{\mathcal{T}_{n+1}}$, and for $0 \le k < m$, $\sup_{u \in [kL_n^2,(k+1)L_n^2]} |X_u - X_{kL_n^2}| < \frac{v}{100}\}$, for some e as above, with (2.12) and (2.34), (2.35):

$$\begin{aligned}\mathcal{E}_{e,m} &\ge \exp\Big\{\frac{c}{\ell_n D_n} \sum_{0 \le j < m} \big[(X_{(j+1)L_n^2} - X_{jL_n^2}) \cdot e - 2\kappa_n L_n^{1-\delta}\big]\Big\} \\ &\ge \exp\Big\{\frac{c}{\ell_n D_n}\Big(\frac{v}{10d} - \kappa_n \ell_n^2 L_n^{1-\delta}\Big)\Big\} \ge \exp\Big\{\frac{c}{\ell_n D_n} v\Big\},\end{aligned}$$

using (1.14), (1.40) and $v \ge D_{n+1}$, in view of (2.11), in the last step. It now follows from (2.37) that the last term of (2.24) is smaller than $2\ell_n^2 \exp\{-\frac{c}{\ell_n D_n} v\}$. Hence we see that when L_0 is large the left-hand side of (2.24) is smaller than $c\ell_n^2(\exp\{-\frac{v}{100D_n}\} + \exp\{-\frac{c}{\ell_n D_n} v\})$.

Using this bound in (2.23) and on the last term of (2.21), (recall that $|z| \le 30\sqrt{d}\, L_{n+1}$ or $z \in K(x_0(\omega))$ in (2.24)), we obtain for large L_0 and $|y| \le 30\sqrt{d}\, L_{n+1}$:

$$\begin{aligned}&P_{y,\omega}[X^*_{L^2_{n+1}} \ge v] \le \\ &c\, L^2_{n+1}\Big(\exp\{-cv^2\} + \ell_n^2 \exp\Big\{-\frac{v}{100D_n}\Big\} + \ell_n^2 \exp\Big\{-\frac{cv}{\ell_n D_n}\Big\}\Big) \le \\ &\exp\Big\{-\frac{10v}{D_{n+1}}\Big\},\end{aligned} \tag{2.38}$$

where we have used in the last step that for large L_0

$$\begin{aligned}\frac{D_{n+1}}{\ell_n D_n} &\overset{(1.15),(1.16)}{\ge} \exp\Big\{c_0\Big[\Big(\log\log L_n + \log\Big(1+\frac{a}{2}\Big)\Big)^2 - (\log\log L_n)^2\Big]\Big\} \\ &\ge \exp\Big\{2c_0 \log\Big(1+\frac{a}{2}\Big)\log\log L_n\Big\},\end{aligned} \tag{2.39}$$

with $2c_0 \log(1+\frac{a}{2}) > 1$, by (1.14), as well as $v \ge D_{n+1}$, in view of (2.11).

We now turn to the simpler case where unlike (2.20)

$$|x_0(\omega)| > \frac{v}{2}. \tag{2.40}$$

Then for $|y| \le 30\sqrt{d}\, L_{n+1}$, L_0 being large, we write:

$$P_{y,\omega}[X^*_{L^2_{n+1}} \ge v] \le P_{y,\omega}\big[T_{B(0,\frac{v}{3})} < H_1 \wedge L^2_{n+1}\big] \le \exp\Big\{-\frac{10v}{D_{n+1}}\Big\}, \tag{2.41}$$

repeating similar bounds as in (2.24), (leading to (2.38)). We now define, cf. (2.15),

$$\Omega_n = \bigcap_{m \ge 0;\, 10^m D_{n+1} < 2KL^2_{n+1}} \Omega_{n,10^m D_{n+1}}, \tag{2.42}$$

and observe that for $\omega \in \Omega_n$, $v \in [D_{n+1}, 2KL^2_{n+1})$, $|y| \le 30\sqrt{d}\, L_{n+1}$,

$$\begin{aligned} P_{y,\omega}[X^*_{L^2_{n+1}} \ge v] \le P_{y,\omega}[X^*_{L^2_{n+1}} \ge v_m] &\overset{(2.38),(2.41)}{\le} \exp\Big\{-\frac{10 v_m}{D_{n+1}}\Big\} \\ &\le \exp\Big\{-\frac{v}{D_{n+1}}\Big\}, \end{aligned} \tag{2.43}$$

where for $m \ge 0$, the notation v_m denotes the unique number $10^m D_{n+1}$, such that $10^m\, D_{n+1} = v_m \le v < 10 v_m$.

In addition from (2.16) we deduce that when L_0 is large

$$\begin{aligned} \mathbb{P}[\Omega^c_n] &\le \Big(\Big[\log\Big(\frac{2K\,L^2_{n+1}}{D_{n+1}}\Big)/\log 10\Big] + 1\Big)\, c\, L^{4d}_{n+1}\, L_n^{-2M_0} \\ &\le \frac{1}{10}\, L^{-M_0}_{n+1}, \end{aligned} \tag{2.44}$$

since $2M_0(1+a)^{-1} > M_0 + 4d + 1$, by (1.14), (1.46). Combining (2.10), (2.43), (2.44), we see that (2.9) is proved. □

We will now conclude this section with an estimate on $\|\chi_{n,x}(S_n - \widetilde{S}_n)\|_n$ that will be repeatedly used in the sequel. We refer to (1.23), (1.30), (1.38) for the notation.

Proposition 2.5. *Given κ^0_n as in* (1.51), *for L_0 large, for any $n \ge 0$, $\omega \in \Omega$, if $x \in L_n\,\mathbb{Z}^d$ is such that for $|y - x| \le 30\sqrt{d}\, L_n$,*

$$P_{y,\omega}\Big[X^*_{L^2_n} \ge \frac{\widetilde{D}_n}{2}\Big] \le e^{-\kappa^0_n}, \tag{2.45}$$

then there exists a κ_n as in (1.51) *such that*

$$\|\chi_{n,x}(S_n - \widetilde{S}_n)\|_n \le e^{-\kappa_n}. \tag{2.46}$$

Proof. We use the shorthand notation

$$\begin{aligned} &\Delta_n = S_n - \widetilde{S}_n, \ \text{so that} \\ &\Delta_n\, g(z) \overset{(1.23)}{=} E_{z,\omega}\big[g(X_{L^2_n}) - g(X_{L^2_n \wedge T_n}),\ T_n < L^2_n\big]. \end{aligned} \tag{2.47}$$

Note that for f with $|f|_{(n)} \le 1$, and x, y as above (2.45),

$$|\Delta_n\, f(y)| \le 2\, P_{y,\omega}\big[T_n < L^2_n\big] \overset{(2.45)}{\le} 2e^{-\kappa^0_n}. \tag{2.48}$$

So when L_0 is large, we find that for y, y' in $B(x, 21\sqrt{d}\, L_n)$, with $|y - y'| \ge e^{-\kappa_n}$,

$$\begin{aligned} |\Delta_n\, f(y) - \Delta_n\, f(y')| &\le 2e^{-\kappa^0_n} \le L^\beta_n \left|\frac{y-y'}{L_n}\right|^\beta e^{-\kappa_n} \\ &\le \left|\frac{y-y'}{L_n}\right| e^{-\kappa_n}, \end{aligned} \tag{2.49}$$

(see above (1.51) for the convention we use, and we are only interested in $y, y' \in B(x, 21\sqrt{d}\, L_n)$ because $\chi_{n,x}$ is supported in $\overline{B}(x, 20\sqrt{d}\, L_n)$, as follows from (1.38)).

We now consider for κ_n as above (2.49),

$$|y - y'| \le e^{-\kappa_n}\,, \tag{2.50}$$

and write

$$\begin{aligned} &|\Delta_n f(y) - \Delta_n f(y')| \le a_1 + a_2, \text{ with} \\ &a_1 = \big|E_{y',\omega}[f(X_{L_n^2 \wedge T_{y'}}) - f(X_{L_n^2 \wedge T_y})]\big|\,, \\ &a_2 = \big|E_{y,\omega}[f(X_{L_n^2}) - f(X_{L_n^2 \wedge T_y})] - E_{y',\omega}[f(X_{L_n^2}) - f(X_{L_n^2 \wedge T_y})]\big|\,, \end{aligned} \tag{2.51}$$

and $T_y = T_{B(y,\widetilde{D}_n)}$, $T_{y'} = T_{B(y',\widetilde{D}_n)}$ in the notation of (1.8). Writing

$$\tau = T_y \wedge T_{y'}\,, \tag{2.52}$$

it follows from the strong Markov property at time τ, with hopefully obvious notation, that

$$\begin{aligned} a_1 \le\; & \big|E_{y',\omega}\big[T_{y'} = \tau < L_n^2 \wedge T_y,\; E_{X_{T_{y'}},\omega}[f(X_{T_y \wedge (L_n^2 - \tau)}) - f(X_0)]\big]\big| + \\ & \big|E_{y',\omega}\big[T_y = \tau < L_n^2 \wedge T_{y'},\; E_{X_{T_y},\omega}[f(X_{T_{y'} \wedge (L_n^2 - \tau)}) - f(X_0)]\big]\big| \\ \stackrel{\text{def}}{=}\; & b_1 + b_2,\; \text{(the inner expectations do not integrate } \tau)\,. \end{aligned} \tag{2.53}$$

We will now bound b_1, b_2 being handled similarly. To this end we consider $z' \in \partial B(y', \widetilde{D}_n) \cap B(y, \widetilde{D}_n)$, ($z'$ plays the role of $X_{T_{y'}}$), $0 \le u \le (L_n^2 - \tau)_+$, and $\mathcal{H}$ the half-space $\{z \in \mathbb{R}^d; z \cdot \ell \ge v\}$, with ℓ the unit vector in the direction $z' - y'$, $v = z' \cdot \ell + |y' - y|$. So $d(\mathcal{H}, \overline{B}(y', \widetilde{D}_n)) = |y - y'|$ in the notation (1.1), and $B(y, D_n) \subset \mathcal{H}^c$. We will use the shorthand notation, cf. (1.8), $H = H_{\mathcal{H}}$, and note that

$$\begin{aligned} &E_{z',\omega}[|X_{T_y \wedge u} - X_0|^\beta \wedge 2] \le \\ &2\, P_{z',\omega}[H > |y' - y|] + E_{z',\omega}[H \le |y' - y|,\; |X_{T_y \wedge u} - X_0|^\beta \wedge 2]\,. \end{aligned} \tag{2.54}$$

To bound the right-hand side of (2.54), we first note that under $P_{z',\omega}$, $(X_s - X_0) \cdot \ell$ admits the semimartingale decomposition

$$(X_s - X_0) \cdot \ell = M_s + A_s,\; s \ge 0\,, \tag{2.55}$$

where in view of (1.3)–(1.5), for some $c > 1$,

$$\frac{1}{c}\, s \le \langle M \rangle_s \le cs, \quad |A_s| \le cs,\; s \ge 0\,. \tag{2.56}$$

Observe also that with c as above,

$$P_{z',\omega}\text{-a.s., } T_y \le H \le \widetilde{H} \stackrel{\text{def}}{=} \inf\{s \ge 0, M_s \ge cs + |y' - y|\}\,. \tag{2.57}$$

As a result we find that

$$
\begin{aligned}
P_{z',\omega}[H \le |y'-y|] &\ge P_{z',\omega}[\widetilde{H} \le |y'-y|] \\
&\ge P_{z',\omega}\Big[\sup_{s\le |y'-y|} M_s \ge (c+1)|y'-y|\Big] \\
&\ge W\Big[\sup_{s\le c|y-y'|} B_s \ge (c+1)\Big] \\
&\overset{\text{scaling}}{=} W\Big[\sup_{s\le 1} B_s \ge c|y-y'|^{\frac12}\Big] \qquad (2.58)\\
&= 1 - \int_{-c|y'-y|^{\frac12}}^{c|y'-y|^{\frac12}} e^{-\frac{v^2}{2}}\,\frac{dv}{\sqrt{v\pi}} \\
&\ge 1 - c\,|y'-y|^{\frac12}\,,
\end{aligned}
$$

where $B_.$ denotes the canonical one-dimensional Brownian motion, W the Wiener measure, and we have used time-change together with (2.56). This yields a bound on the first term in the right-hand side of (2.54). For the second term we note that with c as in (2.56), we can define

$$
\overline{H} = \inf\{s \ge 0,\ M_s = (c+1)|y'-y|\}\,, \qquad (2.59)
$$

and $P_{z',\omega}$-a.s. on the event $\{H \le |y'-y|\}$, one has $T_y \le H \le \overline{H}$, and hence

$$
\begin{aligned}
&E_{z',\omega}\big[H \le |y'-y|,\ |X_{T_y\wedge u} - X_0|^\beta \wedge 2\big] \overset{(1.9)}{\le} \\
&c|y'-y|^\beta + E_{z',\omega}\Big[H \le |y'-y|,\ \sup_{s\le \overline{H}}\Big|\int_0^s \sigma(X_v,\omega)d\beta_v\Big|^\beta\Big] \le \qquad (2.60)\\
&c\,|y'-y|^\beta + c\,E_{z',\omega}[\overline{H}^{\frac{\beta}{2}}] \le c\,|y'-y|^\beta\,,
\end{aligned}
$$

using Burkholder-Davis-Inequalities, cf. Karatzas-Shreve [11, p. 166], then once again a representation of $M_.$ as a time change of Brownian motion together with scaling, and the fact that moments of order less than $\frac12$ of the hitting time of 1 by Brownian motion are finite, cf. [11, p. 96]. We can now collect (2.58), (2.60) to bound the left-hand side of (2.54). Coming back to the first line of (2.53), since $|f|_{(n)} \le 1$, and $\beta \le \frac12$, cf. (1.13), we find (recall τ is not integrated in the inner expectation)

$$
\begin{aligned}
b_1 &\le E_{y',\omega}\Big[T_{y'} = \tau < L_n^2 \wedge T_y,\ E_{X_{T_{y'}},\omega}\big[|X_{T_y\wedge(L_n^2-\tau)} - X_0|^\beta \wedge 2\big]\Big] \\
&\le c\,|y-y'|^\beta\, P_{y',\omega}\big[\tau = T_{y'} < L_n^2\big] \qquad (2.61)\\
&\overset{(2.45)}{\le} c\,|y-y'|^\beta\, e^{-\kappa_n^0} \le \Big|\frac{y-y'}{L_n}\Big|^\beta e^{-\kappa_n}\,.
\end{aligned}
$$

A similar bound can be proved for b_2.

We then turn to the bound on a_2 in (2.51). We use the shorthand notation

$$
t_0 = (\log|y'-y|)^{-2},\ \ (\text{recall (2.50)})\,, \qquad (2.62)
$$

and denote with $q_{t,\omega}(z,z')$ the sub-probability density of the diffusion in the environment ω, killed when exiting the ball $B(y,10)$, at time $t>0$, when starting in $z \in B(y,10)$. We now find that

$$
\begin{aligned}
&1-\int q_{t_0,\omega}(y,z)\wedge q_{t_0,\omega}(y',z)\,dz \le 1-\int q_{t_0,\omega}(y,z)\,dz + \\
&\quad \int |q_{t_0,\omega}(y,z)-q_{t_0,\omega}(y',z)|\,dz \le 1-\int q_{t_0,\omega}(y,z)\,dz + \\
&\quad \int |p_{t_0,\omega}(y,z)-p_{t_0,\omega}(y',z)|\,dz \ + 1-\int q_{t_0,\omega}(y,z)\,dz + \\
&1-\int q_{t_0,\omega}(y',z)\,dz \le c\,e^{-\frac{c}{t_0}} + c\left|\frac{y-y'}{\sqrt{t_0}}\right| \overset{(2.62)}{\le} c\left|\frac{y-y'}{\sqrt{t_0}}\right|,
\end{aligned}
\tag{2.63}
$$

for large L_0, using (1.59) and standard estimates.

With the help of (2.63), we can construct on some auxiliary probability space two processes $Y_.$ and $Y'_.$ with same laws as $X_.$ under $P_{y,\omega}$ and $P_{y',\omega}$ such that

$$
\begin{aligned}
&P[G] \ge 1-c\left|\frac{y-y'}{\sqrt{t_0}}\right|, \text{ with} \\
&G=\big\{Y_u=Y'_u \text{ for } u\ge t_0, \text{ and } Y \text{ and } Y' \text{ do not exit} \\
&\qquad B(y,10) \text{ up to time } t_0\big\}.
\end{aligned}
\tag{2.64}
$$

We now see that with a slight abuse of notation, when L_0 is large:

$$
\begin{aligned}
a_2 &\le \Big|E\Big[G^c, f(Y_{L_n^2})-f(Y_{L_n^2\wedge T_{B(y,\widetilde{D}_n)}(Y)})-\big(f(Y'_{L_n^2})-f(Y'_{L_n^2\wedge T_{B(y,\widetilde{D}_n)}(Y')})\big)\Big]\Big| \\
&\le 4P\big[G^c, T_{B(y,\widetilde{D}_n)}(Y)<L_n^2 \text{ or } T_{B(y,\widetilde{D}_n)}(Y')<L_n^2\big] \\
&\overset{\text{Hölder, (2.45)}}{\le} P[G^c]^{\frac{1+\beta}{2}}\,e^{-\kappa_n} \overset{(2.62),(2.64)}{\le} |y-y'|^\beta\,e^{-\kappa_n} \le \left|\frac{y'-y}{L_n}\right|^\beta e^{-\kappa_n}.
\end{aligned}
\tag{2.65}
$$

Collecting the bounds (2.51), (2.53), (2.61), (2.65), together with (2.49), we see that when L_0 is large, for y, y' in $B(x, 21\sqrt{d}\,L_n)$,

$$
|\Delta_n f(y)-\Delta_n f(y')| \le \left|\frac{y-y'}{L_n}\right|^\beta e^{-\kappa_n}.
\tag{2.66}
$$

This together with (2.48) and (1.38) readily implies (2.46), (see also (A.4)–(A.6) of the Appendix). □

Remark 2.6.

1) We have used the assumption $\beta \le \frac{1}{2}$, cf. (1.13), in the estimate (2.58).
2) Note that the estimates on (2.51), and (1.60) can also be used to show that for $\omega \in \Omega$, $n \ge 0$, $x \in L_n\,\mathbb{Z}^d$,

$$
\|\chi_{n,x}\,\widetilde{S}_n\|_n \le c\,L_n^\beta.
\tag{2.67}
$$

□

3. Controlling traps

We continue the proof of Theorem 1.1. The main objective in this section is to propagate "at level n_0+1" the estimate (1.48), and this comes in Proposition 3.3. As mentioned in the Introduction and in Sect. 1 below Theorem 1.1, the main purpose of the control (1.48) on the tails of the variables in (1.44) measuring the strength of traps, is to later obtain the estimate (5.3), when "repairing defects". This only involves a small portion of (1.48), but (1.48) is there to let the induction proof function. As a preparation for our main task we first construct certain *couplings* of the diffusion in random environment with Brownian motion of variance α_n, cf. (1.22), at times kL_n^2, $k \geq 0$. These couplings will be very handy later in this section when relating Brownian motion estimates to the behavior of the diffusion in a good environment, see (3.51), (3.64), (3.66), as well as in Sect. 6. We begin with some notation. We denote with $d_{n,\beta}(\cdot,\cdot)$ the distance function on $\mathbb{R}^d$:

$$d_{n,\beta}(y,y') = \left|\frac{y-y'}{L_n}\right|^\beta, \quad y, y' \in \mathbb{R}^d, \ n \geq 0\,. \tag{3.1}$$

We define for ν, ν' probabilities on $\mathbb{R}^d$, for which

$$\int |y|^\beta \ \nu(dy) < \infty, \ \int |y|^\beta \ \nu'(dy) < \infty\,, \tag{3.2}$$

$$\begin{aligned} D_{n,\beta}(\nu,\nu') = \sup \Big\{ \Big| \int f d\nu - \int f d\nu' \Big| ;\ &\text{where } f \text{ on } \mathbb{R}^d \text{ is such that} \\ &|f(y) - f(y')| \leq d_{n,\beta}(y,y'), \ \text{for } y, y' \in \mathbb{R}^d \Big\} \\ = \inf \Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} d_{n,\beta}(y,y')\, \rho(dy,dy');\ &\text{with } \rho \text{ a probability} \\ &\text{having } \nu, \nu' \text{ as first and second marginals} \Big\}\,, \end{aligned} \tag{3.3}$$

where the last equality results from the Kantorovich-Rubinstein Theorem, cf. Dudley [8, Theorem 11.8.2]. The function $D_{n,\beta}$ is sometimes called Kantorovich-Rubinstein or Vasserstein distance. We now consider a continuous function h with values in $[0,1]$, and for $\omega \in \Omega$, $n \geq 0$, define the probability kernel on $\mathbb{R}^d$

$$\widetilde{R}_{n,h}(x,dy) = R_n^0(x,dy) + h(x)\,\widetilde{S}_n(x,dy), \ \text{ cf. (1.21), (1.23),} \tag{3.4}$$

(so when $h \equiv 0$, $\widetilde{R}_{n,h} = R_n^0$, and when $h \equiv 1$, $\widetilde{R}_{n,h} = \widetilde{R}_n$).

We are now ready to state and prove the above mentioned result concerning coupling measures.

Proposition 3.1. *Let h be a continuous $[0,1]$-valued function on $\mathbb{R}^d$, $\omega \in \Omega$, and $n \geq 0$ such that $\frac{1}{2\nu} \leq \alpha_n \leq 2\nu$. Then for $y \in \mathbb{R}^d$, there is a measure $Q_{n,y}$ on the canonical space $(\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{N}}$ endowed with the canonical σ-algebra* *and the canonical processes* $\overline{X}_k$, $k \geq 0$, $\overline{X}^0_k$, $k \geq 0$, *such that*

$$\text{(3.5)} \qquad \begin{array}{l} \textit{under } Q_{n,y}, \ \overline{X}_k, \ k \geq 0, \textit{ (resp. } \overline{X}^0_k, \ k \geq 0) \textit{ has the law of the} \\ \textit{Markov chain on } \mathbb{R}^d, \textit{ starting at } y \textit{ with transition kernel} \\ \widetilde{R}_{n,h} \textit{ (resp. } R^0_n) \end{array}$$

and for $k_0 \geq 1$, $\gamma > 0$,

$$\text{(3.6)} \qquad \begin{array}{l} Q_{n,y}\big[|\overline{X}_k - \overline{X}^0_k| \geq \gamma, \ \textit{for some } k \leq k_0\big] \leq \\ k_0^2 \Big(\dfrac{\gamma}{L_n}\Big)^{-\beta} (\kappa_n \, \Gamma_{n,h} + e^{-\kappa_n}) \,, \end{array}$$

with $\Gamma_{n,h} = \sup_{x \in L_n \mathbb{Z}^d : \chi_{n,x} h \not\equiv 0} \| \chi_{n,x} \, \widetilde{S}_n \|_n$.

Remark 3.2. Note that under $Q_{n,y}$ above, $(\overline{X}^0_k)_{k \geq 0}$ has same law as $(X_{\alpha_n k L_n^2})_{k \geq 0}$ under W_y, the Wiener measure starting from y, cf. above (1.21). The inequality (3.6) highlights one of the interests in controlling the norms $\| \cdot \|_n$. □

Proof of Proposition 3.1. For $z \in \mathbb{R}^d$, denote with K_z the non-empty compact subset of $M_1(\mathbb{R}^d \times \mathbb{R}^d)$, the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ endowed with the topology of weak convergence,

$$\text{(3.7)} \qquad \begin{aligned} K_z = \big\{ & \rho \in M_1(\mathbb{R}^d \times \mathbb{R}^d); \, \rho \text{ has marginals } R_{n,h}(z,\cdot) \text{ and } R^0_n(z,\cdot), \\ & \text{and } D_{n,\beta}(\widetilde{R}_{n,h}(z,\cdot), R^0_n(z,\cdot)) = \int d_{n,\beta}(z_1,z_2)\, \rho(dz_1,dz_2) \big\} \,. \end{aligned}$$

Observe that for any sequences z_i, ρ_i, $i \geq 1$, with $\rho_i \in K_{z_i}$, for $i \geq 1$, and z_i converging to z_∞, ρ_i is tight and has a limit point ρ_∞ such that:

$$\text{(3.8)} \qquad \begin{aligned} \int d_{n,\beta}(z_1,z_2)\, \rho_\infty(dz_1,dz_2) & \leq \liminf_i \ D_{n,\beta}\big(\widetilde{R}_{n,h}(z_i,\cdot), R^0_n(z_i,\cdot)\big) \\ & = D_{n,\beta}\big(\widetilde{R}_{n,h}(z_\infty,\cdot), R^0_n(z_\infty,\cdot)\big) \,, \end{aligned}$$

as follows straightforwardly by applying the triangle inequality satisfied by $D_{n,\beta}$, as well as (2.67) and (3.3). This shows that $\rho_\infty \in K_{z_\infty}$. Then with Stroock-Varadhan [26, Lemma 12.1.8 and Theorem 12.1.10, p. 289], we can find a probability kernel $\widetilde{\rho}_z(dz_1,dz_2)$, $z \in \mathbb{R}^d$, such that

$$\text{(3.9)} \qquad \text{for } z \in \mathbb{R}^d, \, \widetilde{\rho}_z(\cdot) \in K_z \,,$$

and define the transition probability $\overline{\rho}_{z,z_0}(dz', dz'_0)$ on $\mathbb{R}^d \times \mathbb{R}^d$:

$$\text{(3.10)} \qquad \int g(z', z'_0)\, \overline{\rho}_{z,z_0}(dz', dz'_0) = \int g(z_1, z_2 - z + z_0)\, \widetilde{\rho}_z(dz_1, dz_2) \,,$$

for g bounded measurable on $\mathbb{R}^d \times \mathbb{R}^d$, and $z, z_0 \in \mathbb{R}^d$.

We then define $Q_{n,y}$ as the canonical law of the Markov chain with transition kernel $\overline{\rho}$ and initial distribution concentrated on (y, y). With (3.7), (3.9), it is straightforward to check that (3.5) holds. To prove (3.6), observe that for $k \geq 1$:

(3.11)
$$E^{Q_{n,y}}\big[d_{n,\beta}\big(\overline{X}_k, \overline{X}^0_k\big)\big] \leq E^{Q_{n,y}}\big[d_{n,\beta}\big(\overline{X}_{k-1}, \overline{X}^0_{k-1}\big)\big] +$$
$$E^{Q_{n,y}}\big[d_{n,\beta}\big(\overline{X}_k, \overline{X}^0_k - \overline{X}^0_{k-1} + X_{k-1}\big)\big] \overset{(3.9),(3.10)}{=}$$
$$E^{Q_{n,y}}\big[d_{n,\beta}\big(\overline{X}_{k-1}, \overline{X}^0_{k-1}\big)\big] + E^{Q_{n,y}}\big[D_{n,\beta}\big(\widetilde{R}_{n,h}(\overline{X}_{k-1}, \cdot), R^0_n(\overline{X}_{k-1}, \cdot)\big)\big]\,.$$

To bound the rightmost term, note that for $z \in \mathbb{R}^d$, when $x \in L_n\mathbb{Z}^d$ is such that $|z - x| \leq \sqrt{d}\,L_n$, and f has Lipschitz constant at most 1 with respect to $d_{n,\beta}(\cdot, \cdot)$, one finds

$$\text{(3.12)} \qquad \begin{aligned} |\widetilde{R}_{n,h}\, f(z) - R^0_n\, f(z)| &\overset{(3.4)}{=} h(z)|\widetilde{S}_n\, f(z)| \\ &= h(z)\,|\widetilde{S}_n\,(f(\cdot) - f(x))(z)| \end{aligned}$$

and since $\widetilde{R}_n(z, \cdot)$ is supported in $\overline{B}(z, \widetilde{D}_n)$ with (1.23), (1.37)

$$\leq h(z)|\,\widetilde{S}_n\,F(z)| + h(z)\big|\,\widetilde{S}_n\big[(1 - \chi_{2\sqrt{d}\,\widetilde{D}_n}(\cdot - x))\,(f(\cdot) - f(x))\big](z)\big|$$
$$\leq h(z)\,|(\chi_{n,x}\,\widetilde{S}_n\,F)(z)| + h(z)\,R^0_n\Big[1_{B(x,2\sqrt{d}\,\widetilde{D}_n)^c}(\cdot)\,\Big|\frac{\cdot - x}{L_n}\Big|^{\beta}\Big](z)\,,$$

with $F(\cdot) = \chi_{2\sqrt{d}\,\widetilde{D}_n}(\cdot - x)(f(\cdot) - f(x))$. Note that

$$\text{(3.13)} \qquad |F|_{(n)} \leq \kappa_n\,,$$

and we now see that the left-hand side of (3.12) is smaller than

$$h(z)\Big(\kappa_n\|\chi_{n,x}\,\widetilde{S}_n\|_n + W_0\big[X_{\alpha_n L_n^2} \notin B(0, 2\sqrt{d}\,\widetilde{D}_n)\big]^{\frac{1}{2}}\,E^{W_0}\Big[\Big|\frac{X_{\alpha_n L_n^2}}{L_n}\Big|^{2\beta}\Big]^{\frac{1}{2}}\Big) \leq$$
$$\kappa_n\,\Gamma_{n,h} + e^{-\kappa_n}\,.$$

With (3.3), we see that we have shown that

$$\text{(3.14)} \qquad \sup_{z\in\mathbb{R}^d} D_{n,\beta}\big(\widetilde{R}_{n,h}(z, \cdot),\ R^0_n(z, \cdot)\big) \leq \kappa_n\,\Gamma_{n,h} + e^{-\kappa_n}\,.$$

Coming back to (3.11), using induction over k, and the fact that $\overline{X}_0 = \overline{X}_0$, $Q_{n,y}$-a.s., we find for $k \geq 0$,

$$\text{(3.15)} \qquad E^{Q_{n,y}}\big[d_{n,\beta}\big(\overline{X}_k, \overline{X}^0_k\big)\big] \leq k(\kappa_n\,\Gamma_{n,h} + e^{-\kappa_n})\,.$$

The application of Chebyshev's inequality now yields for $\gamma > 0$, $k_0 \geq 1$:

$$\begin{aligned} \Big(\frac{\gamma}{L_n}\Big)^{\beta} Q_{n,y}\,\big[|\overline{X}_k - \overline{X}^0_k| \geq \gamma,\ \text{for some } k \leq k_0\big] &\leq \sum_{k=1}^{k_0} k(\kappa_n\,\Gamma_{n,h} + e^{-\kappa_n}) \\ &\leq k_0^2(\kappa_n\,\Gamma_{n,h} + e^{-\kappa_n})\,, \end{aligned}$$

which proves (3.6). □

We can now return to the main object of this section, namely propagating (1.48) "at level n_0+1". The idea is to devise exit strategies from $C_{n_0+1}(x)$ for the path, that show that it is costly for the environment to produce $J_{n_0+1,x,\cdot,\cdot}$ variables above level u_x, for x in a finite collection $\mathcal{A}$. The nature of the exit strategies depends on the level u_x, and there are four regimes, (only three when $d \geq 4$), cf. (3.20). The higher the u_x, the more the exit strategy relies on the control (1.48) at level n_0. The lower the u_x, the more the exit strategy relies on "good-behavior" of the environment around $C'_{n_0+1}(x)$ at the micro-level $n_0 - 1$, in the sense of (2.2), so that good couplings with Brownian motion resulting from Proposition 3.1 can be employed. Good behavior is precisely expressed by the events $\mathcal{C}_x$, cf. (3.24), (3.32), and below (3.33). As one of the first steps, we reduce ourselves to a situation of "only good behavior", cf. (3.36). This involves a certain thinning procedure of $\mathcal{A}$ singling out local high values of u_x and showing that bad behavior of the environment at these sites is costly, cf. (3.36). We then have to control the probability that the variables $J_{n_0+1,x,\cdot,\cdot}$ are bigger than u_x, for x in a thinning of $\mathcal{A}$, in the presence of good-behavior of the environment, cf. Lemma 3.4. This is done with the help of the exit strategies that enable to bound the variables, $J_{n_0+1,x,\cdot,\cdot}$ from above, in terms of $J_{n_0,\cdot,\cdot,\cdot}$ variables, cf. (3.50), (3.58), (3.71), (3.76), and then use the induction assumption, cf. (3.78). The constant ζ, cf. (1.43), (1.44), is important in the treatment of the lower values of u_x, cf. (3.85). We then go back from the estimates on the thinned collection with good-behavior of the environment to the general upper bound in (3.86).

Proposition 3.3. *One can choose a (large enough) positive constant c_2 in* (1.48)*, such that for large L_0 and $n_0 \geq m_0 + 1$, if* (1.49) *holds for n_0 and* (1.47)*,* (1.48) *hold for* $0 \leq n \leq n_0$*, then* (1.48) *holds for* $n_0 + 1$.

Proof. We consider $(u_x, A_x, \gamma_x)_{x\in\mathcal{A}}$, with $\mathcal{A}$ a finite subset of $L_{n_0+1}\,\mathbb{Z}^d$, an (n_0+1)-admissible family, cf. (1.45). From the Definition (1.44), we see that

$$
\tag{3.16}
\begin{aligned}
J_{n,x,A\cup B,\gamma} = J_{n,x,A,\gamma} \vee J_{n,x,B,\gamma}, \ \text{for } n \geq 0,\, x \in L_n\mathbb{Z}^d,\\
A, B \subset C_n(x),\ 1 \leq \gamma \leq 2d\,5^{d-1}\,.
\end{aligned}
$$

As a result we see that

$$
\tag{3.17}
\begin{aligned}
&\mathbb{P}\big[\forall x \in \mathcal{A},\ J_{n_0+1,x,A_x,\gamma_x} \geq u_x\big] \leq \\
&\big(c\ell^d_{n_0-1}\,\ell^d_{n_0}\big)^{|\mathcal{A}|}\,\widetilde{\sup}\,\mathbb{P}\big[\forall x \in \mathcal{A},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \geq u_x\big]
\end{aligned}
$$

where $\widetilde{\sup}$ stands for the supremum over families $z_x \in L_{n_0-1}\,\mathbb{Z}^d$, $x \in \mathcal{A}$, with $C_{n_0-1}(z_x) \subseteq C_{n_0+1}(x)$, and $d_\infty(C_{n_0-1}(z_x), C_{n_0-1}(z_{x'})) \geq 10dL_{n_0}$, for $x \neq x'$, in $\mathcal{A}$.

We will now work on the rightmost term of (3.17). To this end we introduce a thinning $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ as follows. We pick some $x_1 \in \mathcal{A}$ such that $u_{x_1} = \max_x u_x$, and define $\mathcal{N}_1 = \{x \in \mathcal{A}, |x - x_1|_\infty \leq L_{n_0+1}$, and

$(u_x + 1) \log L_{n_0} < (u_{x_1} + 1)\}$, where we recall that $|\cdot|_\infty$ denotes the sup-norm on $\mathbb{R}^d$. So $\mathcal{N}_1$ corresponds to the boxes $C_{n_0+1}(x)$, $x \in \mathcal{A}$, adjacent to $C_{n_0+1}(x_1)$, with value $(u_x + 1)$ smaller than $(u_{x_1} + 1)/\log L_{n_0}$. We define

$$\mathcal{A}_1 = \mathcal{A} \backslash (\mathcal{N}_1 \cup \{x_1\}) .$$

Either $\mathcal{A}_1 = \emptyset$, in which case the process stops, or $\mathcal{A}_1 \neq \emptyset$, and we repeat the same procedure to $\mathcal{A}_1$, and define x_2, $\mathcal{N}_2$ as above, and set $\mathcal{A}_2 = \mathcal{A}_1 \backslash (\mathcal{N}_2 \cup \{x_1\})$, and so on. After p steps, with $p \leq |\mathcal{A}|$, one has $\mathcal{A}_p = \emptyset$, and the process stops. We then write

$$\widetilde{\mathcal{A}} = \{x_1, \ldots, x_p\} = \mathcal{A} \setminus \bigcup_{1 \leq i \leq p} \mathcal{N}_i \,, \tag{3.18}$$

and observe that

$$\begin{aligned} & x, x' \in \widetilde{\mathcal{A}} \text{ and } |x - x'|_\infty \leq L_{n_0+1} \text{ implies} \\ & (\log L_{n_0})^{-1} \leq \frac{u_{x'} + 1}{u_x + 1} \leq \log L_{n_0}, \text{ and} \\ & \sum_{x \in \mathcal{A}} (u_x + 1) \leq \Big(1 + \frac{3^d}{\log L_{n_0}}\Big) \sum_{x \in \widetilde{\mathcal{A}}} (u_x + 1) . \end{aligned} \tag{3.19}$$

We introduce the notation $a_d = \frac{3}{4}(d-2)a$, and partition $\widetilde{\mathcal{A}}$ into four subsets:

$$\begin{aligned} & \widetilde{\mathcal{A}}_1 = \big\{x \in \widetilde{\mathcal{A}};\ u_x \geq L_{n_0}^a\big\}, \widetilde{\mathcal{A}}_2 = \big\{x \in \widetilde{\mathcal{A}};\ L_{n_0}^{a_d} \leq u_x < L_{n_0}^a\big\} , \\ & \widetilde{\mathcal{A}}_3 = \big\{x \in \widetilde{\mathcal{A}};\ \log L_{n_0} \leq u_x < L_{n_0}^{a_d \wedge a}\big\} , \\ & \widetilde{\mathcal{A}}_4 = \big\{x \in \widetilde{\mathcal{A}};\ 0 < u_x < \log L_{n_0}\big\} . \end{aligned} \tag{3.20}$$

Note that $\widetilde{\mathcal{A}}_2 = \emptyset$, whenever $d \geq 4$.

Our aim is to produce an upper bound on quantities of the type $\mathbb{P}[\,\forall x \in \widetilde{\mathcal{A}},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_z} \geq u_x]$. We will in essence show that $\{J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \geq u_x\}$ is unlikely by producing an exit strategy for the process that leads before time $L_{n_0+1}^2 \wedge T_{C'_{n_0+1}(x)}$ from $y \in C_{n_0-1}(z_x) \subseteq C_{n_0+1}(x)$ to the box $C_{n_0+1,\gamma_x}(x)$ with side-length $L_{n_0+1}/5$ that borders $\partial C_{n_0+1}(x)$, cf. below (1.41). The nature of this strategy depends on which $\widetilde{\mathcal{A}}_i$, $1 \leq i \leq 4$, x belongs to. In particular when $x \in \widetilde{\mathcal{A}}_2$, or $x \in \widetilde{\mathcal{A}}_3 \cup \widetilde{\mathcal{A}}_4$, the exit strategy involves certain events describing a "good behavior" of the environment "at level $n_0 - 1$". We first specify these events.

We introduce for $x \in \widetilde{\mathcal{A}}_2$, (recall this only concerns the case of dimension $d = 3$), the numbers α_x, ν_x, ν'_x such that:

$$\begin{aligned} & u_x = L_{n_0}^{\alpha_x}, \ \big(\text{so that by (3.20), } \alpha_x \in \big[\tfrac{3}{4}\, a, a\big)\big), \text{ and} \\ & 0 < \nu_x \overset{\text{def}}{=} \frac{1}{2}\Big(a - \frac{\alpha_x}{2}\Big) < \nu'_x \overset{\text{def}}{=} \frac{5}{8}\,\alpha_x + \frac{a}{4} < \frac{7}{8}\, a . \end{aligned} \tag{3.21}$$

We will now define for $x \in \widetilde{\mathcal{A}}_2$ the event $\mathcal{C}_x$ which in essence specifies the presence in $C_{n_0+1}(x)$ of channels of width $L_{n_0}^{1+\nu_x}$ within distance $\sim L_{n_0}^{1+\nu'_x}$ of any point of $C_{n_0+1}(x)$ where the process easily travels. More precisely call a box $B = z + [0, L_{n_0}^{1+\nu_x}]^d$, $z \in L_{n_0}^{1+\nu_x}\,\mathbb{Z}^d$, of side-length $L_{n_0}^{1+\nu_x}$, n_0-*good* for ω, if all $y \in L_{n_0-1}\,\mathbb{Z}^d$ within $|\cdot|$-distance $30\sqrt{d}\,L_{n_0-1}$ of B belong to $\widetilde{\mathcal{B}}_{n_0-1}(\omega)$, cf. (2.2). Then set

$$
\begin{aligned}
&C^0_{n_0+1}(x) = \big\{z \in C_{n_0+1}(x);\ d(z, C_{n_0+1}(x)^c) > L_{n_0}^{1+\nu'_x}\big\}, \\
&\qquad\qquad \text{and for } e \in \mathbb{Z}^d,\ |e| = 1, \\
&C^e_{n_0+1}(x) = \big(C^0_{n_0+1}(x) + 2eL_{n_0}^{1+\nu'_x}\big)\backslash C_{n_0+1}(x)\,.
\end{aligned}
\tag{3.22}
$$

We now define for $z \in C^0_{n_0+1}(x)$, $z' \in C^e_{n_0+1}(x)$, (e as above), and $s > 0$:

$$
\begin{aligned}
\mathcal{C}_x^{z,z',s} = \big\{\,&\omega \in \Omega; \text{there is a nearest-neighbor path of } n_0\text{-good} \\
&\text{boxes } B_1 = z_1 + \big[0, L_{n_0}^{1+\nu_x}\big], \dots, B_k = z_k + \big[0, L_{n_0}^{1+\nu_x}\big], \\
&k \le 4L_{n_0}^{a-\nu_x}, \text{ moving in the } e\text{-direction after the first} \\
&i \in [1,k], \text{ for which } d_\infty(z_i, C_{n_0+1}(x)^c) \le \tfrac{1}{2}\,L_{n_0}^{1+\nu'_x}, \\
&\text{with } d_\infty(z, B_1) \vee d_\infty(z', B_k) \le sL_{n_0}^{1+\nu'_x}\big\},
\end{aligned}
\tag{3.23}
$$

as well as the event

$$
\begin{aligned}
\mathcal{C}_x = \bigcap_{z,z'} \mathcal{C}_x^{z,z',1}, &\text{ where } z \text{ runs over } C^0_{n_0+1}(x)\,, \\
&z' \text{ runs over } \bigcup_{|e|=1} C^e_{n_0+1}(x)\,,
\end{aligned}
\tag{3.24}
$$

(note that requiring z, z' to have rational coordinates does not change (3.24), and makes clear that $\mathcal{C}_x$ is an event). We now bound $\mathbb{P}[\mathcal{C}_x^c]$. We observe that

$$
\mathbb{P}[\mathcal{C}_x^c] \le c\,L_{n_0}^{2d(a-\nu_x)} \sup_{z,z'} \mathbb{P}\big[(\mathcal{C}_x^{z,z',\frac{1}{2}})^c\big],\ L_0 \text{ large}\,,
\tag{3.25}
$$

where z, z' respectively run over $(L_{n_0}^{1+\nu_x}\,\mathbb{Z}^d) \cap C^0_{n_0+1}(x)$, and $\bigcup_{|e|=1}(L_{n_0}^{1+\nu_x}\,\mathbb{Z}^d) \cap C^e_{n_0+1}(x)$.

We now set $w = L_{n_0}^{-(1+\nu_x)}(z' - z) \in \mathbb{Z}^d$, and for convenience assume that $z' \in C^{e_3}_{n_0+1}(x)$ and $w_i \ge 0$, $1 \le i \le d(=3)$; the other cases being handled in a similar fashion. For $\theta = (\theta_1, \theta_2) \in 2\mathbb{Z}^2$, with $\theta_1, \theta_2 \le 0$, we define $k_\theta = w_1 + w_2 + w_3 + |\theta_1| + |\theta_2|$, and for $0 \le i < k_\theta$,

$$
\begin{aligned}
&p_0^\theta = (0, \theta_1, \theta_2)\,, \\
&p_{i+1}^\theta - p_i^\theta = \begin{cases} (1,0,0), & 0 \le i < w_1 + |\theta_1| \\ (0,1,0), & w_1 + |\theta_1| \le i < w_1 + w_2 + |\theta_1| + |\theta_2| \\ (0,0,1), & w_1 + w_2 + |\theta_1| + |\theta_2| \le i < k_\theta\,, \end{cases}
\end{aligned}
\tag{3.26}
$$

as well as

$$z_{i+1}^{\theta} = z + L_{n_0}^{1+\nu_x}\, p_i^{\theta},\;\; B_{i+1}^{\theta} = z_{i+1}^{\theta} + \big[0, L_{n_0}^{1+\nu_x}\big]^d .$$

Note that for $\theta \neq \theta'$,

$$d_\infty\big(B_i^\theta, B_{i'}^{\theta'}\big) \geq L_{n_0}^{1+\nu_x},\;\; 1 \leq i \leq k_\theta,\;\; 1 \leq i' \leq k_{\theta'}\,, \tag{3.27}$$

and for $|\theta_1|, |\theta_2| \leq \frac{1}{100}\, L_{n_0}^{\nu_x'-\nu_x}$, L_0 large,

$$\begin{cases} k_\theta \leq \big(3L_{n_0+1} + L_{n_0}^{1+\nu_x'}\big)\, L_{n_0}^{-(1+\nu_x)} + \dfrac{2}{100}\, L_{n_0}^{\nu_x'-\nu_x} \overset{(3.21)}{\leq} 4L_{n_0}^{a-\nu_x}\,, \\ d_\infty\big(z, B_1^\theta\big) \vee d_\infty(z', B_{k_\theta}^\theta) < \dfrac{1}{2}\, L_{n_0}^{1+\nu_x'},\;\text{ and for} \\ 1 \leq i < k_\theta, d_\infty\big(z_i^\theta, C_{n_0+1}(x)^c\big) \leq \dfrac{1}{2}\, L_{n_0}^{1+\nu_x'}, \\ \text{implies } z_{i+1}^\theta - z_i^\theta = L_{n_0}^{1+\nu_x}\, e_3\,. \end{cases} \tag{3.28}$$

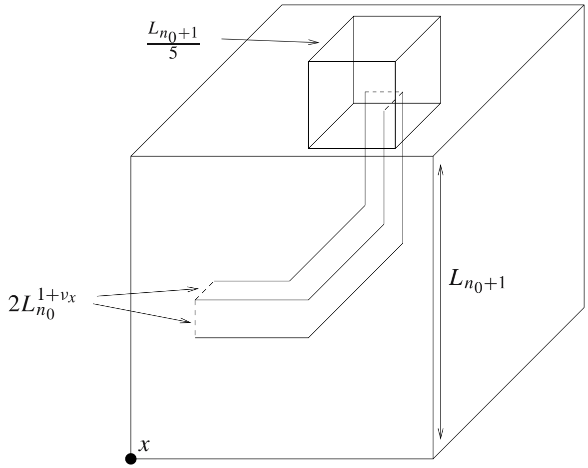

Fig. 2. Candidates for paths of good boxes corresponding to the exit strategy for $\widetilde{\mathcal{A}}_2$. Solid lines are made of boxes of side-length $L_{n_0}^{1+\nu_x}$ and distance between paths of boxes are at least $L_{n_0}^{1+\nu_x}$

So the paths B_i^θ, $1 \leq i \leq k_\theta$, satisfy the requirements set forth in the definition of $\mathcal{C}_x^{z,z',\frac{1}{2}}$.

Then for any such given path B_i^θ, $1 \leq i \leq k_\theta$,

$$\mathbb{P}\big[\text{one of the } B_i^\theta \text{ is not } n_0\text{-good}\big] \leq \Big(\frac{cL_{n_0}^{1+\nu_x}}{L_{n_0-1}}\Big)^d\, 4L_{n_0}^{a-\nu_x}\, L_{n_0-1}^{-M_0} \leq \frac{1}{2}\,, \tag{3.29}$$

when L_0 is large, cf. (1.14), (1.46), (1.47), (3.21), (3.28).

Then using independence, cf. (1.7), (2.3), (3.27), we see that

$$\mathbb{P}\big[(\mathcal{C}_x^{z,z',\frac{1}{2}})^c\big] \le \Big(\frac{1}{2}\Big)^{(cL_{n_0}^{\nu'_x-\nu_x})^2},$$

and using (3.25), we find when L_0 is large, for $x \in \widetilde{\mathcal{A}}_2$,

$$\begin{aligned}\mathbb{P}[\mathcal{C}_x^c] &\le cL_{n_0}^{2d(a-\nu_x)} \exp\big\{-cL_{n_0}^{2(\nu'_x-\nu_x)}\big\} \overset{(3.21)}{\le} \exp\big\{-c\,L_{n_0}^{\frac{a}{16}+\alpha_x}\big\} \\ &< L_{n_0-1}^{-6d\,9^d M(u_x+1)\log L_{n_0}}.\end{aligned} \tag{3.30}$$

When $x \in \widetilde{\mathcal{A}}_3 \cup \widetilde{\mathcal{A}}_4$, (we are back in the case of a general $d \ge 3$), the event $\mathcal{C}_x$ will in place of (3.24) require that there are "few" boxes $C_{n_0-1}(z) \subseteq C'_{n_0+1}(x)$, cf. (1.41), with $z \notin \widetilde{\mathcal{B}}_{n_0-1}(\omega)$. Just as in (3.24), the good behavior of the environment is specified at level n_0-1. More precisely for $x \in \widetilde{\mathcal{A}}_3 \cup \widetilde{\mathcal{A}}_4$ and $\omega \in \Omega$, we introduce the compact sets

$$K_{x,\omega} = \bigcup_z \overline{B}(z, 30\sqrt{d}\,L_{n_0-1}) \supset \widetilde{K}_{x,\omega} = \bigcup_z \overline{B}(z, 29\sqrt{d}\,L_{n_0-1}), \tag{3.31}$$

where the unions run over the set of $z \in L_{n_0-1}\,\mathbb{Z}^d$, with $d(z, C'_{n_0+1}(x)) \le 30\sqrt{d}\,L_{n_0-1}$, such that $z \notin \widetilde{\mathcal{B}}_{n_0-1}(\omega)$. We then define for $x \in \widetilde{\mathcal{A}}_3 \cup \widetilde{\mathcal{A}}_4$,

$$\begin{aligned}\mathcal{C}_x = \big\{&\omega \in \Omega;\ K_{x,\omega} \text{ is contained in the union of } N_x \text{ open balls} \\ &\text{with radius } 4\widetilde{D}_{n_0-1} \text{ and centers in } L_{n_0-1}\,\mathbb{Z}^d\big\},\end{aligned} \tag{3.32}$$

with $N_x = [12d\,9^d(1+a)^2 \frac{M}{M_0}(u_x+1)\log L_{n_0}] + 1$.

For $x \in \widetilde{\mathcal{A}}_3 \cup \widetilde{\mathcal{A}}_4$, on $\mathcal{C}_x^c$, arguing by contradiction we can find N_x disjoint open balls with radius $\frac{3}{2}\widetilde{D}_{n_0-1}$, and centers in $L_{n_0-1}\,\mathbb{Z}^d \cap (x + L_{n_0+1}[-\frac{1}{2},\frac{3}{2}]^d) \cap \widetilde{\mathcal{B}}^c_{n_0-1}(\omega)$. As a result with (1.7), (1.47), (2.3), we find that for large L_0, for $x \in \widetilde{\mathcal{A}}_3 \cup \widetilde{\mathcal{A}}_4$:

$$\begin{aligned}\mathbb{P}[\mathcal{C}_x^c] &\le \big(c(\ell_{n_0-1}\,\ell_{n_0})^d\,L_{n_0-1}^{-M_0}\big)^{N_x} \le \big(cL_{n_0-1}^{da(2+a)-M_0}\big)^{N_x} \\ &\overset{(1.46)}{\le} L_{n_0-1}^{-M_0N_x/2} \le L_{n_0+1}^{-6d\,9^d\,M(u_x+1)(\log L_{n_0})}.\end{aligned} \tag{3.33}$$

For convenience, we set $\mathcal{C}_x = \Omega$, for $x \in \widetilde{\mathcal{A}}_1$. We now come back to the rightmost term of (3.17), and observe that

$$\begin{aligned}&\mathbb{P}[\forall x \in \mathcal{A},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \ge u_x] \le \\ &2^{|\mathcal{A}|} \sup_{\mathcal{G}\subseteq\widetilde{\mathcal{A}}\setminus\widetilde{\mathcal{A}}_1} \mathbb{P}\big[\text{for } x \in \widetilde{\mathcal{A}},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \ge u_x,\ \mathcal{C}_x \\ &\qquad\qquad \text{for } x \in \mathcal{G},\ \mathcal{C}_x^c \text{ for } x \in \widetilde{\mathcal{A}}\setminus(\widetilde{\mathcal{A}}_1 \cup \mathcal{G})\big].\end{aligned} \tag{3.34}$$

For $\mathcal{G}$ as above we chose $\mathcal{M} = \mathcal{M}(\mathcal{G})$ a maximal set of non-adjacent x in $\widetilde{\mathcal{A}}\setminus(\widetilde{\mathcal{A}}_1 \cup \mathcal{G})$, (i.e. with mutual $|\cdot|_\infty$-distance at least L_{n_0+1}), and denote by $\overline{\mathcal{M}}$ the set of $x \in \widetilde{\mathcal{A}}$ adjacent to $\mathcal{M}$. Coming back to the definitions of the events $\mathcal{C}_x$ in (3.24), and the definition of the variables $J_{n_0+1,x,A,\gamma}$, with $A \subseteq C_{n_0+1}(x)$, cf. (1.44), we see with the help of (1.7) that when L_0 is large the collection of events

$$\begin{aligned}&\mathcal{C}_x^c,\ x \in \mathcal{M},\ \big\{\forall x \in \widetilde{\mathcal{A}}\backslash\overline{\mathcal{M}},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \geq u_x,\ \mathcal{C}_x\big\}\\ &\text{are independent}\,.\end{aligned} \tag{3.35}$$

This fact together with (3.30), (3.33), yields that for large L_0

$$\begin{aligned}&P[\forall x \in \mathcal{A},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \geq u_x] \leq\\ &2^{|\mathcal{A}|} \sup_{\mathcal{G}\subseteq\widetilde{\mathcal{A}}\backslash\widetilde{\mathcal{A}}_1} \Big\{ L_{n_0+1}^{-6d9^d M \sum_{x\in\mathcal{M}}(u_x+1)(\log L_{n_0})}\\ &\qquad \mathbb{P}\big[\forall x \in \widetilde{\mathcal{A}}\backslash\overline{\mathcal{M}},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \geq u_x,\ \text{and}\ \mathcal{C}_x\big]\Big\}\,.\end{aligned} \tag{3.36}$$

With the help of (3.19) we also have a lower bound on the exponent in the first term in the right-hand side of (3.36), that we will later use in (3.86):

$$6d\,9^d\,M \sum_{x\in\mathcal{M}} (u_x+1)\ \log L_{n_0} \geq 6d\,3^d\,M \sum_{x\in\overline{\mathcal{M}}} (u_x+1)\,. \tag{3.37}$$

We will now bound the last term in the right-hand side of (3.36):

$$\begin{aligned}&I \stackrel{\text{def}}{=} \mathbb{P}\big[\forall x \in \mathcal{D},\ J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \geq u_x,\ \text{and}\ \mathcal{C}_x\big],\\ &\text{with}\ \mathcal{D} = \widetilde{\mathcal{A}}\backslash\overline{\mathcal{M}}\,.\end{aligned} \tag{3.38}$$

Our main control comes in the next

Lemma 3.4. *For any positive number c_2 there are c', $c(c_2) > 0$, (see above Theorem 1.1 for the convention concerning constants, and c_2 is not yet a constant), such that for $L_0 \geq c(c_2)$, $\prod_{n\geq 0}(1 - c_2(\log L_n)^{-1}) \geq \frac{1}{2}$, and*

$$I \leq L_{n_0+1}^{-\sum_{x\in\widetilde{\mathcal{A}}\backslash\overline{\mathcal{M}}} \overline{M}_{n_0}(1-c'(\log L_{n_0})^{-1})(u_x+1)}\ L_{n_0}^{-\overline{M}_{n_0}\,d3^{d+1}\,a|\widetilde{\mathcal{A}}_4\backslash\overline{\mathcal{M}}|}\,, \tag{3.39}$$

where the notation $\overline{M}_n$ comes from (1.48).

Proof. We define for $1 \leq i \leq 4$, in the notation of (3.20), (3.38),

$$\mathcal{D}_i = \mathcal{D} \cap \widetilde{\mathcal{A}}_i\,. \tag{3.40}$$

The proof involves the construction of "exit strategies" for the process somewhat in the spirit of what was done in [29]. The nature of these exit strategies from $C_{n_0+1}(x)$, leading to $C_{n_0+1,\gamma_x}(x)$ before time $L_{n_0+1}^2 \wedge T_{C'_{n_0+1}(x)}$, when starting in $C_{n_0-1}(z_x)$, depends on which $\mathcal{D}_i$, $1 \leq i \leq 4$, x belongs to.

The exit strategy first uses an "exit path" based on a sequence of nearest-neighbor boxes (of size L_{n_0}), $C_{n_0}(y_{j,x})$, $0 \leq j \leq j_x$, starting at $C_{n_0}(y_{0,x})$, containing or close to $C_{n_0-1}(z_x)$, leading to a final location, the nature of which depends on which $\mathcal{D}_i$, $1 \le i \le 4$, x belongs to.

More precisely we consider a family π_x, $x \in \mathcal{D}$, of finite sequences $\pi_x = (y_{j,x}, \gamma_{j,x})_{0 \le j \le j_x}$ in $L_{n_0}\mathbb{Z}^d \times \{1, \dots, 2d\,5^{(d-1)}\}$, so that writing for simplicity ($0 \le j \le j_x$):

$$\begin{aligned} C^{j,x} &= C_{n_0}(y_{j,x}),\;\; \Delta^{j,x} = C_{n_0,\gamma_{j,x}}(y_{j,x}) \\ \Delta^{-1,x} &= C_{n_0-1}(z_x),\;\; \text{for } x \in \mathcal{D}\backslash\mathcal{D}_4, \\ \Delta^{-1,x} &= C_{n_0}(y_{0,x}),\;\; \text{for } x \in \mathcal{D}_4\,, \end{aligned} \tag{3.41}$$

we have:

$$\begin{cases} C_{n_0-1}(z_x) \subseteq C^{0,x},\;\; C^{j,x} \subseteq C_{n_0+1}(x),\;\; 0 \le j \le j_x,\;\; \text{and} \\ \Delta^{j,x} \subseteq C^{j+1,x},\;\; 0 \le j < j_x,\;\; \text{when } x \in \mathcal{D}\backslash\mathcal{D}_4\,, \\ |y_{0,x} - y_x|_\infty \le L_{n_0},\;\; \text{if } C_{n_0}(y_x) \supseteq C_{n_0-1}(z_x),\;\; \text{when } x \in \mathcal{D}_4\,, \\ \text{(i.e. } C^{0,x} \text{ is adjacent to the } n_0\text{-box containing } C_{n_0-1}(z_x)) \end{cases} \tag{3.42}$$

and moreover the $\Delta^{j,x}$ are spread apart:

$$\begin{aligned} \min\big\{ &d_\infty(\Delta^{j,x}, \Delta^{j',x'});\; (j,x) \neq (j',x'), \\ &-1 \le j \le j_x,\; -1 \le j' \le j_{x'} \big\} \ge 10dL_{n_0-1}\,. \end{aligned} \tag{3.43}$$

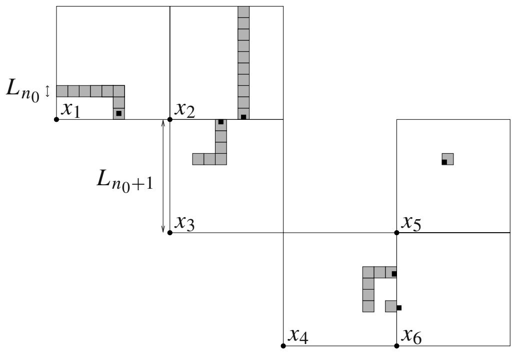

Fig. 3. An example where $\mathcal{D}_1 = \{x_1, x_2\}$, $\mathcal{D}_2 = \{x_3\}$, $\mathcal{D}_3 = \{x_4\}$, $\mathcal{D}_4 = \{x_5, x_6\}$. In black the boxes $C_{n_0-1}(z_x)$, $x \in \mathcal{D}$, and in grey the boxes $C^{j,x}$. The black boxes are at least at mutual $|\cdot|_\infty$-distance $10dL_{n_0}$

We now describe the additional requirements on the π_x involving which $\mathcal{D}_i$, $1 \le i \le 4$, x belongs to. So in addition to the above requirements, π_x are such that:

- when $x \in \mathcal{D}_4$:

$$\begin{aligned} &j_x = 0, \text{ and in addition to the last line of (3.42),} \\ &\gamma_{0,x} \in \{1, \dots, 2d\,5^{(d-1)}\} \text{ is arbitrary}\,. \end{aligned} \tag{3.44}$$

• When $x \in \mathcal{D}_3$:

$$j_x = n_x + 3d \overset{\text{def}}{=} \Big[\frac{(u_x+1)}{3}\Big] + 3d\,, \tag{3.45}$$

and the nearest-neighbor path $(y_{j,x})$ after at most $2d$ steps is such that $C^{j,x}$ remains inside $C_{n_0+1}(x)$ at $|\cdot|_\infty$-distance at least $2L_{n_0}$ from $\partial C_{n_0+1}(x)$, and moves "along some coordinate direction".

• When $x \in \mathcal{D}_2$:

$$j_x \le c\, L_{n_0}^{\nu'_x}\,, \tag{3.46}$$

and the finite sequence $(y_{j,x}, \gamma_{y,x})_{j\le j_x}$ is now such that after at-most $2d$ steps $C^{j,x}$ remains inside $C_{n_0+1}(x)$ at $|\cdot|_\infty$-distance at least $2L_{n_0}$ from $\partial C_{n_0+1}(x)$, and the path ends with $C^{j_x,x}$, $\Delta^{j_x,x} \subset C^0_{n_0+1}(x)$, cf. (3.22).

• When $x \in \mathcal{D}_1$;

$$j_x \le c\,\ell_{n_0}\,, \tag{3.47}$$

after at most $2d$ steps $C^{j,x}$, $j < j_x - 1$, remains at least at $|\cdot|_\infty$-distance $2L_{n_0}$ from $\partial C_{n_0+1}(x)$, and the path ends with $C^{j_x,x}$, $\Delta^{j_x,x}$, so that $\Delta^{j_x,x} \subseteq C_{n_0+1,\gamma_x}(x)$.

We will use the fact that when L_0 is large we can select π_x, when $x \in \mathcal{D}\backslash(\mathcal{D}_2 \cup \mathcal{D}_4)$ and then complete it into π_x, $x \in \mathcal{D}$, so that $\gamma_{j_x,x}$ is arbitrary and $y_{j_x,x}$ is an arbitrary point of, cf. (3.22), $L_{n_0}\mathbb{Z}^d \cap C^0_{n_0+1}(x)$ $\cap\, B_\infty(z_x, 3L_{n_0}^{1+\nu'_x})$, when $x \in \mathcal{D}_2$, while when $x \in \mathcal{D}_4$, $C_{n_0}(y_{0,x})$ is an arbitrary adjacent box of $C_{n_0}(y_x) \supseteq C_{n_0-1}(z_x)$, $\gamma_{0,x}$ is arbitrary in $\{1,\dots,2d\,5^{(d-1)}\}$, and π_x, $x \in \mathcal{D}$ fulfills all the above properties.

We will now derive lower bounds on the exit probabilities of $C_{n_0+1}(x)$ before time $L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}$, via $C_{n_0+1,\gamma_x}(x)$, when starting in $C_{n_0-1}(z_x)$, for $x \in \mathcal{D}$. We only need to consider ω such that $\omega \in \mathcal{C}_x$, for $x \in \mathcal{D}$, cf. (3.38). These lower bounds will yield upper bounds on the variables $J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x}$, $x \in \mathcal{D}$, in terms of $J_{n_0,\cdot,\cdot,\cdot}$ variables to which we will apply the induction assumption (1.48). In what follows π_x, $x \in \mathcal{D}$, always stand for a family of finite sequences satisfying (3.41)–(3.47). We also introduce the shorthand notation

$$J_{j,x} = J_{n_0,y_{j,x},\Delta^{j-1},\gamma_{j,x}},\ \ 0 \le j \le j_x,\ x \in \mathcal{D}\,. \tag{3.48}$$

When $x \in \mathcal{D}_1$: we use the path of boxes $C^{j,x}$ and "boundary boxes" $\Delta^{j,x}$, $0 \le j \le j_x$, to let the path exit. Noting that $c\ell_{n_0} L^2_{n_0} < L^2_{n_0+1}$, when L_0 is large, the strong Markov property implies that for $\omega \in \Omega$:

$$\begin{aligned}&\inf_{y\in C_{n_0-1}(z_x)} P_{y,\omega}\big[H_{C_{n_0+1,\gamma_x}(x)} \le L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}\big] \ge \\ &\prod_{0\le j\le j_x} c_1\, L_{n_0}^{-\zeta J_{j,x}}\,.\end{aligned} \tag{3.49}$$

Using that for large L_0, cf. (1.15), $L_{n_0} \le 2\, L_{n_0+1}^{(1+a)^{-1}}$, we now find the desired upper bound:

$$J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \le c\, \ell_{n_0} (\log L_{n_0+1})^{-1} + (1+a)^{-1} \sum_{0 \le j \le j_x} J_{j,x}\,. \tag{3.50}$$

When $x \in \mathcal{D}_2$, $\omega \in \mathcal{C}_x$: the event $\mathcal{C}_x$, cf. (3.24), ensures the presence of many channels made of at most $4L_{n_0}^{a-\nu_x}$ n_0-good boxes of size $L_{n_0}^{1+\nu_x}$, along which, as we now explain, the diffusion travels well.

Indeed consider B_0 and $B_1 = B_0 + L_{n_0}^{1+\nu_x} e$, with $|e| = 1$, $e \in \mathbb{Z}^d$, two neighboring n_0-good boxes. Denote with U the interior of $B_0 \cup B_1$, with V_0 the concentric sub-cube of B_0 with half-side length, with $V_1 = V_0 + L_{n_0}^{1+\nu_x} e$, the corresponding sub-cube of B_1, and with W_1 the concentric sub-cube of B_1 with quarter side-length. Denote with h a continuous [0, 1]-valued function, equal to 1 on U and vanishing outside an L_{n_0-1}-neighborhood of U. We can consider the coupling measure $Q_{n_0-1,y}$, for $y \in V_0$, constructed in Proposition 3.1. Choosing in the notation of Proposition 3.1:

$$k_0 = \left[\frac{L_{n_0}^{1+\nu_x}}{L_{n_0-1}}\right]^2 \Big(\le L_{n_0-1}^{2(1+\nu_x)(1+a)-2} \overset{(3.21)}{\le} L_{n_0-1}^{4a+a^2}\Big), \text{ and } \gamma = L_{n_0-1}\,,$$

it follows from standard Brownian estimates and Remark 3.2, that

$$\inf_{y \in V_0} Q_{n_0-1,y}\big(\overline{X}^0_{k_0} \in W_1, \text{ and } d(\overline{X}_k, U^c) \ge L_{n_0}, \text{ for } 0 \le k \le k_0\big) \ge c\,. \tag{3.51}$$

By construction, see above (3.22), in the notation of (3.6), we have for large L_0:

$$\begin{aligned} k_0^2\big(\kappa_{n_0-1}\,\Gamma_{n_0-1,h} + e^{-\kappa_{n_0-1}}\big) &\le \kappa_{n_0-1}\, L_{n_0-1}^{8a+4a^2}\, L_{n_0-1}^{-\delta} \\ &\overset{(1.14),(1.40)}{\le} L_{n_0-1}^{-\delta/2}\,. \end{aligned} \tag{3.52}$$

So in the notation of (1.8), (1.19) we find for large L_0:

$$\begin{aligned} &\inf_{y\in V_0} P_{y,\omega}\big[H_{V_1} < T_U \wedge \big(k_0\, L_{n_0-1}^2\big)\big] \ge \\ &\inf_{y\in V_0} P_{y,\omega}\big[X_{kL_{n_0-1}^2} \in V_1, \text{ and for } 0 \le k < k_0, \\ &\qquad d\big(X_{kL_{n_0-1}^2}, U^c\big) \ge \frac{L_{n_0}}{2}, \text{ and } T_{n_0-1}\circ\theta_{kL_{n_0-1}^2} > L_{n_0-1}^2\big] \overset{(2.2)}{\ge} \\ &\inf_{y\in V_0} Q_{n_0-1,y}\,\Big(\overline{X}_{k_0} \in V_1,\, d(\overline{X}_k, U^c) \ge \frac{L_{n_0}}{2}, \\ &\qquad \text{for } 0 \le k \le k_0\Big) - k_0\, e^{-\kappa_{n_0-1}} \overset{(3.6),(3.51),(3.52)}{\ge} c\,. \end{aligned} \tag{3.53}$$

So (3.53) shows in a quantitative way that the diffusion "travels well" from V_0 to V_1 without leaving U. We now explain how this is used to construct an exit strategy from $C_{n_0-1}(z_x)$ to $C_{n_0+1,\gamma_x}(x)$, before time $L_{n_0+1}^2 \wedge T_{C'_{n_0+1}(x)}$.

We use the path of boxes $C^{j,x}$ with boundary boxes $\Delta^{j,x}$, $0 \le j \le j_x$, to go from $C_{n_0-1}(z_x)$ to $\Delta^{j_x,x} \subset B_\infty(z_x, 2L_{n_0}^{1+\nu'_x}) \cap C^0_{n_0+1}(x)$, where $\Delta^{j_x,x}$ is chosen to be inside a channel of n_0-good boxes B_i, $i = 1, \dots, k \le 4L_{n_0}^{a-\nu_x}$, that exit $C_{n_0+1}(x)$ in $C_{n_0+1,\gamma_x}(x)$. More precisely, we define the sequence of stopping times

$$\tau_0 = 0,\;\; \tau_j = \inf\big\{t \ge \tau_{j-1} : X_t \in \Delta^{j,x}\big\},\, j = 1, \dots, j_x\,,$$

and

$$\overline{\tau}_1 = \tau_{j_x},\;\; \overline{\tau}_i = \inf\big\{t \ge \overline{\tau}_{i-1} : X_t \in B_i\big\},\, i = 2, \dots, k\,.$$

We then define the event

$$\begin{aligned}\mathcal{E}_{\pi_x} = \big\{&\tau_j - \tau_{j-1} \le L^2_{n_0},\, j = 1, \dots, j_x\,; \overline{\tau}_i - \overline{\tau}_{i-1} \le \big(L_{n_0}^{1+\nu_x}\big)^2,\\ &i = 2, \dots, k\,;\;\; \overline{\tau}_k < T_{C'_{n_0+1}(x)}\big\}\,.\end{aligned}$$

Note that on the event $\mathcal{E}_{\pi_x}$, the path hits $C_{n_0+1,\gamma_x}(x)$ before exiting $C'_{n_0+1}(x)$, and it does so before time

$$\Big[cL_{n_0}^{\nu'_x} L^2_{n_0} + \frac{cL_{n_0+1}}{L_{n_0}^{1+\nu_x}}\big(L_{n_0}^{1+\nu_x}\big)^2\Big] \le L^2_{n_0+1}\,.$$

Using repeatedly (3.53) and the Markov property to control how the diffusion travels in the channel, and the estimates (3.46) and (1.44), we find that

$$\begin{aligned}&\inf_{y \in C_{n_0-1}(z_x)} P_{y,\omega}\big[H_{C_{n_0+1,\gamma_x}(x)} \le L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}\big] \ge \inf_{y \in C_{n_0-1}(z_x)} P_{y,\omega}[\mathcal{E}_{\pi_x}] \ge\\ &c^{L_{n_0+1}/L_{n_0}^{1+\nu_x}}\; c_1^{\,c\,L_{n_0}^{\nu'_x}}\; L_{n_0}^{-\zeta\sum_{j=0}^{j_x} J_{j,x}}\,.\end{aligned}$$

We can then remove the dependence on the environment entering the choice of the path of boxes $C^{j,x}$, with boundary boxes $\Delta^{j,x}$, $0 \le j \le j_x$, in the above inequality, and write

$$\begin{aligned}&\inf_{y \in C_{n_0-1}(z_x)} P_{y,\omega}\big[H_{C_{n_0+1,\gamma_x}}(x) \le L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}\big] \ge\\ &c^{L_{n_0+1}/L_{n_0}^{1+\nu_x}}\; c_1^{\,c\,L_{n_0}^{\nu'_x}}\;\; \widetilde{\inf}\,\big\{L_{n_0}^{-\zeta\sum_{j=0}^{j_x} J_{j,x}}\big\}\,,\end{aligned} \tag{3.54}$$

where $\widetilde{\inf}$ refers to the fact that one takes the infimum over a collection of finite sequences π_x, with all possible end points $y_{j_x,x} \in L_{n_0}\,\mathbb{Z}^d \cap C^0_{n_0+1}(x)$ $\cap\, B_\infty(z_x, 2L_{n_0}^{1+\nu'_x})$. This is an infimum over a set of cardinality smaller than

$$c\,L_{n_0}^{d\nu'_x} \overset{(3.21)}{\le} L_{n_0}^{da},\;\; L_0\;\text{ large}\,. \tag{3.55}$$

Further from our choice in (3.21), we see that

$$\alpha_x - (a - \nu_x) = \frac{3}{4}\,\alpha_x - \frac{a}{2} \overset{(3.21)}{\geq} \frac{9}{16}\,a - \frac{a}{2} = \frac{a}{16} \tag{3.56}$$

$$\alpha_x - \nu'_x = \frac{3}{8}\,\alpha_x - \frac{a}{4} \overset{(3.21)}{\geq} \frac{9}{32}\,a - \frac{a}{4} = \frac{a}{32}\,. \tag{3.57}$$

As a result of (3.54), analogously to (3.50), we find that for $x \in \mathcal{D}_2$, $\omega \in \mathcal{C}_x$,

$$\begin{aligned} &J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \leq \\ &c\left(L_{n_0}^{(a-\nu_x)} + L_{n_0}^{\nu'_x}\right)(\log L_{n_0+1})^{-1} \;+ (1+a)^{-1}\,\widetilde{\sup}\Big\{\sum_{0\leq j\leq j_x} J_{j,x}\Big\} \leq \\ &c\,L_{n_0}^{\alpha_x - \frac{a}{32}}(\log L_{n_0+1})^{-1} \;+ (1+a)^{-1}\,\widetilde{\sup}\Big\{\sum_{0\leq j\leq x} J_{j,x}\Big\}, \end{aligned} \tag{3.58}$$

and $\widetilde{\sup}$ has a similar meaning as in (3.54) and involves the supremum over a set of cardinality bounded by (3.55).

We now turn to the discussion of $x \in \mathcal{D}_3$ and $x \in \mathcal{D}_4$, beginning with some considerations on $\mathcal{C}_x$, when $x \in \mathcal{D}_3 \cup \mathcal{D}_4$. We thus consider an $x \in \mathcal{D}_3 \cup \mathcal{D}_4$, $\omega \in \mathcal{C}_x$, and $y \in C_{n_0}(y_0)$ with $d_\infty(C_{n_0}(y_0), C_{n_0+1}(x)) \leq L_{n_0}$, such that in the notation of (3.31):

$$d(y, K_{x,\omega}) \overset{\text{def}}{=} r > 0\,. \tag{3.59}$$

For $m \geq 1$, we define

$$\begin{aligned} D_m &= \widehat{y}_0 + 2^m\Big(\Big[-\tfrac{L_{n_0}}{2}, \tfrac{L_{n_0}}{2}\Big]^d \setminus \Big(-\tfrac{L_{n_0}}{4}, \tfrac{L_{n_0}}{4}\Big)^d\Big), \text{ with} \\ &\quad \widehat{y}_0 \text{ the center of } C_{n_0}(y_0)\,, \end{aligned} \tag{3.60}$$

$$K_m = K_{x,\omega} \cap D_m,\;\; K_0 = K_{x,\omega} \cap C_{n_0}(y_0)\,. \tag{3.61}$$

Keeping in mind L_{n_0+1} as a unit scale, we consider for $m \geq 0$, the Newtonian capacity of $L_{n_0+1}^{-1}\,K_m$:

$$\text{cap}_m = \text{cap}\big(L_{n_0+1}^{-1}\,K_m\big) \overset{(3.32)}{\leq} \kappa_{n_0-1}\,\frac{N_x}{(\ell_{n_0-1}\,\ell_{n_0})^{d-2}}\,. \tag{3.62}$$

We now consider an arbitrary continuous, $[0, 1]$-valued, function h such that:

$$\begin{aligned} &h = 1 \text{ on } C'_{n_0+1}(x)\backslash \widetilde{K}_{x,\omega} \overset{(3.31)}{\supseteq} C'_{n_0+1}(x)\backslash K_{x,\omega}, \text{ and} \\ &h\,\chi_{n_0-1,z} \equiv 0,\;\text{ for all } z \in L_{n_0-1}\,\mathbb{Z}^d \cap \widetilde{\mathcal{B}}^c_{n_0-1}(\omega)\,. \end{aligned} \tag{3.63}$$

We can now consider the coupling measure $Q_{n_0-1,y}$ from Proposition 3.1. Keeping in mind that under this measure $\overline{X}^0_k$, $k \geq 0$, is a Brownian motion starting from y sampled at times $\alpha_{n_0-1}\, k\, L^2_{n_0-1}$, we see from an analogous calculation as for the classical Wiener test, cf. [28, p. 72–74], that

$$\begin{aligned}
&Q_{n_0-1,y}\big[\overline{X}^0_k \in K_{x,\omega},\ \text{for some } k \geq 0\big] \leq \\
(3.64)\quad &c\Big(\sum_{m\geq 2} \mathrm{cap}_m (2^m\, \ell^{-1}_{n_0})^{-(d-2)} + \sum_{m=0,1} \mathrm{cap}_m \left(\frac{r}{L_{n_0+1}}\right)^{-(d-2)}\Big) \overset{(3.62)}{\leq} \\
&\kappa_{n_0-1}\, N_x \left(\ell^{-(d-2)}_{n_0-1} + \left(\frac{r}{L_{n_0-1}}\right)^{-(d-2)}\right),
\end{aligned}$$

where we recall the notation (3.59).

We now proceed in a similar fashion as in (3.53), with the help of Proposition 3.1, choosing in (3.6) $\gamma = L_{n_0-1}$, and

$$k_0 = \left[\frac{1}{10}\left(\frac{L_{n_0+1}}{L_{n_0-1}}\right)^2\right] \leq L^{4a+2a^2}_{n_0-1}\,. \tag{3.65}$$

We find that for large L_0:

$$\begin{aligned}
&P_{y,\omega}\Big[H_{C_{n_0+1,\gamma_x}(x)} < \Big(\frac{1}{5}\, L^2_{n_0+1}\Big) \wedge T_{C'_{n_0+1}(x)}\Big] \geq \\
&P_{y,\omega}\Big[X_{k_0 L^2_{n_0-1}} \in C_{n_0+1,\gamma_x}(x),\, d\big(X_{kL^2_{n_0-1}}, C'_{n_0+1}(x)^c\big) \geq \frac{L_{n_0}}{2}, \\
&\qquad d\big(X_{kL^2_{n_0-1}},\ \widetilde{\mathcal{B}}^c_{n_0-1}(\omega) \cap L_{n_0-1}\,\mathbb{Z}^d\big) \geq 29\sqrt{d}\, L_{n_0-1}, \\
(3.66)\quad &\qquad \text{for } 0 \leq k \leq k_0,\ T_{n_0-1} \circ \theta_{kL^2_{n_0-1}} > L^2_{n_0-1},\ \text{for } 0 \leq k < k_0\Big] \geq \\
&Q_{n_0-1,y}\Big[\overline{X}_{k_0} \in C_{n_0+1,\gamma_x}(x),\ d\big(\overline{X}_k,\ C'_{n_0+1}(x)^c\big) \geq \frac{L_{n_0}}{2}\,, \\
&\qquad d\big(\overline{X}_k,\ \widetilde{\mathcal{B}}^c_{n_0-1}(\omega) \cap L_{n_0-1}\,\mathbb{Z}^d\big) \geq 29\sqrt{d}\, L_{n_0-1}, \\
&\qquad \text{for } 0 \leq k \leq k_0\Big] - k_0\, e^{-\kappa_{n_0-1}},
\end{aligned}$$

where we used that $h = 1$ on $C'_{n_0+1}(x)\backslash \widetilde{K}_{x,\omega}$, cf. (3.63), as well as the localization part of (2.2). Then with (3.6), denoting with $\widetilde{C}_{n_0+1,\gamma_x}(x)$ the concentric box to $C_{n_0+1,\gamma_x}(x)$, with half-size, we find

$$\begin{aligned}
\geq\ &Q_{n_0-1,u}\Big[\overline{X}^0_{k_0} \in \widetilde{C}_{n_0+1,\gamma_x}(x),\ d\big(\overline{X}^0_k, C'_{n_0+1}(x)^c\big) \geq L_{n_0}\ \text{ for } 0 \leq k \leq k_0\,, \\
&\quad \overline{X}^0_k \notin K_{x,\omega},\ \text{for } 0 \leq k \leq k_0\Big] - \\
&\quad k_0\, e^{-\kappa_{n_0-1}} - k_0^2\big(\kappa_{n_0-1}\, L^{-\delta}_{n_0-1} + e^{-\kappa_{n_0-1}}\big)\,,
\end{aligned}$$

where we have used that $h\, \chi_{n_0-1,z} \equiv 0$, for $z \in L_{n_0-1}\,\mathbb{Z}^d \backslash \widetilde{\mathcal{B}}_{n_0-1}(\omega)$, as well as (2.2) in estimating $\Gamma_{n_0-1,h}$ of (3.6).

Combining this with (3.64), (3.65), and the inequality

$$Q_{n_0-1,y}\Big[\overline{X}^0_{k_0} \in \widetilde{C}_{n_0+1,\gamma_x}(x),\ d\big(\overline{X}^0_k, C'_{n_0+1}(x)^c\big) \geq L_{n_0}, \\ \text{for } 0 \leq k \leq k_0\Big] \geq 4c_1$$

that follows from the definition of c_1 below (1.44), and (1.49), we conclude with (1.14), (1.40) that

$$(3.67)\qquad \begin{aligned} &P_{y,\omega}\Big[H_{C_{n_0+1,\gamma_x}(x)} < \Big(\frac{1}{5}\,L^2_{n_0+1}\Big) \wedge T_{C'_{n_0+1}(x)}\Big] \geq \\ &4c_1 - \kappa_{n_0-1}\,N_x\Big(\ell_{n_0-1}^{-(d-2)} + \Big(\frac{r}{L_{n_0-1}}\Big)^{-(d-2)}\Big). \end{aligned}$$

This will be a crucial estimate to control exit strategies of the path starting in $C_{n_0-1}(z_x)$ and landing in $C_{n_0+1,\gamma_x}(x)$ before time $L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}$, when x belongs to $\mathcal{D}_3 \cup \mathcal{D}_4$.

When $x \in \mathcal{D}_3$, $\omega \in \mathcal{C}_x$: we describe the exit strategy. First consider the boxes $C^{j,x}$, with boundary boxes $\Delta^{j,x}$, $0 \leq j \leq j_x \overset{(3.45)}{=} n_x + 3d$. Consider a path of the diffusion starting in $C_{n_0-1}(z_x)$ successively entering the $\Delta^{j,x} \subset C^{j+1,x}$ before time $L^2_{n_0} \wedge T_{C'_{n_0}(y_j;x)}$, $0 \leq j \leq j_x$. From the time it enters $C^{2d,x}$ until it enters $\Delta^{j_x,x}$, the path remains in $C_{n_0+1}(x)$, and has diameter at least $n_x\,L_{n_0}$.

If $\theta > 0$ is such that the above mentioned portion of the path remains in the open set

$$(3.68)\qquad U_\theta = \big\{y \in \mathbb{R}^d,\ d(y, K_{x,\omega}) < \theta\big\},$$

in view of (3.32), the fact that $\omega \in \mathcal{C}_x$ then implies

$$n_x\,L_{n_0} < 2N_x(4\,\widetilde{D}_{n_0-1} + \theta)\,.$$

Choosing

$$(3.69)\qquad r = \frac{n_x\,L_{n_0}}{2N_x} - 4\,\widetilde{D}_{n_0-1} > 0,\ \text{when } L_0 \text{ is large, cf. (3.32), (3.45)},$$

the path enters $C_{n_0+1}(x) \cap U^c_r$ before time $(j_x+1)\,L^2_{n_0} \wedge T_{C'_{n_0+1}(x)} \leq \frac{1}{4}\,L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}$. Letting this entrance point in $C_{n_0+1}(x) \cap U^c_r$ play the role of y in (3.67), we can use the strong Markov property and find that for large L_0:

$$(3.70)\qquad \begin{aligned} &\inf_{w \in C_{n_0-1}(z_x)} P_{w,\omega}\big[H_{C_{n_0+1,\gamma_x}(x)} \leq L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}\big] \geq \\ &c_1^{j_x+1}\,L_{n_0}^{(-\zeta\,\sum_{0\leq j\leq j_x} J_{j,x})} 2c_1\,, \end{aligned}$$

where we used that thanks to (1.14), (3.20), (3.32), (3.45), (3.69), the last term of (3.67) is arbitrarily small, when L_0 is large. As a result we thus see that when L_0 is large:

$$(3.71)\qquad J_{n_0+1,x,C_{n_0-1}(z_x),\gamma_x} \leq c\,n_x(\log L_{n_0+1})^{-1} + (1+a)^{-1} \sum_{0\leq j\leq j_x} J_{j,x}\,.$$

When $x \in \mathcal{D}_4$, $\omega \in \mathcal{C}_x$*:* we denote with $\widetilde{C}_{n_0,x}$ the union, (we recall that $C_{n_0}(y_x) \supseteq C_{n_0-1}(z_x)$, cf. (3.42)):

$$\widetilde{C}_{n_0,x} = \bigcup_{|\overline{y}-y_x|_\infty \le L_{n_0}} \overline{C'_{n_0}}(\overline{y})\,. \tag{3.72}$$

By the same argument as below (3.68), a path of the diffusion inside $\widetilde{C}_{n_0,x}$ starting in $C_{n_0-1}(z_x)$, which has diameter at least $\frac{1}{2}\,L_{n_0}$ before time $(\frac{1}{2}\,L^2_{n_0+1}) \wedge T_{C'_{n_0+1}(x)}$, enters before that time the set $\widetilde{C}_{n_0,x} \cap U_r^c$, with

$$r = \frac{L_{n_0}}{4N_x} - 4\,\widetilde{D}_{n_0-1} > 0, \text{ when } L_0 \text{ is large}\,. \tag{3.73}$$

If this entrance point in $\widetilde{C}_{n_0,x} \cap U_r^c$ plays the role of y, (3.67) provides a lower bound on the probability that the path then reaches $C_{n_0+1,\gamma_x}(x)$ before $(\frac{1}{5}\,L^2_{n_0+1}) \wedge T_{C'_{n_0+1}(x)}$.

Note that when starting at u in $C_{n_0}(\overline{y})$, with $|\overline{y} - y_x|_\infty \le L_{n_0}$:

$$P_{u,\omega}\Big[X^*_{L^2_{n_0} \wedge T_{C'_{n_0}(\overline{y})}} \ge \frac{1}{2}\,L_{n_0}\Big] \ge c_1\,L_{n_0}^{-\zeta J_x}, \text{ where} \tag{3.74}$$

$$\begin{aligned} J_x = \sup\big\{&J_{n_0,y',C_{n_0}(y'),\gamma'};\ |y' - y_x|_\infty \le L_{n_0},\\ &\gamma' \in \{1,\dots,2d\,5^{(d-1)}\}\big\}\,. \end{aligned} \tag{3.75}$$

With the strong Markov property, we thus see that for large L_0,

$$\begin{aligned} &\inf_{y \in C_{n_0-1}(z_x)} P_{y,\omega}\big[H_{C_{n_0+1,\gamma_x}(x)} \le L^2_{n_0+1} \wedge T_{C'_{n_0+1}(x)}\big] \ge \\ &2c_1\big(1 - \big(1 - c_1\,L_{n_0}^{-\zeta J_x}\big)^{[\ell^2_{n_0}/3]}\big) \ge 2c_1\big(1 - \big(1 - c_1\,L_{n_0}^{-\zeta J_x}\big)^{\ell^2_{n_0}/4}\big) = \\ &\inf_{y',\gamma'} 2c_1\big(1 - \big(1 - c_1\,L_{n_0}^{-\zeta J_{n_0,y',C_{n_0}(y'),\gamma'}}\big)^{\ell^2_{n_0}/4}\big)\,, \end{aligned} \tag{3.76}$$

where the infimum is over the same set as in (3.75).

We will now employ the bounds (3.50), (3.58), (3.71), (3.76) to bound I in (3.38) and prove the claim (3.39). To keep track of the supremum and infimum that respectively enter (3.58), (3.76), we introduce a set Π of $\pi = (\pi_x)_{x \in \mathcal{D}}$, such that for any $x_0 \in \mathcal{D}_2$, $\pi \in \Pi$, the set of $\pi' \in \Pi$ that coincide with π for $x \neq x_0$ is such that all points of $L_{n_0}\,\mathbb{Z}^d \cap C^0_{n_0+1}(x) \cap B_\infty(x, 3\,L_{n_0}^{1+\nu'_x})$ and all γ in $\{1,\dots,2d\,5^{(d-1)}\}$ occur as $y_{j_{x_0,x_0}}$ and $\gamma_{j_{x_0,x}}$, and similarly for any $x_0 \in \mathcal{D}_4$, $\pi \in \Pi$, the set of $\pi' \in \Pi$ that coincide with π for $x \neq x_0$ is such that all $y' \in L_{n_0}\,\mathbb{Z}^d$ with $C_{n_0}(y') \subset \widetilde{C}_{n_0,x}$ and $\gamma' \in \{1,\dots,2d\,5^{(d-1)}\}$ occur as $y_{0,x}$ and $\gamma_{0,x}$. With (3.55) we see that when L_0 is large we can choose such a Π with cardinality

$$|\Pi| \le L_{n_0}^{da\,|\mathcal{D}_2|}\,c^{|\mathcal{D}_4|}\,. \tag{3.77}$$

Note that for any $\pi \in \Pi$, the sets $\Delta^{j,x}$, $-1 \le j < j_x$, $x \in \mathcal{D}$, lie at mutual $|\cdot|_\infty$-distance at least $10d\, L_{n_0-1}$, cf. (3.43), so that in view of (1.48), for any choice of $v_{j,x} \ge 0$, where $(j,x) \in \mathcal{J} \stackrel{\text{def}}{=} \{(j',x');\ x' \in \mathcal{D},\ 0 \le j' \le j_x\}$,

$$\mathbb{P}[\text{for all } (j,x) \in \mathcal{J},\ \ J_{j,x} \ge v_{j,x}] \le \prod_{(j,x)\in\mathcal{J}} P[Z \ge v_{j,x}]\,, \tag{3.78}$$

where $Z = Z_1$, and $Z_k, k \ge 1$, is an i.i.d. family of non-negative random variables defined in some auxiliary probability space such that

$$P[Z > v] = L_{n_0}^{-\overline{M}_{n_0}(1+v)} \text{ for } v > 0\,, \tag{3.79}$$

(so $P[Z = 0] = 1 - L_{n_0}^{-\overline{M}_{n_0}}$, and we assume from now on that $L_0 \ge \text{const}\,(c_2)$ so that $\prod_{n\ge 0}(1 - c_2(\log L_n)^{-1}) \ge \frac{1}{2}$). Let us mention that (3.78) can be rephrased in terms of upper orthant order, see Shaked-Shanthikumar [24, p. 140]. We denote with Σ_k, $k \ge 0$, the partial sums

$$\Sigma_0 = 0,\ \ \Sigma_k = Z_1 + \cdots + Z_k,\ \ \text{for } k \ge 1\,. \tag{3.80}$$

Note that for $0 \le \lambda < \overline{M}_{n_0} \log L_{n_0}$, one has

$$\begin{aligned} E[e^{\lambda Z}] &= 1 + \int_0^\infty \lambda e^{\lambda v}\, L_{n_0}^{-\overline{M}_{n_0}(v+1)}\, dv \\ &= 1 + \frac{\lambda}{(\overline{M}_{n_0}\log L_{n_0} - \lambda)}\, L_{n_0}^{-\overline{M}_{n_0}}\,. \end{aligned} \tag{3.81}$$

Analogously for an arbitrary collection $v_x \ge 0, x \in \mathcal{D}$, and $\lambda_x \in [0, \overline{M}_{n_0}\log L_{n_0})$, $x \in \mathcal{D}\backslash\mathcal{D}_4$, it follows from (3.78), see also [24, Theorem 5.G.1, p. 141], that:

$$\begin{aligned} &\mathbb{P}\Big[\sum_{0\le j\le j_x} J_{j,x} \ge v_x,\ \text{ for } x \in \mathcal{D}\Big] \le \\ &\exp\Big\{-\sum_{x\in\mathcal{D}\backslash\mathcal{D}_4} \lambda_x\, v_x\Big\} E\Big[\exp\Big\{\sum_{x\in\mathcal{D}\backslash\mathcal{D}_4} \lambda_x \sum_{0\le j\le j_x} J_{j,x}\Big\}, \\ &\qquad\qquad\qquad \text{for } x \in \mathcal{D}_4,\ J_{0,x} \ge v_x\Big] \stackrel{(3.78)}{\le} \\ &\exp\Big\{-\sum_{x\in\mathcal{D}\backslash\mathcal{D}_4} \lambda_x\, v_x\Big\} \prod_{x\in\mathcal{D}\backslash\mathcal{D}_4} E[e^{\lambda_x \Sigma_{(j_x+1)}}] \prod_{x\in\mathcal{D}_4} P[Z \ge v_x] \stackrel{(3.81)}{=} \\ &\exp\Big\{-\sum_{x\in\mathcal{D}\backslash\mathcal{D}_4} \lambda_x\, v_x\Big\} \prod_{x\in\mathcal{D}\backslash\mathcal{D}_4} \Big(1 + \frac{\lambda_x\, L_{n_0}^{-\overline{M}_{n_0}}}{\overline{M}_{n_0}\log L_{n_0} - \lambda_x}\Big)^{j_x+1} \\ &\prod_{x\in\mathcal{D}_4} P[Z \ge v_x]\,. \end{aligned} \tag{3.82}$$

We will now use (3.82) to bound I in (3.38). Indeed for large L_0, with (3.50), (3.58), (3.71), (3.76) we have

$$
\begin{aligned}
I \le \mathbb{P}\Big[\bigcup_{\pi\in\Pi}\Big\{ & c\ell_{n_0}(\log L_{n_0+1})^{-1} + (1+a)^{-1}\sum_{0\le j\le j_x} J_{j,x} \ge u_x, \\
& \qquad\qquad \text{for } x \in \mathcal{D}_1\,, \\
& c\,L_{n_0}^{\alpha_x-\frac{a}{32}}(\log L_{n_0+1})^{-1} + (1+a)^{-1}\sum_{0\le j\le j_x} J_{j,x} \ge u_x, \\
& \qquad\qquad \text{for } x \in \mathcal{D}_2\,, \\
& c\,n_x(\log L_{n_0+1})^{-1} + (1+a)^{-1}\sum_{0\le j\le j_x} J_{j,x} \ge u_x, \\
& \qquad\qquad \text{for } x \in \mathcal{D}_3\,, \\
& 1-\frac{1}{2}\,L_{n_0+1}^{-\zeta u_x} \le \big(1-c_1\,L_{n_0}^{-\zeta J_{0,x}}\big)^{\ell_{n_0}^2/4},\ \text{for } x\in\mathcal{D}_4\Big\}\Big].
\end{aligned} \tag{3.83}
$$

From (3.45)–(3.47), $j_x \le c\ell_{n_0}$, for $x \in \mathcal{D}\backslash\mathcal{D}_4$, so using (3.77) and (3.82) with $\lambda_x = \lambda_* \overset{\text{def}}{=} \overline{M}_{n_0}\log L_{n_0} - 1$, for all $x \in \mathcal{D}\backslash\mathcal{D}_4$, and $(1+u) \le e^u$, we find

$$
\begin{aligned}
I \le\ & L_{n_0}^{da|\mathcal{D}_2|}\,c^{|\mathcal{D}_4|} \\
& \exp\Big\{-\sum_{x\in\mathcal{D}\backslash\mathcal{D}_4}\overline{M}_{n_0}(\log L_{n_0+1})(u_x+1)\big(1-c(\log L_{n_0})^{-1}\big)\Big\} \\
& \prod_{x\in\mathcal{D}_4} P\Big[1-\frac{1}{2}\,L_{n_0+1}^{-\zeta u_x} \le \big(1-c_1\,L_{n_0}^{-\zeta Z}\big)^{\ell_{n_0}^2/4}\Big].
\end{aligned} \tag{3.84}
$$

Note that with L_0 large and $\Pi_{n\ge 0}(1-c_2(\log L_n)^{-1}) \ge \frac{1}{2}$, cf. below (3.79), each individual term of the last product is smaller than

$$
\begin{aligned}
& P\big[c\,\ell_{n_0}^{-2}\,L_{n_0+1}^{-\zeta u_x} \ge L_{n_0}^{-\zeta Z}\big] \le P\big[\zeta Z \ge \zeta u_x(1+a) + 2a - c(\log L_{n_0})^{-1}\big] \overset{(3.79)}{\le} \\
& \exp\Big\{-(\log L_{n_0})\,\overline{M}_{n_0}\Big(u_x(1+a) + \frac{2}{\zeta}\,a - c(\log L_{n_0})^{-1} + 1\Big)\Big\} \le \\
& \exp\Big\{-(\log L_{n_0})\,\overline{M}_{n_0}\Big[(1+a)(u_x+1) + \frac{3}{4}\Big(\frac{2}{\zeta}-1\Big)\,a\Big]\Big\}.
\end{aligned} \tag{3.85}
$$

Coming back to (3.84), we obtain

$$
I \le L_{n_0+1}^{[-\sum_{x\in\mathcal{D}}\overline{M}_{n_0}(1-c(\log L_{n_0})^{-1})(u_x+1)]}\;L_{n_0}^{-|\mathcal{D}_4|(\frac{1}{\zeta}-\frac{1}{2})a\overline{M}_{n_0}}\,,
$$

and in view of (1.43), this proves (3.39). $\square$

We can now conclude the proof of Proposition 3.3. Coming back to (3.17), (3.36), (3.37), (3.39), we observe that when L_0 is large,

$$\big(2c\,\ell_{n_0-1}^d\,\ell_{n_0}^d\big)^{|\mathcal{A}|}\le L_{n_0}^{3da|\mathcal{A}|}\overset{(3.18)}{\le} L_{n_0}^{d\,3^{d+1}a|\widetilde{\mathcal{A}}|}\,,$$

and hence

$$\begin{aligned}
&\mathbb{P}\big[\forall x\in\mathcal{A},\ J_{n_0+1,x,A_x,\gamma_x}\ge u_x\big]\overset{(3.36)}{\le}\\
&L_{n_0}^{d3^{d+1}a|\widetilde{\mathcal{A}}|}\,L_{n_0+1}^{-\overline{M}_{n_0}(1-c'/\log L_{n_0})\sum_{x\in\widetilde{\mathcal{A}}\setminus\overline{\mathcal{M}}}(u_x+1)}\\
&\cdot L_{n_0}^{-d3^{d+1}a\overline{M}_{n_0}|\widetilde{\mathcal{A}}_4\setminus\overline{\mathcal{M}}|}\,L_{n_0}^{-2d\,3^{d+1}M\sum_{x\in\overline{\mathcal{M}}}(u_x+1)}\le\\
&L_{n_0}^{d\,3^{d+1}a|\widetilde{\mathcal{A}}\setminus\widetilde{\mathcal{A}}_4|}\,L_{n_0+1}^{-\overline{M}_{n_0}(1-c'/\log L_{n_0})\sum_{x\in\widetilde{\mathcal{A}}}(u_x+1)}\le\\
&L_{n_0+1}^{-\overline{M}_{n_0}(1-c''/\log L_{n_0})\sum_{x\in\widetilde{\mathcal{A}}}(u_x+1)}\overset{(3.19)}{\le}\\
&L_{n_0+1}^{-\overline{M}_{n_0}(1-c'''/\log L_{n_0})\sum_{x\in\widetilde{\mathcal{A}}}(u_x+1)}
\end{aligned}\tag{3.86}$$

where $L_0\ge\text{const}(c_2)$, so that $\prod_{n\ge0}(1-c_2(\log L_n)^{-1})\ge\frac12$ and in particular $\overline{M}_{n_0}\ge 1$. We then see that if c_2 is chosen to be constant bigger than the constant c''' that appears in the last member of (3.86), then (1.48) holds for $n=n_0+1$. This proves Proposition 3.3. □

4. Surgery and contraction of Hölder-norms

We continue the proof of Theorem 1.1. The aim is now to propagate the part of (1.47) that concerns Hölder-norms at level n_0+1, cf. (1.39). The part of (1.47) that concerns localization controls has been taken care of in Proposition 2.2. The control of Hölder-norms will be carried out in the present and in the next section. Here we first perform "surgery" and remove "Hölder-norm defects" at level n_0-m_0-1 that occur in the large box $5\mathcal{T}_{n_0+1}$, see (2.1). We show that with overwhelming $\mathbb{P}$-probability the kernel R_n of the diffusion in the modified environment, when starting in $\mathcal{T}_{n_0+1}$, gets closer and closer in $\|\cdot\|_n$-norm to R_n^0 as n goes from n_0-m_0-1 to n_0, cf. Proposition 4.11. The crucial step comes in Proposition 4.1, where Hölder-norm estimates are derived on what is in essence the linearization of the evolution after surgery at level $n+1$, when expressed in terms of the one at level n, as n varies from n_0-m_0-1 to n_0.

As a shorthand notation, we write, cf. Theorem 1.1,

$$n_0'=n_0-m_0-1\ge 0\,.\tag{4.1}$$

Keeping in mind the notation (1.51) and the convention on constants above Theorem 1.1, we will repeatedly use in the sequel that when L_0 is large, for

$n_0' \le n \le n_0+1$, $e^{-\kappa_n} \le e^{-\kappa_{n_0+1}}$, and $2e^{-\kappa_{n_0+1}} \le e^{-\kappa_{n_0+1}}$, where of course the various constants entering the various occurrences of κ_n and κ_{n_0+1} vary (but do not depend on the particular value of n_0).

We introduce the event, cf. (2.2)

$$\begin{aligned} G = \big\{\omega \in \Omega;\ & L_{n_0'}\,\mathbb{Z}^d \cap \widetilde{\mathcal{B}}_{n_0'}(\omega)^c \cap (5\,\mathcal{T}_{n_0+1}) \text{ is contained} \\ & \text{in the union of at most } \tilde{\ell}_0 \text{ open balls with radius } 3\widetilde{D}_{n_0'} \\ & \text{and center in } L_{n_0'}\,\mathbb{Z}^d\big\}, \\ \text{where } \tilde{\ell}_0 = & \Big[\frac{2M_0}{M_0(1+a)^{-(m_0+2)} - 2d}\Big] + 1\,. \end{aligned} \tag{4.2}$$

By analogous considerations as in (3.31), (3.32), we see that on G^c, we can find $\tilde{\ell}_0$ disjoint open balls with centers in $L_{n_0'}\,\mathbb{Z}^d \cap \widetilde{\mathcal{B}}_{n_0'}(\omega)^c \cap (5\,\mathcal{T}_{n_0+1})$ and radius $\frac{3}{2}\,\widetilde{D}_{n_0'}$, so that with (2.3), (1.7), (1.46), (1.47), we see that when L_0 is large

$$\begin{aligned} \mathbb{P}[G^c] & \le c\left(\frac{L^2_{n_0+1}}{L_{n_0'}}\right)^{\tilde{\ell}_0 d} L_{n_0'}^{-M_0\tilde{\ell}_0} \\ & \le c\, L_{n_0+1}^{\tilde{\ell}_0 d(2-(1+a)^{-(m_0+2)}) - M_0\tilde{\ell}_0(1+a)^{-(m_0+2)}} \\ & \le (100(m_0+2))^{-1}\, L_{n_0+1}^{-M_0}\,. \end{aligned} \tag{4.3}$$

We introduce the set of finite sequences of length at most $\tilde{\ell}_0$:

$$\begin{aligned} \Sigma = \big\{\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{\tilde{\ell}});\ & \text{with } 0 \le \tilde{\ell} \le \tilde{\ell}_0,\ \sigma_i \in L_{n_0'}\,\mathbb{Z}^d, \\ & B(\sigma_i, 3\widetilde{D}_{n_0'}) \cap 5\,\mathcal{T}_{n_0+1} \ne \emptyset,\ \text{for } 1 \le i \le \tilde{\ell}\big\}, \end{aligned} \tag{4.4}$$

we denote with $\emptyset$ the only element of Σ with length $\tilde{\ell} = 0$. We can now write

$$\begin{aligned} & G \subset \bigcup_{\sigma \in \Sigma} G_\sigma,\ \text{with} \\ & G_\sigma = \Big\{\omega \in \Omega;\ \big(5\,\mathcal{T}_{n_0+1} \cap L_{n_0'}\,\mathbb{Z}^d\big) \setminus \bigcup_{i=1}^{\tilde{\ell}} B(\sigma_i, 3\widetilde{D}_{n_0'}) \subseteq \widetilde{\mathcal{B}}_{n_0'}(\omega)\Big\}, \end{aligned} \tag{4.5}$$

for $\sigma = (\sigma_1, \dots, \sigma_{\tilde{\ell}})$, with $0 \le \tilde{\ell} \le \tilde{\ell}_0$.

Loosely speaking, on G_σ the defects at level n_0' occurring within $5\,\mathcal{T}_{n_0+1}$ are contained in the "small set" $\bigcup_1^{\tilde{\ell}} B(\sigma_i, 3\widetilde{D}_{n_0'})$. We are now going to perform surgery on these defects. To this end for each $\sigma \in \Sigma$, we choose a $[0,1]$-valued function g_σ such that with $\sigma = (\sigma_1, \dots, \sigma_\ell)$, $0 \le \tilde{\ell} \le \tilde{\ell}_0$,

$$\begin{cases} g_\sigma = 0 \quad \text{on} \quad \displaystyle\bigcup_{1 \le i \le \tilde{\ell}} \overline{B}(\sigma_i,\, 5\,\widetilde{D}_{n_0'}) \cup (5\,\mathcal{T}_{n_0+1})^c\,, \\ \qquad = 1 \quad \text{on} \quad \big\{d_\infty(\cdot, (5\,\mathcal{T}_{n_0+1})^c) \ge 2\widetilde{D}_{n_0'}\big\} \setminus \displaystyle\bigcup_{1 \le i \le \tilde{\ell}} B\big(\sigma_i, 7\widetilde{D}_{n_0'}\big)\,, \\ |g_\sigma(y) - g_\sigma(z)| \le c\left|\dfrac{y-z}{L_{n_0'}}\right|,\ \text{for all } y, z \in \mathbb{R}^d\,, \end{cases} \tag{4.6}$$

(with the $\beta = 1$ analogue to (1.29), one can for instance construct g_σ as a product of functions attached to each σ_i, $1 \le i \le \tilde{\ell}$, when $\tilde{\ell} \ge 1$).

One can then define the corrected transition kernels for $\sigma \in \Sigma$, $\omega \in \Omega$:

$$R^*_{n_0',\sigma} = \widetilde{R}^0_{n_0'} + g_\sigma\big(\widetilde{R}_{n_0'} - \widetilde{R}^0_{n_0'}\big), \text{ cf. (1.20), (1.21)}, \tag{4.7}$$

and by induction for $n \in [n_0', n_0]$:

$$\begin{aligned} &R^*_{n+1,\sigma} = (R^0_n + h_n\, S^*_{n,\sigma})^{\ell_n^2}, \text{ with } S^*_{n,\sigma} = R^*_{n,\sigma} - R^0_n, \\ &\text{and } h_n \text{ functions with values in } [0,1], \text{ taking the value 1 on} \\ &\big\{d_\infty\big(\cdot, (5\mathcal{T}_{n_0+1})^c\big) \ge 2L^2_{n+1}\big\}, \text{ the value 0 on} \\ &\big\{d_\infty\big(\cdot, (5\mathcal{T}_{n_0+1})^c\big) \le L^2_{n+1}\big\}, \text{ such that } \sup_{n_0' \le n \le n_0} |h_n|_{(n)} \le c\,. \end{aligned} \tag{4.8}$$

Note that $R^*_{n_0',\sigma}(x,dy)$ is supported in $\overline{B}(x, \widetilde{D}_{n_0'})$, and when L_0 is large, it follows by induction that for $n_0' \le n \le n_0$:

$$\begin{aligned} &R^*_{n+1,\sigma}(x,dy) = (R^*_{n_0',\sigma})^{L^2_{n+1}/L^2_{n_0'}}(x,dy), \\ &\text{if } d_\infty(x, (5\mathcal{T}_{n_0+1})^c) \ge 3L^2_{n+1}\,. \end{aligned} \tag{4.9}$$

It is also convenient to introduce some further kernels $\widetilde{R}^*_{n,\sigma}$ that have a well-localized dependence on the environment, and intuitively are "stopped versions" of the kernels $R^*_{n,\sigma}$. For our purpose the difference between these two kernels will be "negligible", cf. (4.140), and (4.12). More precisely, for $x \in \mathbb{R}^d$, we consider, (see (1.14) for the notation)

$$\begin{aligned} &\psi_{n,x}(z) \text{ a piecewise linear function of } |z-x|, \\ &\text{with value 1 for } |z-x| \le D^*_n \overset{\text{def}}{=} L_n\, e^{3c_0(\log\log L_n)^2}, \\ &\text{and value 0 for } |z-x| \ge D^*_n + 1\,. \end{aligned} \tag{4.10}$$

We define the probability kernels $\widetilde{R}^*_{n,\sigma}$, for $n_0' \le n \le n_0$, as

$$\begin{aligned} (\widetilde{R}^*_{n,\sigma} f)(x) = &\sum_{0 \le m < L^2_n/L^2_{n_0'}} \big[(\psi_{n,x}\, R^*_{n_0',\sigma})^m (1-\psi_{n,x}) f\big](x) \\ &+ \big[(\psi_{n,x}\, R^*_{n_0',\sigma})^{L^2_n/L^2_{n_0'}} f\big](x), \text{ with } f \text{ bounded measurable}\,. \end{aligned} \tag{4.11}$$

The kernel $\widetilde{R}^*_{n,\sigma}(x,dy)$ corresponds to a "soft stopping" with the function $\psi_{n,x}$ of the Markov chain with kernel $R^*_{n_0',\sigma}$ starting at x, at time $L^2_n/L^2_{n_0'}$, see also (4.138) for a trajectorial interpretation. In particular $\widetilde{R}^*_{n_0',\sigma}$ coincides with $R^*_{n_0',\sigma}$ and $\widetilde{R}^*_{n,\sigma}(x,dy)$ is supported in $\overline{B}(x, D^*_n + 1 + \widetilde{D}_{n_0'})$. It is also convenient to introduce

$$\widetilde{S}^*_{n,\sigma} = \widetilde{R}^*_{n,\sigma} - R^0_n\,, \tag{4.12}$$

and we now see that for $n_0' \le n \le n_0$, $x \in \mathbb{R}^d$,

$$\text{(4.13)} \qquad \begin{array}{l} \widetilde{R}^*_{n,\sigma}(x,dy) \text{ or } \widetilde{S}^*_{n,\sigma}(x,dy) \text{ depend} \\ \text{in a } \mathcal{G}_{\overline{B}(x,D^*_n+1+\widetilde{D}_{n_0'})}\text{-measurable fashion in } \omega\,. \end{array}$$

In analogy with (1.24), we also define for $\sigma \in \Sigma$, $\omega \in \Omega$, $n_0' \le n \le n_0$, $x \in \mathbb{R}^d$,

$$\text{(4.14)} \qquad \begin{aligned} \widetilde{d}^*_{n,\sigma}(x,\omega) &= \int (y-x)\, \widetilde{R}^*_{n,\sigma}(x,dy) = \int (y-x)\, \widetilde{S}^*_{n,\sigma}(x,dy) \\ (\widetilde{\gamma}^*_{n,\sigma})^{i,j}(x,\omega) &= \int (y-x)_i\, (y-x)_j\, \widetilde{S}^*_{n,\sigma}(x,dy),\ 1 \le i,j \le d\,. \end{aligned}$$

We want to compare $R^*_{n,\sigma}$ with R^0_n on the event G_σ, when starting reasonably away from $(5\,\mathcal{T}_{n_0+1})^c$, for $n_0' \le n \le n_0+1$. Note that with (4.8), using perturbation expansion for $n_0' \le n \le n_0$:

(4.15)

$$\begin{aligned} S^*_{n+1,\sigma} = &\ \big(R^0_n + h_n\, S^*_{n,\sigma}\big)^{\ell_n^2} - \big(R^0_n\big)^{\ell_n^2} + \big(R^0_n\big)^{\ell_n^2} - R^0_{n+1} \\ \overset{(1.21),(1.54)}{=} &\ \sum_{\substack{k_0+\dots+k_m+m=\ell_n^2 \\ k_i \ge 0, m \ge 1}} \big(R^0_n\big)^{k_0}\, h_n\, S^*_{n,\sigma} \big(R^0_n\big)^{k_1}\, h_n\, S^*_{n,\sigma} \dots h_n\, S^*_{n,\sigma} \big(R^0_n\big)^{k_m} \\ &+\ P_{\alpha_n L^2_{n+1}} - P_{\alpha_{n+1} L^2_{n+1}}\,. \end{aligned}$$

In essence we are going to first study the "linearized" term corresponding to $m=1$ in the above series, however replacing $S^*_{n,\sigma}$, with the more convenient $\widetilde{S}^*_{n,\sigma}$, due to their better localization properties. With this in mind, we introduce for $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $v \in L_{n+1}\,\mathbb{Z}^d$, with the notation (1.38), the operator

$$\text{(4.16)} \qquad \widetilde{\mathcal{L}}_{\sigma,n,v} = \sum_{0 \le k < \ell_n^2} \chi_{n+1,v} (R^0_n)^k\, h_{n,v}\, \widetilde{S}^*_{n,\sigma} (R^0_n)^{\ell_n^2-k-1}\, \widetilde{\chi}_{n+1,v}\,,$$

where we have used the shorthand notation $h_{n,v}(\cdot) = h_n(\cdot)\,\chi_{D_{n+1}}(\cdot - v)$, and $\widetilde{\chi}_{n+1,v}(\cdot) = \chi_{\widetilde{D}_{n+1}}(\cdot - v)$, cf. (1.37). We also introduce, cf. (1.13), (1.40),

$$\text{(4.17)} \qquad \nu_n = 2\,\overline{\kappa}_{n_0'} (L_{n_0'})^{-\delta} \Big(\frac{L_n}{L_{n_0'}}\Big)^{-\beta/4},\ \text{ for } n_0' \le n \le n_0+1\,,$$

where it should be observed that $\frac{\beta}{4} > \delta$, and $\overline{\kappa}_n$ is defined in (2.4). Our first important step comes with

Proposition 4.1. *When L_0 is large, if for some $n \in [n_0', n_0]$,*

$$\text{(4.18)} \qquad \mathbb{P}\Big[\sup_{y \in [0,L_n]^d} \Big\{ \Big| \frac{\widetilde{d}^*_{n,\sigma=\emptyset}}{L_n}(y,\omega) \Big| \vee \Big| \frac{\widetilde{\gamma}^*_{n,\sigma=\emptyset}}{L_n^2}(y,\omega) \Big| > \nu_n \Big\} \Big] \le L_{n_0}^{-2}\,,$$

then for any $\sigma \in \Sigma$, $v \in L_{n+1}\,\mathbb{Z}^d$, *and event* $G_{\sigma,n,v} \subseteq G_\sigma$ *on which*

$$\sup_{x\in\mathcal{S}_{n,v}} \|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\|_n \le \nu_n, \quad \text{with } \mathcal{S}_{n,v} \stackrel{\text{def}}{=} L_n\,\mathbb{Z}^d \cap \{d(\cdot, \operatorname{Supp} h_{n,v}) \le 20\sqrt{d}\, L_n\}, \tag{4.19}$$

one has

$$\mathbb{P}\Big[G_{\sigma,n,v} \cap \Big\{\big\|\widetilde{\mathcal{L}}_{\sigma,n,v}\big\|_{n+1} > \frac{\kappa_n\,\nu_n}{\ell_n^{\beta/3}}\Big\}\Big] \le e^{-\kappa_{n_0}}. \tag{4.20}$$

Proof. Without loss of generality we assume that

$$h_{n,v} \text{ is not identically } 0, \tag{4.21}$$

otherwise there is nothing to prove. We decompose $\widetilde{\mathcal{L}}_{\sigma,n,v}$ into

$$\widetilde{\mathcal{L}}_{\sigma,n,v} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C + \mathcal{L}_D, \tag{4.22}$$

where the operators on the right-hand side of (4.22) are respectively obtained by restricting the summation over k in (4.16) to

$$\begin{aligned} &I_A = \{0\},\ I_B = \Big\{k:\ 0 < k \le \frac{\ell_n^2}{2}\Big\},\\ &I_C = \Big\{k: \frac{\ell_n^2}{2} < k \le \ell_n^2 - \ell_n^{\frac{2}{3}\beta}\Big\},\\ &I_D = \Big\{k: \ell_n^2 - \ell_n^{\frac{2}{3}\beta} < k \le \ell_n^2 - 1\Big\}. \end{aligned} \tag{4.23}$$

We will obtain controls like (4.20) on each term of the decomposition, with the role of $\ell_n^{\beta/3}$ replaced with $\ell_n^{1-\beta}$ for $\mathcal{L}_A$, $\ell_n^{(1-\beta)\wedge(\frac{d}{2}-1)}$ for $\mathcal{L}_B$, $\ell_n^{\beta/3} = \ell_n^{\beta/3\wedge\beta\wedge(\frac{d}{2}-1)}$ for $\mathcal{L}_C$, and $\ell_n^{\beta/3}$ for $\mathcal{L}_D$, cf. Lemmas 4.2, 4.3, 4.5, 4.6. We begin with the control of $\mathcal{L}_A$.

Lemma 4.2. *When* L_0 *is large, for* $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $v \in L_{n+1}\,\mathbb{Z}^d$, *with* (4.21), $\omega \in \Omega$*:*

$$\|\mathcal{L}_A\|_{n+1} \le \frac{\kappa_n}{\ell_n^{1-\beta}}\Big(\sup_{x\in\mathcal{S}_{n,v}} \|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big). \tag{4.24}$$

Proof. By construction, cf. (4.19), $\operatorname{Supp} h_{n,v} \subseteq \bigcup_{x\in\mathcal{S}_{n,v}} \overline{B}(x, \sqrt{d}L_n)$, and for $x \in \mathcal{S}_{n,v}$, $y \in \overline{B}(x, 20\sqrt{d}L_n)$, f with $|f|_{(n+1)} \le 1$, one has

$$\begin{aligned} &\big(\widetilde{S}^*_{n,\sigma}\big(R_n^0\big)^{\ell_n^2-1}\,\widetilde{\chi}_{n+1,v}f\big)(y) = \big(\widetilde{S}^*_{n,\sigma}H\big)(y), \text{ with}\\ &H(z) = (P_{\alpha_n(\ell_n^2-1)L_n^2}\,\widetilde{\chi}_{n+1,v}f)(z) - (P_{\alpha_n(\ell_n^2-1)L_n^2}\widetilde{\chi}_{n+1,v}f)(x), \end{aligned} \tag{4.25}$$

simply because $\widetilde{S}^*_n 1 = 0$. With the help of (1.49), (1.56), we find

$$|\nabla H| \le c\,L_{n+1}^{-1}, \text{ and } H(x) = 0. \tag{4.26}$$

Using a cut-off function and (A.6) from the Appendix, we can thus find $\widetilde{H}$ such that

$$\text{(4.27)}\qquad \begin{aligned} &\operatorname{Supp} \widetilde{H} \subset B(x, 4D_n^*),\ |\widetilde{H}| \le |H|,\ \widetilde{H} = H \text{ on } \overline{B}(x, 3D_n^*), \\ &\text{and } |\widetilde{H}|_{(n)} \le \frac{\kappa_n}{\ell_n}\,. \end{aligned}$$

With the remark above (4.12) on the support of $\widetilde{R}^*_{n,\sigma}(y,\cdot)$, we see that

$$\text{(4.28)}\qquad \chi_{n,x}\, \widetilde{S}^*_{n,\sigma}(H - \widetilde{H}) = -\chi_{n,x}\, R_n^0(H - \widetilde{H})\,,$$

and with (1.49), (1.56) and (4.27), we find

$$\text{(4.29)}\qquad |\chi_{n,x}\,(\widetilde{S}^*_{n,\sigma}\, H - \widetilde{S}^*_{n,\sigma}\, \widetilde{H})|_{(n)} \le e^{-\kappa_n}\,.$$

As a result of (4.25), (4.29), we obtain

$$\begin{aligned} \big|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\big(R_n^0\big)^{\ell_n^2-1}\, \widetilde{\chi}_{n+1,v}\, f\big|_{(n)} &\le |\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\, \widetilde{H}|_{(n)} + e^{-\kappa_n} \\ &\overset{(4.27)}{\le} \|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\|_n\, \frac{\kappa_n}{\ell_n} + e^{-\kappa_n}\,. \end{aligned}$$

Letting the family of functions $h_{n,v}\, \chi_{n,x}\, \widetilde{S}^*_{n,\sigma}(R_n^0)^{\ell_n^2-1}\, \widetilde{\chi}_{n+1,v}\, f$, $x \in \mathcal{S}_{n,v}$, play the role of the $(g_i)_{i\in I}$ in Lemma A.1 of the Appendix, with (1.29) we find for large L_0:

$$\text{(4.30)}\qquad \begin{aligned} &\big|\chi_{n+1,v}\, h_{n,v}\, \widetilde{S}^*_{n,\sigma}\big(R_n^0\big)^{\ell_n^2-1}\, \widetilde{\chi}_{n+1,v}\, f\big|_{(n)} \le \\ &\frac{\kappa_n}{\ell_n}\Big(\sup_{x\in\mathcal{S}_{n,v}} \|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big)\,, \end{aligned}$$

and since $\|\mathcal{L}_A\|_{n+1} \le \ell_n^\beta \|\mathcal{L}_A\|_n$, (4.24) follows. □

We now turn to the control of $\mathcal{L}_D$.

Lemma 4.3. *When L_0 is large, for $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $v \in L_{n+1}\,\mathbb{Z}^d$ with* (4.21)*, $\omega \in \Omega$:*

$$\text{(4.31)}\qquad \|\mathcal{L}_D\|_{n+1} \le \frac{\kappa_n}{\ell_n^{\beta/3}}\Big(\sup_{x\in\mathcal{S}_{n,v}} \|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big)\,.$$

Proof. Note that for $k < \ell_n^2$, cf. (4.22), and f with $|f|_{(n+1)} \le 1$, with (1.55), (1.49), $|(R_n^0)^{\ell_n^2-k-1}\, \widetilde{\chi}_{n+1,v}\, f|_{(n+1)} \le c$. Hence for $x \in \mathcal{S}_{n,v}$, repeating the construction used in Lemma 4.2, we can find $\widetilde{H}$ with $\operatorname{Supp} \widetilde{H} \subset B(x, 4D_n^*)$, $|\widetilde{H}|_{(n)} \le \kappa_n\, \ell_n^{-\beta}$ such that

$$\text{(4.32)}\qquad \big|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\big(R_n^0\big)^{\ell_n^2-k-1}\, \widetilde{\chi}_{n+1,v}\, f - \chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\, \widetilde{H}\big|_{(n)} \le e^{-\kappa_n}\,.$$

With L_0 large we thus find that

$$\begin{aligned} &\big|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\big(R^0_n\big)^{\ell_n^2-k-1}\,\widetilde{\chi}_{n+1,v}\,f\big|_{(n)} \le \\ &\frac{\kappa_n}{\ell_n^\beta}\big(\sup_{x'\in\mathcal{S}_{n,v}}\|\chi_{n,x'},\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\big)\,. \end{aligned} \tag{4.33}$$

Note also that with (1.49), (1.56), for $t \ge \alpha_n\, L^2_{n+1}/2$,

$$|P_t\, g|_{(n+1)} \le c\, |g|_\infty,\ \text{when } g \text{ is bounded measurable}\,, \tag{4.34}$$

so that for each $k \in I_D$,

$$\begin{aligned} &\big|\chi_{n+1,v}\,\big(R^0_n\big)^k\,h_{n,v}\,\widetilde{S}^*_{n,\sigma}\big(R^0_n\big)^{\ell_n^2-k-1}\,\widetilde{\chi}_{n+1,v}\,f\big|_{(n+1)} \le \\ &\frac{\kappa_n}{\ell_n^\beta}\big(\sup_{x\in\mathcal{S}_{n,v}}\|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\big)\,. \end{aligned} \tag{4.35}$$

Since $|I_D| \le \ell_n^{\frac{2}{3}\beta}$, summing over $k \in I_D$, we obtain (4.31). □

We continue with the analysis of $\mathcal{L}_C$ and $\mathcal{L}_B$. We first need to recall some facts related to Taylor's formula. For g a C^2-function on $\mathbb{R}^d$, Taylor's formula with integral remainder of order 2 states that for $y, z \in \mathbb{R}^d$:

$$g(y+z) = g(y) + \sum_{|\gamma|\le 2}\frac{1}{\gamma!}\,D^\gamma g(y)\,z^\gamma + r_g(y,z) \tag{4.36}$$

where $\gamma = (\gamma_1,\dots,\gamma_d)$ is a multi-index, $|\gamma| = \gamma_1+\dots+\gamma_d$, $\gamma! = \gamma_1!\dots\gamma_d!$, $z^\gamma = z_1^{\gamma_1}\dots z_d^{\gamma_d}$, and

$$r_g(y,z) = \int_0^1 3(1-t)^2\sum_{|\gamma|=3}\frac{1}{\gamma!}\,D^\gamma\,g(y+tz)\,z^\gamma\,dt\,, \tag{4.37}$$

and otherwise hopefully obvious notation. We recall the Definition (4.14), and the notation (1.54). Also we denote with D and $D^{(2)}$ the first and second differential of a function.

Lemma 4.4. *When L_0 is large, $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $\omega \in \Omega$, for $1 \le j \le \ell_n^2$, $x \in L_n\,\mathbb{Z}^d$, $|y-x| \le 10\sqrt{d}L_n$, f bounded measurable,*

$$\begin{aligned} \int \widetilde{S}^*_{n,\sigma}(y,dz)\big[\big(R^0_n\big)^j\,f\big](z) = {}& \widetilde{d}^*_{n,\sigma}(y,\omega)\cdot(DP_{\alpha_n\,jL_n^2}\,f)(y) \\ &+ \frac{1}{2}\,\widetilde{\gamma}^*_{n,\sigma}(y,\omega)\cdot\big(D^{(2)}P_{\alpha_n jL_n^2}\,f\big)(y) \\ &+ H_{j,f}(y), \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} |H_{j,f}(y)| \le c\Big(1+\frac{(D_n^*)^{1-\beta}}{\sqrt{j}\,L_n^{1-\beta}}\Big)\,\frac{D_n^{*3}}{j^{\frac{3}{2}}\,L_n^3}\,\Big(\frac{|f|_1}{(\sqrt{j}\,L_n)^d}\wedge|f|_\infty\Big) \\ \cdot\big(\|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\big)\,. \end{aligned} \tag{4.39}$$

Proof. With (4.36), (1.21), we can write:

$$\left(R_n^0\right)^j f(y+z) = P_{\alpha_n j L_n^2} f(y) + (D P_{\alpha_n j L_n^2} f)(y) \cdot z + \frac{1}{2}\left(D^{(2)} P_{\alpha_n j L_n^2} f\right)(y) \cdot z \otimes z + r_{j,f,y}(z),$$

and $r_{j,f,y}(\cdot - y)$ coincides in $\overline{B}(y, 3D_n^*)$ with $\widetilde{r}(\cdot)$ which is supported in $B(y, 4D_n^*)$, and such that

$$|\widetilde{r}|_{(n)} \le c\, \frac{D_n^{*3}}{j^{\frac{3}{2}} L_n^3} \left[\frac{|f|_1}{(\sqrt{j}\, L_n)^d} \wedge |f|_\infty\right]\left(1 + \frac{D_n^{*(1-\beta)}}{\sqrt{j}\, L_n^{(1-\beta)}}\right). \tag{4.40}$$

Indeed with (4.37), (1.49), (1.56):

$$\sup_{x \in B(y, 5D_n^*)} |r_{j,f,y}(x-y)| \le a_{j,n} \overset{\text{def}}{=} c\, \frac{D_n^{*3}}{j^{\frac{3}{2}} L_n^3} \left[\frac{|f|_1}{(\sqrt{j}\, L_n)^d} \wedge |f|_\infty\right], \tag{4.41}$$

and for $w, w' \in B(y, 5D_n^*)$,

$$\begin{aligned}
&|r_{j,f,y}(w-y) - r_{j,f,y}(w'-y)| \le \\
&c \sup_{0 \le t \le 1, |\gamma|=3} \left|\left(D^\gamma P_{\alpha_n j L_n^2} f\right)(y + t(w-y)) \cdot (w-y)^\gamma - \right. \\
&\left.\left(D^\gamma P_{\alpha_n j L_n^2} f\right)(y + t(w'-y)) \cdot (w'-y)^\gamma\right| \le \\
&c \sup_{\substack{z \in B(y, 5D_n^*) \\ |\gamma|=3}} \left|D^\gamma P_{\alpha_n j L_n^2} f(z)\right| D_n^{*2} |w-w'| + \\
&c|w-w'| \sup_{\substack{z \in B(y, 5D_n^*) \\ |\gamma|=4}} \left|D^\gamma P_{\alpha_n j L_n^2} f(z)\right| D_n^{*3} \overset{(1.56)}{\le} \\
&\frac{c\,|w-w'|}{D_n^*} \left(\frac{D_n^{*3}}{j^{\frac{3}{2}}\, L_n^3} + \frac{D_n^{*4}}{j^2 L_n^4}\right)\left(\frac{|f|_1}{(\sqrt{j}\, L_n)^d} \wedge |f|_\infty\right) \le \\
&c \left|\frac{w-w'}{D_n^*}\right|^\beta a_{j,n}\left(1 + \frac{D_n^*}{\sqrt{j} L_n}\right).
\end{aligned} \tag{4.42}$$

So using a cut-off function, we obtain the claim (4.40). Since $\widetilde{R}^*_{n,\sigma}(y, dz)$ is supported in $\overline{B}(y, 3D_n^*)$, cf. above (4.12),

$$\begin{aligned}
&\left|\int \widetilde{S}^*_{n,\sigma}(y, dz)(r_{j,f,y}(z-y) - \widetilde{r}(z))\right| = \\
&\left|\int R_n^0(y, dz)(r_{j,f,y}(z-y) - \widetilde{r}(z))\right| \le c a_{j,n}\, e^{-\kappa_n},
\end{aligned} \tag{4.43}$$

using Cauchy-Schwarz's inequality, (1.49), (1.56) in the last step.

Taking into account that $\chi_{n,x}(y) = 1$, and

$$H_{j,f}(y) = \int \widetilde{S}^*_{n,\sigma}(y, dz)\, r_{j,f,y}(z-y),$$

the claim (4.39) now follows from the above inequality and (4.40). □

We now decompose $\mathcal{L}_C$, cf. (4.22), into

$$\mathcal{L}_C = \mathcal{L}_C^1 + \mathcal{L}_C^2\,, \tag{4.44}$$

where in the notation of (4.14), (4.38)

$$\begin{aligned}\mathcal{L}_C^1\, f(y) = \\ \sum_{k\in I_C} \chi_{n+1,v}(y)\Big\{\big(R_n^0\big)^k\Big(h_{n,v}(\cdot)\Big[\widetilde{d}^*_{n,\sigma}(\cdot,\omega)\cdot\big(DP_{\alpha_n(\ell_n^2-k-1)}\,\widetilde{\chi}_{n+1,v}\,f\big)(\cdot) \\ + \tfrac{1}{2}\,\widetilde{\gamma}^*_{n,\sigma}(\cdot,\omega)\cdot\big(D^{(2)}P_{\alpha_n(\ell_n^2-k-1)}\,\widetilde{\chi}_{n+1,v}\,f\big)(\cdot)\Big]\Big)\Big\}\,(y)\end{aligned}$$

and

$$\mathcal{L}_C^2\, f(y) = \sum_{k\in I_C} \chi_{n+1,v}(y)\int \big(R_n^0\big)^k(y,dz)\,h_{n,v}(z)\,H_{\ell_n^2-k-1,\widetilde{\chi}_{n+1,v}\,f}(z)\,.$$

Our next step comes with

Lemma 4.5. *When L_0 is large, $\sigma\in\Sigma$, $n_0'\le n\le n_0$, $v\in L_{n+1}\,\mathbb{Z}^d$ with* (4.21)*, $\omega\in\Omega$:*

$$\|\mathcal{L}_C^2\|_{n+1} \le \frac{\kappa_n}{\ell_n^{\beta/3}}\,\Big(\sup_{x\in\mathcal{S}_{n,v}}\|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big)\,. \tag{4.45}$$

Moreover, if n is as in (4.18)*, with the notation* (4.17) *and above* (4.19)

$$\mathbb{P}\Big[G_{\sigma,n,v}\cap\Big\{\|\mathcal{L}_C^1\|_{n+1}\ge\frac{\kappa_n}{\ell_n^{\beta\wedge(\frac{d}{2}-1)}}\,v_n\Big\}\Big]\le e^{-\kappa_{n_0}}\,. \tag{4.46}$$

Proof. We begin with the proof of (4.45). We choose f with $|f|_{(n+1)}\le 1$, and deduce from (4.39) and (4.34), that

$$\|\mathcal{L}_C^2\|_{n+1} \le \sum_{k\in I_C}\kappa_n\big(\ell_n^2-k-1\big)^{-\frac{3}{2}}\Big(\sup_{x\in\mathcal{S}_{n,v}}\|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big)\,. \tag{4.47}$$

Noting that $\sum_{j\ge\ell_n^{2\beta/3}} j^{-3/2}\le c\ell_n^{-\beta/3}$, we find (4.45).

We then turn to the proof of (4.46). We further decompose $\mathcal{L}_C^1$ into

$$\mathcal{L}_C^1 = \sum_{\gamma\in\{0,1\}^d}\mathcal{L}_{C,\gamma} + \mathcal{L}_C', \tag{4.48}$$

where

$$\mathcal{L}_{C,\gamma}\, f(y) = \sum_{q\in\Lambda_\gamma,k\in I_C}\chi_{n+1,v}(y)\,\Phi_{q,k}(f)(y)\,, \tag{4.49}$$

$$\mathcal{L}_C'\, f(y) = \sum_{q\in\Lambda',k\in I_C}\chi_{n+1,v}(y)\,\Phi_{q,k}(f)(y)\,, \tag{4.50}$$

and we have used the notation for $q \in \mathbb{Z}^d$, $k \geq 0$,

$$\Phi_{q,k}(f)(y) = \tag{4.51}$$
$$\int_{B_q} P_{\alpha_n k L_n^2}(y, dz)\, h_{n,v}(z)\Big[\widetilde{d}^*_{n,\sigma}(z,\omega) \cdot \big(D P_{\alpha_n(\ell_n^2-k-1)L_n^2}\, \widetilde{\chi}_{n+1,v}\, f\big)(z)$$
$$+\ \frac{1}{2}\, \widetilde{\gamma}^*_{n,\sigma}(z,\omega) \cdot \big(D^2 P_{\alpha_n(\ell_n^2-k-1)L_n^2}\, \widetilde{\chi}_{n+1,v}\, f\big)(z)\Big],$$

$$B_q = \ 10 D_n^*(q + [0,1]^d), \tag{4.52}$$

$$\begin{aligned} \Lambda' = \ & \Big\{q \in \mathbb{Z}^d;\ B_q \cap \Big(\textstyle\bigcup_{1 \leq i \leq \tilde{\ell}}\, \overline{B}(\sigma_i, 20\sqrt{d}\, D_n^*)\Big) \neq \emptyset\Big\}\Big),\ \text{with} \\ & \sigma = (\sigma_1, \dots, \sigma_{\tilde{\ell}}),\, 0 \leq \tilde{\ell} \leq \tilde{\ell}_0\,, \end{aligned} \tag{4.53}$$

$$\begin{aligned} \Lambda_\gamma = \ & \big\{q \in \mathbb{Z}^d \backslash \Lambda';\ q_i = \gamma_i \bmod 2, \\ & \text{for } 1 \leq i \leq d,\ \text{and } B_q \cap \operatorname{Supp} h_{n,v} \neq \emptyset\big\}\,. \end{aligned} \tag{4.54}$$

Note that in view of (1.7) and (4.13), for f bounded measurable and $\gamma \in \{0,1\}^d$,

$$\{(\Phi_{q,k}(f))_{0 \leq k \leq \ell_n^2}\} \text{ are independent under } \mathbb{P}, \text{ as } q \text{ varies over } \Lambda_\gamma\,. \tag{4.55}$$

Note also that when L_0 is large, with σ, n, v as above (4.45) and $\gamma \in \{0,1\}^d$, by the properties of the support of $h_{n,v}$, cf. below (4.16),

$$|\Lambda_\gamma| \leq c \left(\frac{D_{n+1}}{D_n^*}\right)^d \leq \kappa_n\, \ell_n^d,\ \ |\Lambda'| \leq c\,. \tag{4.56}$$

We use wavelets, see (1.34), to control $\|\mathcal{L}_C^1\|_{n+1}$, and recall from Proposition A.2 in the Appendix that for $\gamma \in \{0,1\}^d$:

$$\|\mathcal{L}_{C,\gamma}\|_{n+1} \leq c \sup_{\alpha,\ell,p} \sum_{\alpha',\ell',p'} \frac{2^{\beta\ell'}}{2^{\beta\ell}}\, \frac{1}{2^{d\ell}}\, \big|\big\langle \theta_{\alpha,\ell,p},\ \mathcal{L}_{C,\gamma}\, \theta_{\alpha',\ell',p'}\big\rangle\big|\,, \tag{4.57}$$

where the supremum runs over $\alpha \in \{0,1\}^d$, $\ell \leq J_{n+1}$, cf. (A.7), $p \in \mathbb{Z}^d$, with $\alpha \neq 0$, when $\ell < J_{n+1}$, and similar constraints for α', ℓ', p' in the sum. An analogous inequality holds for $\mathcal{L}'_C$ in place of $\mathcal{L}_{C,\gamma}$. From now we consider triplets

$$\begin{aligned} & (\alpha, \ell, p), (\alpha', \ell', p') \text{ satisfying the above conditions} \\ & \text{and such that } \operatorname{Supp} \theta_{\alpha,\ell,p} \cap \operatorname{Supp} \chi_{n+1,v} \neq \emptyset, \\ & \text{and } \operatorname{Supp} \theta_{\alpha',\ell',p'} \cap \operatorname{Supp} \widetilde{\chi}_{n+1,v} \neq \emptyset\,, \end{aligned} \tag{4.58}$$

cf. below (4.16) for the notation.

Given $\gamma \in \{0,1\}^d$, we introduce an enumeration q_j, $1 \le j \le |\Lambda_\gamma|$, of Λ_γ. We then define for $0 \le j \le |\Lambda_\gamma|$

$$M_j = 2^{-d\ell} \sum_{k \in I_C, j' \le j} \big\langle \theta_{\alpha,\ell,p}, \chi_{n+1,v}\, \psi_{j',k}\big\rangle, \text{ for } j \ge 1,\ M_0 = 0\,, \tag{4.59}$$

where in the notation of (4.51),

$$\psi_{j,k}(y) \stackrel{\text{def}}{=} \Phi_{q_j,k}(\theta_{\alpha',\ell',p'})(y)\,. \tag{4.60}$$

We now bound $|M_j - M_{j-1}|$, first when $\omega \in G_{\sigma,n,v}$, cf. above (4.19), and then for a general $\omega \in \Omega$. Note that with analogous arguments as in the proof of Lemma 2.1, in view of (4.14), (4.19), (1.49), for $\omega \in G_{\sigma,n,v}$,

$$\text{for } y \in \operatorname{Supp} h_{n,v},\ |\widetilde{d}^*_{n,\sigma}(y,\omega)| \le \kappa_n\, \nu_n\, L_n,\ |\widetilde{\gamma}^*_{n,\sigma}(y,\omega)| \le \kappa_n\, \nu_n\, L_n^2\,. \tag{4.61}$$

In addition to (4.58), let us first assume that

$$2^{\ell'} \le L_n\,. \tag{4.62}$$

Then for $y, y' \in B(v, 20\sqrt{d}\, L_{n+1}) \cap \operatorname{Supp} \theta_{\alpha,\ell,p}$, $\omega \in G_{\sigma,n,v}$, $1 \le j \le |\Lambda_\gamma|$, with the help of (1.49), (1.56), (4.61), in view of (4.51), (4.60), we find when L_0 is large:

$$\begin{aligned}
&\sum_{k \in I_C} |\psi_{j,k}(y) - \psi_{j,k}(y')| \le \frac{c\,2^\ell}{L_{n+1}} \Big(\frac{D_n^*}{L_{n+1}}\Big)^d \\
&\sum_{k \in I_C} \Big[\kappa_n\, \nu_n\, L_n \big(\ell_n^2 - k - 1\big)^{-\frac12} L_n^{-1} \exp\Big\{-\frac{c|2^{\ell'} p' - 10 D_n^* q_j|^2}{(\ell_n^2 - k - 1)L_n^2}\Big\} \\
&\big(2^{\ell'}\big(\ell_n^2 - k - 1\big)^{-\frac12} L_n^{-1}\big)^d + \kappa_n\, \nu_n\, L_n^2 \big(\ell_n^2 - k - 1\big)^{-1} L_n^{-2} \\
&\exp\Big\{-\frac{c|2^{\ell'} p' - 10 D_n^* q_j|^2}{(\ell_n^2 - k - 1)L_n^2}\Big\}\big(2^{\ell'}\big(\ell_n^2 - k - 1\big)^{-\frac12} L_n^{-1}\big)^d\Big],
\end{aligned} \tag{4.63}$$

where in the expression inside the exponential we made use of (4.62), of $\operatorname{Supp} \theta_{\alpha',\ell',p'} \stackrel{(1.34)}{\subseteq} B(2^{\ell'} p', c 2^{\ell'})$, and of $(\ell_n^2 - k - 1)^{1/2} L_n \ge D_n^*$, for $k \in I_C$. Hence the left-hand side of (4.63) is smaller than:

$$\begin{aligned}
&\frac{c\,2^\ell}{L_{n+1}}\, \kappa_n\, \nu_n \Big(\frac{D_n^*}{L_{n+1}}\Big)^d \sum_{k \in I_C} \big(\ell_n^2 - k - 1\big)^{-\frac12} \\
&\exp\Big\{-\frac{c|2^{\ell'} p' - 10 D_n^* q_j|^2}{(\ell_n^2 - k - 1)L_n^2}\Big\}\big(2^{\ell'}\big(\ell_n^2 - k - 1\big)^{-\frac12} L_n^{-1}\big)^d\,.
\end{aligned} \tag{4.64}$$

Using a comparison with $\int_0^\infty s^{-\rho}\, e^{-u/s} ds$, we find that

$$\text{for } \rho > 1, u > 0,\ \sum_{1 \le k < \infty} k^{-\rho} \exp\Big\{-\frac{u}{k}\Big\} \le c(\rho)(u^{-(\rho-1)} \wedge 1)\,, \tag{4.65}$$

so that

$$\sum_{k\in I_C} \big(\ell_n^2-k-1\big)^{-\frac{(d+1)}{2}} \exp\Big\{-\frac{c\big|2^{\ell'}p'-10D_n^*\,q_j\big|^2}{\big(\ell_n^2-k-1\big)L_n^2}\Big\} \le c\Big[\Big(\frac{L_n}{\big|2^{\ell'}p'-10D_n^*q_j\big|}\Big)^{d-1}\wedge 1\Big],$$

and coming back to (4.63), (4.64), we find that for $\omega\in G_{\sigma,n,v}$, $1\le j\le|\Lambda_\gamma|$, $y,y'\in B(v,20\sqrt{d}L_{n+1})\cap \operatorname{Supp}\theta_{\alpha,\ell,p}$:

$$\sum_{k\in I_C} |\psi_{j,k}(y)-\psi_{j,k}(y')| \le \kappa_n\,\nu_n\,\frac{2^\ell}{L_{n+1}}\Big(\frac{D_n^*}{L_{n+1}}\Big)^d\Big(\frac{2^{\ell'}}{L_n}\Big)^d\Big[\Big(\frac{L_n}{|2^{\ell'}p'-10D_n^*q_j|}\Big)^{d-1}\wedge 1\Big], \tag{4.66}$$

and with entirely analogous bounds

$$\sum_{k\in I_C} |\psi_{j,k}(y)| \le \kappa_n\,\nu_n\Big(\frac{D_n^*}{L_{n+1}}\Big)^d\Big(\frac{2^{\ell'}}{L_n}\Big)^d\Big[\Big(\frac{L_n}{|2^{\ell'}p'-10D_n^*q_j|}\Big)^{d-1}\wedge 1\Big]. \tag{4.67}$$

We now replace (4.62) with:

$$L_n < 2^{\ell'} \le L_{n+1}\,. \tag{4.68}$$

We then write for $y,y'\in B(v,20\sqrt{d}L_{n+1})\cap\operatorname{Supp}\theta_{\alpha,\ell,p}$, $\omega\in G_{\sigma,n,v}$, $1\le j\le|\Lambda_\gamma|$:

(4.69)

$$\begin{aligned}\sum_{k\in I_C} |\psi_{j,k}(y)-\psi_{j,k}(y')| &\le \frac{c\,2^\ell}{L_{n+1}}\Big(\frac{D_n^*}{L_{n+1}}\Big)^d\kappa_n\,\nu_n\\ &\cdot\Big[\sum_{\substack{k\in I_C,\\ 2^{\ell'}\le(\ell_n^2-k-1)^{1/2}L_n}}\big(\ell_n^2-k-1\big)^{-\frac{(d+1)}{2}}\exp\Big\{-\frac{c|2^{\ell'}p'-10D_n^*\,q_j|^2}{(\ell_n^2-k-1)L_n^2}\Big\}\Big(\frac{2^{\ell'}}{L_n}\Big)^d+\\ &\qquad\sum_{\substack{k\in I_C,\\ 2^{\ell'}>(\ell_n^2-k-1)^{1/2}L_n}}\big(\ell_n^2-k-1\big)^{-\frac{1}{2}}\exp\Big\{-\frac{c(|2^{\ell'}p'-10D_n^*\,q_j|-c2^{\ell'})_+^2}{(\ell_n^2-k-1)L_n^2}\Big\}\Big],\end{aligned}$$

where we omitted the intermediary step, cf. (4.63), where terms corresponding to $\widetilde{d}_{n,\sigma}^*$ and $\widetilde{\gamma}_{n,\sigma}^*$ are separately bounded. Note that

$$\begin{aligned}&\sum_{\substack{k\in I_C,\\ 2^{\ell'}\le(\ell_n^2-k-1)^{1/2}L_n}}\big(\ell_n^2-k-1\big)^{-\frac{(d+1)}{2}}\le c\Big(\frac{L_n}{2^{\ell'}}\Big)^{d-1},\\ &\sum_{\substack{k\in I_C,\\ 2^{\ell'}>(\ell_n^2-k-1)^{1/2}L_n}}\big(\ell_n^2-k-1\big)^{-\frac{1}{2}}\le c\frac{2^{\ell'}}{L_n}\,.\end{aligned} \tag{4.70}$$

These inequalities together with (4.65) show that for $\omega \in G_{\sigma,n,v}$, $y, y' \in B(v, 20\sqrt{d}\, L_{n+1}) \cap \operatorname{Supp} \theta_{\alpha,\ell,p}$, $1 \le j \le |\Lambda_\gamma|$, with (4.68) we have:

$$
\begin{aligned}
&\sum_{k \in I_C} |\psi_{j,k}(y) - \psi_{j,k}(y')| \le \\
&\kappa_n\, \nu_n \Big(\frac{2^\ell}{L_{n+1}}\Big)\Big(\frac{D_n^*}{L_{n+1}}\Big)^d \Big[\Big(\frac{2^{\ell'}}{L_n}\Big)^d \Big\{\Big(\frac{L_n}{2^{\ell'}}\Big)^{d-1} \wedge \Big(\frac{L_n}{|2^{\ell'} p' - 10 D_n^* q_j|}\Big)^{d-1}\Big\} + \\
&\qquad \Big(\frac{2^{\ell'}}{L_n}\Big) \exp\Big\{- c\, \frac{\big(|2^{\ell'} p' - 10 D_n^* q_j| - c\, 2^{\ell'}\big)_+^2}{2^{2\ell'}}\Big\}\Big] \le \qquad (4.71)\\
&\kappa_n \nu_n \Big(\frac{2^\ell}{L_{n+1}}\Big)\Big(\frac{D_n^*}{L_{n+1}}\Big)^d \frac{2^{\ell'}}{L_n}\Big[1 \wedge \Big(\frac{2^{\ell'}}{|2^{\ell'} p' - 10 D_n^* q_j|}\Big)^{d-1} + \\
&\qquad \exp\Big\{- c\Big(\frac{|2^{\ell'} p' - 10 D_n^* q_j|}{2^{\ell'}}\Big)^2\Big\}\Big],
\end{aligned}
$$

and with entirely similar estimates we also have in this situation

$$
\begin{aligned}
\sum_{k \in I_C} |\psi_{j,k}(y)| \le \kappa_n\, \nu_n \Big(\frac{D_n^*}{L_{n+1}}\Big)^d \frac{2^{\ell'}}{L_n} \Big[1 \wedge \Big(\frac{2^{\ell'}}{|2^{\ell'} p' - 10 D_n^* q_j|}\Big)^{d-1} + \\
\exp\Big\{- c\Big(\frac{|2^{\ell'} p' - 10 D_n^* q_j|}{2^{\ell'}}\Big)^2\Big\}\Big].
\end{aligned} \tag{4.72}
$$

Using the fact that $\int \theta_{\alpha,\ell,p}(y)\, dy = 0$, when $\alpha \neq 0$, cf. (A.12), we see collecting (4.66), (4.67), (4.71), (4.72) that for large L_0, $\omega \in G_{\sigma,n,v}$, $\gamma \in \{0,1\}^d$, (α, ℓ, p), (α', ℓ', p') as in (4.58),

$$
|M_j - M_{j-1}| \le \delta_{\ell,\ell'}(j),\ 1 \le j \le |\Lambda_\gamma|, \tag{4.73}
$$

where up to a constant multiplicative factor, $\delta_{\ell,\ell'}(j)$ is given by the right-hand side of (4.66) when $2^{\ell'} \le L_n$, and by the last member of (4.71) when $L_n < 2^{\ell'} \le L_{n+1}$.

Observe that when we consider a general ω in place of $\omega \in G_{\sigma,n,v}$, as above, we can use analogous bounds with the only difference that (4.61) is now replaced with:

$$
\begin{aligned}
&|\widetilde{d}^*_{n,\sigma}(y,\omega)| \le \kappa_n\, L_n, \\
&|\widetilde{\gamma}^*_{n,\sigma}(y,\omega)| \le \kappa_n\, L_n^2,\ \text{ for } \sigma \in \Sigma,\ y \in \mathbb{R}^d, \omega \in \Omega\,.
\end{aligned} \tag{4.74}
$$

Hence we find that for $\omega \in \Omega$, $\gamma \in \{0,1\}^d$, (α, ℓ, p), (α', ℓ', p') as in (4.58),

$$
|M_j - M_{j-1}| \le \kappa_n\, \nu_n^{-1}\, \delta_{\ell',\ell'}(j),\ 1 \le j \le |\Lambda_\gamma|\,. \tag{4.75}
$$

Now for $\gamma \in \{0,1\}^d$, (α, ℓ, p), (α', ℓ', p') as in (4.58), we introduce the conditional probability:

$$
\widetilde{\mathbb{P}}(\cdot) = \mathbb{P}\big[\ \cdot \mid |M_j - M_{j-1}| \le \delta_{\ell,\ell'}(j),\ 1 \le j \le |\Lambda_\gamma|\big], \tag{4.76}
$$

and denote with $\widetilde{\mathbb{E}}$ the corresponding expectation. We note that thanks to the independence under $\mathbb{P}$ of the increments $M_j - M_{j-1}$, $1 \le j \le |\Lambda_\gamma|$, cf. (4.55), these increments are independent under $\widetilde{\mathbb{P}}$ as well. We will now bound

$$\Delta_j \stackrel{\text{def}}{=} \widetilde{\mathbb{E}}[M_j - M_{j-1}],\ 1 \le j \le |\Lambda_\gamma|\,. \tag{4.77}$$

First note that for $y \in \bigcup_{q\in\Lambda_\gamma} B_q$, cf. (4.52), (4.54), with L_0 large, we can replace $R^*_{n_0',\sigma}$ with $\widetilde{R}_{n_0'}$ in the right-hand side of (4.11), when calculating $\widetilde{d}^*_{n,\sigma}(y,\omega), \widetilde{\gamma}^*_{n,\sigma}(y,\omega)$ in (4.14). So by isotropy, cf. (1.12), for $y \in \bigcup_{q\in\Lambda_\gamma} B_q$:

$$\mathbb{E}[\widetilde{d}^*_{n,\sigma}(y,\omega)] = 0\,. \tag{4.78}$$

Moreover for y in the same set, with $1 \le i, j \le d$, we have

$$\mathbb{E}[(\widetilde{\gamma}^*_{n,\sigma})^{i,j}(y,\omega)] \stackrel{(1.25)}{=} \mathbb{E}\big[(\widetilde{\gamma}^*_{n,\sigma})^{i,j}(y,\omega) - (\widetilde{\gamma}_n)^{i,j}(y,\omega)\big]\,. \tag{4.79}$$

On the event where for all $x \in L_n\,\mathbb{Z}^d$ with Supp $\chi_{n,x} \cap B(y, 3D^*_n) \neq \emptyset$, $x \in \widetilde{B}_n(\omega)$, and all $x' \in L_{n_0'}\,\mathbb{Z}^d$ with Supp $\chi_{n_0',x} \cap B(y, 3D^*_n) \neq \emptyset$, $x' \in \mathcal{B}_{n_0'}(\omega)$, the integrand in the right-hand side of (4.79), using the remark above (4.78), the strong Markov property, and the localization estimate in (2.2), is bounded in absolute value by

$$c\, D_n^{*2}\left(\left(\frac{L_n}{L_{n_0'}}\right)^2 e^{-\kappa_{n_0'}} + e^{-\kappa_n}\right) \le e^{-\kappa_{n_0}},\ \text{with } L_0 \text{ large}\,.$$

Bounding with (1.47) the probability of the complement of this event, we see that for large L_0, $\gamma \in \{0,1\}^d$, $y \in \bigcup_{q\in\Lambda_\gamma} B_q$,

$$\begin{aligned} |\mathbb{E}[\widetilde{\gamma}^*_{n,\sigma}(y,\omega)]| &\le c\, D_n^{*2}\left[\left(\frac{D_n^*}{L_{n_0'}}\right)^d L_{n_0'}^{-M_0} + \kappa_n\, L_n^{-M_0}\right] + e^{-\kappa_{n_0}} \\ &\le \kappa_{n_0}\, L_{n_0}^{(2+d)-M_0(1+a)^{-(m_0+1)}} \stackrel{(1.46)}{\le} L_{n_0}^{-10}\,. \end{aligned} \tag{4.80}$$

We then observe that the bounds we derived below (4.61) until (4.73) show that when $1 \le j \le |\Lambda_\gamma|$, with κ_n as in (4.61), (α, ℓ, p), (α', ℓ', p') as in (4.58),

$$\begin{aligned} &\text{if } |\widetilde{d}^*_{n,\sigma}(y,\omega)| \le \kappa_n\, \nu_n\, L_n,\ \ |\widetilde{\gamma}^*_{n,\sigma}(y,\omega)| \le \kappa_n\, \nu_n L_n^2, \\ &\text{for all } y \in B_{q_j}, \text{then} |M_j - M_{j-1}| \le \delta_{\ell,\ell'}(j)\,. \end{aligned} \tag{4.81}$$

Hence on the event $\{|M_j - M_{j-1}| > \delta_{\ell,\ell'}(j)\}$, for some $y \in B_{q_j}$, (4.81) does not hold, and by the remark above (4.78), we can replace σ with $\emptyset (\in \Sigma)$, when negating (4.81). We thus find with (4.18) that when L_0 is large, for

$\gamma \in \{0,1\}^d$, (α,ℓ,p), (α',ℓ',p') with (4.58), for $1 \le j \le |\Lambda_\gamma|$:

$$\mathbb{P}[|M_j - M_{j-1}| > \delta_{\ell,\ell'}(j)] \le c\,\frac{|B_{q_j}|}{L_n^d}\, L_{n_0}^{-2} \overset{(4.52)}{\le} \kappa_n\, L_{n_0}^{-2}\,. \tag{4.82}$$

Coming back to (4.78), (4.80), to replace (4.61), the estimates (4.61) until (4.73) now show that with $1 \le j \le |\Lambda_\gamma|$:

$$|\mathbb{E}[M_j - M_{j-1}]| \le \big(\kappa_n\,\nu_n\,L_n^2\big)^{-1} L_{n_0}^{-10}\,\delta_{\ell,\ell'}(j) \le L_{n_0}^{-10}\,\delta_{\ell,\ell'}(j) \tag{4.83}$$

and noting that

$$\begin{aligned}\mathbb{E}[M_j - M_{j-1}] \overset{(4.77)}{=} \Delta_j\,\mathbb{P}\big[|M_j - M_{j-1}| \le \delta_{\ell,\ell'}(j)\big] + \\ \mathbb{E}\big[M_j - M_{j-1}, |M_j - M_{j-1}| > \delta_{\ell,\ell'}(j)\big]\,,\end{aligned}$$

we obtain from (4.75), (4.82), (4.83), that for $\gamma \in \{0,1\}^d$, (α,ℓ,p), (α',ℓ',p') as in (4.58), $1 \le j \le |\Lambda_j|$:

$$|\Delta_j| \le 2\big(L_{n_0}^{-10} + \kappa_n\,L_{n_0}^{-2}\,\nu_n^{-1}\big)\,\delta_{\ell,\ell'}(j) \le L_{n_0}^{-1}\,\delta_{\ell,\ell'}(j) \overset{\text{def}}{=} \widetilde{\delta}_{\ell,\ell'}(j)\,. \tag{4.84}$$

Observe that under $\widetilde{\mathbb{P}}$, $M_{|\Lambda_\gamma|} - \sum_{j=1}^{|\Lambda_\gamma|} \Delta_j$ is a sum of $|\Lambda_\gamma|$ independent variables respectively bounded by $2\,\delta_{\ell,\ell'}(j)$. Note also that when $2^{\ell'} \le L_n$, by (4.66), (4.67)

$$\begin{aligned}\Big(\sum_{1\le j\le|\Lambda_\gamma|} \delta_{\ell,\ell'}(j)^2\Big)^{\frac12} &\le \kappa_n\,\nu_n\,\frac{2^\ell}{L_{n+1}}\,\Big(\frac{D_n^*}{L_{n+1}}\Big)^d\,\Big(\frac{2^{\ell'}}{L_n}\Big)^d \\ &\le \kappa_n\,\nu_n\,\frac{2^\ell}{L_{n+1}}\,\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d \overset{\text{def}}{=} \sigma_n(\ell,\ell')\,,\end{aligned} \tag{4.85}$$

whereas for $L_n < 2^{\ell'} \le L_{n+1}$, with (4.71), (4.72)

$$\begin{aligned}\Big(\sum_{1\le j\le|\Lambda_\gamma|} \delta_{\ell',\ell'}(j)^2\Big)^{\frac12} &\le \kappa_n\,\nu_n\,\frac{2^\ell}{L_{n+1}}\,\Big(\frac{D_n^*}{L_{n+1}}\Big)^d\,\frac{2^{\ell'}}{L_n}\,\Big[\Big(\frac{2^{\ell'}}{D_n^*}\Big)^d + \Big(\frac{2^{\ell'}}{D_n^*}\Big)^d\Big]^{\frac12} \\ &\le \kappa_n\,\nu_n\,\ell_n^{-d}\,\frac{2^\ell}{L_{n+1}}\,\Big(\frac{2^{\ell'}}{L_n}\Big)^{\frac{d}{2}+1} \overset{\text{def}}{=} \sigma_n(\ell,\ell')\,.\end{aligned} \tag{4.86}$$

Note also that when L_0 is large, for $\ell,\ell' \le J_{n+1}$, $\gamma \in \{0,1\}^d$:

$$\begin{aligned}\sum_{1\le j\le|\Lambda_\gamma|} \widetilde{\delta}_{\ell,\ell'}(j) &\underset{\text{Cauchy-Schwarz}}{\overset{(4.84)}{\le}} |\Lambda_\gamma|^{\frac12}\,L_{n_0}^{-1}\,\Big(\sum_{1\le j\le|\Lambda_\gamma|} \sigma_{\ell,\ell'}(j)^2\Big)^{\frac12} \\ &\overset{(4.56),(1.14)}{\le} \frac12\,\sigma_n(\ell,\ell')\,.\end{aligned} \tag{4.87}$$

We thus see that for $u \geq \sigma_n(\ell, \ell')$, with a slight variation of Azuma's inequality, cf. [1], or [30], p. 308,

$$\begin{aligned}
&\mathbb{P}\big[|M_{|\Lambda_\gamma|}| \geq u,\ G_{\sigma,n,v}\big] \leq \widetilde{\mathbb{P}}\big[|M_{|\Lambda_\gamma|}| \geq u\big] \leq \\
&\widetilde{\mathbb{P}}\big[|M_{|\Lambda_\gamma|} - \sum_{1\leq j\leq |\Lambda_\gamma|} \Delta_j| \geq u \ - \sum_{1\leq j\leq |\Lambda_\gamma|} \widetilde{\delta}_{\ell,\ell'}(j)\big] \leq \\
&2\exp\Big\{-\frac{1}{32}\Big(\frac{u}{\sigma_n(\ell,\ell')}\Big)^2\Big\}.
\end{aligned} \tag{4.88}$$

If we define for $\gamma \in \{0,1\}^d$ the event

$$\begin{aligned}
G_{\sigma,n,v,C,\gamma} = G_{\sigma,n,v} \cap \Big\{&\text{for } (\alpha,\ell,p), (\alpha',\ell',p'), \ \text{as in (4.58)}, \\
&\frac{1}{2^{d\ell}}\big|\langle\theta_{\alpha,\ell,p}, \mathcal{L}_{C,\gamma}\,\theta_{\alpha',\ell',p'}\rangle\big| \leq \\
&\sigma_n(\ell,\ell')(1+\ell_- + \ell'_-)\, e^{(\log\log L_n)^2}\Big\},
\end{aligned} \tag{4.89}$$

(ℓ_-, ℓ'_- denote the respective negative parts of ℓ, ℓ'), we see that when L_0 is large,

$$\begin{aligned}
&\mathbb{P}\big[G_{\sigma,n,v}\backslash G_{\sigma,n,v,C,\gamma}\big] \leq \\
&\sum_{\ell,\ell'\leq J_{n+1}} c\Big(\frac{D_{n+1}}{2^\ell}\Big)^d \Big(\frac{\widetilde{D}_{n+1}}{2^{\ell'}}\Big)^d \exp\Big\{-\frac{1}{32}\, e^{2(\log\log L_n)^2}(1+\ell_-+\ell'_-)^2\Big\} \leq \\
&c\Big(\sum_{\ell\leq J_{n+1}} \Big(\frac{\widetilde{D}_{n+1}}{2^\ell}\Big)^d \exp\Big\{-\frac{1}{64}\, e^{2(\log\log L_n)^2}(1+\ell_-^2)\Big\}\Big)^2 \leq e^{-\kappa n_0}.
\end{aligned} \tag{4.90}$$

Observe that on $G_{\sigma,n,v,C,\gamma}$ in view of (4.57) one has

$$\|\mathcal{L}_{C,\gamma}\|_{n+1} \leq \Gamma' \stackrel{\text{def}}{=} c \sup_{\alpha,\ell,p} \sum_{\alpha',\ell',p'} \frac{2^{\beta\ell'}}{2^{\beta\ell}}\, \sigma_n(\ell,\ell')\, e^{(\log\log L_n)^2}(1+\ell_-+\ell'_-)\,, \tag{4.91}$$

with (α, ℓ, p), (α', ℓ', p') varying over the set described in (4.58). We now write:

$$\Gamma' \leq \Gamma'_1 + \Gamma'_2\,, \tag{4.92}$$

where Γ'_1 corresponds to the expression in the right-hand side of (4.91) with $2^{\ell'} \leq L_n$, and Γ'_2 to the expression with $L_n < 2^{\ell'} \leq L_{n+1}$. We thus see that for large L_0,

$$\begin{aligned}
\Gamma'_1 &\stackrel{(4.85)}{\leq} \kappa_n\, \nu_n \sup_{2^\ell\leq L_{n+1}} \frac{2^\ell}{L_{n+1}} \sum_{2^{\ell'}\leq L_n, p'} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d (1+\ell_-+\ell'_-)\, \frac{2^{\beta\ell'}}{2^{\beta\ell}} \\
&\leq \kappa_n\, \nu_n \sup_{2^\ell\leq L_{n+1}} \Big(\frac{2^\ell}{L_{n+1}}\Big)^{1-\beta}(1+\ell_-) \sum_{2^{\ell'}\leq L_n} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^{\beta}(1+\ell'_-) \\
&\leq \frac{\kappa_n\, \nu_n}{\ell_n^\beta}\,.
\end{aligned} \tag{4.93}$$

On the other hand, (recall $\ell'_- = 0$, when $L_n < 2^{\ell'} \le L_{n+1}$):

$$\begin{aligned}
\Gamma'_2 &\overset{(4.86)}{\le} \kappa_n\, \nu_n \sup_{2^\ell \le L_{n+1}} \frac{2^\ell}{L_{n+1}}\,(1+\ell_-) \sum_{L_n < 2^{\ell'} \le L_{n+1}, p'} \ell_n^{-d} \Big(\frac{2^{\ell'}}{L_n}\Big)^{\frac{d}{2}+1} \frac{2^{\beta\ell'}}{2^{\beta\ell}} \\
&\le \kappa_n\, \nu_n \sup_{2^\ell \le L_{n+1}} \Big(\frac{2^\ell}{L_{n+1}}\Big)^{1-\beta} (1+\ell_-) \\
&\quad \cdot \sum_{L_n < 2^{\ell'} \le L_{n+1}} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^{\beta} \ell_n^{-d} \Big(\frac{2^{\ell'}}{L_n}\Big)^{\frac{d}{2}+1} \Big(\frac{\widetilde{D}_{n+1}}{2^{\ell'}}\Big)^{d} \qquad (4.94)\\
&\le \kappa_n\, \nu_n \sup_{2^\ell \le L_{n+1}} \Big(\frac{2^\ell}{L_{n+1}}\Big)^{1-\beta} (1+\ell_-) \sum_{L_n < 2^{\ell'} \le L_{n+1}} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^{\beta} \Big(\frac{2^{\ell'}}{L_n}\Big)^{-\frac{d}{2}+1} \\
&\le \frac{\kappa_n\, \nu_n}{\ell_n^{\beta \wedge (\frac{d}{2}-1)}}\,.
\end{aligned}$$

Combining (4.93), (4.94), we see that when L_0 is large, for $\gamma \in \{0,1\}^d$, on $G_{\sigma,n,v,C,\gamma}$:

$$\|\mathcal{L}_{C,\gamma}\|_{n+1} \le \frac{\kappa_n\, \nu_n}{\ell_n^{\beta \wedge (\frac{d}{2}-1)}}\,. \qquad (4.95)$$

We now turn to the study of $\mathcal{L}'_C$. Keeping in mind that $|\Lambda'| \le c$, cf. (4.53), using similar estimates as in (4.66), (4.67), (4.71), (4.72), we see that for large L_0, with (α, ℓ, p), (α', ℓ', p') as in (4.58), and for $\omega \in G_{\sigma,n,v}$:

$$\begin{aligned}
&\frac{1}{2^{d\ell}} \left|\langle \theta_{\alpha,\ell,p}, \mathcal{L}'_C\, \theta_{\alpha',\ell',p'}\rangle\right| \le \\
&\begin{cases} \kappa_n\, \nu_n \dfrac{2^\ell}{L_{n+1}} \Big(\dfrac{D^*_n}{L_{n+1}}\Big)^d \Big(\dfrac{2^{\ell'}}{L_n}\Big)^d, \ \text{for } 2^{\ell'} \le L_n\,, \\ \kappa_n\, \nu_n \dfrac{2^\ell}{L_{n+1}} \Big(\dfrac{D^*_n}{L_{n+1}}\Big)^d \dfrac{2^{\ell'}}{L_n}, \ \text{for } L_n < 2^{\ell'} \le L_{n+1}\,. \end{cases}
\end{aligned} \qquad (4.96)$$

By direct inspection in (4.85), (4.86), we see that the above right-hand side is bounded by $\kappa_n\, \sigma_n(\ell, \ell')$. Hence the analogous bound as in (4.57), for $\mathcal{L}'_C$, as well as (4.91)–(4.94), now prove that when L_0 is large, for $\omega \in G_{\sigma,n,v}$:

$$\|\mathcal{L}'_C\|_{n+1} \le \frac{\kappa_n\, \nu_n}{\ell_n^{\beta \wedge (\frac{d}{2}-1)}}\,. \qquad (4.97)$$

Collecting (4.90), (4.95), (4.97), we have completed the proof of (4.46). □

We continue with the analysis of $\mathcal{L}_B$. In analogy with (4.44), and with I_B replacing I_C there, we write:

$$\mathcal{L}_B = \mathcal{L}^1_B + \mathcal{L}^2_B\,, \qquad (4.98)$$

Lemma 4.6. *When L_0 is large, $\sigma \in \Sigma$, $n'_0 \le n \le n_0$, $v \in L_{n+1}\,\mathbb{Z}^d$ with* (4.21)*, $\omega \in \Omega$:*

$$\big\|\mathcal{L}_B^2\big\|_{n+1} \le \frac{\kappa_n}{\ell_n}\,\Big(\sup_{x\in\mathcal{S}_{n,v}} \|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big)\,. \tag{4.99}$$

Moreover if n is as in (4.18) *with the notation* (4.17) *and above* (4.19),

$$\mathbb{P}\Big[G_{\sigma,n,v} \cap \Big\{\big\|\mathcal{L}_B^1\big\|_{n+1} \ge \frac{\kappa_n\,\nu_n}{\ell_n^{(1-\beta)\wedge(\frac{d}{2}-1)}}\Big\}\Big] \le e^{-\kappa_{n_0}}\,. \tag{4.100}$$

Proof. We begin with the proof of (4.99). Note that with (1.49), (1.56), for g bounded measurable,

$$|\chi_{n+1,v}\,P_{\alpha_n k L_n^2}\,g|_{(n+1)} \le \frac{c\,\ell_n}{\sqrt{k}}\,|g|_\infty,\ \text{for } 1 \le k \le \ell_n^2\,, \tag{4.101}$$

hence with (4.39) we find

$$\begin{aligned}\big\|\mathcal{L}_B^2\big\|_{n+1} &\le \sum_{k\in I_B} \frac{c\,\ell_n}{\sqrt{k}}\,\frac{\kappa_n}{\ell_n^3}\,\Big(\sup_{x\in\mathcal{S}_{n,v}} \|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big)\\ &\le \frac{\kappa_n}{\ell_n}\Big(\sup_{x\in\mathcal{S}_{n,v}} \|\chi_{n,x}\,\widetilde{S}^*_{n,\sigma}\|_n + e^{-\kappa_n}\Big)\,.\end{aligned} \tag{4.102}$$

This proves (4.99).

We continue with the proof of (4.100). In analogy to (4.48), and with I_B replacing I_C there, we decompose $\mathcal{L}_B^1$ into:

$$\mathcal{L}_B^1 = \sum_{\gamma\in\{0,1\}^d} \mathcal{L}_{B,\gamma} + \mathcal{L}'_B\,, \tag{4.103}$$

For $\gamma \in \{0,1\}^d$, (α,ℓ,p), (α',ℓ',p') satisfying (4.58), we introduce in full analogy with (4.59), with I_B replacing I_C there, M_j, $0 \le j \le |\Lambda_\gamma|$. With the Definition (4.60), we observe that for large L_0, when

$$2^\ell \le L_n\,, \tag{4.104}$$

for $y, y' \in B(v, 20\sqrt{d}\,L_{n+1}) \cap \operatorname{Supp}\theta_{\alpha,\ell,p}$, $\omega \in G_{\sigma,n,v}$, $1 \le j \le |\Lambda_j|$, with the help of (1.56), (4.61),

$$\begin{aligned}&\sum_{k\in I_B} |\psi_{j,k}(y) - \psi_{j,k}(y')| \le\\ &\sum_{k\in I_B} c\,\frac{D_n^{*d}}{L_n^d}\,\frac{1}{k^{\frac{d+1}{2}}}\,\frac{|y-y'|}{L_n}\exp\Big\{-\frac{c\,A_j(y,y')^2}{k\,L_n^2}\Big\}\,\frac{\kappa_n\,\nu_n}{\ell_n}\,\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d,\end{aligned} \tag{4.105}$$

with

(4.106)
$$A_j(y,y') = \inf\big\{|w-\widetilde{w}|,\ w \in B_{q_j},\ \widetilde{w} = \lambda y + (1-\lambda)y',\ 0 \le \lambda \le 1\big\}\,.$$

As a result of (4.65), under the above assumptions:

$$\sum_{k\in I_B} |\psi_{j,k}(y)-\psi_{j,k}(y')| \le \frac{\kappa_n\,\nu_n}{\ell_n}\Big[\Big(\frac{L_n}{A_j(y,y')}\Big)^{d-1}\wedge 1\Big]\,\frac{|y-y'|}{L_n}\,, \tag{4.107}$$

and by an analogous calculation

$$\sum_{k\in I_B} |\psi_{j,k}(y)| \le \frac{\kappa_n\,\nu_n}{\ell_n}\Big[\Big(\frac{L_n}{A_j(y)}\Big)^{d-2}\wedge 1\Big], \text{ with } A_j(y)\stackrel{\text{def}}{=} d(y,B_{q_j})\,. \tag{4.108}$$

If we now turn to the case

$$L_n < 2^\ell \le L_{n+1}\,, \tag{4.109}$$

under the same conditions as stated above (4.105), we find

$$\begin{aligned}\sum_{k\in I_B,\sqrt{k}L_n>2^\ell} |\psi_{j,k}(y)-\psi_{j,k}(y')| &\le \frac{\kappa_n\,\nu_n}{\ell_n}\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d\,\Big|\frac{y-y'}{L_n}\Big| \\ &\quad\cdot \sum_{2^\ell<\sqrt{k}L_n\le L_{n+1}} k^{-\frac{(d+1)}{2}}\;\exp\Big\{-\frac{c\,A_j(y,y')^2}{k\,L_n^2}\Big\}.\end{aligned} \tag{4.110}$$

Note that one has the following refinement of (4.65):

$$\sum_{v<k} k^{-\rho}\exp\Big\{-\frac{u}{k}\Big\} \le c(\rho)\big\{(u\vee v)^{-(\rho-1)}\wedge 1\big\},\ \text{ for } u,v>0,\ \rho>1\,, \tag{4.111}$$

that is obtained by considering the case $u=0$, and using (4.65). Hence for large L_0, when (4.109) holds, for $y,y'\in B(v,20\sqrt{d}L_{n+1})\cap \operatorname{Supp}\theta_{\alpha,\ell,p}$, $\omega\in G_{\sigma,n,v}$, $1\le j\le|\Lambda_\gamma|$:

$$\begin{aligned}&\sum_{k\in I_B,\sqrt{k}L_n>2^\ell} |\psi_{j,k}(y)-\psi_{j,k}(y')| \le \\ &\frac{\kappa_n\,\nu_n}{\ell_n}\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d\,\frac{|y-y'|}{L_n}\,\Big\{\Big(\frac{L_n}{2^\ell\vee A_j(y,y')}\Big)^{d-1}\wedge 1\Big\}\end{aligned} \tag{4.112}$$

and in an analogous fashion:

$$\sum_{k\in I_B,\sqrt{k}L_n>2^\ell} |\psi_{j,k}(y)| \le \frac{\kappa_n\,\nu_n}{\ell_n}\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d\,\Big\{\Big(\frac{L_n}{2^\ell\vee A_j(y)}\Big)^{d-2}\wedge 1\Big\}. \tag{4.113}$$

On the other hand with (4.60), (4.51):

$$
\begin{aligned}
&\sum_{k\in I_B,\sqrt{k}L_n\le 2^\ell} \frac{1}{2^{\ell d}} \int_{\mathrm{Supp}\,\theta_{\alpha,\ell,p}} |\psi_{j,k}(y)|\,dy \le \\
&\frac{\kappa_n\,\nu_n}{\ell_n} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d \sum_{k\in I_B,\sqrt{k}L_n\le 2^\ell} \frac{1}{2^{\ell d}} \int_{B_{q_j}} dz \int_{B(2^\ell p, c2^\ell)} dy \\
&\frac{c}{(k\,L_n^2)^{d/2}}\ \exp\Big\{-\frac{c(z-y)^2}{k\,L_n^2}\Big\} \le \\
&\frac{\kappa_n\,\nu_n}{\ell_n} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d \Big(\frac{2^\ell}{L_n}\Big)^2 \Big(\frac{D_n^*}{2^\ell}\Big)^d \exp\Big\{-c\Big(\frac{A_j(2^\ell p)}{2^\ell}\Big)^2\Big\}.
\end{aligned}
\tag{4.114}
$$

Collecting our bounds, we thus see that when L_0 is large, for $\gamma \in \{0,1\}^d$, (α,ℓ,p), (α',ℓ',p') as in (4.58), $\omega \in G_{\sigma,n,v}$, $1 \le j \le |\Lambda_\gamma|$:

$$
|M_j - M_{j-1}| \le \delta_{\ell,p,\ell'}(j) \tag{4.115}
$$

where for $2^\ell \le L_n$, $2^{\ell'} \le L_{n+1}$, $1 \le j \le |\Lambda_\gamma|$:

$$
\begin{aligned}
\delta_{\ell,p,\ell'}(j) = \frac{\kappa_n\,\nu_n}{\ell_n} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d \Big[&\frac{2^\ell}{L_{n+1}} \Big\{\Big(\frac{L_n}{A_{j,\ell,p}}\Big)^{d-2} \wedge 1\Big\} + \\
&\frac{2^\ell}{L_n}\Big\{\Big(\frac{L_n}{A_{j,\ell,p}}\Big)^{d-1} \wedge 1\Big\}\Big],
\end{aligned}
\tag{4.116}
$$

where

$$
A_{j,\ell,p} = \inf\{|w-\widetilde{w}|,\ w \in B_{q_j},\ \widetilde{w} \in B(2^\ell p, c\,2^\ell)\}\,,
$$

with c such that $\mathrm{Supp}\ \theta_\alpha(\cdot) \subseteq B(0,c)$, for all $\alpha \in \{0,1\}^d$, and we have made use of the fact that since $2^\ell \le L_n$, $\alpha \ne 0$, and in view of (A.12), $\int \theta_{\alpha,\ell,p}(y)dy = 0$.

On the other hand when $L_n < 2^\ell \le L_{n+1}$, $2^{\ell'} \le L_{n+1}$, $1 \le j \le |\Lambda_\gamma|$:

$$
\begin{aligned}
\delta_{\ell,p,\ell'}(j) = \frac{\kappa_n\,\nu_n}{\ell_n} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d \Big[&\frac{2^\ell}{L_{n+1}} \Big(\frac{L_n}{2^\ell \vee A_{j,\ell,p}}\Big)^{d-2} + \frac{2^\ell}{L_n}\Big(\frac{L_n}{2^\ell \vee A_{j,\ell,p}}\Big)^{d-1} + \\
&\Big(\frac{L_n}{2^\ell}\Big)^{d-2} \exp\Big\{-c\Big(\frac{A_{j,\ell,p}}{2^\ell}\Big)^2\Big\}\Big].
\end{aligned}
\tag{4.117}
$$

Arguing as above (4.75), we see that when L_0 is large, for $\gamma \in \{0,1\}^d$, (α,ℓ,p), (α',ℓ',p') as in (4.58), for $\omega \in \Omega$, $1 \le j \le |\Lambda_\gamma|$:

$$
|M_j - M_{j-1}| \le \kappa_n\,\nu_n^{-1}\,\delta_{\ell,p,\ell'}(j)\,. \tag{4.118}
$$

Keeping the same notation $\widetilde{\mathbb{P}}$ and Δ_j, $1 \le j \le |\Lambda_\gamma|$, as in (4.76), (4.77), with the only difference that $\delta_{\ell,p,\ell'}(j)$ replaces $\delta_{\ell,\ell'}(j)$ in (4.76), repeating the argument leading to (4.84), we see that for large L_0, under the same conditions as above (4.118)

$$|\Delta_j| \le L_{n_0}^{-1}\, \delta_{\ell,p,\ell'}(j) \stackrel{\text{def}}{=} \widetilde{\delta}_{\ell,p,\ell'}(j),\ 1 \le j \le |\Lambda_\gamma|\,. \tag{4.119}$$

Keeping in mind the objective of deriving bounds that parallel (4.88), we now bound $(\sum_{1\le j\le|\Lambda_\gamma|} \delta^2_{\ell,p,\ell'}(j))^{1/2}$. To this end note first that for $2^\ell \le L_n$, p, ℓ' compatible with (4.58), cf. (4.116),

$$\begin{aligned}&\Big(\sum_{1\le j\le|\Lambda_\gamma|} \delta_{\ell,p,\ell'}(j)^2\Big)^{\frac12} \le \\ &\frac{\kappa_n\,\nu_n}{\ell_n}\,\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d \Big[\frac{2^\ell}{L_{n+1}}\Big(\sum_{1\le j\le|\Lambda_\gamma|}\Big(\frac{L_n}{A_{j,\ell,p}}\Big)^{2(d-2)} \wedge 1\Big)^{\frac12} + \\ &\qquad\Big(\frac{2^\ell}{L_n}\Big)\Big(\sum_{1\le j\le|\Lambda_\gamma|}\Big(\frac{L_n}{A_{j,\ell,p}}\Big)^{2(d-1)} \wedge 1\Big)^{\frac12}\Big].\end{aligned} \tag{4.120}$$

Observe that with (4.54), and the notation below (4.116),

(4.121)

$$\begin{aligned}\text{i)}\ &\sum_{1\le j\le|\Lambda_\gamma|}\Big(\frac{L_n}{A_{j,\ell,p}}\Big)^{2(d-2)} \wedge 1 \le \kappa_n\,\ell_n^{2\nu(d)},\ \text{with } \nu(d) = \frac12,\ \text{when } d=3\,,\\ &\hspace{20em} = 0,\ \text{when } d\ge 4\,,\\ \text{ii)}\ &\sum_{1\le j\le|\Lambda_\gamma|}\Big(\frac{L_n}{A_{j,\ell,p}}\Big)^{2(d-1)} \wedge 1 \le c\,.\end{aligned}$$

As a result we obtain that for $2^\ell \le L_n$, p, ℓ' compatible with (4.58):

$$\Big(\sum_{1\le j\le|\Lambda_\gamma|} \delta_{\ell,p,\ell'}(j)^2\Big)^{\frac12} \le \sigma_n(\ell,\ell') \stackrel{\text{def}}{=} \frac{\kappa_n\,\nu_n}{\ell_n}\,\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d\,\frac{2^\ell}{L_n}\,. \tag{4.122}$$

To handle the case $L_n < 2^\ell \le L_{n+1}$, observe that:

(4.123)

$$\begin{aligned}\text{i)}\ &\sum_{1\le j\le|\Lambda_\gamma|} \exp\Big\{-c\Big(\frac{A_{j,\ell,p}}{2^\ell}\Big)^2\Big\} \le c\,\Big(\frac{2^\ell}{L_n}\Big)^d\\ \text{ii)}\ &\sum_{1\le j\le|\Lambda_\gamma|}\Big(\frac{L_n}{2^\ell \vee A_{j,\ell,p}}\Big)^{2(d-1)} \le c\,\Big(\frac{L_n}{2^\ell}\Big)^{2(d-1)}\Big(\frac{2^\ell}{D_n^*}\Big)^d + c\,\Big(\frac{L_n}{2^\ell}\Big)^{d-2}\\ &\hspace{15em}\le c\,\Big(\frac{L_n}{2^\ell}\Big)^{d-2}\\ \text{iii)}\ &\sum_{1\le j\le|\Lambda_\gamma|}\Big(\frac{L_n}{2^\ell \vee A_{j,\ell,p}}\Big)^{2(d-2)} \le c\,\Big(\frac{L_n}{2^\ell}\Big)^{2(d-2)}\Big(\frac{2^\ell}{D_n^*}\Big)^d + \\ &\hspace{15em}\sum_{c2^\ell < iD_n^* \le c\widetilde{D}_{n+1}} c\,i^{-(d-3)}\\ &\hspace{15em}\le \kappa_n\,\ell_n^{2\nu(d)},\ \text{with the notation of (4.121)}\,.\end{aligned}$$

Coming back to (4.117), we obtain for $L_n < 2^\ell \le L_{n+1}$, p, ℓ' compatible with (4.58):

$$\Big(\sum_{1\le j\le|\Lambda_\gamma|} \delta_{\ell,p,\ell'}(j)^2\Big)^{\frac{1}{2}} \le$$

$$\frac{\kappa_n\, \nu_n}{\ell_n}\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d\Big[\Big(\frac{2^\ell}{L_{n+1}}\Big)\,\ell_n^{\nu(d)} + \frac{2^\ell}{L_n}\Big(\frac{L_n}{2^\ell}\Big)^{\frac{d}{2}-1} + \Big(\frac{L_n}{2^\ell}\Big)^{\frac{d}{2}-2}\Big] \le \tag{4.124}$$

$$\frac{\kappa_n\, \nu_n}{\ell_n}\Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d\Big[\Big(\frac{2^\ell}{L_{n+1}}\Big)\,\ell_n^{\nu(d)} + \Big(\frac{2^\ell}{L_n}\Big)^{\nu(d)}\Big] \overset{\text{def}}{=} \sigma_n(\ell,\ell')\,.$$

The same argument leading to (4.87), (4.88) shows that when L_0 is large, $\ell, \ell' \le J_{n+1}$, $\gamma \in \{0,1\}^d$:

$$\sum_{1\le j\le|\Lambda_\gamma|} \widetilde{\delta}_{\ell,p,\ell'}(j) \le \frac{1}{2}\,\sigma_n(\ell,\ell')\,, \tag{4.125}$$

and for $u \ge \sigma_n(\ell,\ell')$,

$$\mathbb{P}\big[|M_{|\Lambda_\gamma|}| \ge u, G_{\sigma,n,v}\big] \le 2\,\exp\Big\{-\frac{1}{32}\Big(\frac{u}{\sigma_n(\ell,\ell')}\Big)^2\Big\}\,. \tag{4.126}$$

We can now introduce for $\gamma \in \{0,1\}^d$ the event

$$\begin{aligned} G_{\sigma,n,v,B,\gamma} = G_{\sigma,n,v} \cap \Big\{&\text{for } (\alpha,\ell,p), (\alpha',\ell',p') \text{ as in (4.58)}\,,\\ &\frac{1}{2^{d\ell}}\,\big|\big\langle\theta_{\alpha,\ell,p}, \mathcal{L}^1_{B,\gamma}\theta_{\alpha',\ell',p'}\big\rangle\big| \le\\ &\sigma_n(\ell,\ell')(1+\ell_- + \ell'_-)\,e^{(\log\log L_n)^2}\Big\}\,, \end{aligned} \tag{4.127}$$

and find that when L_0 is large, for $\gamma \in \{0,1\}^d$, similarly to (4.90),

$$\mathbb{P}[G_{\sigma,n,v}\backslash G_{\sigma,n,v,B,\gamma}] \le e^{-\kappa_{n_0}}\,. \tag{4.128}$$

Moreover on the event $G_{\sigma,n,v,B,\gamma}$, we have

$$\big\|\mathcal{L}^1_{B,\gamma}\big\|_{n+1} \le \Gamma \overset{\text{def}}{=} c\,\sup_{\alpha,\ell,p}\,\sum_{\alpha',\ell',p'}\,\frac{2^{\beta\ell'}}{2^{\beta\ell}}\,\sigma_n(\ell,\ell')(1+\ell_-+\ell'_-)\,e^{(\log\log L_n)^2}\,, \tag{4.129}$$

with $(\alpha,\ell,p), (\alpha',\ell',p')$ varying over the set described in (4.58). We now write:

$$\Gamma \le \Gamma_1 \vee \Gamma_2\,, \tag{4.130}$$

with Γ_1 defined as Γ with the additional requirement $2^\ell \le L_n$, and Γ_2 with the additional requirement $L_n < 2^\ell \le L_{n+1}$, instead. With (4.122), we find for large L_0:

$$\begin{aligned}
\Gamma_1 &\le \frac{\kappa_n\,\nu_n}{\ell_n} \sup_{2^\ell \le L_n} \sum_{\alpha',\ell',p'} \Big(\frac{2^{\ell'}}{L_{n+1}}\Big)^d \Big(\frac{2^\ell}{L_n}\Big)(1+\ell_- + \ell'_-)\,\frac{2^{\beta\ell'}}{2^{\beta\ell}} \\
&\le \frac{\kappa_n\,\nu_n}{\ell_n} \sup_{2^\ell \le L_n} \Big(\frac{2^\ell}{L_n}\Big)^{1-\beta}(1+\ell_-) \sum_{2^{\ell'} \le L_{n+1}} (1+\ell'_-)\Big(\frac{2^{\ell'}}{L_n}\Big)^\beta \\
&\le \frac{\kappa_n\,\nu_n}{\ell_n^{(1-\beta)}}\,,
\end{aligned} \tag{4.131}$$

whereas with (4.124), we find, (recall $\ell_- = 0$, when $L_n < 2^\ell \le L_{n+1}$):

$$\begin{aligned}
\Gamma_2 &\le \frac{\kappa_n\,\nu_n}{\ell_n} \sup_{L_n < 2^\ell \le L_{n+1}} \sum_{2^{\ell'} \le L_{n+1}} \frac{2^{\beta\ell'}}{2^{\beta\ell}} \Big[\frac{2^\ell}{L_{n+1}}\,\ell_n^{\nu(d)} + \Big(\frac{2^\ell}{L_n}\Big)^{\nu(d)}\Big](1+\ell'_-) \\
&\le \frac{\kappa_n\,\nu_n}{\ell_n} \sup_{L_n < 2^\ell \le L_{n+1}} \Big[\ell_n^{\nu(d)}\Big(\frac{2^\ell}{L_{n+1}}\Big)^{1-\beta} + \Big(\frac{L_{n+1}}{2^\ell}\Big)^\beta \Big(\frac{2^\ell}{L_n}\Big)^{\nu(d)}\Big] \\
&\le \frac{\kappa_n\,\nu_n}{\ell_n^{1-(\beta\vee\nu(d))}} = \frac{\kappa_n\,\nu_n}{\ell_n^{(1-\beta)\wedge(\frac{d}{2}-1)}}\,.
\end{aligned} \tag{4.132}$$

Coming back to (4.129), we see that when L_0 is large, for $\gamma \in \{0,1\}^d$, on $G_{\sigma,n,v,B,\gamma}$, we have

$$\big\|\mathcal{L}^1_{B,\gamma}\big\|_{n+1} \le \frac{\kappa_n\,\nu_n}{\ell_n^{(1-\beta)\wedge(\frac{d}{2}-1)}}\,. \tag{4.133}$$

We now turn to the study of $\mathcal{L}'_B$. Keeping in mind that $|\Lambda'| \le c$, cf. (4.53), using similar estimates as in (4.115), (4.116), (4.117), we see that for large L_0, with (α,ℓ,p), (α',ℓ',p') as in (4.58), and for $\omega \in G_{\sigma,n,v}$:

$$\begin{aligned}
&2^{-d\ell}\big|\big\langle \theta_{\alpha,\ell,p}, \mathcal{L}'_B\,\theta_{\alpha',\ell',p'}\big\rangle\big| \le \\
&\begin{cases}
\dfrac{\kappa_n\,\nu_n}{\ell_n}\Big(\dfrac{2^{\ell'}}{L_{n+1}}\Big)^d \dfrac{2^\ell}{L_n}, & \text{if } 2^\ell \le L_n\,, \\[2ex]
\dfrac{\kappa_n\,\nu_n}{\ell_n}\Big(\dfrac{2^\ell}{L_{n+1}}\Big)^d \Big(\dfrac{2^\ell}{L_{n+1}} + \Big(\dfrac{L_n}{2^\ell}\Big)^{d-2}\Big), & \text{if } L_n < 2^\ell \le L_{n+1}.
\end{cases}
\end{aligned} \tag{4.134}$$

By direct inspection, cf. (4.122), (4.124), we see that the right-hand side above is bounded by $\kappa_n\,\sigma_n(\ell,\ell')$. Hence the analogous bound to (4.129) for $\mathcal{L}'_B$, as well as (4.131), (4.132) show that when L_0 is large, for $\omega \in G_{\sigma,n,v}$:

$$\|\mathcal{L}'_B\|_{n+1} \le \frac{\kappa_n\,\nu_n}{\ell_n^{(1-\beta)\wedge(\frac{d}{2}-1)}}\,. \tag{4.135}$$

Combining (4.128), (4.133), (4.135), we have proved (4.100). □

Collecting Lemmas 4.2, 4.3, 4.5, 4.6, we see that we have proved Proposition 4.1. □

Remark 4.7. As a result of Lemmas 4.2, 4.3, 4.5, 4.6, we see that with high probability on $G_{\sigma,n,v}$, $\|\widetilde{\mathcal{L}}_{\sigma,n,v}\|_{n+1}$ is smaller than $\kappa_n\, \nu_n$ by the crucial contraction factor $\ell_n^{-\beta/3\wedge(1-\beta)\wedge(d/2+1)}$ ($= \ell_n^{-\beta/3}$, with our choice $\beta \in (0, \frac{1}{2}]$ in (1.13)). In the proof of Proposition 4.1, there is an asymmetry in the role of k close to 0 and k close to $\ell_n^2 - 1$ in the decomposition (4.22), which stems from the use of Taylor's formula to second order, cf. (4.38). In a loose sense, if the $\widetilde{S}^*_{n,\sigma}$ in the definition of $\widetilde{\mathcal{L}}_{\sigma,n,v}$ in (4.16) had been centered under $\mathbb{P}$, we could have avoided Taylor's expansion, and chosen in (4.22), $I_A = \{0\}$, $I_B = \{k : 0 < k \le \ell_n^2/2\}$, $I_C = \{k : \ell_n^2/2 < k < \ell_n^2 - 1\}$, $I_D = \{\ell_n^2 - 1\}$. With the proper assumptions, the role of $\ell_n^{-\beta/3\wedge(1-\beta)\wedge(d/2-1)}$ would then have been replaced with $\ell_n^{-\beta\wedge(1-\beta)\wedge(\frac{d}{2}-1)}$, displaying a higher symmetry between the role of small k and k close to $\ell_n^2 - 1$. Ultimately the asymmetry in the proof results from the fact that we work with $\widetilde{S}_n$ which compares $\widetilde{R}_n$ to the Gaussian kernel R_n^0, rather than separately analyzing $\widetilde{R}_n - \mathbb{E}[\widetilde{R}_n]$ and $\mathbb{E}[\widetilde{R}_n] - R_n^0$. □

Our next objective, see the comments above (4.10), is to control $\|h_n(S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma})\|_n$. To this end we introduce the event, cf. (4.2):

$$\widetilde{G} = G \cap \bigcap_{n_0' < n \le n_0} \big\{\omega \in \Omega;\ L_n\, \mathbb{Z}^d \cap \widetilde{\mathcal{B}}_n(\omega)^c \cap (5\mathcal{T}_{n_0+1}) \text{ is contained in the union of at most } \tilde{\ell}_0 \text{ open balls with radius } 3\widetilde{D}_n \text{ and center in } L_n\, \mathbb{Z}^d\big\}\,. \tag{4.136}$$

The same estimates as in (4.3), show that for large L_0,

$$\mathbb{P}[\widetilde{G}^c] \le (n_0 - n_0' + 1)(100(m_0+2))^{-1}\, L_{n_0+1}^{-M_0} \le \frac{1}{100}\, L_{n_0+1}^{-M_0}\,. \tag{4.137}$$

It is also convenient for $\sigma \in \Sigma$, $\omega \in \Omega$, to introduce the laws $P^\sigma_{y,\omega}$, $y \in \mathbb{R}^d$, of the canonical Markov chain on $(\mathbb{R}^d)^{\mathbb{N}}$, with transition kernel $R^*_{n_0',\sigma}$, cf. (4.7). We denote with $E^\sigma_{y,\omega}$ the corresponding expectation and with Z_k, $k \ge 0$, the canonical process on $(\mathbb{R}^d)^{\mathbb{N}}$. So for instance for bounded measurable f and $n \in [n_0', n_0]$, $y \in \mathbb{R}^d$, in view of (4.11),

$$\widetilde{R}^*_{n,\sigma} f(y) = E^\sigma_{y,\omega}\Big[\sum_{0\le m<k_n} \prod_{0\le k<m} \psi_{n,y}(Z_k)(1-\psi_{n,y}(Z_m))\, f(Z_m) + \prod_{0\le k<k_n} \psi_{n,y}(Z_k)\, f(Z_{k_n})\Big]\,, \tag{4.138}$$

with $k_n = (L_n/L_{n_0'})^2$.

Lemma 4.8. *When L_0 is large, for $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $y \in \{d(\cdot, \mathrm{Supp}\, h_n) \le 50\sqrt{d} L_n\}$, $x \in L_n \mathbb{Z}^d \cap \{d(\cdot, \mathrm{Supp}\, h_n) \le 20\sqrt{d} L_n\}$, $\omega \in \widetilde{G}$,*

$$P^\sigma_{y,\omega}\Big[\sup_{0 \le k \le k_n} |Z_k - Z_0| \ge 30 \tilde{\ell}_0 \widetilde{D}_n\Big] \le e^{-\kappa_{n_0}}, \tag{4.139}$$

$$\|\chi_{n,x}(S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma})\|_n \le e^{-\kappa_{n_0}}, \tag{4.140}$$

and

$$\|\chi_{n,x}\, S^*_{n,\sigma}\|_n \le c\, L_n^\beta. \tag{4.141}$$

Proof. We begin with the proof of (4.139). The case $n = n_0'$ is obvious since $k_n = 1$, and the steps of $Z_.$ have length at most $\widetilde{D}_{n_0'}$, $P^\sigma_{y,\omega}$-a.s. cf. (4.7). Since $\omega \in \widetilde{G}$, we can find a collection $w_i \in L_n \mathbb{Z}^d$, $1 \le i \le \tilde{\ell}_0$, with $B(w_i, 3\widetilde{D}_n) \cap 5\mathcal{T}_{n_0+1} \neq \emptyset$, such that

$$\widetilde{\mathcal{B}}_n(\omega) \supseteq \big((5\mathcal{T}_{n_0+1}) \cap L_n \mathbb{Z}^d\big) \setminus \bigcup_{1 \le i \le \tilde{\ell}_0} B(w_i, 3\widetilde{D}_n). \tag{4.142}$$

Let us write $\sigma = (\sigma_1, \dots, \sigma_{\tilde{\ell}})$, where $0 \le \tilde{\ell} \le \tilde{\ell}_0$, and introduce the open set

$$U = \Big(\bigcup_{1 \le i \le \tilde{\ell}_0} B(w_i, 6\widetilde{D}_n)\Big) \cup \Big(\bigcup_{1 \le i \le \tilde{\ell}} B(\sigma_i, 6\widetilde{D}_n)\Big).$$

Since $P^\sigma_{y,\omega}$-a.s., $Z_.$ has steps of length at most $\widetilde{D}_{n_0'}$, and U is a union of at most $2\tilde{\ell}_0$ balls of radius $6\widetilde{D}_n$, using a connectedness argument we see that $P^\sigma_{y,\omega}$-a.s., on the event $\bigcap_{0 \le k \le k_n}\{Z_k \in U\}$, one has $\sup_{0 \le k \le k_n} |Z_k - Z_0| \le 7 \times 2\tilde{\ell}_0 \widetilde{D}_n$. Therefore $P^\sigma_{y,\omega}$-a.s., on the event in (4.139), $Z_.$ exits U before times k_n. If we now define:

$$\begin{aligned} &\tau = \inf\{k \ge 0;\ \inf_z d(Z_k, z) \ge 4\widetilde{D}_n\}, \\ &(z \text{ runs over } \{w_1, \dots, w_{\tilde{\ell}_0}, \sigma_1, \dots, \sigma_\ell\}) \end{aligned} \tag{4.143}$$

we see that the probability in (4.139) is smaller than:

$$E^\sigma_{y,\omega}\Big[\tau < k_n,\ P^\sigma_{Z_\tau,\omega}\Big[\sup_{0 \le k \le k_n} |Z_k - Z_0| > \frac{\widetilde{D}_n}{2}\Big]\Big], \tag{4.144}$$

where we have used the strong Markov property. With our choice of y, see also below (4.8), we see that $P^\sigma_{y,\omega}$-a.s., on $\{\tau < k_n\}$, $d_\infty(Z_\tau, (5\mathcal{T}_{n_0+1})^c) \ge L^2_{n+1} - c\, L_n - (L_n/L_{n_0'})^2\, \widetilde{D}_{n_0'} \ge \widetilde{D}_n + 2\widetilde{D}_{n_0'}$, when L_0 is large.

So in view of (4.6), with the notation (1.18), we obtain that $P^\sigma_{y,\omega}$-a.s., on $\{\tau < k_n\}$,

$$P^\sigma_{Z_\tau,\omega}\Big[\sup_{0 \le k \le k_n} |Z_k - Z_0| > \frac{\widetilde{D}_n}{2}\Big] \le P_{Z_\tau,\omega}\Big[X^*_{L_n^2} > \frac{\widetilde{D}_n}{2}\Big] \overset{(2.2),(4.142)}{\le} e^{-\kappa_n}.$$

Coming back to (4.144), we obtain (4.139).

We now prove (4.140). Once again the case $n = n_0'$ is immediate since $\widetilde{R}^*_{n_0',\sigma}$ coincides with $R^*_{n_0',\sigma}$. We thus assume $n_0' < n \le n_0$, and choose f with $|f|_{(n)} \le 1$, $\omega \in \widetilde{G}$. With large L_0, we see that, cf. (4.9), (4.138), for x as in (4.140), $y \in \mathbb{R}^d$,

(4.145)

$$\chi_{n,x}(y)(S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma})\, f(y) \stackrel{\text{def}}{=} \chi_{n,x}(y)\,\Delta_n\, f(y), \text{ with}$$
$$\Delta_n\, f(y) = E^\sigma_{y,\omega}\Big[f(Z_{k_n}) - \sum_{0\le m<k_n} \prod_{0\le k<m} \psi_{n,y}(Z_k)(1-\psi_{n,y}(Z_m))\, f(Z_m) - \prod_{0\le k<k_n} \psi_{n,y}(Z_k)\, f(Z_{k_n})\Big],$$

and hence by the choice of $\psi_{n,y}$, cf. (4.10),

$$|\chi_{n,x}(y)(S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma})\, f(y)| \le 2\chi_{n,x}(y)\, P^\sigma_{y,\omega}\Big[\sup_{0\le k\le k_n} |Z_k - Z_0| \ge D^*_n\Big]$$
$$\stackrel{(4.139)}{\le} e^{-\kappa n_0}\,. \tag{4.146}$$

Then for y, y' in $\{d(\cdot, \text{Supp}\,\chi_{n,x}) \le L_n\}$, we see that when $|y-y'| \ge e^{-\kappa n_0}$,

$$|\chi_{n,x}(y)\Delta_n\, f(y) - \chi_{n,x}(y')\,\Delta_n\, f(y')| \le e^{-\kappa n_0} \le \left|\frac{y-y'}{L_n}\right|^\beta e^{-\kappa n_0}\,. \tag{4.147}$$

We thus consider y, y' in $\{d(\cdot, \text{Supp}\,\chi_{n,x}) \le L_n\}$, with

$$|y-y'| \le e^{-\kappa n_0}\,, \tag{4.148}$$

and write in analogy with (2.51):

$$|\Delta_n\, f(y) - \Delta_n\, f(y')| \le a_1 + a_2, \text{ where} \tag{4.149}$$

$$a_1 = \Big|E^\sigma_{y',\omega}\Big[\sum_{0\le m<k_n} \prod_{0\le k<m} \psi_{n,y'}(Z_k)(1-\psi_{n,y'}(Z_m))\, f(Z_m) + \prod_{0\le k<k_n} \psi_{n,y'}(Z_k)\, f(Z_{k_n}) - \sum_{0\le m<k_n} \prod_{0\le k<m} \psi_{n,y}(Z_k)(1-\psi_{n,y}(Z_m))\, f(Z_m) - \prod_{0\le k<k_n} \psi_{n,y}(Z_k)\, f(Z_{k_n})\Big]\Big|$$

and with hopefully obvious notation

$$a_2 = \Big|\Big(E^\sigma_{y,\omega} - E^\sigma_{y',\omega}\Big)\Big[f(Z_{k_n}) - \sum_{0\le m<k_n} \psi_{n,y}(Z_k)(1-\psi_{n,y}(Z_m))\, f(Z_m) - \prod_{0\le k<k_n} \psi_{n,y}(Z_k)\, f(Z_{k_n})\Big]\Big|.$$

In view of (4.10), $|\psi_{n,y}(\cdot) - \psi_{n,y'}(\cdot)| \le |y - y'|$, and we see that with (4.148) and (1.13),

$$a_1 \le \big(k_n^2 + k_n\big)|y - y'| \le \big(k_n^2 + k_n\big) e^{-\kappa_{n_0}} \Big|\frac{y - y'}{L_n}\Big|^\beta \le e^{-\kappa_{n_0}} \Big|\frac{y - y'}{L_n}\Big|^\beta , \tag{4.150}$$

using (4.148), and (1.13). Then using the fact that, cf. (4.6), (4.7),

$$R^*_{n_0',\sigma} = (1 - g_\sigma)\, \widetilde{R}^0_{n_0'} + g_\sigma\, \widetilde{R}_{n_0'} \,,$$

we can write

$$\begin{aligned} R^*_{n_0',\sigma} &= A + B, \text{ with } A = (1 - g_\sigma)\, R^0_{n_0'} + g_\sigma\, R_{n_0'}, \ \text{ and} \\ &\qquad B = (1 - g_\sigma)\,(\widetilde{R}^0_{n_0'} - R^0_{n_0'}) + g_\sigma(\widetilde{R}_{n_0'} - R_{n_0'})\,. \end{aligned} \tag{4.151}$$

With (1.60), (1.29), (2.46), we find

$$\begin{aligned} &\text{i)}\ \ \|A\|_{L^\infty \to (n)} \le \Big(\frac{L_n}{L_{n_0'}}\Big)^\beta \|A\|_{L^\infty \to (n_0')} \le c\, L_n^\beta, \ \text{ and} \\ &\text{ii)}\ \|B\|_n \le \Big(\frac{L_n}{L_{n_0'}}\Big)^\beta \|B\|_{n_0'} \le e^{-\kappa_{n_0}}\,. \end{aligned} \tag{4.152}$$

Denoting with $g(\cdot)$ the function $\chi_{L_n}(\cdot - y)$, cf. (1.37), we have

$$a_2 = \Big|R^*_{n_0',\sigma}\big(g\, E^\sigma_{\cdot,\omega}[H]\big)(y) - R^*_{n_0',\sigma}\big(g\, E^\sigma_{\cdot,\omega}[H]\big)(y')\Big| , \tag{4.153}$$

where $|H| \le 2\, 1_{\{\sup_{0 \le k \le k_n} |Z_k - Z_0| \ge D^*_n/2\}}$, and

$$\begin{aligned} E^\sigma_{z,\omega}[H] = (R^*_{n_0',\sigma})^{k_n - 1}\, f(z) &- \\ \sum_{0 \le m < k_n - 1} (\psi_{n,y} R^*_{n_0',\sigma})^m (1 - \psi_{n,y})\, f(z) &- (\psi_{n,y} R^*_{n_0',\sigma})^{k_n - 1} f(z)\,. \end{aligned}$$

Using (4.151) in (4.153), as well as (4.152) i), we thus find

$$\begin{aligned} a_2 &\le \Big|\frac{y - y'}{L_n}\Big|^\beta c\, L_n^\beta \sup_{z \in B(y, 2L_n)} P^\sigma_{z,\omega}\Big[\sup_{0 \le k \le k_n} |Z_k - Z_0| \ge \frac{D^*_n}{2}\Big] + a_2' \\ &\overset{(4.139)}{\le} \Big|\frac{y - y'}{L_n}\Big|^\beta e^{-\kappa_{n_0}} + a_2', \ \text{ where} \\ a_2' &= |B(gE_{\cdot,\omega}[H])(y) - B(gE_{\cdot,\omega}[H])(y')|\,. \end{aligned} \tag{4.154}$$

In view of (4.152) ii), (4.147)–(4.150), the claim (4.140) will follow once we show that

$$|g\, E_{\cdot,\omega}[H]|_{(n)} \le c\, L_n^\beta\, k_n\,. \tag{4.155}$$

To this end observe that for $m \ge 1$, with (4.151), using perturbation expansion

$$(R^*_{n_0',\sigma})^m = B^m + \sum_{0 \le m' < m} B^{m'} A (R^*_{n_0',\sigma})^{m-m'-1}, \tag{4.156}$$

so that with (4.152)

$$\big|(R^*_{n_0',\sigma})^{k_n-1} f\big|_{(n)} \le \|B\|_n^{k_n-1} + \sum_{0 \le m' < k_n-1} \|B\|_n^{m'} c\, L_n^\beta \le c\, L_n^\beta .$$

Analogously, we see that with $0 \le m < k_n - 1$,

$$\begin{aligned} \big|(\psi_{n,y} R^*_{n_0',\sigma})^m (1-\psi_{n,y}) f\big|_{(n)} \le \|\psi_{n,y} B\|_n^m |1-\psi_{n,y}|_{(n)} + \\ \sum_{0 \le m' < m} \|\psi_{n,y} B\|_n^{m'} c\, L_n^\beta \\ \overset{(4.10),(4.152)}{\le} c\, L_n^\beta, \text{ and} \\ |(\psi_{n,y} R^*_{n_0',\sigma})^{k_n-1} f|_{(n)} \le c\, L_n^\beta . \end{aligned}$$

The claim (4.155) follows, and this finishes the proof of (4.140).

Let us finally prove (4.141). For large L_0, $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $\omega \in \widetilde{G}$, and x as in (4.141), as a result of (4.8), (4.9):

$$\chi_{n,x} S^*_{n,\sigma} = \chi_{n,x} (R^*_{n_0',\sigma})^{k_n} - \chi_{n,x} R_n^0 .$$

Using (4.156) and (4.152), the claim (4.141) immediately follows. □

Keeping in mind the expansion (4.15), it is convenient to modify (4.16), and introduce for $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $v \in L_{n+1}\mathbb{Z}^d$ the operator

$$\mathcal{L}_{\sigma,n,v} = \sum_{0 \le k < \ell_n^2} \chi_{n+1,v} \big(R_n^0\big)^k h_n S^*_{n,\sigma} \big(R_n^0\big)^{\ell_n^2-k-1} . \tag{4.157}$$

As an application of the previous lemma we have

Lemma 4.9. *When L_0 is large, for $\sigma \in \Sigma$, $n_0' \le n \le n_0$, $v \in L_{n+1}\mathbb{Z}^d$, $\omega \in \widetilde{G}$*

$$\|\mathcal{L}_{\sigma,n,v} - \widetilde{\mathcal{L}}_{\sigma,n,v}\|_{n+1} \le e^{-\kappa_{n_0}} . \tag{4.158}$$

Proof. We write, (recall that $h_{n,v}(\cdot) = \chi_{D_{n+1}}(\cdot - v)\, h_n(\cdot)$),

$$\begin{aligned} &\mathcal{L}_{\sigma,n,v} - \widetilde{\mathcal{L}}_{\sigma,n,v} = \mathcal{L}^1 + \mathcal{L}^2 + \mathcal{L}^3, \text{ with} \\ &\mathcal{L}^1 = \sum_{0 \le k < \ell_n^2} \chi_{n+1,v} \big(R_n^0\big)^k (h_n - h_{n,v}) S^*_{n,\sigma} \big(R_n^0\big)^{\ell_n^2-k-1} \\ &\mathcal{L}^2 = \sum_{0 \le k < \ell_n^2} \chi_{n+1,v} \big(R_n^0\big)^k h_{n,v} (S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma}) \big(R_n^0\big)^{\ell_n^2-k-1} \\ &\mathcal{L}^3 = \sum_{0 \le k < \ell_n^2} \chi_{n+1,v} \big(R_n^0\big)^k h_{n,v} \widetilde{S}^*_{n,\sigma} \big(R_n^0\big)^{\ell_n^2-k-1} (1 - \widetilde{\chi}_{n+1,v}) . \end{aligned} \tag{4.159}$$

Keeping in mind (4.140), (4.141), together with (1.55), (1.56), (1.49), (1.29), we see that

$$\|\mathcal{L}^1\|_n \le \ell_n^2\, c\, e^{-\kappa_{n_0}}\; c\, L_n^\beta \le e^{-\kappa_{n_0}}, \quad \|\mathcal{L}^2\|_n \le \ell_n^2\, c\, e^{-\kappa_{n_0}} \le e^{-\kappa_{n_0}}\,.$$

Noting that $h_{n,v}\, \widetilde{S}^*_{n,\sigma}\, g = -h_{n,v}\, R_n^0\, g$, when g is supported in $B(v, 3D_{n+1})^c$, with L_0 large, we also find

$$\|\mathcal{L}^3\|_n \le \ell_n^2\, c\, L_n^\beta\, e^{-\kappa_{n_0}} \le e^{-\kappa_{n_0}}\,.$$

Since we also have $\|\mathcal{L}^i\|_{n+1} \le \ell_n^\beta \|\mathcal{L}^i\|_n$, for $i = 1, 2, 3$, the claim (4.158) follows. □

Proposition 4.10. *When L_0 is large, for $n_0' \le n \le n_0$,* (4.18) *is satisfied.*

Proof. We use induction over $n \in [n_0', n_0]$. First observe that with the notation (4.5) and in analogy with (4.3)

$$\begin{aligned}\mathbb{P}[G_\emptyset] &\ge 1 - c\left(\frac{L^2_{n_0+1}}{L_{n_0'}}\right)^d L_{n_0'}^{-M_0} \ge 1 - c\, L_{n_0+1}^{2d - M_0(1+a)^{-(m_0+2)}}\\ &\overset{(1.46)}{\ge} 1 - c\, L_{n_0+1}^{-98d}\,.\end{aligned}$$

Hence with (4.137), we find for large L_0

$$\mathbb{P}[G_{\emptyset,n_0'}] \ge 1 - L_{n_0+1}^{-97d}, \text{ with } G_{\emptyset,n_0'} \overset{\text{def}}{=} G_\emptyset \cap \widetilde{G}\,. \tag{4.160}$$

We introduce the notation

$$\mathcal{S}_n = L_n\, \mathbb{Z}^d \cap \{d(\cdot, \operatorname{Supp} h_n) \le 20\sqrt{d}\, L_n\}, \text{ for } n_0' \le n < n_0\,. \tag{4.161}$$

Note for later use that with the notation (4.19), for $n_0' \le n < n_0$,

$$\mathcal{S}_{n+1} \subseteq \left\{v \in L_{n+1}\, \mathbb{Z}^d;\ \mathcal{S}_{n,v} \ne \emptyset\right\} = \{v \in L_{n+1}\, \mathbb{Z}^d;\ h_{n,v} \not\equiv 0\}\,. \tag{4.162}$$

Further when L_0 is large, for all $\omega \in G_{\emptyset,n_0'}$, $x \in \mathcal{S}_{n_0'}$, with (4.7)

$$\|\chi_{n_0',x}\, \widetilde{S}^*_{n_0',\emptyset}\|_{n_0'} = \|\chi_{n_0',x}\, \widetilde{S}_{n_0'}\|_{n_0'} \overset{(2.2)}{\le} \nu_{n_0'}\,, \tag{4.163}$$

and for all $y \in [0, L_{n_0'}]^d$, using (2.2), (2.4)

$$\begin{aligned}&\left|\frac{\widetilde{d}^*_{n_0',\emptyset}}{L_{n_0'}}\,(y,\omega)\right|\left(= \left|\frac{\widetilde{d}_{n_0'}}{L_{n_0'}}\,(y,\omega)\right|\right) \le \nu_{n_0'},\\ &\left|\frac{\widetilde{\gamma}^*_{n_0',\emptyset}}{L^2_{n_0'}}\,(y,\omega)\right|\left(= \left|\frac{\widetilde{\gamma}_{n_0'}}{L^2_{n_0'}}\,(y,\omega)\right|\right) \le \nu_{n_0'}\,.\end{aligned} \tag{4.164}$$

Let us assume that for n_1 with $n_0' \le n_1 < n_0$, we have a decreasing sequence of events $G_{\emptyset,n}$, $n_0' \le n \le n_1$, such that for $n_0' \le n < n_1$

$$\mathbb{P}[G_{\emptyset,n} \backslash G_{\emptyset,n+1}] \le e^{-\kappa_{n_0}}\,, \tag{4.165}$$

and for $\omega \in G_{\emptyset,n}$, $x \in \mathcal{S}_n$, (4.163), (4.164) hold with n in place of n_0' (the expressions in parenthesis in (4.164) being now disregarded). With (4.160), we see that (4.18) is satisfied with $n = n_1$, and with (4.20) of Proposition 4.1, where we have set $G_{\emptyset,n_1,v} \equiv G_{\emptyset,n_1}$, we obtain since $\mathcal{S}_{n_1,v} \subseteq \mathcal{S}_{n_1}$, for all $v \in L_{n+1}\,\mathbb{Z}^d$,

$$\begin{aligned}&\mathbb{P}\Big[G_{\emptyset,n_1} \cap \Big\{ \sup_{v \in L_{n_1+1}\mathbb{Z}^d : \mathcal{S}_{n_1,v} \neq \emptyset} \|\widetilde{\mathcal{L}}_{\emptyset,n_1,v}\|_{n_1+1} > \frac{\kappa_{n_1}\,\nu_{n_1}}{\ell_{n_1}^{\beta/3}}\Big\}\Big] \le \\ &c\,\Big(\frac{L^2_{n_0+1}}{L_{n_1+1}}\Big)^d\, e^{-\kappa_{n_0}} \le e^{-\kappa_{n_0}}\,.\end{aligned} \tag{4.166}$$

We then define

$$G_{\emptyset,n_1+1} = G_{\emptyset,n_1} \cap \Big\{ \sup_{v \in L_{n_1+1}\mathbb{Z}^d : \mathcal{S}_{n_1,v} \neq \emptyset} \|\widetilde{\mathcal{L}}_{\emptyset,n_1,v}\|_{n_1+1} \le \frac{\kappa_{n_1}\,\nu_{n_1}}{\ell_{n_1}^{\beta/3}}\Big\}\,, \tag{4.167}$$

and note from the above that (4.165) is true for $n = n_1$. Then with Lemma 4.9, since $G_{\emptyset,n_1+1} \subseteq \widetilde{G}$, we have for $\omega \in G_{\emptyset,n_1+1}$

$$\sup_{v \in L_{n_1+1}\mathbb{Z}^d : \mathcal{S}_{n_1,v} \neq \emptyset} \|\mathcal{L}_{\emptyset,n_1,v}\|_{n_1+1} \le 2\,\frac{\kappa_{n_1}\,\nu_{n_1}}{\ell_{n_1}^{\beta/3}}\,. \tag{4.168}$$

Coming back to (4.15), we see that for $\omega \in G_{\emptyset,n_1+1}$, $v \in \mathcal{S}_{n_1+1}$:

(4.169)

$$\begin{aligned}&\|\chi_{n_1+1,v}\, S^*_{n_1+1,\emptyset}\|_{n_1+1} \le \|\mathcal{L}_{\emptyset,n_1,v}\|_{n_1+1} + \\ &\Big\| \sum_{\substack{k_0+\dots+k_m+m=\ell^2_{n_1}\\ k_i \ge 0, m \ge 2}} \chi_{n_1+1,v}\big(R^0_{n_1}\big)^{k_0} h_{n_1}\, S^*_{n_1,\emptyset}\big(R^0_{n_1}\big)^{k_1} \dots h_{n_1}\, S^*_{n_1,\emptyset}\big(R^0_{n_1}\big)^{k_m} \Big\|_{n_1+1} + \\ &c\,\Big\| P_{\alpha_{n_1} L^2_{n_1+1}} - P_{\alpha_{n_1+1} L^2_{n_1+1}} \Big\|_{n_1+1} \stackrel{\text{def}}{=} a_1 + a_2 + a_3\,.\end{aligned}$$

With (4.162), from (4.168) we find

$$a_1 \le \kappa_{n_1}\,\nu_{n_1}\,\ell_{n_1}^{-\beta/3}\,. \tag{4.170}$$

Then with (4.163), with n_1 in place of n_0', (A.3) of the Appendix, and (4.140), we see that for $\omega \in G_{\emptyset,n_1+1} \subseteq G_{\emptyset,n_1}$:

$$\|h_{n_1}\, S^*_{n_1,\emptyset}\|_{n_1} \le \|h_{n_1}\, \widetilde{S}^*_{n_1,\emptyset}\|_{n_1} + e^{-\kappa_{n_0}} \le c\,\nu_{n_1} + e^{-\kappa_{n_0}} \le c_3\,\nu_{n_1}\,. \tag{4.171}$$

As a result with the help of (1.55) and the fact that $\|\cdot\|_{n+1} \le \ell_n^\beta \|\cdot\|_n$, we obtain

$$
(4.172)\qquad
\begin{aligned}
a_2 &\le c\,\ell_{n_1}^\beta \sum_{\substack{k_0+\dots+k_m+m=\ell_{n_1}^2\\ k_i\ge 0, m\ge 2}} (c_3\,\nu_{n_1})^m \\
&= c\,\ell_{n_1}^\beta\big[(1+c_3\,\nu_{n_1})^{\ell_{n_1}^2} - 1 - \ell_{n_1}^2\,c_3\,\nu_{n_1}\big] \\
&\le c\,\ell_{n_1}^{\beta+4}\,\nu_{n_1}^2\,\exp\big\{c\,\nu_{n_1}\,\ell_{n_1}^2\big\} \overset{(1.14),(1.40),(4.17)}{\le} c\,L_{n_1}^{5a}\,\nu_{n_1}^2\,,
\end{aligned}
$$

where we used the inequalities $(1+u)^\ell \le e^{u\ell}$ and $e^v - 1 - v \le v^2 e^v$, for ℓ, u, v positive numbers. To bound a_3, we use the heat equation satisfied by the Brownian semigroup, which implies that for f with $|f|_{(n_1+1)} \le 1$,

$$
(4.173)\qquad
\begin{aligned}
&|P_{\alpha_{n_1} L_{n_1+1}^2} f - P_{\alpha_{n_1+1} L_{n_1+1}^2} f|_{(n_1+1)} = \\
&\Big|\int_{\alpha_{n_1+1} L_{n_1+1}^2}^{\alpha_{n_1} L_{n_1+1}^2} \frac{1}{2}\,\Delta\,P_s\,f\,ds\Big|_{(n_1+1)} = \\
&\Big|\int_{\alpha_{n_1} L_{n_1+1}^2}^{\alpha_{n_1+1} L_{n_1+1}^2} \frac{1}{2}\,P_{s/2}\,\Delta\,P_{s/2}\,f\,ds\Big|_{(n_1+1)} \overset{(1.56),(1.49)i)}{\le} \\
&c|\alpha_{n_1+1} - \alpha_{n_1}| \overset{(1.49)ii)}{\le} c\,L_{n_1}^{-\frac{19}{10}\delta} \overset{(1.14)}{\le} L_{n_1+1}^{-\frac{18}{10}\delta}\,.
\end{aligned}
$$

We have thus shown that when L_0 is large

$$
(4.174)\qquad a_3 \le c\,L_{n_1+1}^{-\frac{18}{10}\delta}\,.
$$

Collecting (4.170), (4.172), (4.174), we see that when L_0 is large, for $\omega \in G_{\emptyset,n_1+1}$, $v \in \mathcal{S}_{n_1+1}$:

$$
(4.175)\qquad \|\chi_{n_1+1,v}\,S^*_{n_1+1,\emptyset}\|_{n_1+1} \le c\Big(\kappa_{n_1}\,\nu_{n_1}\,\ell_{n_1}^{-\beta/3} + L_{n_1}^{5a}\,\nu_{n_1}^2 + L_{n_1+1}^{-\frac{18}{10}\delta}\Big)\,,
$$

and thank to (4.140), a similar inequality is satisfied by $\chi_{n_1+1,v}\,\widetilde{S}^*_{n_1+1,\emptyset}$. If we now choose $v=0$, analogous controls as in the derivation of (2.4), using (4.14), and (1.49) i) with $n = n_1+1 \le n_0$, and the remark below (4.11), show that

$$
(4.176)\qquad
\begin{aligned}
&\sup_{y\in[0,L_{n_1+1}]^d}\Big(\Big|\frac{\widetilde{d}^*_{n_1+1}}{L_{n_1+1}}\,(y,\omega)\Big| + \Big|\frac{\widetilde{\gamma}^*_{n_1+1}}{L_{n_1+1}^2}\,(y,\omega)\Big|\Big) \le \\
&\kappa_{n_1+1}\Big(\kappa_{n_1}\,\nu_{n_1}\,\ell_{n_1}^{-\beta/3} + L_{n_1}^{5a}\,\nu_{n_1}^2 + L_{n_1+1}^{-\frac{18}{10}\delta}\Big) \le \nu_{n_1+1}\,,
\end{aligned}
$$

using (1.14), (1.40), (4.17) in the last step.

We thus see that (4.163), (4.164) are satisfied for $\omega \in G_{\emptyset,n_1+1}$, $v \in \mathcal{S}_{n_1+1}$, with n_1+1, in place of n_0'. This completes the induction step, and with (4.160), this is more than enough to prove the claim of Proposition 4.10. $\square$

We are now ready to state and prove the main result of this section. We recall the notation introduced in (4.4), (4.5), (4.136).

Proposition 4.11. *When L_0 is large, for each $\sigma \in \Sigma$ there is an event $G_{\sigma,n_0+1} \subseteq G_\sigma \cap \widetilde{G}$, such that:*

$$\sup_{\sigma\in\Sigma} \mathbb{P}[(G_\sigma \cap \widetilde{G})\backslash G_{\sigma,n_0+1}] \le e^{-\kappa_{n_0}}, \tag{4.177}$$

$$\mathbb{P}\Big[\Big(\bigcup_{\sigma\in\Sigma} G_{\sigma,n_0+1}\Big)^c\Big] \le \frac{1}{20}\, L_{n_0+1}^{-M_0}, \tag{4.178}$$

and on G_{σ,n_0+1}, for all $n_0' \le n \le n_0$, (cf. (4.17), (4.162) *for the notation),*

$$\sup_{x\in\mathcal{S}_n} \big(\|\chi_{n,x}\, S^*_{n,\sigma}\|_n \vee \|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\|_n\big) \le \nu_n, \tag{4.179}$$

and

$$\big\|\chi_{n_0+1,0}\big(R^*_{n_0+1,\sigma} - \big(R^0_{n_0}\big)^{\ell^2_{n_0}}\big)\big\|_{n_0+1} \le \nu_{n_0+1}. \tag{4.180}$$

Proof. The argument is similar to the proof of Proposition 4.10. We define for $\sigma \in \Sigma$,

$$G_{\sigma,n_0'} = G_\sigma \cap \widetilde{G}, \tag{4.181}$$

(this is consistent with (4.160), when $\sigma = \emptyset$). We then observe with (4.7), (4.11), that when L_0 is large, for $\sigma \in \Sigma$, $\omega \in G_{\sigma,n_0'}$, $v \in \mathcal{S}_{n_0'}$:

$$\begin{aligned} \|\chi_{n_0',x}\, \widetilde{S}^*_{n_0',\sigma}\|_{n_0'} &= \|\chi_{n_0',x}\, S^*_{n_0',\sigma}\|_{n_0} \\ &= \big\|\chi_{n_0',x}\big(g_\sigma\, \widetilde{S}_{n_0'} + (1-g_\sigma)\big(\widetilde{R}^0_{n_0'} - R^0_{n_0'}\big)\big)\big\|_{n_0'} \\ &\overset{(4.6),(2.2),(2.46)}{\le} c\,\big(L_{n_0'}^{-\delta} + e^{-\kappa_{n_0}}\big) \overset{(4.17)}{\le} \nu_{n_0'}. \end{aligned} \tag{4.182}$$

Let us now assume that for n_1 with $n_0' \le n_1 < n_0$, and $\sigma \in \Sigma$, we have a decreasing sequence of events, $n_0' \le n \le n_0$, such that

$$\sup_{\sigma\in\Sigma} \mathbb{P}[G_{\sigma,n}\backslash G_{\sigma,n+1}] \le e^{-\kappa_{n_0}}, \ \text{for } n_0' \le n < n_1, \tag{4.183}$$

and such that on $G_{\sigma,n}$:

$$\sup_{x\in\mathcal{S}_n} (\|\chi_{n,x}\, S^*_{n,\sigma}\|_n \vee \|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\|_n) \le \nu_n. \tag{4.184}$$

Then with Proposition 4.1, for $\sigma \in \Sigma$,

$$\begin{aligned} &\mathbb{P}\Big[G_{\sigma,n_1} \cap \Big\{\sup_{v\in L_{n_1+1}\mathbb{Z}^d:\mathcal{S}_{n_1,v}\neq\emptyset} \|\widetilde{\mathcal{L}}_{\sigma,n_1,v}\|_{n_1+1} > \frac{\kappa_n\, \nu_{n_1}}{\ell_{n_1}^{\beta/3}}\Big\}\Big] \le \\ &c\left(\frac{L^2_{n_0+1}}{L_{n_1+1}}\right)^d e^{-\kappa_{n_0}} \le e^{-\kappa_{n_0}}. \end{aligned} \tag{4.185}$$

We then define for $\sigma \in \Sigma$, (this is consistent with (4.167)):

$$G_{\sigma,n_1+1} = G_{\sigma,n_1} \cap \Big\{ \sup_{v \in L_{n_1+1}\mathbb{Z}^d : \mathcal{S}_{n_1,v} \neq \emptyset} \|\widetilde{\mathcal{L}}_{\sigma,n_1,v}\|_{n_1+1} \le \frac{\kappa_n\, \nu_{n_1}}{\ell_{n_1}^{\beta/3}} \Big\}, \tag{4.186}$$

and see that (4.183) holds with $n_1 + 1$ in place of n_1. Moreover in a parallel fashion to (4.169), for $\sigma \in \Sigma$, $\omega \in G_{\sigma,n_1+1}$, $v \in \mathcal{S}_{n_1+1}$,

$$\|\chi_{n_1+1,v}\, S^*_{n_1+1,\sigma}\|_{n_1+1} \le a_1 + a_2 + a_3\,, \tag{4.187}$$

where a_i, $1 \le i \le 3$, are just as in (4.169), with σ replacing $\emptyset$ in the expressions entering a_1, a_2. The same reasoning (4.170)–(4.174) shows that when L_0 is large, for $\sigma \in \Sigma$, $\omega \in G_{\sigma,n_1+1}$, and $v \in \mathcal{S}_{n_1+1}$:

$$\|\chi_{n_1+1,v}\, S^*_{n_1+1,\sigma}\|_{n_1+1} \le c\left(\kappa_{n_1}\, \nu_{n_1}\, \ell_{n_1}^{-\beta/3} + L_{n_1}^{5a}\, \nu_{n_1}^2 + L_{n_1+1}^{-\frac{18}{10}\delta}\right), \tag{4.188}$$

and that with (4.140) a similar inequality holds for $\chi_{n_1+1,v}\, \widetilde{S}^*_{n_1+1,\sigma}$. This implies that (4.184) is true for $n = n_1 + 1$. This proves by induction (4.183) for $n_0' \le n < n_0$ and (4.184) for $n_0' \le n \le n_0$. We can then define for $\sigma \in \Sigma$, G_{σ,n_0+1} via (4.186) with n_0 in place of n_1. We then obtain (4.177), (4.180) by writing the analogue of (4.15) for $R^*_{n_0+1} - (R^0_{n_0})^{\ell^0_{n_0}}$, i.e. without the bottom line of (4.15), (incidentally we recall that (1.50) remains to be proved, cf. Proposition 5.7 below). The claim (4.178) is now a straightforward consequence of (4.5), (4.137), (4.177). This concludes the proof of Proposition 4.11. □

5. Repairing defects

We conclude the proof of Theorem 1.1 in this section. The main remaining task is to propagate the part of (1.47) concerning Hölder-norm controls at level $n_0 + 1$. In Sect. 4 we have performed surgery on the environment and removed defects occurring at level $n_0' = n_0 - m_0 - 1$. We have shown that the kernels $R^*_{n,\sigma}$, $n_0' \le n \le n_0 + 1$, $\sigma \in \Sigma$, cf. (4.7), (4.8), describing the evolution at level n "after surgery", were typically well-behaved for Hölder-norms, when $\omega \in G_{\sigma,n_0+1}$, and that the complement of $\bigcup_{\sigma\in\Sigma} G_{\sigma,n_0+1}$, was "negligible" for our purpose, cf. Proposition 4.11. We now have to show that on "most" of G_{σ,n_0+1}, $R^*_{n_0+1,\sigma}$ and R_{n_0+1}, the true object of our interest, are close in the Hölder-norm sense. To this end we will in essence use the smoothing effect of the kernels "after surgery" to repair defects, as well as (1.48) to prevent any trapping effect of the defects. The main step comes with Proposition 5.1. We will also prove (1.50), cf. Proposition 5.7, and thereby complete the proof of Theorem 1.1.

We first introduce some additional notation. We recall that Z_k, $k \ge 0$, denotes the canonical process on $(\mathbb{R}^d)^{\mathbb{N}}$, and that the laws $P^\sigma_{y,\omega}$, for $\sigma \in \Sigma$, $\omega \in \Omega$, $y \in \mathbb{R}^d$, with corresponding expectation $E^\sigma_{y,\omega}$, have been defined above (4.138). We let $P^e_{y,\omega}$ stand for the canonical law on $(\mathbb{R}^d)^{\mathbb{N}}$ of the Markov chain starting at $y \in \mathbb{R}^d$, with transition kernel $R_{n_0'}$. It describes the diffusion in the environment (whence the superscript e) $\omega \in \Omega$, viewed at times $k L^2_{n_0'}$, $k \geq 0$, originating from y. We let $E^e_{y,\omega}$ stand for the corresponding expectation. When no confusion with (1.8) arises, we use the notation

$$H_C = \inf\{k \geq 0, Z_k \in C\}, \;\; T_C = \inf\{k \geq 0, Z_k \notin C\}\,. \tag{5.1}$$

Likewise we still denote with θ_k, $k \geq 0$, the canonical shift on $(\mathbb{R}^d)^{\mathbb{N}}$. With the notation of (1.44), we introduce the event

$$\begin{aligned} \overline{G} = \big\{\omega \in \Omega;\; & J_{n,x,C_n(x),\gamma} = 0, \text{ for all } n_0' \leq n \leq n_0+1, \\ & x \in L_n\,\mathbb{Z}^d \cap (5\mathcal{T}_{n_0+1})\,,\; \gamma \in \{1,\dots,2d\,5^{(d-1)}\}\big\}\,. \end{aligned} \tag{5.2}$$

This is the place where we use the control on traps to make sure that $\overline{G}^{\,c}$ has negligible probability. With (1.48), for $n \leq n_0$ and Proposition 3.3 when $n = n_0+1$, (we in fact only need in these controls the case of $\mathcal{A}$ singleton and $u_x \to 0$) we see that when L_0 is large,

$$\begin{aligned} \mathbb{P}\big[\overline{G}^c\big] & \leq \sum_{n_0' \leq n \leq n_0+1} c\left(\frac{L^2_{n_0+1}}{L_n}\right)^d L_n^{-\overline{M}_n} \\ & \leq c(m_0+2)\, L_{n_0+1}^{2d-(1+a)^{-(m_0+2)}M/2} \underset{(1.46)}{\overset{(1.14),(1.17)}{\leq}} L_{n_0+1}^{-2M_0}\,. \end{aligned} \tag{5.3}$$

With the notation of Proposition 4.11, (4.5), (4.136), we define for each $\sigma \in \Sigma$:

$$\overline{G}_{\sigma,n_0+1} = G_{\sigma,n_0+1} \cap \overline{G} \subseteq G_\sigma \cap \widetilde{G} \cap \overline{G}\,. \tag{5.4}$$

When L_0 is large with (4.178), (5.3), we find:

$$\mathbb{P}\Big[\Big(\bigcup_{\sigma\in\Sigma} \overline{G}_{\sigma,n_0+1}\Big)^c\Big] \leq \mathbb{P}\Big[\Big(\bigcup_{\sigma\in\Sigma} G_{\sigma,n_0+1}\Big)^c\Big] + \mathbb{P}\big[\overline{G}^c\big] \leq \frac{1}{10}\, L_{n_0+1}^{-M_0}\,. \tag{5.5}$$

The next proposition is an important step in our program of "defects repairs". Some elements are reminiscent of Sidoravicius-Sznitman [25], cf. below (2.33) of [25].

Proposition 5.1. *When L_0 is large, for $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, f with $|f|_{(n_0+1)} \leq 1$,*

$$\begin{aligned} & \sup_{|y| \leq \widetilde{D}_{n_0+1}} |E^e_{y,\omega}[f(Z_T)] - E^\sigma_{y,\omega}[f(Z_T)]| \leq L_{n_0+1}^{-(\beta+\delta+a)}, \text{ with} \\ & T = \left(\frac{L_{n_0+1}}{L_{n_0'}}\right)^2 - 1 \overset{(4.138)}{=} k_{n_0+1} - 1\,. \end{aligned} \tag{5.6}$$

Proof. We break the difference in (5.6) into three terms that will be separately bounded. Recall from (4.4) that $\sigma = (\sigma_1, \dots, \sigma_{\tilde\ell})$, where $0 \le \tilde\ell \le \tilde\ell_0$. We introduce

$$K_\sigma = \bigcup_{i=1}^{\tilde\ell} \overline{B}(\sigma_i, 10\widetilde{D}_{n_0'}),\ \ U_\sigma = \bigcup_{i=1}^{\ell} B\Big(\sigma_i, \frac{1}{5\tilde\ell_0}\, L_{n_0'+2}\Big)\,, \tag{5.7}$$

and write for $y \in B(0, \widetilde{D}_{n_0+1})$, (cf. (5.6)),

$$\begin{aligned} A_1 &= E^e_{y,\omega}\big[f(Z_T),\ H_{K_\sigma} > T\big] - E^\sigma_{y,\omega}\big[f(Z_T),\ H_{K_\sigma} > T\big]\,,\\ A_2 &= E^e_{y,\omega}\Big[f(Z_T),\ \frac{T}{2} < H_{K_\sigma} \le T\Big] - E^\sigma_{y,\omega}\Big[f(Z_T),\ \frac{T}{2} < H_{K_\sigma} \le T\Big]\,,\\ A_3 &= E^e_{y,\omega}\Big[f(Z_T),\ H_{K_\sigma} \le \frac{T}{2}\Big] - E^\sigma_{y,\omega}\Big[f(Z_T),\ H_{K_\sigma} \le \frac{T}{2}\Big]\,, \end{aligned} \tag{5.8}$$

(incidentally note that $A_2 = A_3 = 0$, when $\sigma = \emptyset$). We thus have

$$E^e_{y,\omega}[f(Z_T)] - E^\sigma_{y,\omega}[f(Z_T)] = A_1 + A_2 + A_3\,. \tag{5.9}$$

We first bound A_1. Note that when L_0 is large, for $y \in B(0, \widetilde{D}_{n_0+1})$, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$,

$$P^\sigma_{y,\omega}\text{-a.s.},\ \ T < T_{\frac{1}{5}\mathcal{T}_{n_0+1}}\,, \tag{5.10}$$

indeed, $T \le (L_{n_0+1}/L_{n_0'})^2 < L^2_{n_0+1}/10\widetilde{D}_{n_0'}$, when L_0 is large, see also (4.7). Coming back to the diffusion process, we can write, cf. (4.7):

$$\begin{aligned} A_1 = {} & E_{y,\omega}\big[f(X_{TL^2_{n_0'}}),\ X_{kL^2_{n_0'}} \notin K_\sigma,\ \text{for } 0 \le k \le T\big] - \\ & E_{y,\omega}\big[f(X_{V_T}),\ X_{V_k} \notin K_\sigma,\ \text{for } 0 \le k \le T\big]\,, \end{aligned} \tag{5.11}$$

where V_k, $k \ge 0$, are the iterates of the stopping time $L^2_{n_0'} \wedge T_{n_0'}$ on $C(\mathbb{R}_+, \mathbb{R}^d)$, cf. (1.19), that is:

$$V_0 = 0,\ V_1 = L^2_{n_0'} \wedge T_{n_0'},\ \text{and } V_{k+1} = V_1 \circ \theta_{V_k} + V_k,\ \text{for } k \ge 1\,, \tag{5.12}$$

(here of course $(\theta_t)_{t\ge 0}$ stands for the canonical shift on $C(\mathbb{R}_+, \mathbb{R}^d)$). With (5.10), (5.11), we see that:

$$\begin{aligned} |A_1| \le 2 \sum_{0\le k<T} P_{y,\omega}\big[& T_{n_0'} \circ \theta_{mL^2_{n_0'}} > L^2_{n_0'},\ \text{for } 0 \le m < k,\\ & T_{n_0'} \circ \theta_{kL^2_{n_0'}} \le L^2_{n_0'}\,,\\ & \text{and } X_{mL^2_{n_0'}} \in \mathcal{T}_{n_0+1}\backslash K_\sigma,\ \text{for } 0 \le m \le k\big]\\ \overset{(2.2),(4.5)}{\le} {} & 2T\, e^{-\kappa_{n_0'}} \le e^{-\kappa_{n_0+1}}\,. \end{aligned} \tag{5.13}$$

We now bound A_2, and by the remark following (5.8), we may and will assume that $\sigma \neq \emptyset$. Note that:

$$A_2 \le P^e_{y,\omega}\Big[\frac{T}{2} < H_{K_\sigma} \le T\Big] + P^\sigma_{y,\omega}\Big[\frac{T}{2} < H_{K_\sigma} \le T\Big]. \tag{5.14}$$

We can express both probabilities in the right member of (5.14) in terms of the diffusion process in a similar fashion as in (5.11). Using analogous bounds we see that

$$\Big|P^e_{y,\omega}\Big[\frac{T}{2} < H_{K_\sigma} \le T\Big] - P^\sigma_{y,\omega}\Big[\frac{T}{2} < H_{K_\sigma} \le T\Big]\Big| \le e^{-\kappa_{n_0+1}}\,. \tag{5.15}$$

Further since $\omega \in \overline{G}_{\sigma,n_0+1} \subseteq \overline{G}$, see (5.4), it follows from (5.2), (1.44) with $n = n_0$, and the Markov property that for y as in (5.6),

$$P_{y,\omega}\Big[\sup_{0\le u\le v\le \frac{T}{4}L^2_{n_0'}} |X_v - X_u| < \frac{L_{n_0}}{2}\Big] \le (1-c_1)^{\ell^2_{n_0}/8} \le e^{-\kappa_{n_0+1}}\,. \tag{5.16}$$

With a similar argument as in (3.68), one sees that on the complement of the event that appears in the above probability, $X_.$ must have exited the open set $\bigcup_{i=1}^{\tilde\ell} B(\sigma_i, \frac{L_{n_0}}{4\tilde\ell_0})$ by time $\frac{T}{4}L^2_{n_0'}$. We hence find that

$$\begin{aligned}
&P^\sigma_{y,\omega}\Big[\frac{T}{2} < H_{K_\sigma} \le T\Big] \le \\
&P_{y,\omega}\Big[X_{V_m} \notin K_\sigma,\ \text{for all } 0 \le m \le \frac{T}{2},\ \text{and } X_{V_k} \in K_\sigma, \text{for some } \frac{T}{2} < k \le T,\\
&\qquad \text{and} \sup_{0\le u\le \frac{T}{4}L^2_{n_0'}} d(X_u, K_\sigma) \ge \frac{L_{n_0}}{4\tilde\ell_0} - 10\widetilde{D}_{n_0'}\Big] + e^{-\kappa_{n_0+1}}\,.
\end{aligned}$$

Introducing the open set:

$$\mathcal{U} = \Big\{z \in \mathbb{R}^d;\ d(z, K_\sigma) < \frac{L_{n_0}}{4\tilde\ell_0} - 11\widetilde{D}_{n_0'}\Big\}, \tag{5.17}$$

we see with a similar argument as in (5.13), using (5.10), that

$$\begin{aligned}
&P^\sigma_{y,\omega}\Big[\frac{T}{2} < H_{K_\sigma} \le T\Big] \le \\
&P^\sigma_{y,\omega}\Big[\frac{T}{2} < H_{K_\sigma} \le T \wedge T_{\mathcal{T}_{n_0+1}}, T_{\mathcal{U}} < \frac{T}{2}\Big] + e^{-\kappa_{n_0+1}} \le \\
&\sup_{z\in\mathcal{T}_{n_0+1}\setminus\mathcal{U}} P^\sigma_{z,\omega}\big[H_{K_\sigma} < T \wedge T_{\mathcal{T}_{n_0+1}}\big] + e^{-\kappa_{n_0+1}}\,.
\end{aligned} \tag{5.18}$$

Coming back to (5.14), (5.15), we find

$$A_2 \le 2 \sup_{z\in\mathcal{T}_{n_0+1}\setminus\mathcal{U}} P^\sigma_{z,\omega}\big[H_{K_\sigma} < T \wedge T_{\mathcal{T}_{n_0+1}}\big] + e^{-\kappa_{n_0+1}}\,. \tag{5.19}$$

The next step is to bound the first expression in the right-hand side of (5.19). To this end for $w \in 2\mathcal{T}_{n_0+1}$, we introduce the function:

$$(5.20)\quad n_w(z) = \begin{cases} n_0', & \text{if } D^*_{n_0'+2} \geq |z-w|\,, \\ \sup\{n \in [n_0', n_0];\ |z-w| > D^*_{n+1}\}, & \text{else}\,, \end{cases}$$

and the stopping time (for $Z_.$):

$$(5.21)\qquad \tau_w = \begin{cases} 1, \ \text{when } n_w(Z_0) = n_0'\,, \\ k_n \wedge \inf\{k \geq 0:\ |Z_k - Z_0| \geq D^*_{n_w(Z_0)}\}, \ \text{else}\,, \end{cases}$$

(recall $k_n = (L_n/L_{n_0'})^2$, cf. (4.138), and D_n^* is defined in (4.10)). We write below $n(z)$ for $n_w(z)$. We also introduce the function

$$(5.22)\qquad f_w(z) = \left|\frac{z-w}{D^*_{n_0+1}}\right|^{-\gamma} \wedge 1,\ \ z \in \mathbb{R}^d,\ \text{ with } \gamma = d-2-\frac{1}{100}\,.$$

Lemma 5.2. *When L_0 is large, for $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $w \in 2\mathcal{T}_{n_0+1}$, $z \in (2\mathcal{T}_{n_0+1}) \cap B(w, L_{n_0}^{(1+\delta/2)})$, (cf.* (1.40) *for the definition of δ), we have*

$$(5.23)\qquad E^\sigma_{z,w}[f_w(Z_{\tau_w})] \leq f_w(z)\,.$$

Proof. When $|z-w| \leq D^*_{n_0+1}$, (5.23) is immediate. We thus assume that

(5.24)

$$z_0 \overset{\text{def}}{=} z-w \text{ satisfies } |z_0| > D^*_{n_0+1},\ \text{ and } z \in (2\mathcal{T}_{n_0+1}) \cap B\big(w, L_{n_0}^{(1+\frac{\delta}{2})}\big)\,.$$

Consider $x \in \mathbb{R}^d$, such that $|x| \leq \frac{1}{2}\,|z_0|$. Writing $\widehat{z}_0 = \frac{z_0}{|z_0|}$, we have

$$(5.25)\quad \begin{aligned} |z_0+x|^{-\gamma} &= |z_0|^{-\gamma}\left|\widehat{z}_0 + \frac{x}{|z_0|}\right|^{-\gamma} = |z_0|^{-\gamma}\Big(1 + 2\widehat{z}_0 \cdot \frac{x}{|z_0|} + \frac{|x|^2}{|z_0|^2}\Big)^{-\frac{\gamma}{2}} \\ &= |z_0|^{-\gamma}\Big(1 - \frac{\gamma}{2}\,\Big(2\widehat{z}_0 \cdot \frac{x}{|z_0|} + \frac{|x|^2}{|z_0|^2}\Big) + \\ &\qquad \frac{1}{2}\,(\gamma^2+2\gamma)\,\Big(\frac{\widehat{z}_0 \cdot x}{|z_0|}\Big)^2 + r(z_0,x)\Big), \\ &\text{with } |r(z_0,x)| \leq c\,\Big(\frac{|x|}{|z_0|}\Big)^3, \end{aligned}$$

after the application of Taylor's formula to second order in the neighborhood of 0, to the function $(1+u)^{-\gamma/2}$, $u \in (-1,1)$. Coming back to (5.21), with (5.24) in force, we see that

$$(5.26)\quad \begin{aligned} &E^\sigma_{z,\omega}[f_w(Z_{\tau_w})] \leq \\ &f_w(z)\Big(1 - \frac{\gamma}{|z_0|}\,\widehat{z}_0 \cdot E^\sigma_{z,\omega}\,[Z_{\tau_w} - Z_0] - \frac{\gamma}{2|z_0|^2}\,E^\sigma_{z,\omega}\big[|Z_{\tau_w} - Z_0|^2\big] + \\ &\qquad \frac{1}{2}\,\frac{(\gamma^2+2\gamma)}{|z_0|^2}\,E^\sigma_{z,\omega}\big[\{\widehat{z}_0 \cdot (Z_{\tau_w} - Z_0)\}^2\big] + c\,\Big(\frac{D^*_{n(z)}}{|z_0|}\Big)^3\Big). \end{aligned}$$

Comparing the law of Z_{τ_w} under $P^\sigma_{z,\omega}$ with $\widetilde{R}^*_{n(z),\sigma}(z,\cdot)$, cf. (4.138), with (4.139), and $\omega \in \overline{G}_{\sigma,n_0+1}$, we see that when L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $w, z \in 2\mathcal{T}_{n_0+1}$, with (5.24):

$$\tag{5.27}
\begin{aligned}
&\big|E^\sigma_{z,\omega}[Z_{\tau_w} - Z_0] - \widetilde{d}^*_{n(z),\sigma}(z,\omega)\big| \le e^{-\kappa_{n_0}}\,,\\
&\big|E^\sigma_{z,\omega}[(Z_{\tau_w} - Z_0)_i\,(Z_{\tau_\omega} - Z_0)_j] - \\
&\alpha_{n(z)}\,\delta_{ij}\,L^2_{n(z)} - (\widetilde{\gamma}^*_{n(z),\sigma})^{i,j}(z,\omega)\big| \le e^{-\kappa_{n_0}}\,,
\end{aligned}$$

for $1 \le i, j \le d$, with the notation of (4.14). Using (4.179), (1.49), and once again an analogous calculation as in Lemma 2.1, we see that under the same conditions as in (5.27)

$$\tag{5.28} |\widetilde{d}^*_{n(z),\sigma}(z,\omega)| \le \kappa_{n_0}\,L_{n(z)}\,\nu_{n(z)},\ |\widetilde{\gamma}^*_{n(z),\sigma}(z,\omega)| \le \kappa_{n_0}\,L^2_{n(z)}\,\nu_{n(z)}\,.$$

As a result we obtain, (recall $\gamma + 2 - d = -\frac{1}{100}$):

$$\begin{aligned}
&(\gamma+2)\,E^\sigma_{z,\omega}\big[\{\widehat{z}_0\cdot(Z_{\tau_w} - Z_0)\}^2\big] - E^\sigma_{z,\omega}\big[|Z_{\tau_w} - Z_0|^2\big] \le\\
&-\frac{1}{100}\,\alpha_{n(z)}\,L^2_{n(z)} + \kappa_{n_0}\,L^2_{n(z)}\,\nu_{n(z)}\,.
\end{aligned}$$

Therefore for large L_0, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $w, z \in 2\mathcal{T}_{n_0+1}$, with (5.24), we find

$$\tag{5.29}
\begin{aligned}
&E^\sigma_{y,\omega}[f_w(Z_{\tau_w})] \le\\
&f_w(z)\Big[1 + \frac{\kappa_{n_0}}{|z_0|}\,L_{n(z)}\,\nu_{n(z)} - \frac{\gamma}{2|z_0|^2}\,L^2_{n(z)}\Big(\frac{\alpha_{n(z)}}{100} - \kappa_{n_0}\,\nu_{n(z)}\Big) + c\,\Big(\frac{D^*_{n(z)}}{|z_0|}\Big)^3\Big]\\
&\overset{(5.24),(5.20)}{\le} f_w(z)\Big[1 + \frac{L_{n(z)}}{|z_0|}\,\Big(\kappa_{n_0}\,\nu_{n(z)} - \frac{c\,L_{n(z)}}{|z_0|}\Big)\Big] \le f_w(z)\,,
\end{aligned}$$

using (5.26), (5.28), and (4.17). The claim (5.23) now follows. □

Coming back to (5.19), (5.7), we see that

$$\tag{5.30}
\begin{aligned}
&A_2 \le\\
&2\tilde{\ell}_0 \sup_{1\le i\le\tilde{\ell}}\ \sup_{z\in\mathcal{T}_{n_0+1}:|z-\sigma_i|\ge\frac{L_{n_0}}{4\tilde{\ell}_0}-\widetilde{D}_{n_0'}} P^\sigma_{z,\omega}[H_{\overline{B}(\sigma_i,10\widetilde{D}_{n_0'})} < T\wedge T_{\mathcal{T}_{n_0+1}}] + e^{-\kappa_{n_0+1}} \le\\
&2\tilde{\ell}_0 \sup_{1\le i\le\tilde{\ell}}\ \sup_{z\in\mathcal{T}_{n_0+1}:\frac{L_{n_0}}{4\tilde{\ell}_0}-\widetilde{D}_{n_0'}\le|z-\sigma_i|\le\frac{L_{n_0}}{4\tilde{\ell}_0}} P^\sigma_{z,\omega}[H_{\overline{B}(\sigma_i,10\widetilde{D}_{n_0'})} < T\wedge T_{\mathcal{T}_{n_0+1}}] + e^{-\kappa_{n_0+1}}
\end{aligned}$$

using the strong Markov property in the last step.

With (4.139), $n = n_0$, and the Markov property, we observe that for large L_0, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $z \in \mathcal{T}_{n_0+1}$,

$$\tag{5.31} P^\sigma_{z,\omega}\Big[\sup_{0\le k\le T} |Z_k - Z_0| > \ell^2_{n_0}\,30\tilde{\ell}_0\,\widetilde{D}_{n_0}\Big] \le e^{-\kappa_{n_0+1}}\,.$$

As a result when $z \in \mathcal{T}_{n_0+1}$ is such that for some $1 \le i \le \ell$, $\frac{L_{n_0}}{4\tilde{\ell}_0} - \widetilde{D}_{n'_0} \le |z-\sigma_i| \le \frac{L_{n_0}}{4\tilde{\ell}_0}$, with (1.14), (1.40), we find

$$\begin{aligned} &P^\sigma_{z,\omega}\big[H_{\overline{B}(\sigma_i,10\widetilde{D}_{n'_0})} < T \wedge T_{\mathcal{T}_{n_0+1}}\big] \le \\ &P^\sigma_{z,\omega}\big[H_{\overline{B}(\sigma_i,10\widetilde{D}_{n'_0})} < T_{B(\sigma_i,L_{n_0}^{(1+\delta/2)})}\big] + e^{-\kappa_{n_0+1}}\,. \end{aligned} \tag{5.32}$$

We can then introduce $\tau^k_{\sigma_i}$, $k \ge 0$, the iterates of the stopping time τ_{σ_i}, cf. (5.21) with $w = \sigma_i$.

$$\tau^0_{\sigma_i} = 0, \quad \tau^1_{\sigma_i} = \tau_{\sigma_i}, \quad \tau^{k+1}_{\sigma_i} = \tau_{\sigma_i} \circ \theta_{\tau^k_{\sigma_i}} + \tau^k_{\sigma_i}, \text{ for } k \ge 1\,, \tag{5.33}$$

as well as

$$N = \inf\big\{k \ge 0;\ Z_{\tau^k_{\sigma_i}} \in \overline{B}(\sigma_i, 10\widetilde{D}_{n'_0}) \cup B(\sigma_i, L_{n_0}^{(1+\delta/2)})^c\big\}\,. \tag{5.34}$$

Using induction over k, the strong Markov property and (5.23), we see that

$$E^\sigma_{z,\omega}[f_{\sigma_i}(Z_{\tau^{N\wedge k}_{\sigma_i}})] \text{ is a decreasing function of } k \ge 0\,. \tag{5.35}$$

Further observe that for z as above (5.32), $P^\sigma_{z,\omega}$-a.s., on the event $\{H_{\overline{B}(\sigma_i,10\widetilde{D}_{n'_0})} < T_{B(\sigma_i,L_{n_0}^{(1+\delta/2)})}\}$, it holds that $Z_{\tau^N_{\sigma_i}} \in \overline{B}(\sigma_i, 10\widetilde{D}_{n'_0})$, as follows from (5.21), (5.33), (5.34). Hence with Fatou's lemma, we find

$$\begin{aligned} P^\sigma_{z,\omega}\big[H_{\overline{B}(\sigma_i,10\widetilde{D}_{n'_0})} < T_{B(\sigma_i,L_{n_0}^{(1+\delta/2)})}\big] &\le E^\sigma_{z,\omega}\big[f_{\sigma_i}(Z_{\tau^N_{\sigma_i}}), N < \infty\big] \\ &\le f_{\sigma_i}(z)\,. \end{aligned} \tag{5.36}$$

The above inequality together with (5.22), (5.30), shows that when L_0 is large,

$$\begin{aligned} A_2 &\le \kappa_{n_0+1}\Big(\frac{L_{n_0}}{L_{n'_0+1}}\Big)^{-(d-2-\frac{1}{100})} + e^{-\kappa_{n_0+1}} \\ &\overset{(4.1)}{\le} \kappa_{n_0+1}\, L_{n_0+1}^{-\frac{99}{100}((1+a)^{-1}-(1+a)^{-(m_0+1)})} \overset{(1.14),(1.17)}{\le} L_{n_0+1}^{-\frac{8}{10}}\,. \end{aligned} \tag{5.37}$$

We now bound A_3. As in the case of A_2, we only need to consider the case $\sigma \ne \emptyset$, see below (5.8). We first introduce some notation. We consider the functions, (with $\omega \in \overline{G}_{\sigma,n_0+1}$, and f as in (5.9)):

$$\begin{aligned} &F^e(k,z) = E^e_{z,\omega}[f(Z_{T-k})], \\ &F^\sigma(k,z) = E^\sigma_{z,\omega}[f(Z_{T-k})],\ z \in \mathbb{R}^d,\ 0 \le k \le T\,. \end{aligned} \tag{5.38}$$

We also introduce the probability kernels:

$$(5.39)\qquad \begin{aligned} Q^e\, G(k,z) &= E^e_{z,\omega}\big[G((k+T_{U_\sigma}\wedge t_0)\wedge T,\ Z_{T_{U_\sigma}\wedge t_0\wedge(T-k)})\big],\\ &\qquad\qquad 0\le k\le T, z\in\mathbb{R}^d\,,\\ Q^\sigma\, G(k,z) &= E^\sigma_{z,\omega}\big[G((k+T_{U_\sigma}\wedge t_0)\wedge T,\ Z_{T_{U_\sigma}\wedge t_0\wedge(T-k)})\big],\\ &\qquad\qquad 0\le k\le T, z\in\mathbb{R}^d\,, \end{aligned}$$

with G bounded measurable on $\{0,\dots,T\}\times\mathbb{R}^d$, U_σ as in (5.7), and

$$(5.40)\qquad t_0 = k_{n_0'+3} \overset{(4.138)}{=} (L_{n_0'+3}/L_{n_0'})^2\,.$$

Loosely speaking, these kernels describe for the Markov chain in the true environment or in the environment after surgery how the process initiated at time $k\le T$, and stopped at the deterministic time $T\wedge(k+t_0)$ quits U_σ. We also introduce sub-probability kernels describing returns to K_σ prior to T or exit from $\frac34\,\mathcal{T}_{n_0+1}$:

$$(5.41)\qquad \begin{aligned} R^e\, G(k,z) &= E^e_{z,\omega}\big[G((k+H_{K_\sigma})\wedge T,\ Z_{H_{K_\sigma}\wedge(T-k)}),\\ &\qquad H_{K_\sigma} < (T-k)\wedge T_{\frac34\mathcal{T}_{n_0+1}}\big]\\ R^\sigma\, G(k,z) &= E^\sigma_{z,\omega}\big[G((k+H_{K_\sigma})\wedge T,\ Z_{H_{K_\sigma}\wedge(T-k)}),\\ &\qquad H_{K_\sigma} < (T-k)\wedge T_{\frac34\mathcal{T}_{n_0+1}}\big]\,, \end{aligned}$$

with $0\le k\le T$, $z\in\mathbb{R}^d$, and G as below (5.39).

Coming back to the definition of A_3 in (5.8), we see using the strong Markov property at time H_{K_σ}, analogous considerations as in the control of A_1 and (5.10), that for large L_0, $\sigma\in\Sigma$, $\omega\in\overline{G}_{\sigma,n_0+1}$, $y\in B(0,\widetilde{D}_{n_0+1})$:

(5.42)

$$|A_3 - A_3'| \le e^{-\kappa_{n_0+1}},\ \text{with}$$

$$A_3' \overset{\text{def}}{=} E^\sigma_{y,\omega}\Big[H_{K_\sigma}\le\frac{T}{2}\wedge T_{\frac15\mathcal{T}_{n_0+1}},\ F^e(H_{K_\sigma}, Z_{H_{K_\sigma}}) - F^\sigma(H_{K_\sigma}, Z_{H_{K_\sigma}})\Big]\,.$$

Applying the strong Markov property, we see that for $0\le k\le T$, $z\in\mathbb{R}^d$:

$$(5.43)\qquad \begin{aligned} F^e(k,z) - F^\sigma(k,z) &= Q^e\,F^e(k,z) - Q^\sigma\,F^\sigma(k,z)\\ &= Q^e(F^e - F^\sigma)(k,z) + (Q^e - Q^\sigma)\,F^\sigma(k,z)\,. \end{aligned}$$

The next lemma will provide an analogue of (4.139) for the Markov chain in the true environment (i.e. under $P^e_{z,\omega}$).

Lemma 5.3. *When L_0 is large, for $\sigma\in\Sigma$, $\omega\in\overline{G}_{\sigma,n_0+1}$, $z\in 3\mathcal{T}_{n_0+1}$, $n_0'\le n\le n_0$:*

$$(5.44)\qquad P^e_{z,\omega}\big[\sup_{0\le k\le k_n}|Z_k - Z_0| \ge 30\tilde{\ell}_0\,\widetilde{D}_n\big] \le e^{-\kappa_{n_0+1}}\,.$$

with $k_n \overset{(4.138)}{=} (L_n/L_{n_0'})^2$, and $\tilde{\ell}_0$ as below (4.2).

Proof. The argument is similar to the proof of (4.139). The probability in (5.44) coincides with

$$P_{z,\omega}\Big[\sup_{0\le k\le k_n} |X_{kL^2_{n'_0}} - X_0| \ge 30\tilde{\ell}_0\, \widetilde{D}_n\Big]. \tag{5.45}$$

On the event inside the above probability, $X_.$ exits the open set U defined below (4.142):

$$U = \Big(\bigcup_{1\le i\le \tilde{\ell}_0} B(w_i, 6\widetilde{D}_n)\Big) \cup \Big(\bigcup_{1\le i\le \tilde{\ell}} B(\sigma_i, 6\widetilde{D}_n)\Big),$$

where the w_i are omitted when $n = n'_0$. We denote with S the stopping time on $C(\mathbb{R}_+, \mathbb{R}^d)$:

$$S = \inf\big\{s \ge 0,\ |X_s - x| \ge 4\widetilde{D}_n,\ \text{ for all } x \in \{w_1, \dots, w_{\tilde{\ell}_0}, \sigma_1, \dots, \sigma_{\tilde{\ell}}\}\big\},$$

where the w_i are omitted when $n = n'_0$. From the discussion above, with the notation (1.18), the expression in (5.45) is smaller than:

$$\begin{aligned} &E_{z,\omega}\big[S < L_n^2,\, P_{X_S,\omega}[X^*_{L_n^2} \ge \widetilde{D}_n]\big] \overset{(2.10)}{\le} \\ &E_{z,\omega}\big[S < L_n^2 \wedge T_{4\mathcal{T}_{n_0+1}},\, P_{X_S,\omega}[X^*_{L_n^2} \ge \widetilde{D}_n]\big] + e^{-\kappa_{n_0+1}} \\ &\quad \le e^{-\kappa_n} + e^{-\kappa_{n_0+1}} \le e^{-\kappa_{n_0+1}}, \end{aligned} \tag{5.46}$$

using the definition of U, and (2.2) in the last step, together with the notation (1.51) and the remark below (4.1). This proves the lemma. □

We now work on the quantities that appear in the last line of (5.43). For $0 \le k \le T$, $z \in \frac{1}{2}\,\mathcal{T}_{n_0+1}$, we can write:

$$\begin{aligned} F^e(k,z) - F^\sigma(k,z) = &E^e_{z,\omega}\big[H_{K_\sigma} < (T-k) \wedge T_{\frac{3}{4}\mathcal{T}_{n_0+1}},\, f(Z_{T-k})\big] - \\ &E^\sigma_{z,\omega}\big[H_{K_\sigma} < (T-k) \wedge T_{\frac{3}{4}\mathcal{T}_{n_0+1}},\, f(Z_{T-k})\big] + \\ &E^e_{z,\omega}\big[H_{K_\sigma} \ge (T-k) \wedge T_{\frac{3}{4}\mathcal{T}_{n_0+1}},\, f(Z_{T-k})\big] - \\ &E^\sigma_{z,\omega}\big[H_{K_\sigma} \ge (T-k) \wedge T_{\frac{3}{4}\mathcal{T}_{n_0+1}},\, f(Z_{T-k})\big]. \end{aligned}$$

Note that when L_0 is large $\ell^2_{n_0}\, \widetilde{D}_{n_0} < \frac{1}{8}\, L^2_{n_0+1}$, so that with (4.139) and (5.44) when $n = n_0$, difference of the last two terms of the above equality is bounded in absolute value by

$$\begin{aligned} &\big|E^e_{z,\omega}\big[H_{K_\sigma} \wedge T_{\frac{3}{4}\mathcal{T}_{n_0+1}} \ge T-k,\, f(Z_{T-k})\big] - \\ &E^\sigma_{z,\omega}\big[H_{K_\sigma} \wedge T_{\frac{3}{4}\mathcal{T}_{n_0+1}} \ge T-k,\, f(Z_{T-k})\big]\big| + e^{-\kappa_{n_0+1}} \\ &\quad \le 2e^{-\kappa_{n_0+1}} \le e^{-\kappa_{n_0+1}}, \end{aligned}$$

using in the last step analogous estimates as for A_1, cf. (5.13), and the remark below (4.1). Further the terms in the first line of the right-hand side of the above equality are seen to coincide with $R^e F^e(k,z) - R^\sigma F^\sigma(k,z)$, after application of the Markov property at time $H_{K_\sigma} \wedge (T-k)$. Using once again estimates as in the control of A_1, or in the derivation of (5.42), we see that $R^e F^e(k,z) - R^\sigma F^\sigma(k,z)$ differs at most by $e^{-\kappa_{n_0+1}}$ from $R^e(F^e - F^\sigma)(k,z)$. Collecting our bounds, we see that when L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $0 \le k \le T$, $z \in \frac{1}{2}\,\mathcal{T}_{n_0+1}$:

$$|(F^e - F^\sigma)(k,z) - R^e(F^e - F^\sigma)(k,z)| \le e^{-\kappa_{n_0+1}}\,. \tag{5.47}$$

Letting $y' \in \frac{1}{4}\mathcal{T}_{n_0+1}$ be a dummy variable playing the role of $Z_{H_{K_\sigma}}$ in (5.42), and noting that in view of (5.39), (5.44), when $0 \le k' \le T$, $Q^e((k',y'), \{0,\dots,T\} \times (\frac{1}{2}\,\mathcal{T}_{n_0+1})^c) \le e^{-\kappa_{n_0+1}}$, we see with (5.43) and (5.47) that for $0 \le k' \le T$:

$$\begin{aligned}&|(F^e - F^\sigma)(k',y') - Q^e\,R^e(F^e - F^\sigma)(k',y') -\\ &(Q^e - Q^\sigma)\,F^\sigma(k',y')| \le e^{-\kappa_{n_0+1}}\,.\end{aligned} \tag{5.48}$$

Thanks to (5.43) the expression under the absolute value coincides with

(5.49)

$$\Big[F^e - F^\sigma - Q^e\,R^e\,Q^e(F^e - F^\sigma) - \sum_{m=0}^{1} (Q^e\,R^e)^m (Q^e - Q^\sigma) F^\sigma\Big](k',y')\,.$$

Using the strong Markov property, (5.39), (5.41), (2.1)

$$\begin{aligned}&Q^e\,R^e\,Q^e\big((k',y'), \{0,\dots,T\} \times \big(\tfrac{1}{2}\,\mathcal{T}_{n_0+1}\big)^c\big) \le\\ &P^e_{y',\omega}\big[\sup_{k\le T} |Z_k - Z_0| \ge \frac{1}{4}\,L^2_{n_0+1}\big] \overset{(5.44)}{\le} e^{-\kappa_{n_0+1}}\,.\end{aligned} \tag{5.50}$$

Hence using (5.47) to transform (5.49), we deduce from (5.48), (5.50) that

$$\begin{aligned}&\big|\big[F^e - F^\sigma - (Q^e R^e)^2\,(F^e - F^\sigma) -\\ &\sum_{m=0} (Q^e R^e)^m (Q^e - Q^\sigma) F^\sigma\big](k',y')\big| \le e^{-\kappa_{n_0+1}}\,.\end{aligned} \tag{5.51}$$

Note that (5.50) holds for $(Q^e R^e)^m Q^e$, $m \ge 0$, arbitrary in place of $(Q^e\,R^e)\,Q^e$, as follows from the strong Markov property. We can then repeat the above manipulation finitely many times and find that when L_0 is large, for $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $y' \in \frac{1}{4}\,\mathcal{T}_{n_0+1}$, $0 \le k' \le T$:

$$\begin{aligned}&\Big|\Big[F^e - F^\sigma - (Q^e R^e)^{m_*}\,(F^e - F^\sigma) -\\ &\sum_{0\le m<m_*} (Q^e\,R^e)^m (Q^e - Q^\sigma) F^\sigma\Big](k',y')\Big| \le e^{-\kappa_{n_0+1}}\,,\end{aligned} \tag{5.52}$$

with in the notation of (1.14), (1.17):

$$m_* = [a^{-1}(1+a)^{m_0+1}] + 1\,. \tag{5.53}$$

Keeping in mind that y' plays the role of $Z_{H_{K_\sigma}}$ and letting k' play the role of H_{K_σ} in (5.42), we are now going to bound $[(Q^e R^e)^{m_*} 1](k', y')$, for $0 \le k' \le \frac{T}{2}$, $y' \in \frac{1}{4}\,\mathcal{T}_{n_0+1}$.

Lemma 5.4. *When L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, for $0 \le k' \le \frac{T}{2}$, $y' \in \frac{1}{4}\,\mathcal{T}_{n_0+1}$,*

$$\sup_{0 \le m \le m_*} (Q^e R^e)^m \Big((k', y'),\ \Big[\frac{3}{4}\,T, T\Big] \times \mathbb{R}^d\Big) \le L_{n_0+1}^{-\frac{8}{10}}\,, \tag{5.54}$$

$$(Q^e R^e)^{m_*} \big((k', y'),\ [0, T] \times \mathbb{R}^d\big) \le 2L_{n_0+1}^{-\frac{8}{10}}\,. \tag{5.55}$$

Proof. We first prove (5.54). When $m = 0$, the expression that appears in (5.54) vanishes, and we can restrict to the case $1 \le m \le m_*$. We can rewrite the quantity in (5.54) using the strong Markov property, (5.39), (5.41), as the $P^e_{y,\omega}$-probability of a certain event (loosely speaking expressing the occurrence of m successive possibly truncated departures from U_σ and returns to K_σ prior to exit of $\frac{3}{4}\,\mathcal{T}_{n_0+1}$, with the m-th return taking place sometimes during $[\frac{3}{4}\,T - k', T - k'))$. On this event since truncated departures have at most a duration of t_0, cf. (5.39), at least one of the return periods has a duration of at least

$$\Big(\frac{3}{4}\,T - k' - m_*\,t_0\Big) \big/ m_* \ge \frac{T}{4m_*} - t_0\,.$$

As a result we have:

$$\begin{aligned}
&(Q^e R^e)^m \Big((k', y'), \Big[\frac{3}{4}\,T, T\Big] \times \mathbb{R}^d\Big) \le \\
&m \sup_{z \in \frac{3}{4}\,\mathcal{T}_{n_0+1}} P^e_{z,\omega}\Big[\frac{T}{4m_*} - t_0 \le H_{K_\sigma} < T\Big].
\end{aligned} \tag{5.56}$$

The probability that appears in the right-hand side of (5.56) is similar to the first probability that appears in (5.14), ($y \in B(0, \widetilde{D}_{n_0+1})$ is now replaced with $z \in \frac{3}{4}\,\mathcal{T}_{n_0+1}$, and $\frac{T}{2}$ with $\frac{T}{4m_*} - t_0$). The same estimates leading to (5.37) now yield for L_0 large:

$$m_* \sup_{z \in \frac{3}{4}\,\mathcal{T}_{n_0+1}} P^e_{z,\omega}\Big[\frac{T}{4m_*} - t_0 \le H_{K_\sigma} < T\Big] \le L_{n_0+1}^{-\frac{8}{10}}\,, \tag{5.57}$$

thus proving (5.54).

We now turn to the proof of (5.55). With (5.54) and using the strong Markov property in the second inequality, we find

$$(5.58)\qquad \begin{aligned} &(Q^e R^e)^{m_*}((k', y'), [0, T] \times \mathbb{R}^d) \le \\ &L_{n_0+1}^{-\frac{8}{10}} + (Q^e R^e)^{m_*}((k', y'), \Big(0, \frac{3T}{4}\Big) \times \mathbb{R}^d\Big) \le \\ &L_{n_0+1}^{-\frac{8}{10}} + \big(\sup_{z \in \frac{3}{4} \mathcal{T}_{n_0+1}} P_{z,\omega}\big[H_{K_\sigma} \circ \theta_{T_{U_\sigma} \wedge t_0} < T_{\mathcal{T}_{n_0+1}} \wedge T\big]\big)^{m_*} . \end{aligned}$$

The same argument employed in (5.16)–(5.18), shows that for $z \in \frac{3}{4} \mathcal{T}_{n_0+1}$, (recall $t_0 \overset{(5.40)}{=} k_{n_0'+3}$):

$$(5.59)\qquad \begin{aligned} &P^e_{z,\omega}\Big[d(Z_k, K_\sigma) \ge \frac{L_{n_0'+2}}{4\tilde{\ell}_0} - 11\widetilde{D}_{n_0'},\ \text{for some } 0 \le k < t_0\Big] \ge \\ &1 - e^{-\kappa_{n_0+1}} , \end{aligned}$$

so that we find with (5.7)

$$(5.60)\qquad \begin{aligned} &P_{z,\omega}\big[H_{K_\sigma} \circ \theta_{T_{U_\sigma} \wedge t_0} < T_{\mathcal{T}_{n_0+1}} \wedge T\big] \le \\ &e^{-\kappa_{n_0+1}} + E^e_{z,\omega}\big[T_{U_\sigma} < t_0, P^e_{Z_{T_{U_\sigma} \wedge t_0},\omega}[H_{K_\sigma} < T_{\mathcal{T}_{n_0+1}} \wedge T]\big] . \end{aligned}$$

But for $\overline{z} \in \mathcal{T}_{n_0+1} \backslash U_\sigma$ playing the role of $Z_{T_{U_\sigma} \wedge t_0},\omega$ in the last term of (5.60), we find just as for (5.15):

$$(5.61)\qquad P^e_{\overline{z},\omega}\big[H_{K_\sigma} < T_{\mathcal{T}_{n_0+1}} \wedge T\big] \le P^\sigma_{\overline{z},\omega}\big[H_{K_\sigma} < T_{\mathcal{T}_{n_0+1}} \wedge T\big] + e^{-\kappa_{n_0+1}} .$$

The first term on the right-hand side of (5.61) can be bounded in the same way as in (5.30)–(5.37), to obtain with L_0 large:

$$(5.62)\qquad P^\sigma_{\overline{z},\omega}[H_{K_\sigma} < T_{\mathcal{T}_{n_0+1}} \wedge T] \le \ell\Big(\frac{c\, L_{n_0'+2}}{D^*_{n_0'+1}}\Big)^{-\frac{99}{100}} + e^{-\kappa_{n_0+1}} \le \ell_{n_0'+1}^{-\frac{9}{10}} .$$

Coming back to (5.58), (5.60), we see that when L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $0 \le k' \le \frac{T}{2}$, $y' \in \frac{1}{4} \mathcal{T}_{n_0+1}$:

$$(5.63)\qquad \begin{aligned} (Q^e R^\sigma)^{m_*}\big((k', y'), [0, T] \times \mathbb{R}^d\big) &\le L_{n_0+1}^{-\frac{8}{10}} + \big(\ell_{n_0'+1}^{-\frac{9}{10}} + e^{-\kappa_{n_0+1}}\big)^{m_*} \\ &\overset{(1.15),(5.53)}{\le} 2L_{n_0+1}^{-\frac{8}{10}} . \end{aligned}$$

This proves the claim (5.55). □

We return to (5.52), and observe with the help of the above lemma that when L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma_{n_0+1}}$, for $0 \le k' \le \frac{T}{2}$, $y' \in K_\sigma \cap \big(\frac{1}{4} \mathcal{T}_{n_0+1}\big)$,

$$(5.64)\qquad \begin{aligned} &|(F^e - F^\sigma)(k', y')| \le \\ &c\Big(L_{n_0+1}^{-\frac{8}{10}} + \sup_{k \le \frac{3}{4} T,\, z \in K_\sigma \cap (\frac{3}{4} \mathcal{T}_{n_0+1})} |(Q^e - Q^\sigma)\, F^\sigma(k, z)|\Big) , \end{aligned}$$

where we used that $y' \in K_\sigma$ when handling the term corresponding to $m = 0$, in (5.52). We now bound the last term of (5.64). We consider $k \le \frac{3}{4}\,T$, $z \in K_\sigma \cap (\frac{3}{4}\,\mathcal{T}_{n_0+1})$, as above and introduce (recall $t_0 \overset{(5.40)}{=} k_{n_0'+3}$)

$$\widetilde{k} = \inf\{m \in t_0\,\mathbb{Z} + T;\ m \ge k + t_0\}\,. \tag{5.65}$$

With (5.39), and the Markov property in (5.38), we can write

$$\begin{aligned} Q^e\,F^\sigma(k,z) &= E^e_{z,\omega}\big[F^\sigma(k + T_{U_\sigma} \wedge t_0,\, Z_{T_{U_\sigma}\wedge t_0})\big] \\ &= E^e_{z,\omega}\big[E^\sigma_{Z_{T_{U_\sigma}\wedge t_0},\omega}\big[F^\sigma\big(\widetilde{k},\, Z_{\widetilde{k}-\overline{k}}\big)\big]\big]\,, \end{aligned} \tag{5.66}$$

where $\overline{k} = k + T_{U_\sigma} \wedge t_0$ is not part of the inner expectation. The same calculation for $Q^\sigma\,F^\sigma(k,z)$ and the strong Markov property yield:

$$Q^\sigma\,F^\sigma(k,z) = E^\sigma_{z,\omega}\big[F^\sigma\big(\widetilde{k},\, Z_{\widetilde{k}-k}\big)\big]\,. \tag{5.67}$$

Using controls on the size of displacements of $Z_.$ in a time interval of length t_0 or $2t_0$, under $P^\sigma_{z,\omega}$ or $P^e_{z,\omega}$, cf. (4.139), (5.44), we see that:

$$\sup_{k \le \frac{3}{4}\,T,\ z \in K_\sigma \cap (\frac{3}{4}\,\mathcal{T}_{n_0+1})} |(Q^e - Q^\sigma)\,F^\sigma(k,z)| \le e^{-\kappa_{n_0+1}} + \operatorname{var} F^\sigma, \text{ where} \tag{5.68}$$

$$\begin{aligned} \operatorname{var} F^\sigma \overset{\text{def}}{=} \sup\Big\{&|F^\sigma(\widetilde{k}, z_1) - F^\sigma(\widetilde{k}, z_2)|,\ z_1, z_2 \in \mathcal{T}_{n_0+1}, \\ &|z_1 - z_2| \le D^*_{n_0'+3}\,,\ \widetilde{k} \in (t_0\,\mathbb{Z} + T) \cap \Big[0, \frac{4}{5}\,T\Big]\Big\}\,. \end{aligned} \tag{5.69}$$

We will bound var F^σ with the help of the smoothness properties resulting from (4.179) and (5.38). We introduce a cut-off function h with values in $[0,1]$ such that with (2.1):

$$\begin{aligned} &h = 1 \text{ on } 2\mathcal{T}_{n_0+1},\ h = 0 \text{ on } \Big(\frac{5}{2}\,\mathcal{T}_{n_0+1}\Big)^c, \text{ and} \\ &|h|_{(n_0+1)} \le 1 + \frac{c}{L^\beta_{n_0+1}}\,. \end{aligned} \tag{5.70}$$

Lemma 5.5. *For large L_0, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, $n_0' \le n \le n_0$,*

$$\|h R^*_{n,\sigma}\|_{n_0+1} = \big\|h(R^*_{n_0',\sigma})^{k_n}\big\|_{n_0+1} \le 1 + \kappa_n\,\nu_n\,, \tag{5.71}$$

with ν_n defined in (4.17)*, and $k_n \overset{(4.138)}{=} L_n^2/L^2_{n_0'}$.*

Proof. The equality in (5.70) follows from (4.9), (5.69). Then with (4.9), (5.70), we can write

$$\begin{aligned} h(R^*_{n_0',\sigma})^{k_n} &= h\,R^*_{n,\sigma} \overset{(4.8)}{=} h\,R^0_n + h\,S^*_{n,\sigma} \\ &= h\,R^0_n + h\,\widetilde{S}^*_{n,\sigma} + h(S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma})\,. \end{aligned} \tag{5.72}$$

From (1.29), (1.55), (5.70) we have

$$\| h\, R_n^0 \|_{n_0+1} \le 1 + \frac{c}{L_{n_0+1}^\beta}\,, \tag{5.73}$$

and from (4.140) we deduce

$$\begin{aligned} \| h(S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma})\|_{n_0+1} &\le \Big(\frac{L_{n_0+1}}{L_n}\Big)^\beta \, \| h(S^*_{n,\sigma} - \widetilde{S}^*_{n,\sigma})\|_n \\ &\le \Big(\frac{L_{n_0+1}}{L_n}\Big)^\beta \, e^{-\kappa_{n_0}} \le e^{-\kappa_{n_0+1}}\,. \end{aligned} \tag{5.74}$$

If g is such that $|g|_{(n_0+1)} = 1$, and $x \in L_n\, \mathbb{Z}^d$ such that $\chi_{n,x}\, h \neq 0$, we can find $\widetilde{G}$ such that:

$$\begin{aligned} &\mathrm{Supp}\, \widetilde{G} \subseteq B(x, 4D_n^*), \\ &\widetilde{G} = g - g(x) \text{ on } B(x, 3D_n^*),\ |\widetilde{G}|_{(n)} \le \kappa_n \Big(\frac{L_n}{L_{n_0+1}}\Big)^\beta . \end{aligned} \tag{5.75}$$

We thus see, cf. above (4.12), that with (1.49)

$$\begin{aligned} &|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\, g|_{(n)} \le |\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\, \widetilde{G}|_{(n)} + e^{-\kappa_n} \le \\ &\|\chi_{n,x}\, \widetilde{S}^*_{n,\sigma}\|_n\, \kappa_n \Big(\frac{L_n}{L_{n_0+1}}\Big)^\beta + e^{-\kappa_n} \overset{(4.179)}{\le} \kappa_n\, \nu_n \Big(\frac{L_n}{L_{n_0+1}}\Big)^\beta . \end{aligned} \tag{5.76}$$

As a consequence we see with (A.3) from the Appendix and (5.70) that

$$|h\, \widetilde{S}^*_{n,\sigma}\, g|_{(n_0+1)} \le \Big(\frac{L_{n_0+1}}{L_n}\Big)^\beta |h\, \widetilde{S}^*_{n,\sigma}\, g|_{(n)} \le \kappa_n\, \nu_n = \kappa_n\, \nu_n\, |g|_{n_0+1}\,. \tag{5.77}$$

Collecting (5.72), (5.73), (5.74), (5.77) we obtain (5.71). □

We return to the task of bounding (5.69). With $\widetilde{k}$ as in (5.69) we have $T - \widetilde{k} - k_{n_0} \in t_0\, \mathbb{N}$, and hence we can write

$$T - \widetilde{k} - k_{n_0} = \sum_{n_0'+3 \le n \le n_0} u_n\, k_n, \ \text{with } u_n \text{ suitable integers in } [0, \ell_n^2 - 1)\,. \tag{5.78}$$

Then for $z \in \mathcal{T}_{n_0+1}$, f as in (5.6), (or (5.9)), we have:

$$\begin{aligned} F^\sigma(\widetilde{k}, z) &\overset{(5.38)}{=} (R^*_{n_0',\sigma})^{T-\widetilde{k}}\, f(z) \\ &= (R^*_{n_0',\sigma})^{k_{n_0}} (R^*_{n_0',\sigma})^{u_{n_0'+3} k_{n_0'+3}} \dots (R^*_{n_0',\sigma})^{u_n k_n}\, f(z)\,. \end{aligned}$$

Using (4.9) and $(T - \widetilde{k})\, \widetilde{D}_{n_0'} < \frac{1}{10}\, L^2_{n_0+1}$, cf. below (5.10), we find

$$\begin{aligned} F^\sigma(\widetilde{k}, z) &= R^*_{n_0,\sigma} (h R^*_{n_0'+3,\sigma})^{u_{n_0'+3}} \dots (h R^*_{n,\sigma})^{u_n} \dots (h R^*_{n_0,\sigma})^{u_{n_0}}\, f(z) \\ &= R^*_{n_0,\sigma}\, \widetilde{f}(z)\,, \end{aligned} \tag{5.79}$$

where in view of (5.71), (5.78)

(5.80)

$$|\widetilde{f}|_{(n_0+1)} \le \prod_{n_0'+3\le n\le n_0} (1+\kappa_n\,\nu_n)^{\ell_n^2} \le \exp\Big\{\sum_{n_0'+3\le n\le n_0} \kappa_n\,\nu_n\,\ell_n^2\Big\} \overset{(1.15),(4.17)}{\le} c\,.$$

So we see that for $z_1, z_2 \in \mathcal{T}_{n_0+1}$, with $|z_1 - z_2| \le D^*_{n_0'+3}$,

$$\begin{aligned}
&|F^\sigma(\widetilde{k}, z_1) - F^\sigma(\widetilde{k}, z_2)| \overset{(5.79)}{\le}\\
&\big|R^0_{n_0}\widetilde{f}(z_1) - R^0_{n_0}\widetilde{f}(z_2)\big| + |S^*_{n_0,\sigma}\widetilde{f}(z_1) - S^*_{n_0,\sigma}\widetilde{f}(z_2)| \underset{(4.179)}{\overset{(1.49),(1.56)}{\le}}\\
&\frac{c\,D^*_{n_0'+3}}{L_{n_0}} + c\Big(\frac{D^*_{n_0'+3}}{L_{n_0}}\Big)^\beta \nu_{n_0} \overset{(4.17),(4.1)}{\le}\\
&\kappa_{n_0}\Big(L_{n_0+1}^{-(\frac{1}{1+a}-(1+a)^{-(m_0-1)})} +\\
&\qquad L_{n_0+1}^{-\big(\frac{\beta}{1+a}+\frac{\beta}{4(a+1)}-\beta(1+a)^{-(m_0-1)}-(\frac{\beta}{4}-\delta)(1+a)^{-(m_0+2)}\big)}\Big) \underset{(1.40)}{\overset{(1.14),(1.17)}{\le}}\\
&L_{n_0+1}^{-(\beta+\delta)}\Big(L_{n_0+1}^{-2a} + L_{n_0+1}^{-\big(\frac{\beta}{4(1+a)}-\delta-a\,\frac{\beta}{1+a}-\frac{\beta}{100}-\frac{1}{100}(\frac{\beta}{4}-\delta)\big)}\Big) \le\\
&c\,L_{n_0+1}^{-(\beta+\delta+2a)}\,.
\end{aligned} \tag{5.81}$$

So we have shown that when L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$,

$$\operatorname{var} F^\sigma \le c\,L_{n_0+1}^{-(\beta+\delta+2a)}\,. \tag{5.82}$$

Collecting (5.42), (5.64), (5.68), we obtain since $\beta+\delta+2a < \frac{8}{10}$,

$$A_3 \le c\,L_{n_0+1}^{-(\beta+\delta+2a)}\,. \tag{5.83}$$

Substituting in (5.9) the bounds (5.13), (5.37), (5.83) we now obtain (5.6) and this concludes the proof of Proposition 5.1. □

As an application of Proposition 4.11 and 5.1, we have

Proposition 5.6. *When L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$,*

$$\big\|\chi_{n_0+1,0}\big(R_{n_0+1} - \big(R^0_{n_0}\big)^{\ell^2_{n_0}}\big)\big\|_{n_0+1} \le c\,L_{n_0+1}^{-(\delta+a)}\,. \tag{5.84}$$

Proof. We have

$$\begin{aligned}
&\big\|\chi_{n_0+1,0}\big(R_{n_0+1} - \big(R^0_{n_0}\big)^{\ell^2_{n_0}}\big)\big\|_{n_0+1} \le\\
&\big\|\chi_{n_0+1,0}\big(R_{n_0+1} - R^*_{n_0+1,\sigma}\big)\big\|_{n_0+1} +\\
&\big\|\chi_{n_0+1,0}\big(R^*_{n_0+1,\sigma} - \big(R^0_{n_0}\big)^{\ell^2_{n_0}}\big)\big\|_{n_0+1} \overset{(4.180)}{\le}\\
&\big\|\chi_{n_0+1,0}\big(R_{n_0+1} - R^*_{n_0+1,\sigma}\big)\big\|_{n_0+1} + \nu_{n_0+1}\,.
\end{aligned} \tag{5.85}$$

With the notation of (5.6), and with (4.9), we also find that:

$$\begin{aligned}\chi_{n_0+1,0}(R_{n_0+1} - R^*_{n_0+1,\sigma}) &= \chi_{n_0+1,0}\big(R_{n_0'}(R_{n_0'})^T - R^*_{n_0',\sigma}(R^*_{n_0',\sigma})^T\big) = \\ \chi_{n_0+1,0}\, R_{n_0'}\big((R_{n_0'})^T - (R^*_{n_0',\sigma})^T\big) &+ \chi_{n_0+1,0}(R_{n_0'} - R^*_{n_0',\sigma})(R^*_{n_0',\sigma})^T\,.\end{aligned} \tag{5.86}$$

With (1.60) and (5.6), we see that

$$\begin{aligned}&\big\| R_{n_0'}\, 1_{B(0,\widetilde{D}_{n_0+1})}\big((R_{n_0'})^T - (R^*_{n_0',\sigma})^T\big)\big\|_{n_0+1} \le \\ &\Big(\frac{L_{n_0+1}}{L_{n_0'}}\Big)^\beta c\, L^\beta_{n_0'}\, L^{-(\beta+\delta+a)}_{n_0+1} \le c\, L^{-(\delta+a)}_{n_0+1}\,.\end{aligned} \tag{5.87}$$

Also note that when $|g|_\infty \le 2$ and $g\, 1_{B(0,\widetilde{D}_{n_0+1})} = 0$, then with the notation (1.57), $\chi_{n_0+1,0}\, R_{n_0'}\, g = \chi_{n_0+1,0}\, P_{1,\omega}\, P_{L^2_{n_0'}-1,\omega}\, g$, and from inequalities such as in (2.10), and from (1.17), we see that $|1_{B(0,D_{n_0+1})}\, P_{L^2_{n_0'}-1,\omega}\, g|_\infty \le e^{-c\, L_{n_0'}}$, so that using (1.59) as in the proof of (1.60), we find that $|\chi_{n_0+1,0}\, R_{n_0'}\, g|_{(n_0+1)} \le e^{-c\, L_{n_0'}}$. Coming back to (5.87), we hence obtain:

$$\big\| \chi_{n_0+1,0}\, R_{n_0'}\big((R_{n_0'})^T - (R^*_{n_0',\sigma})^T\big)\big\|_{n_0+1} \le c\, L^{-(\delta+a)}_{n_0+1}\,. \tag{5.88}$$

We now turn to the last term of (5.86) and observe that:

$$R_{n_0'} - R^*_{n_0',\sigma} \overset{(4.7)}{=} (1-g_\sigma)(R_{n_0'} - \widetilde{R}^0_{n_0'}) + g_\sigma(R_{n_0'} - \widetilde{R}_{n_0'})\,.$$

With the same argument employed above (5.88), cf. (1.20), (1.37), for the notation, applied to the last expression of the following identity

$$\begin{aligned}&\chi_{n_0+1,0}\, g_\sigma(R_{n_0'} - \widetilde{R}_{n_0'})(R^*_{n_0',\sigma})^T = \chi_{n_0+1,0}\, g_\sigma(R_{n_0'} - \widetilde{R}_{n_0'})\, \chi_{D_{n_0+1}}(R^*_{n_0',\sigma})^T + \\ &\chi_{n_0+1,0}\, g_\sigma\, R_{n_0'}(1-\chi_{D_{n_0+1}})(R^*_{n_0',\sigma})^T \overset{\text{def}}{=} A_1 + A_2\,,\end{aligned}$$

we see that $\|A_2\|_{n_0+1}$ is smaller than $e^{-cL_{n_0'}}$. Further just as in (5.80) we see that:

$$\big\|\chi_{D_{n_0+1}}(R^*_{n_0',\sigma})^T\big\|_{n_0+1} \le c$$

and together with (4.6), (2.2), (2.46) we obtain:

$$\begin{aligned}\|A_1\|_{n_0+1} &\le \big\|\chi_{n_0+1,0}\, g_\sigma\, (S_{n_0'} - \widetilde{S}_{n_0'})\, \chi_{D_{n_0+1}}(R^*_{n_0',\sigma})^T\big\|_{n_0+1} + e^{-\kappa_{n_0+1}} \\ &\le e^{-\kappa_{n_0+1}}\,.\end{aligned} \tag{5.89}$$

In view of the identity below (5.88), to control the rightmost expression in (5.86), it remains to bound $\|\chi_{n_0+1,0}(1-g_\sigma)(R_{n_0'} - \widetilde{R}^0_{n_0'})(R^*_{n_0',\sigma})^T\|_{n_0+1}$. To this end in analogy with (1.20) we define the probability kernel

$$(5.90)\quad \begin{aligned} &R^*_{n'_0}(x,dy) = P_{x,\omega}[X_{L^2_{n'_0}\wedge T^*_{n'_0}} \in dy],\ x \in \mathbb{R}^d, \omega \in \Omega, \text{ with} \\ &T^*_{n'_0} = \inf\{u \ge 0, X^*_u \ge D^*_{n'_0}\},\ \text{cf. (4.10), (1.18) for the notation}\,. \end{aligned}$$

As in Lemma 5.3, see in particular (5.46), we see that when L_0 is large, $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma,n_0+1}$, for $y \in B(0, \widetilde{D}_{n_0+1})$,

$$P_{y,\omega}\Big[X^*_{L^2_{n'_0}} \ge \frac{D^*_{n'_0}}{2}\Big] \le e^{-\kappa_{n'_0}}\,.$$

Then with a slight variation on the proof of Proposition 2.5, for $x \in L_{n'_0}\mathbb{Z}^d \cap B(0, D_{n_0+1})$,

$$(5.91)\quad \big\| \chi_{n'_0,x}(R^*_{n'_0} - R_{n'_0})\big\|_{n'_0} \le e^{-\kappa_{n'_0}}\,.$$

Employing a similar identity as above (5.89) in the first inequality, and (5.91) in the second, we find

$$\begin{aligned} &\big\| \chi_{n_0+1,0}(1-g_\sigma)\big(R_{n'_0} - \widetilde{R}^0_{n'_0}\big)(R^*_{n'_0,\sigma})^T \big\|_{n_0+1} \le \\ &\big\| \chi_{n_0+1,0}(1-g_\sigma)\big(R^*_{n'_0} - \widetilde{R}^0_{n'_0}\big)\chi_{D_{n_0+1}}(R^*_{n'_0,\sigma})^T \big\|_{n_0+1} + \\ &\big\| \chi_{n_0+1,0}(1-g_\sigma)R_{n'_0}(1-\chi_{D_{n_0+1}})(R^*_{n'_0,\sigma})^T \big\|_{n_0+1} + e^{-\kappa_{n'_0}} \le \\ &\big\| \chi_{n_0+1,0}(1-g_\sigma)\big(R^*_{n'_0} - \widetilde{R}^0_{n'_0}\big)\, \chi_{D_{n_0+1}}(R^*_{n'_0,\sigma})^T \big\|_{n_0+1} + e^{-\kappa_{n'_0}}\,, \end{aligned}$$

with the same argument as applied above (5.88). Note that thanks to (1.60), (4.6), (5.91), $\|(1-g_\sigma)\,\chi_{n'_0,x}(R^*_{n'_0} - \widetilde{R}^0_{n'_0})\|_{n'_0} \le c\,L^\beta_{n'_0}$, with x as above (5.91). For f with $|f|_{(n_0+1)} \le 1$, and writing $Q = (1-g_\sigma)(R^*_{n'_0} - \widetilde{R}^0_{n'_0})$, we also find

$$(5.92)\quad \chi_{n_0+1,0}\, Q\, \chi_{D_{n_0+1}}(R^*_{n'_0,\sigma})^T f = \chi_{n_0+1,0}\, Q\, \chi_{D_{n_0+1}}(R^*_{n'_0,\sigma})^{k_{n_0}} \widetilde{f}$$

where $\widetilde{f}$ just as in (5.79), (5.80) satisfies

$$(5.93)\quad |\widetilde{f}|_{(n_0+1)} \le c\,.$$

Further if $x \in L_{n'_0}\mathbb{Z}^d$ is such that $d(x, \mathrm{Supp}\,\chi_{n_0+1,0}) \le 30\sqrt{d}\,L_{n'_0}$, we can use a cut-off function and construct $\widetilde{H}_1$, $\widetilde{H}_2$ supported in $B(x, 3D^*_{n'_0})$ (where $\chi_{D_{n_0+1}}(\cdot) = 1$), such that in $\overline{B}(x, 2D^*_{n'_0})$

$$(5.94)\quad \begin{aligned} &\widetilde{H}_1 \text{ coincides with } R^0_{n_0}\,\widetilde{f}(\cdot) - R^0_{n_0}\,\widetilde{f}(x)\,, \\ &\widetilde{H}_2 \text{ coincides with } S^*_{n_0,\sigma}\,\widetilde{f}(\cdot) - S^*_{n_0,\sigma}\,\widetilde{f}(x) \overset{(4.8),(4.9)}{=} (R^*_{n'_0,\sigma})^{k_{n_0}}\,\widetilde{f}(\cdot) \\ &\qquad - (R^*_{n'_0,\sigma})^{k_{n_0}}\,\widetilde{f}(x) - R^0_{n_0}\,\widetilde{f}(\cdot) + R^0_{n_0}\,\widetilde{f}(x)\,, \end{aligned}$$

and so that they satisfy the bounds

$$|\widetilde{H}_1|_{(n_0')} \overset{(1.56)}{\leq} \kappa_{n_0'} \frac{L_{n_0'}}{L_{n_0}}, \quad |\widetilde{H}_2|_{(n_0')} \overset{(4.179)}{\leq} \kappa_{n_0'} \Big(\frac{L_{n_0'}}{L_{n_0}}\Big)^\beta \nu_{n_0}. \tag{5.95}$$

As a result we obtain

$$\begin{aligned}\big|\chi_{n_0',x}\, Q\, \chi_{D_{n_0+1}} (R^*_{n_0',\sigma})^T f\big|_{(n_0')} &\leq |\chi_{n_0',x}\, Q\, \widetilde{H}_1|_{(n_0')} + |\chi_{n_0',x}\, Q\, \widetilde{H}_2|_{(n_0')} \\ &\underset{(5.95)}{\overset{(2.2),(4.6)}{\leq}} \kappa_{n_0'} L^\beta_{n_0'} \Big(\frac{L_{n_0'}}{L_{n_0}} + \Big(\frac{L_{n_0'}}{L_{n_0}}\Big)^\beta \nu_{n_0}\Big).\end{aligned}$$

We thus find

$$\begin{aligned}&\|\chi_{n_0+1,0}(1-g_\sigma)(R_{n_0'} - \widetilde{R}_{n_0'})(R^*_{n_0',\sigma})^T\|_{n_0+1} \leq \\ &\kappa_{n_0'} L^\beta_{n_0'} \Big(\frac{L_{n_0+1}}{L_{n_0}}\Big)^\beta \Big(\Big(\frac{L_{n_0'}}{L_{n_0}}\Big)^{1-\beta} + \nu_{n_0}\Big) + e^{-\kappa_{n_0'}} \leq L_{n_0+1}^{-(\delta+a)},\end{aligned} \tag{5.96}$$

using similar calculations as in the bottom lines of (5.81). Collecting (5.88), (5.89), (5.96), we obtain (5.84). □

Before concluding the proof of Theorem 1.1, we yet have to control the difference $\alpha_{n_0+1} - \alpha_{n_0}$.

Proposition 5.7. *Under the assumptions of Theorem 1.1, when L_0 is large,*

$$|\alpha_{n_0+1} - \alpha_{n_0}| \leq L_{n_0}^{-(1+\frac{9}{10})\delta}. \tag{5.97}$$

Proof. Recall the definition of α_n in (1.22). In analogy with (2.5) we consider the function, cf. (1.37) for the notation:

$$f(z) = \chi_{2\widetilde{D}_{n_0+1}}(z) \frac{|z|^2}{L^2_{n_0+1}}, \; z \in \mathbb{R}^d, \tag{5.98}$$

so that $|f|_{(n_0+1)} \leq \kappa_{n_0+1}$, and:

$$\alpha_{n_0+1} \overset{(1.22)}{=} \mathbb{E}[\widetilde{R}_{n_0+1} f(0)]. \tag{5.99}$$

We denote with $\widetilde{\Omega}$ the event

$$\begin{aligned}\widetilde{\Omega} = \Big\{&\omega \in \Omega;\; \text{for } |y| \leq 30\sqrt{d}\, L_{n_0+1}, \\ &P_{y,\omega}[X^*_{L^2_{n_0+1}} \geq v] \leq \exp\Big\{-\frac{v}{D_{n_0+1}}\Big\}, \text{for all } v \geq D_{n_0+1}\Big\} \cap \\ &\big\{\omega \in \Omega;\; \text{for all } x \in L_{n_0}\mathbb{Z}^d \cap (5\mathcal{T}_{n_0+1}), x \in \widetilde{B}_{n_0}(\omega)\big\}.\end{aligned} \tag{5.100}$$

With (2.9) and (1.47), we see that when L_0 is large,

$$\mathbb{P}[\widetilde{\Omega}^c] \leq \frac{1}{10} L_{n_0+1}^{-M_0} + c\Big(\frac{L^2_{n_0+1}}{L_{n_0}}\Big)^d L_{n_0}^{-M_0} \underset{(1.46)}{\overset{(1.14),(1.15)}{\leq}} L_{n_0+1}^{-10}. \tag{5.101}$$

Then for $\omega \in \widetilde{\Omega}$, we see that (cf. (1.37) for the notation):

$$\big|\widetilde{R}_{n_0+1}\, f(0) - \big(R^0_{n_0} + \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\big)^{\ell^2_{n_0}} f(0)\big| \le |\widetilde{R}_{n_0+1}\, f(0) - R_{n_0+1}\, f(0)| +$$
$$\big|\big(R^0_{n_0} + S_{n_0}\big)^{\ell^2_{n_0}} f(0) - \big(R^0_{n_0} + \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\big)^{\ell^2_{n_0}} f(0)\big| \le e^{-\kappa_{n_0+1}} +$$
$$\big|\sum_{0\le k<\ell^2_{n_0}} (R_{n_0})^k\,(1-\chi_{\widetilde{D}_{n_0+1}})\, S_{n_0}\, \big(R^0_{n_0} + \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\big)^{\ell^2_{n_0}-k-1} f(0)\big|$$

using (2.46) with $n = n_0 + 1$, and perturbation expansion in the last step. Since $R^0_{n_0} + \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0} = (1-\chi_{\widetilde{D}_{n_0+1}})\, R^0_{n_0} + \chi_{\widetilde{D}_{n_0}}\, R_{n_0}$ contracts the sup-norm, we see with (5.100), that when L_0 is large, for $\omega \in \widetilde{\Omega}$:

$$\big|\widetilde{R}_{n_0+1}\, f(0) - \big(R^0_{n_0} + \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\big)^{\ell^2_{n_0}} f(0)\big| \le e^{-\kappa_{n_0+1}}\,. \tag{5.102}$$

Using perturbation expansion as in (4.15) we find that for all $\omega \in \Omega$:

$$\begin{aligned}&\big(R^0_{n_0} + \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\big)^{\ell^2_{n_0}} f(0) - \big(R^0_{n_0}\big)^{\ell^2_{n_0}} f(0) = \\ &\sum_{0\le k<\ell^2_{n_0}} \big(R^0_{n_0}\big)^k \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0} (R^0_{n_0})^{\ell^2_{n_0}-k-1} f(0) + \\ &\sum_{\substack{k_0+\dots+k_m+m=\ell^2_{n_0}\\ k_i\ge 0, m\ge 2}} \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\big(R^0_{n_0}\big)^{k_1} \dots \chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\big(R^0_{n_0}\big)^{k_m} f(0)\,.\end{aligned} \tag{5.103}$$

Further for $\omega \in \widetilde{\Omega}$, $\|\chi_{\widetilde{D}_{n_0+1}}\, S_{n_0}\| \overset{(2.2),(2.46)}{\le} c\, L_{n_0}^{-\delta}$, so that the term in the last line of (5.103) is smaller in absolute value than:

$$\begin{aligned}\sum_{\substack{k_0+\dots+k_m+m=\ell^2_{n_0}\\ k_i\ge 0, m\ge 2}} \big(c\, L^{-\delta}_{n_0}\big)^m\, \kappa_{n_0+1} &= \kappa_{n_0+1}\big[\big(1 + c\, L^{-\delta}_{n_0}\big)^{\ell^2_{n_0}} - 1 - c\, \ell^2_{n_0}\, L^{-\delta}_{n_0}\big] \\ &\le \kappa_{n_0+1}\, L^{-2\delta+4a}_{n_0}\,,\end{aligned} \tag{5.104}$$

with c denoting the same constant in both members of the equality, and using a similar argument as in (4.172).

Coming back to (5.102), (5.103), noting that $(R^0_{n_0})^{\ell^2_{n_0}} f(0) = P_{\alpha_{n_0}\, L^2_{n_0+1}}\, f(0)$, cf. (1.21), (1.54), and that in view of (1.49) i) and (5.98) this quantity differs at most by $e^{-\kappa_{n_0+1}}$ from $d\alpha_{n_0}$, we see that for $\omega \in \widetilde{\Omega}$:

$$\begin{aligned}&\big|\widetilde{R}_{n_0+1} f(0) - d\alpha_{n_0} - \sum_{0\le k<\ell^2_{n_0}} \big(R^0_{n_0}\big)^k \chi_{\widetilde{D}_{n_0+1}} \widetilde{S}_{n_0}\big(R^0_{n_0}\big)^{\ell^2_{n_0}-k-1} f(0)\big| \le \\ &\kappa_{n_0+1}\, L^{-2\delta+4a}_{n_0},\end{aligned} \tag{5.105}$$

where we used (2.46) with $n = n_0$.

Observe that for $z \in B(0, \frac{3}{2}\, \widetilde{D}_{n_0+1})$, with (1.49) i) and (5.98),

$$\sup_{0 \le k < \ell_{n_0}^2} \left| \left(R_{n_0}^0\right)^{\ell_{n_0}^2 - k - 1} (f - g)(z) \right| \le e^{-\kappa_{n_0+1}}, \text{ with } g(\cdot) = \frac{|\cdot|^2}{L_{n_0+1}^2}\,.$$

Hence with (5.105) we see that when L_0 is large, for $\omega \in \widetilde{\Omega}$:

(5.106)

$$\begin{aligned} &\Big| \widetilde{R}_{n_0+1}\, f(0) - d\alpha_{n_0} - \\ &\sum_{0 \le k < \ell_{n_0}^2} \int P_{\alpha_{n_0} k L_{n_0}^2}(0, dz)\, \chi_{\widetilde{D}_{n_0+1}}(z) \Big(\frac{2\widetilde{d}_{n_0}(z,\omega)}{L_{n_0+1}^2} \cdot z + \sum_{i=1}^d \frac{\widetilde{\gamma}_{n_0}^{i,i}(z,\omega)}{L_{n_0+1}^2} \Big) \Big| \le \\ &\kappa_{n_0+1}\, L_{n_0}^{-2\delta+4a}\,. \end{aligned}$$

In view of (1.24), (1.25), the $\mathbb{P}$-expectation of the sum in (5.106) vanishes. Hence with (5.101) we see that for large L_0:

(5.107)

$$\begin{aligned} &\Big| \mathbb{E}\Big[\widetilde{\Omega}, \sum_{0 \le k < \ell_{n_0}^2} \int P_{\alpha_{n_0} k L_{n_0}^2}(0, dz)\, \chi_{\widetilde{D}_{n_0+1}}(z) \Big(\frac{2\widetilde{d}_{n_0}(z,\omega)}{L_{n_0+1}^2} \cdot z + \sum_{i=1}^d \frac{\widetilde{\gamma}_{n_0}^{i,i}(z,\omega)}{L_{n_0+1}^2} \Big) \Big] \Big| \le \\ &\kappa_{n_0}\, \ell_{n_0}^2\, L_{n_0+1}^{-10} \le L_{n_0+1}^{-9}\,. \end{aligned}$$

So using (5.101), (5.105), (5.107), we see that when L_0 is large

$$\begin{aligned} d|\alpha_{n_0+1} - \alpha_{n_0}| &\le \big| \mathbb{E}\big[\widetilde{R}_{n_0+1}\, f(0) - d\alpha_{n_0}, \widetilde{\Omega}^c \big] \big| + \mathbb{E}[\widetilde{R}_{n_0+1}\, f(0) - d\alpha_{n_0}, \widetilde{\Omega}]| \\ &\le \kappa_{n_0+1} L_{n_0}^{-2\delta+4a} \overset{(1.14),(1.40)}{\le} L_{n_0}^{-(1+\frac{9}{10})\delta}\,, \end{aligned}$$

and (5.97) is proved. □

We can now conclude the proof of Theorem 1.1. We have just shown (1.50) and there remains to complete the proof of (1.47) with $n = n_0 + 1$. With (5.84), we see that when L_0 is large, for $\sigma \in \Sigma$, $\omega \in \overline{G}_{\sigma, n_0+1}$,

$$\begin{aligned} \| \chi_{n_0+1,0}\, S_{n_0+1} \|_{n_0+1} &\le c\, L_{n_0+1}^{-(\delta+a)} + \| P_{\alpha_{n_0} L_{n_0+1}^2} - P_{\alpha_{n_0+1} L_{n_0+1}^2} \|_{n_0+1} \\ &\le c\, L_{n_0+1}^{-(\delta+a)} + c\, |\alpha_{n_0+1} - \alpha_{n_0}| \overset{(5.97)}{\le} c\, L_{n_0+1}^{-(\delta+a)}\,, \end{aligned}$$

using in the second inequality a similar bound as in (4.173). Further with (5.5) we find $\mathbb{P}[(\bigcup_{\sigma \in \Sigma} \overline{G}_{\sigma, n_0+1})^c] \le \frac{1}{10}\, L_{n_0+1}^{-M_0}$. These bounds together with (2.9) and (2.46) show that

$$\mathbb{P}[0 \notin \mathcal{B}_{n_0+1}(\omega)] \le \Big(\frac{1}{10} + \frac{1}{10} \Big) L_{n_0+1}^{-M_0} \le L_{n_0+1}^{-M_0}\,.$$

This concludes the proof of (1.47) for $n = n_0 + 1$, and hence of Theorem 1.1. □

6. Invariance principle, transience and homogenization

In this section as mentioned in the introduction, we apply Theorem 1.1 and prove an invariance principle and transience for isotropic diffusions in random environment that are small perturbations of Brownian motion, cf. Theorem 6.3. We also provide an application to homogenization, cf. Theorem 6.4. But the heart of the matter really comes with Proposition 6.2, where a sequence of good couplings of the diffusion in random environment with Brownian motion of variance α_n is constructed. We begin with a lemma that is helpful when applying Theorem 1.1.

Lemma 6.1. *When L_0 is large, for $\omega \in \Omega$, $0 \le n \le m_0 + 1$,*

$$\|\chi_{n,0}(P_{1,\omega} - P_1)\, P_{L_n^2-1}\|_n \le \frac{1}{10}\, L_n^{-\delta}\,, \tag{6.1}$$

cf. (1.17), (1.38), (1.40), (1.54), (1.57) *for the notation.*

Proof. We recall the convention $L_{-1} = 1$, see below (1.15), and extend using this convention the definitions $|\cdot|_{(n)}$, $\|\cdot\|_n$, $\chi_{n,x}$, to the case $n = -1$, cf. (1.28), (1.30), (1.38). We also introduce the probability kernels, see above (1.21) for the notation

$$\begin{aligned} \widetilde{P}_{1,\omega}(z, dy) &= P_{x,\omega}[X_{1\wedge T_{-1}} \in dy], \\ \widetilde{P}_1(x, dy) &= W_x[X_{1\wedge T_{-1}} \in \cdot],\ x \in \mathbb{R}^d,\ \text{where} \end{aligned} \tag{6.2}$$

$$T_{-1} = \inf\left\{u \ge 0,\ X_u^* \ge L_0^{\frac{1}{10}}\right\}. \tag{6.3}$$

With the same proof as in Proposition 2.5, using exponential inequalities, cf. [23, p. 145], in place of (2.45), we see that for large L_0, for $\omega \in \Omega$, $x \in \mathbb{Z}^d$,

$$\|\chi_{-1,x}(P_{1,\omega} - \widetilde{P}_{1,\omega})\|_{-1} \vee \|\chi_{-1,x}(P_1 - \widetilde{P}_1)\|_{-1} \le e^{-c\,L_0^{1/10}}\,. \tag{6.4}$$

Hence it follows that for $0 \le n \le m_0 + 1$,

$$\begin{aligned} &\|\chi_{-1,x}(P_{1,\omega} - P_1) P_{L_n^2-1}\|_{-1} \le \\ &\|\chi_{-1,x}(P_{1,\omega} - \widetilde{P}_{1,\omega})\, P_{L_n^2-1}\|_{-1} \ + \\ &\|\chi_{-1,x}(\widetilde{P}_{1,\omega} - \widetilde{P}_1) P_{L_n^2-1}\|_{-1} + \|\chi_{-1,x}(\widetilde{P}_1 - P_1) P_{L_n^2-1}\|_{-1} \le \\ &c\, e^{-cL_0^{1/10}} \ + \|\chi_{-1,x}(\widetilde{P}_{1,\omega} - \widetilde{P}_1) P_{L_n^2-1}\|_{-1}\,. \end{aligned} \tag{6.5}$$

With a similar argument as in (5.94), for $0 \le n \le m_0 + 1$, and f with $|f|_{(n)} \le 1$, we can construct with a cut-off function, a function $\widetilde{H}$ supported in $B(x, 3L_0^{1/10})$, such that:

$$\begin{aligned} &\widetilde{H} \text{ agrees with } P_{L_n^2-1}\, f - P_{L_n^2-1}\, f(x) \text{ in } B\big(x, 2L_0^{1/10}\big) \\ &\text{and } |\widetilde{H}|_{(-1)} \le c\, \frac{L_0^{\frac{1}{10}}}{L_n}\,, \end{aligned} \tag{6.6}$$

where (1.56) has been used for the last inequality. We hence find that with large L_0

$$|\chi_{-1,x}(\widetilde{P}_{1,\omega}-\widetilde{P}_1)P_{L_n^2-1}\,f|_{(-1)} = |\chi_{-1,x}(\widetilde{P}_{1,\omega}-\widetilde{P}_1)H|_{(-1)} \overset{(1.62),(6.4),(6.6)}{\leq} c\,L_0^{\frac{1}{10}}\,L_n^{-1}\,,$$

and hence with (6.5), (6.6):

$$\begin{aligned}\|\chi_{n,0}(P_{1,\omega}-P_1)\,P_{L_n^2-1}\|_n &\leq L_n^{\beta}\,\|\chi_{n,0}(P_{1,\omega}-P_1)\,P_{L_n^2-1}\|_{-1}\\ &\leq L_n^{\beta}\big(c\,e^{-cL_0^{\frac{1}{10}}}+c\,L_0^{\frac{1}{10}}\,L_n^{-1}\big)\\ &\overset{(1.17)}{\leq} \frac{1}{10}\,L_n^{-\delta}\,.\end{aligned} \tag{6.7}$$

This proves our claim. □

The next proposition is instrumental and enables to construct good couplings of the diffusion in random environment with Brownian motion. From now on we specify the choices of $\nu = 2$, $\beta = \frac{1}{2}$, $a, c_0, \varphi, \psi, \zeta, M_0, M$, cf. (1.5), (1.13), (1.14), (1.32), (1.43), (1.46). In accordance with the convention concerning constants started above Theorem 1.1, constants will solely depend on d, K, R in view of the choices we just made. We denote with $\widetilde{X}_t$, $t \geq 0$, and $\widetilde{X}_t^0$, $t \geq 0$, the canonical processes on $C(\mathbb{R}_+, \mathbb{R}^d)^2$, the space on which we will construct the coupling measures.

Proposition 6.2. *($d \geq 3$)*
Given $K > 1$, $R > 0$, there exists $\eta_0 > 0$, depending only on d, K, R, such that for $a(x,\omega)$, $b(x,\omega)$ as in (1.2)*, satisfying* (1.4), (1.7), (0.4)*, and*

$$|a(x,\omega)-I| \leq \eta_0,\ \ |b(x,\omega)| \leq \eta_0,\ \ \text{for } x \in \mathbb{R}^d,\ \omega \in \Omega\,, \tag{6.8}$$

then there is an event $\overline{\Omega}$ with full $\mathbb{P}$-measure and a finite $N(\cdot)$ on $\overline{\Omega}$, such that for $\omega \in \overline{\Omega}$, when $n \geq N(\omega)$:

$$\begin{aligned}&\text{for all } x \in L_n\,\mathbb{Z}^d \cap (4\mathcal{T}_{n+3}),\ x \in \mathcal{B}_n(\omega),\\ &\text{(cf. (1.39), (2.1) for notation)}\,,\end{aligned} \tag{6.9}$$

and for any $y \in \mathbb{R}^d$ there is a coupling measure $\widetilde{Q}_{n,y,\omega}$ on $C(\mathbb{R}_+, \mathbb{R}^d)^2$ such that under $\widetilde{Q}_{n,y,\omega}$,

$$\widetilde{X}^0_{\cdot} \text{ is distributed as } X_{\alpha_n \cdot} \text{ under } W_y\,, \tag{6.10}$$

$$\widetilde{X}_{\cdot\wedge T_{2\mathcal{T}_{n+3}}(\widetilde{X})} \text{ is distributed as } X_{\cdot\wedge T_{2\mathcal{T}_{n+3}}} \text{ under } P_{y,\omega}\,, \tag{6.11}$$

$$\widetilde{Q}_{n,y,\omega}\Big[\sup_{u\leq L_{n+3}^2}\big|\widetilde{X}_u-\widetilde{X}_u^0\big| \geq 3\widetilde{D}_n\Big] \leq L_n^{-\delta/2}\,,\ \text{when } y \in \mathcal{T}_{n+3} \text{ and} \tag{6.12}$$

$$\begin{aligned}&\text{for } n \geq 0,\ \alpha_n \in \Big[\frac{1}{4},4\Big],\ |\alpha_{n+1}-\alpha_n| \leq L_n^{-(1+\frac{9}{10})\,\delta},\\ &\text{(in particular } (\alpha_n) \text{ is a convergent sequence)}\,.\end{aligned} \tag{6.13}$$

Proof. In the sequel we use the expression "small enough η_0", in place $\eta_0 \le c$, with c a constant, with the meaning explained above Proposition 6.2. From now on we assume $\eta_0 < 1$ small enough so that (1.3), (1.5) are satisfied. We now choose constants L_0 and c_2 according to Theorem 1.1, Lemma 6.1, and such that for all $n \ge 0$, (recall W_0 denotes the Wiener measure)

$$
\text{(6.14)} \quad
\begin{aligned}
&\text{i)} \ \begin{array}{l} \text{if in (2.45)}, \ \kappa_n^0 = \frac{1}{2}\,(\widetilde{D}_n/D_n), \\ \text{then } e^{-\kappa_n} \text{ in (2.46) is smaller than } \frac{1}{10}\, L_n^{-\delta}\,, \end{array} \\
&\text{ii)} \ W_0[X^*_{L_n^2} \ge v] \le \frac{1}{10}\ \exp\Big\{-\frac{4v}{D_n}\Big\}, \ \text{for } v \ge \frac{1}{4}\, D_n\,, \\
&\text{iii)} \ \big(E^{W_0}\big[|X_{L_n^2}|^4\big]^{\frac{1}{2}} + \widetilde{D}_n^2\big)\, W_0\Big[X^*_{L_n^2} > \frac{\widetilde{D}_n}{4}\Big]^{\frac{1}{2}} \le \frac{1}{100}\,, \\
&\text{iv)} \ |\chi_{n,0}|_{(n)} \sup_{\frac{1}{2} \le \alpha \ne \alpha' \le 4} \frac{\|P_{\alpha L_n^2} - P_{\alpha' L_n^2}\|_n}{|\alpha - \alpha'|} \le L_0, \ \text{cf. (4.173)}\,,
\end{aligned}
$$

and

$$
\text{(6.15)} \qquad \sum_{n \ge 0} L_n^{-(1+\frac{9}{10})\delta} < \frac{1}{10}\,.
$$

We have now specified L_0, and we will first see that:

$$
\text{(6.16)} \qquad \begin{array}{l} \text{for } \eta_0 \text{ small enough, (1.47), (1.48), (1.49) hold for all} \\ n_0 \ge m_0 + 1, \text{ and } |\alpha_0 - 1| < \frac{1}{10}\,. \end{array}
$$

To this end, first recall from (1.9) that for $\omega \in \Omega$, $x \in \mathbb{R}^d$, there is an $(\mathcal{F}_t)$-Brownian motion $\beta_.$ such that $P_{x,\omega}$-a.s., for all $t \ge 0$,

$$
\text{(6.17)} \qquad \begin{array}{l} X_t = x + \displaystyle\int_0^t \sigma(X_s,\omega)\, d\beta_s + \int_0^t b(X_s,\omega)\, ds, \\ \text{with } \sigma(\cdot,\omega) = a(\cdot,\omega)^{\frac{1}{2}}\,. \end{array}
$$

Note that for $y \in \mathbb{R}^d$, $\omega \in \Omega$, $\sigma(y,\omega) - I = (a(y,\omega) - I)(\sigma(y,\omega) + I)^{-1}$, so for small η_0, $y \in \mathbb{R}^d$, $\omega \in \Omega$, with (6.8),

$$
\text{(6.18)} \qquad |\sigma(y,\omega) - I| \le c\, \eta_0\,.
$$

Further from the exponential martingale inequalities, cf. [23], p. 145,

$$
\text{(6.19)} \qquad \begin{array}{l} P_{x,\omega}\Big[\displaystyle\sup_{v \le t} \Big|\int_0^v \sigma(X_s,\omega)\, d\beta_s - \beta_v\Big| \ge u\Big] \le c\ \exp\Big\{-\frac{cu^2}{\eta_0^2\, t}\Big\}, \\ \text{for } u, t > 0,\, x \in \mathbb{R}^d,\, \omega \in \Omega\,. \end{array}
$$

Choosing η_0 small, with (6.8), (6.17), (6.19), we see that for $\omega \in \Omega$, $0 \le n \le m_0 + 1$, $x \in L_n Z^d$, $A \subseteq C_n(x)$, $\gamma \in \{1, \dots, 2d\, 5^{d-1}\}$, and the notation (1.44),

$$J_{n,x,A,\gamma}(\omega) = 0\,, \tag{6.20}$$

so that (1.48) holds for $0 \le n \le m_0 + 1$. Likewise with (6.14) ii), we see that choosing η_0 small we can make sure that for $\omega \in \Omega$, $0 \le n \le m_0 + 1$, $y \in \mathbb{R}^d$,

$$P_{y,\omega}[X^*_{L_n^2} \ge v] \le \exp\Big\{ - \frac{v}{D_n}\Big\}, \text{ for all } v \ge D_n\,. \tag{6.21}$$

Further we have

$$\chi_{n,0}(R_n - P_{L_n^2}) = \chi_{n,0}\, P_{1,\omega}(P_{L_n^2-1,\omega} - P_{L_n^2-1}) + \chi_{n,0}(P_{1,\omega} - P_1)\, P_{L_n^2-1}\,,$$

and with (1.60), (6.1), (6.19), it follows that choosing η_0 small, for $\omega \in \Omega$, and $0 \le n \le m_0 + 1$,

$$\|\chi_{n,0}(R_n - P_{L_n^2})\|_n \le \frac{1}{5}\, L_n^{-\delta}\,. \tag{6.22}$$

Recall that, cf. (1.22)

$$\alpha_n = \frac{1}{d\, L_n^2}\, E_0\big[|X_{L_n^2 \wedge T_n}|^2\big]\,,$$

and note that for small η_0, with (6.14) iii), (6.19), for $0 \le n \le m_0 + 1$,

$$\begin{aligned}
&\big|E_0\big[|X_{L_n^2}|^2\big] - E_0\big[|X_{L_n^2\wedge T_n}|^2\big]\big| \le \\
&E_0\big[\big(|X_{L_n^2}|^2 + \widetilde{D}_n^2\big),\ T_n < L_n^2\big] \le \\
&\big(E_0\big[|X_{L_n^2}|^4\big]^{\frac12} + \widetilde{D}_n^2\big)\Big(P_0\Big[\sup_{s\le L_n^2} |\beta_s| \ge \frac{\widetilde{D}_n}{4}\Big]^{\frac12} + \\
&\qquad\qquad P_0\Big[\sup_{0\le s\le L_n^2}\Big|\int_0^s (\sigma(X_s,\omega) - I)d\beta_s\Big| \ge \frac{\widetilde{D}_n}{4}\Big]^{\frac12}\Big) \le \frac{1}{20}\,.
\end{aligned}$$

So when η_0 is small enough, for $0 \le n \le m_0 + 1$,

$$|\alpha_n - 1| \le \frac{1}{20d\, L_n^2} + \frac{1}{d\, L_n^2}\big|E_0\big[|X_{L_n^2}|^2\big] - E_0\big[|\beta_{L_n^2}|^2\big]\big| \overset{(6.17),(6.19)}{\le} \frac{1}{10\, L_n^2}\,, \tag{6.23}$$

and hence

$$\begin{aligned}
&\text{i)}\quad |\alpha_n - \alpha_{n+1}| \le L_n^{-(1+\frac{9}{10})\delta}, 0 \le n \le m_0, \text{ and}\\
&\text{ii)}\quad \alpha_n \in \Big[\frac14, 4\Big]\Big(= \Big[\frac{1}{2\nu}, 2\nu\Big]\Big), \text{ for } 0 \le n \le m_0 + 1\,.
\end{aligned} \tag{6.24}$$

This proves that (1.49) holds for $0 \le n \le m_0 + 1$. Then observe that for $0 \le n \le m_0 + 1$, $\omega \in \Omega$,

$$\begin{aligned} \|\chi_{n,0}\, \widetilde{S}_n\|_n \le \|\chi_{n,0}(\widetilde{S}_n - S_n)\|_n + \big\|\chi_{n,0}\big(R^n - P_{L_n^2}\big)\big\|_n \\ + \|\chi_{n,0}(P_{\alpha_n L_n^2} - P_{L_n^2})\|_n\,, \end{aligned}$$

so that using (6.21), (2.46), (6.14) i) to bound the first term in the right-hand side, (6.22) to bound the second term, (6.14) iv), (6.23), (6.24) ii) to bound the last term, we see that when η_0 is small, for $\omega \in \Omega$, $0 \le n \le m_0 + 1$,

$$\|\chi_{n,0}\, \widetilde{S}_n\|_n \le \frac{1}{10}\, L_n^{-\delta} + \frac{1}{5}\, L_n^{-\delta} + \frac{1}{5}\, L_0\, L_n^{-2} \le L_n^{-\delta}\,. \tag{6.25}$$

Hence with (6.21), we see that for small η_0, when $\omega \in \Omega$, $0 \le n \le m_0 + 1$,

$$0 \in \mathcal{B}_n(\omega)\,. \tag{6.26}$$

We can now apply Theorem 1.1, and with (6.15) note that $|\alpha_0 - 1| < \frac{1}{10}$ implies that (1.49) remains also satisfied by induction, so that (6.16) is proved.

As a next step observe that for $n \ge m_0 + 1$,

$$\begin{aligned} &\mathbb{P}\big[\text{for some } x \in L_n\, \mathbb{Z}^d \cap (4\mathcal{T}_{n+3}),\ x \notin \mathcal{B}_n(\omega)\big] \le \\ &c\left(\frac{L_{n+3}^2}{L_n}\right)^d L_n^{-M_0} \overset{(1.46)}{\le} c\, L_n^{2d(1+a)^3 - 100d(1+a)^{m_0+2}} \le c\, L_n^{-98d}\,, \end{aligned}$$

and this last quantity is the general term of a convergent series. With Borel-Cantelli's lemma, we see that there is an event $\overline{\Omega}$ with full $\mathbb{P}$-measure, and a finite $N(\cdot)$ on $\overline{\Omega}$, such that when $n \ge N(\omega)$, (6.9) holds.

Let us now fix $\omega \in \overline{\Omega}$. Given $n \ge N(\omega)$, we denote with h some $[0,1]$-valued continuous function with value 1 on $2\mathcal{T}_{n+3}$, and 0 on $(3\mathcal{T}_{n+3})^c$. Consider the Markov chains with respective transition kernels $\widetilde{R}_{n,h}$, cf. (3.4), and $R_{n,h}$, as in (3.4) with S_n in place of $\widetilde{S}_n$. They can be coupled in a natural fashion up to the first time either one exits the set $\{h = 1\}$ using their respective interpretations in terms of the diffusion in the random environment ω. The coupling can then be extended using from then on independent moves. With Proposition 3.1, we thus naturally obtain for $y \in \mathbb{R}^d$ a coupling measure still denoted by $Q_{n,y}$ on $(\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{N}}$, under which the canonical processes $\overline{X}_k$, $k \ge 0$, and $\overline{X}_k^0$, $k \ge 0$, have the laws of the Markov chains on $\mathbb{R}^d$ starting at y with respective transitions $R_{n,h}$, and R_n^0. Let $P_{z,z',\omega}^{L_n^2}$ denote the bridge measure in time L_n^2 between z and z' for the diffusion in random environment. Similarly, let $P_{z,z'}^{L_n^2}$ denote the analogous bridge measure in time L_n^2 for the Brownian motion with covariance $\alpha_n I$. Let

$$\begin{aligned} Q_{z,z'}^{L_n^2} = (h(z)p_{L_n^2,\omega}(z,z') + (1-h(z))p_{\alpha_n L_n^2}(z,z'))^{-1}\big(h(z)p_{L_n^2}(z,z')P_{z,z',\omega}^{L_n^2} \\ + (1-h(z))p_{\alpha_n L_n^2}(z,z')\, P_{z,z'}^{L_n^2}\big) \end{aligned}$$

and define the bridge measure $\tilde{Q}_{z,z',z_0,z_0',\omega,n} = Q^{L_n^2}_{z,z'} \otimes P^{L_n^2}_{z_0,z_0'}$ on $C([0, L_n^2]; \mathbb{R}^d)^2$. Use now the bridge measure $\tilde{Q}_{\bar{X}_k,\bar{X}_{k+1},\bar{X}_k^0,\bar{X}_{k+1}^0}$ to interpolate the chains $\bar{X}_k$ and $\bar{X}_k^0$ to diffusion processes, whose joint law is the coupling measure $\tilde{Q}_{n,y,\omega}$ on $C^2(\mathbb{R}_+, \mathbb{R}^d)$ (note that conditioned on $\bar{X}_k, k \ge 0, \bar{X}_k^0, k \ge 0$, all the interpolating bridges are independent). Now (6.10) and (6.11) hold. Then using (3.6), (6.9), (1.39), we find for $y \in \mathcal{T}_{n+3}$:

$$
\begin{aligned}
&\widetilde{Q}_{n,y,\omega}\Big[\sup_{u \le L_{n+3}^2} \big|\widetilde{X}_u - \widetilde{X}_u^0\big| \ge 3\widetilde{D}_n\Big] \le \\
&\Big(\frac{L_{n+3}}{L_n}\Big)^4 (\kappa_n L_n^{-\delta} + e^{-\kappa_n}) + 2\Big(\frac{L_{n+3}}{L_n}\Big)^2 e^{-\kappa_n} \le L_n^{-\delta/2},
\end{aligned}
\tag{6.27}
$$

when n is large enough. Hence increasing $N(\cdot)$ if necessary, we see that for $\omega \in \overline{\Omega}$, (6.10), (6.11), (6.12), (6.13) holds, and this finishes the proof of Proposition 6.2. $\square$

We are now ready to state and prove our main applications.

Theorem 6.3. $(d \ge 3)$
With $\eta_0(d, K, R) > 0$, as in Proposition 6.2, when $a(x,\omega), b(x,\omega)$, as in (1.2), satisfy (1.4), (1.7), (0.4) as well as (6.8), i.e.

$$
|a(x,\omega) - I| \le \eta_0, \ |b(x,\omega)| \le \eta_0, \ \text{for } x \in \mathbb{R}^d, \omega \in \Omega,
$$

then $\mathbb{P}$-a.s.,

$$
\begin{aligned}
&\frac{1}{\sqrt{t}}\, X_{\cdot t} \text{ converges in } P_{0,\omega}\text{-law, as } t \to \infty, \text{ to a Brownian motion} \\
&\text{on } \mathbb{R}^d \text{ with deterministic variance } \sigma^2 > 0,
\end{aligned}
\tag{6.28}
$$

$$
\text{for all } x \in \mathbb{R}^d,\ P_{x,\omega}\text{-a.s.}, \lim_{t\to\infty} |X_t| = \infty. \tag{6.29}
$$

Proof. We keep the notation of Proposition 6.2. We first prove (6.28). From (6.13) we know that α_n converges and we write

$$
\sigma^2 \overset{\text{def}}{=} \lim_n \alpha_n \Big(\in \Big[\frac{1}{4}, 4\Big]\Big). \tag{6.30}
$$

The claim (6.28) will follow once we prove that for any ω in $\overline{\Omega}$, in the notation of Proposition 6.2,

$$
\lim_{t\to\infty} E_{0,\omega}\Big[F\Big(\frac{1}{\sqrt{t}}\, X_{\cdot t}\Big)\Big] = E^{W_0}[F(X_{\sigma^2 \cdot})], \tag{6.31}
$$

for any F on $C([0,T], \mathbb{R}^d)$, $T > 0$, bounded by 1, Lipschitz relative to the distance function

$$
D_T(w, w') = \sup_{s \le T} |w(s) - w'(s)| \wedge 1, \ w, w' \in C([0,T], \mathbb{R}^d), \tag{6.32}
$$

with Lipschitz constant 1, with a slight abuse of notation in (6.31). For t large we define the integer $n(t) \ge 0$, such that

$$L^2_{n(t)+1} \le t < L^2_{n(t)+2}\,, \tag{6.33}$$

and observe that for $\omega \in \overline{\Omega}$, F as above and large t

$$\Big| E_{0,\omega}\Big[F\Big(\frac{1}{\sqrt{t}}\, X_{\cdot\, t}\Big)\Big] - E^{W_0}[F(X_{\sigma^2_{\cdot}})]\Big| \le a_1 + a_2 + a_3, \text{ where}$$

$$\begin{aligned} a_1(t) &= \Big|E_{0,\omega}\Big[F\Big(\frac{1}{\sqrt{t}}\, X_{\cdot\, t}\Big)\Big] - E_{0,\omega}\Big[F\Big(\frac{1}{\sqrt{t}}\, X_{(\cdot\, t)\wedge T_{2\mathcal{T}_{n(t)+3}}}\Big)\Big]\Big|\,, \\ a_2(t) &= \Big|E_{0,\omega}\Big[F\Big(\frac{1}{\sqrt{t}}\, X_{(\cdot\, t)} \wedge T_{2\mathcal{T}_{n(t)+3}}\Big)\Big] - E^{W_0}\Big[F\Big(\frac{1}{\sqrt{t}}\, X_{\alpha_{n(t)}\cdot}\Big)\Big]\Big|\,, \\ a_3(t) &= \big|E^{W_0}\big[F\big(\sqrt{\alpha}_{n(\cdot)}\, X_{\cdot}\big)\big] - E^{W_0}[F(\sigma X_{\cdot})]\big|\,, \end{aligned} \tag{6.34}$$

and we have used Brownian scaling for $a_3(\cdot)$. From (6.30) and dominated convergence, we see that

$$\lim_{t\to\infty} a_3(t) = 0\,. \tag{6.35}$$

Further when t is large,

$$\begin{aligned} a_1(t) \le 2P_{0,\omega}[T_{2\mathcal{T}_{n(t)+3}} < T\,t] &\overset{(6.33)}{\le} 2P_{0,\omega}\big[T_{2\mathcal{T}_{n(t)+3}} < T\,L^2_{n(t)+2}\big] \\ &\overset{(2.10)}{\le} c\,\exp\big\{-c\,L^2_{n(t)+3}\big\}, \text{ so that} \end{aligned}$$

$$\lim_{t\to\infty} a_1(t) = 0\,. \tag{6.36}$$

As for $a_2(t)$, using the coupling measure $\widetilde{Q}_{n(t),0,\omega}$ from Proposition 6.2, we find with (6.10), (6.11), that for large t

$$\begin{aligned} a_2(t) &= \Big|E^{\widetilde{Q}_{n(t),0,\omega}}\Big[F\Big(\frac{1}{\sqrt{t}}\, \widetilde{X}_{(\cdot\, t)\wedge T_{2\mathcal{T}_{n(t)+3}}(\widetilde{X})}\Big) - F\Big(\frac{1}{\sqrt{t}}\, \widetilde{X}^0_{\cdot\, t}\Big)\Big]\Big| \\ &\le E^{\widetilde{Q}_{n(t),0,\omega}}\Big[\sup_{u\le T\,t} \frac{|\widetilde{X}_{u\wedge T_{2\mathcal{T}_{n(t)+3}}(\widetilde{X})} - \widetilde{X}^0_u|}{\sqrt{t}} \wedge 1\Big] \\ &\underset{(2.10)}{\overset{(6.12),(6.33)}{\le}} \frac{3\widetilde{D}_{n(t)}}{\sqrt{t}} + L_{n(t)}^{-\delta/2} + c\,e^{-cL^2_{n(t)+3}}, \end{aligned}$$

so that

$$\lim_{t\to\infty} a_2(t) = 0\,. \tag{6.37}$$

Combining (6.35)–(6.37), the claim (6.31) follows. This proves (6.28).

We now prove (6.29). When n is large, it follows from standard estimates on Brownian motion and (1.49) that for $|z| = L_{n+1}$,

$$\begin{aligned} &W_z\big[X_{\alpha_n\,\cdot} \text{ exits } B(0, 2L_{n+2}) \text{ before time } L^2_{n+3} \text{ or} \\ &\quad \text{entering } \overline{B}(0, 4\widetilde{D}_n)\big] \ge 1 - \frac{\kappa_n}{\ell_n}\,. \end{aligned} \tag{6.38}$$

Then for $\omega \in \overline{\Omega}$, with Proposition 6.2 and (6.38) we see that for large n and $|z| = L_{n+1}$,

$$\begin{aligned}&\widetilde{Q}_{n,z,\omega}\big[\widetilde{X}_{\cdot} \text{ enters } \overline{B}(0, L_n) \text{ before exiting } B(0, L_{n+2})\big] \leq \\ &L_n^{-\delta/2} + \frac{\kappa_n}{\ell_n} \leq \frac{\kappa_n}{\ell_n}\,.\end{aligned} \tag{6.39}$$

With (6.11), we thus see that for large n and $|z| = L_{n+1}$,

$$P_{z,\omega}[H_{\overline{B}(0,L_n)} < T_{B(0,L_{n+2})}] \leq \frac{\kappa_n}{\ell_n} \leq \ell_n^{-1/2}\,,$$

so that with the strong Markov property we find:

$$P_{z,\omega}[H_{\overline{B}(0,L_n)} = \infty] \geq \prod_{k \geq 0} \big(1 - \ell_{n+k}^{-1/2}\big) \underset{n \to \infty}{\longrightarrow} 1\,. \tag{6.40}$$

It now follows in a standard way that when $\omega \in \overline{\Omega}$,

$$\text{for } x \in \mathbb{R}^d,\ \ P_{x,\omega}\big[\lim_{t \to \infty} |X_t| = \infty\big] = 1\,, \tag{6.41}$$

and this proves (6.29). □

We conclude this section with an application to homogenization in random media. Given f, g bounded functions on $\mathbb{R}^d$ respectively continuous and Hölder continuous, under the assumptions of Theorem 6.3, for $\omega \in \Omega$ and $\epsilon > 0$, there is a unique bounded solution of the Cauchy problem

$$\begin{cases} \partial_t\, u_\epsilon = L_\epsilon\, u_\epsilon + g \text{ in } (0,\infty) \times \mathbb{R}^d\,, \\ u_\epsilon|_{t=0} = f\,, \end{cases} \tag{6.42}$$

where

$$L_\epsilon = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\Big(\frac{x}{\epsilon}, \omega\Big)\, \partial^2_{ij} + \sum_{i=1}^{d} \frac{1}{\epsilon}\, b_i\Big(\frac{x}{\epsilon}, \omega\Big)\, \partial_i\,, \tag{6.43}$$

see for instance [9, Theorem 12, p. 25], and [10, Theorem 5.3]. The asymptotic behavior of u_ϵ, as $\epsilon \to 0$, is intimately related to the invariance principle proved in Theorem 6.3.

Theorem 6.4. *($d \geq 3$)*
Under the same assumptions as in Theorem 6.3, *on a set of full $\mathbb{P}$-measure, for any f, g as above, the solution u_ϵ of* (6.42) *converges uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{R}^d$ to the solution of the Cauchy problem*

$$\begin{cases} \partial_t\, u_0 = \dfrac{\sigma^2}{2}\, \Delta\, u_0 + g \textit{ in } (0,\infty) \times \mathbb{R}^d\,, \\ u_\epsilon|_{t=0} = f\,, \end{cases} \tag{6.44}$$

with σ^2 as in (6.28).

Proof. Consider $\omega \in \overline{\Omega}$, (cf. Proposition 6.2), and $\epsilon > 0$, with [10, Theorem 5.3], we can write

$$(6.45) \quad u_\epsilon(s,x) = E_{x/\epsilon,\omega}\Big[f(\epsilon X_{s/\epsilon^2}) - \int_0^s g(\epsilon\, X_{v/\epsilon^2})\, dv\Big], \text{ for } s \geq 0, x \in \mathbb{R}^d .$$

Letting ϵ^{-1} play the role of t in (6.33), very similar bounds as in (6.34)–(6.37), with some obvious modifications for the bound above (6.37) yield that as $\epsilon \to 0$,

$$(6.46) \quad \begin{array}{l} u_\epsilon \text{ converges uniformly on compact subsets of } \mathbb{R}_+ \times \mathbb{R}^d \text{ to} \\ u_0(s,x) = E^{W_x}\Big[f(X_{\sigma^2 s}) - \displaystyle\int_0^s g(X_{\sigma^2 v})\, dv\Big], \end{array}$$

and our claim now follows. □

The proofs of the last two theorems illustrate the fact that the measures constructed in Proposition 6.2 offer a very quantitative and handy comparison of the isotropic diffusion in random environment with Brownian motion.

A. Appendix

This appendix collects several results concerning the Hölder-norms $|\cdot|_{(n)}$, $\|\cdot\|_n$, cf. (1.28), (1.30). In particular the effective control of these norms with the help of wavelets is discussed in Proposition A.2. We begin with the convenient

Lemma A.1. *($n \geq 0$, L_n as in* (1.15)*, $\beta \in (0,1)$)*
Consider a non-empty index set I, f, $(g_i)_{i\in I}$, scalar functions on $\mathbb{R}^d$, $(x_i)_{i\in I}$, points of $\mathbb{R}^d$, such that

$$(A.1) \quad f = g_i, \text{ on } B(x_i, 2L_n),\ i \in I, \text{ and}$$

$$(A.2) \quad \mathrm{Supp}\, f \subseteq \bigcup_{i\in I} \overline{B}(x_i, L_n), \text{ then}$$

$$(A.3) \quad |f|_{(n)} \leq 3 \sup_{i\in I} |g_i|_{(n)} .$$

Moreover if f is a scalar function, $\Gamma > 0$, and

$$(A.4) \quad \sup_{x\in\mathbb{R}^d} |f(x)| \leq \Gamma ,$$

$$(A.5) \quad \begin{array}{l} |f(x) - f(y)| \leq \Gamma \left|\dfrac{x-y}{L_n}\right|^\beta, \text{ for } x, y \text{ in the open} \\ L_n\text{-neighborhood of the support of } f \text{ and } |x-y| < L_n , \end{array}$$

then

$$(A.6) \quad |f|_{(n)} \leq 3\Gamma .$$

Proof. We first prove (A.3). Note that

$$|f|_\infty \le \sup_{i\in I} |g_i|_\infty ,$$

and for x, y in $\mathbb{R}^d$ with $|x-y| \ge L_n$,

$$L_n^\beta \frac{|f(x)-f(y)|}{|x-y|^\beta} \le 2 \sup_i |g_i|_\infty .$$

On the other hand, when x, y are distinct points of $\mathbb{R}^d$, with $|x-y| < L_n$ and say $x \in \operatorname{Supp} f$, then $x \in \overline{B}(x_{i_0}, L_n)$, for some $i_0 \in I$. One then has

$$L_n^\beta \frac{|f(x)-f(y)|}{|x-y|^\beta} \overset{(A.1)}{=} L_n^\beta \frac{|g_{i_0}(x)-g_{i_0}(y)|}{|x-y|^\beta} ,$$

whereas when none of x, y belongs to $\operatorname{Supp} f$, the left member vanishes. The claim (A.3) now follows.

We now prove (A.6). Note that when x, y are such that $|x-y| \ge L_n$, then

$$L_n^\beta \frac{|f(x)-f(y)|}{|x-y|^\beta} \le 2\,|f|_\infty \overset{(A.4)}{\le} 2\Gamma .$$

On the other hand when x, y are distinct points of $\mathbb{R}^d$ with $|x-y| < L_n$, and either some or none of them belongs to $\operatorname{Supp} f$, we find with (A.5)

$$L_n^\beta \frac{|f(x)-f(y)|}{|x-y|^\beta} \le \Gamma ,$$

and the claim (A.6) now follows. □

The next result will provide an effective control of the Hölder-norms (1.28), (1.30), with the help of the expansion in an orthonormal basis of wavelets. The fact that such bases give rise to a handy control of the Hölder-property is well known, cf. Daubechies [6, p. 199–203], Mallat [16, p. 169–173]. The proposition we will now prove, gives a version of these results useful for the calculations of Sect. 4. We introduce the sequence of non-negative integers $J_n, n \ge 0$, such that

$$2^{J_n} \le L_n < 2^{J_n+1} , \tag{A.7}$$

and recall the $L^2(\mathbb{R}^d)$-orthogonal expansion in (1.35).

Proposition A.2. *($d \ge 1, 0 < \beta < 1, \varphi, \psi$)*
There is a constant $\Gamma > 1$, depending on d, β, φ, ψ, such that for $n \ge 0$, and f compactly supported bounded measurable function, one has, cf. (1.35) *for the notation,*

$$\frac{1}{\Gamma}\,|f|_{(n)} \le \sup_{\substack{\alpha,\ell\le J_n, p\in\mathbb{Z}^d\\ \alpha\ne 0,\ \text{for } \ell<J_n}} 2^{\beta(J_n-\ell)} \big|c^{J_n}_{\alpha,\ell,p}\big| \le \Gamma\,|f|_{(n)} . \tag{A.8}$$

Moreover, when A is a bounded linear operator mapping bounded measurable functions on $\mathbb{R}^d$ into bounded measurable compactly supported functions on $\mathbb{R}^d$, and A vanishes for functions supported in the complement of some compact subset of $\mathbb{R}^d$, then

$$\text{(A.9)} \quad \begin{aligned} \frac{1}{\Gamma}\,\|A\|_n \le & \sup_{\substack{\alpha,\ell\le J_n, p\in\mathbb{Z}^d \\ \alpha\neq 0, \text{ when } \ell<J_n}} \; \sum_{\substack{\alpha',\ell'\le J_n, p'\in\mathbb{Z}^d \\ \alpha'\neq 0, \text{ when } \ell'<J_n}} \frac{2^{\beta\ell'}}{2^{\beta\ell}}\,\frac{1}{2^{d\ell}}\,\big|\langle\theta_{\alpha,\ell,p},\, A\theta_{\alpha',\ell',p'}\rangle\big| \\ \le & \;\Gamma\|A\|_n\,, \end{aligned}$$

with the notation $\langle h, g\rangle = \int h(x)\, g(x)\, dx.$

Proof. We begin with the proof of (A.8). For f as in the statement, $\alpha \in \{0,1\}^d$, $\ell \le J_n$, $p \in \mathbb{Z}^d$, with $\alpha \neq 0$, when $\ell < J_n$, the coefficients $c^{J_n}_{\alpha,\ell,p}$ of (A.8), are expressed in view of (1.35), as

$$\text{(A.10)} \qquad c^{J_n}_{\alpha,\ell,p} = \frac{1}{2^{\ell d}} \int_{\mathbb{R}^d} f(x)\,\theta_\alpha\Big(\frac{x}{2^\ell} - p\Big)\,dx\,,$$

(note incidentally that for $n \ge 0$, $\ell \le J_n$, $\alpha \neq 0$, $c^{J_n}_{\alpha,\ell,p} = c^{J_{n+1}}_{\alpha,\ell,p}$). Denoting throughout the proof with c a positive constant changing from place to place and solely depending on d, β, φ, ψ, we find that for $\ell \le J_n$, $p \in \mathbb{Z}^d$, $\alpha \in \{0,1\}^d$, with $\alpha \neq 0$ if $\ell < J_n$:

$$\text{(A.11)} \qquad \big|c^{J_n}_{\alpha,\ell,p}\big| \le c\,|f|_\infty \le c\,|f|_{(n)}\,.$$

Note that when $\alpha \neq 0$, $\theta_{\alpha_i} = \psi$, for some $1 \le i \le d$, in (1.33), hence

$$\text{(A.12)} \qquad \int \theta_\alpha(x)\,dx = 0, \ \text{ for } \alpha \neq 0\,.$$

We see that for $\ell < J_n$, $p \in \mathbb{Z}^d$, $\alpha \neq 0$:

$$\text{(A.13)} \quad \big|c^{J_n}_{\alpha,\ell,p}\big| = 2^{-\ell d} \int_{2^\ell(p+\mathrm{Supp}\,\theta_\alpha)} \big(f(x) - f(2^\ell p)\big)\,\theta_\alpha\Big(\frac{x}{2^\ell} - p\Big)\,dx\,,$$

and hence

$$\text{(A.14)} \qquad \big|c^{J_n}_{\alpha,\ell,p}\big| \le c\,\Big(\frac{2^\ell}{L_n}\Big)^\beta\,|f|_{(n)} \le c\,2^{\beta(\ell-J_n)}\,|f|_{(n)}\,.$$

The right inequality in (A.8) now follows from (A.11), (A.14).

Conversely, expanding f as in (1.35), assume that

$$\text{(A.15)} \quad \begin{aligned} \rho_f \stackrel{\text{def}}{=} \sup\Big\{&\big|c^{J_n}_{\alpha,\ell,p}\big|\,2^{\beta(J_n-\ell)}; \\ &\alpha\in\{0,1\}^d,\ \ell\le J_n,\, p\in\mathbb{Z}^d, \alpha\neq 0 \text{ when } \ell<J_n\Big\} < \infty\,. \end{aligned}$$

Observe that for $\overline{\ell}_1 \le \overline{\ell}_0 \le J_n$ and $x \in \mathbb{R}^d$,

$$\text{(A.16)}\quad \begin{aligned} \Big| \sum_{\substack{\alpha,p\\ \overline{\ell}_1\le\ell\le\overline{\ell}_0}} c^{J_n}_{\alpha,\ell,p}\theta_\alpha\Big(\frac{x}{2^\ell}-p\Big)\Big| &\le \rho_f \sum_{\substack{\alpha,p\\ \overline{\ell}_1\le\ell\le\overline{\ell}_0}} 2^{\beta(\ell-J_n)}\Big|\theta_\alpha\Big(\frac{x}{2^\ell}-p\Big)\Big| \\ &\le c\rho_f \sum_{\overline{\ell}_1\le\ell\le\overline{\ell}_0} 2^{\beta(\ell-J_n)} \le c\rho_f 2^{\beta(\overline{\ell}_0-J_n)}, \end{aligned}$$

since for each $\ell \le J_n$, at most c of the summands in the expression after the first inequality do not vanish. In particular $\sum_{\substack{\alpha,p\\ \overline{\ell}_1\le\ell\le J_n}} c^{J_n}_{\alpha,\ell,p}\theta_{\alpha,\ell,p}$ converges uniformly (and of course in L^2) towards f, which is continuous and satisfies:

$$\text{(A.17)}\quad |f|_\infty \le c\,\rho_f\,.$$

Note that when $|x-y| \ge 2^{J_n}$, one has

$$\text{(A.18)}\quad |f(x)-f(y)| \le 2\,|f|_\infty \le 2\,c\,\rho_f \le c\,\rho_f\,\Big|\frac{x-y}{L_n}\Big|^\beta\,.$$

On the other hand, when $|x-y| < 2^{J_n}$, so that

$$\text{(A.19)}\quad 2^{\overline{\ell}_0} < |x-y| \le 2^{\overline{\ell}_0+1},\ \text{with } \overline{\ell}_0 < J_n\,,$$

we introduce $\widetilde{f} = \sum_{\substack{\alpha,p\\ \overline{\ell}_0\le\ell\le J_n}} c^{J_n}_{\alpha,\ell,p}\theta_{\alpha,\ell,p}$, and find

$$\text{(A.20)}\quad \begin{aligned} &|f(x)-f(y)| \le 2\,|f-\widetilde{f}|_\infty + |\widetilde{f}(x)-\widetilde{f}(y)| \overset{(A.16)}{\le} \\ &c\,\rho_f\,2^{\beta(\overline{\ell}_0-J_n)} + \Big|\sum_{\substack{\alpha,p\\ \overline{\ell}_0\le\ell\le J_n}} c^{J_n}_{\alpha,\ell,p}\ \Big(\theta_\alpha\Big(\frac{x}{2^\ell}-p\Big)-\theta_\alpha\Big(\frac{y}{2^\ell}-p\Big)\Big)\Big| \le \\ &c\,\rho_f\,2^{\beta(\overline{\ell}_0-J_n)} + c\,\rho_f \sum_{\overline{\ell}_0\le\ell\le J_n} 2^{\beta(\ell-J_n)}\,\Big|\frac{x-y}{2^\ell}\Big| \overset{(A.19)}{\le} \\ &c\,\rho_f\Big(\Big|\frac{x-y}{L_n}\Big|^\beta + |x-y| \sum_{\ell_0\le\ell\le J_n} 2^{-(1-\beta)\ell-\beta J_n}\Big) \le \\ &c\,\rho_f\Big(\Big|\frac{x-y}{L_n}\Big|^\beta + |x-y|\,2^{-(1-\beta)\overline{\ell}_0-\beta J_n}\Big) \overset{(A.19)}{\le} c\,\rho_f\Big|\frac{x-y}{L_n}\Big|^\beta\,. \end{aligned}$$

Combining (A.17), (A.18), (A.20), the proof of (A.8) is completed.

We now turn to the proof of (A.9). We begin with the proof of the left-hand inequality. We denote with Φ_A the middle expression of (A.9), which we assume finite. We pick a [0, 1]-valued function h, compactly supported such that

$$\text{(A.21)}\quad |h|_{(n)} \le 3,\ \text{and}$$

$$\text{(A.22)}\quad A(hg) = A(g)\ \text{for any bounded measurable } g\,.$$

Indeed given our assumptions on A, we can for instance pick h of the form (1.37), with u large, and use (A.6). For g with $|g|_{(n)} \le 1$, we define

$$f = hg\,, \tag{A.23}$$

so that expanding f as in (1.35) with (J_n in place of j_0), and keeping the notation (A.15) for ρ_f, we find:

$$\rho_f \overset{(A.8)}{\le} c\,|f|_{(n)} \overset{(1.29),(A.21)}{\le} c\,|g|_{(n)} \le c\,. \tag{A.24}$$

Since $A(g) = A(f)$ is bounded measurable and compactly supported, we find:

$$A(g) = A(f) \overset{(1.35),(A.10)}{=} \sum_{\substack{\alpha,\ell \le J_n, p \\ \alpha \neq 0,\ \text{for}\ \ell < J_n}} \frac{1}{2^{\ell d}}\,\langle \theta_{\alpha,\ell,p}, A(f)\rangle \theta_{\alpha,\ell,p}\,. \tag{A.25}$$

We also know that the partial sums $\widetilde{f}$, cf. above (A.20), converge uniformly to f, as ℓ_0 tends to $-\infty$, and only finitely many terms in the sum defining $\widetilde{f}$ do not identically vanish on the support of h. Using the continuity of A for the sup-norm, we find that for $\alpha \in \{0,1\}^d$, $\ell \le J_n$, $p \in \mathbb{Z}^d$, with $\alpha \neq 0$, for $\ell < J_n$, with hopefully obvious notation:

$$\begin{aligned} &2^{\beta(J_n-\ell)}\,\frac{1}{2^{\ell d}}\,|\langle \theta_{\alpha,\ell,p}, A(f)\rangle| \le \\ &2^{\beta(J_n-\ell)-\ell d} \sum_{\alpha',\ell',p'} \big|c^{J_n}_{\alpha',\ell',p'}(f)\big|\,\big|\big\langle \theta_{\alpha,\ell,p}, A(\theta_{\alpha',\ell',p'})\big\rangle\big| \overset{(A.15)}{\le} \\ &\rho_f \sum_{\alpha',\ell',p'} \frac{2^{\beta\ell'}}{2^{\beta\ell}}\,\frac{1}{2^{\ell d}}\big|\big\langle \theta_{\alpha,\ell,p}, A(\theta_{\alpha',\ell',p'})\big\rangle\big|\,. \end{aligned} \tag{A.26}$$

Keeping in mind (A.24), we see coming back to (A.25) with the help of (A.8) that $A(g)$ is a β-Hölder continuous function and:

$$|A(g)|_{(n)} \le c\,\Phi_A,\ \text{(cf. above (A.21) for the notation)}\,. \tag{A.27}$$

This proves the left inequality of (A.9).

We now prove the right inequality of (A.9). Without loss of generality we assume $\|A\|_n$ finite, i.e. A maps boundedly the set of bounded β-Hölder continuous functions endowed with $|\cdot|_{(n)}$, into itself. Consider $\alpha_0 \in \{0,1\}^d$, $\ell_0 \le J_n$, $p_0 \in \mathbb{Z}^d$, with $\alpha_0 \neq 0$, if $\ell_0 < J_n$, and $\mathcal{J}'$ a finite set of (α', ℓ', p') satisfying analogous constraints. Using the convention sign $(0) = 1$, we define

$$f = \sum_{\mathcal{J}'} \text{sign}\,\Big(\big\langle \theta_{\alpha_0,\ell_0,p_0}, A(\theta_{\alpha',\ell',p'})\big\rangle\Big)\, 2^{\beta\ell'}\,\theta_{\alpha',\ell',p'}\,. \tag{A.28}$$

From (A.8), we deduce that

$$|f|_{(n)} \le c\, 2^{\beta J_n}, \text{ and that} \tag{A.29}$$

$$\begin{aligned} |A(f)|_{(n)} &\overset{(A.8),(A.10)}{\ge} c\, 2^{\beta(J_n-\ell_0)}\, \frac{1}{2^{\ell_0 d}}\, \big|\langle \theta_{\alpha_0,\ell_0,p_0}, A(f)\rangle\big| \\ &\overset{(A.28)}{=} c\, 2^{\beta(J_n-\ell_0)} \sum_{\mathcal{J}'} \frac{2^{\beta\ell'}}{2^{\ell_0 d}}\, \big|\langle \theta_{\alpha_0,\ell_0,p_0}, A(\theta_{\alpha',\ell',p'})\rangle\big| \\ &\overset{(A.29)}{\ge} c\, |f|_{(n)} \sum_{\mathcal{J}'} \frac{2^{\beta\ell'}}{2^{(d+\beta)\ell_0}}\, \big|\langle \theta_{\alpha_0,\ell_0,p_0}, A(\theta_{\alpha',\ell',p'})\rangle\big|\,. \end{aligned} \tag{A.30}$$

Since f in (A.28) is not identically zero and α_0, ℓ_0, p_0, and $\mathcal{J}'$ are arbitrary, we find that

$$\|A\|_n \ge c\, \Phi_A, \text{ (cf. above (A.21) for the notation)}\,. \tag{A.31}$$

This finishes the proof of (A.9), and of Proposition A.2. □

References

1. Alon, N., Spencer, J., Erdös, P.: The probabilistic method. New York: John Wiley & Sons 1992
2. Anshelevich, V.V., Khanin, K.M., Sinai, Ya.G.: Symmetric random walks in random environments. Commun. Math. Phys. **85**, 449–470 (1982)
3. Boivin, D., Depauw, J.: Spectral homogeneization of reversible random walks on $\mathbb{Z}^d$ in a random environment. Stochastic Processes Appl. **104**, 29–56 (2003)
4. Bolthausen, E., Sznitman, A.S., Zeitouni, O.: Cut points and diffusive random walks in random environment. Ann. Inst. Henri Poincaré **39**, 527–555 (2003)
5. Bricmont, J., Kupiainen, A.: Random walks in asymmetric random environments. Commun. Math. Phys. **142**, 345–420 (1991)
6. Daubechies, I.: Ten lectures on wavelets, vol. 61. CBMS-NSF Regional Conference Series in Applied Mathematics. Philadelphia: SIAM 1992
7. De Masi, A., Ferrari, P.A., Goldstein, S., Wick, W.D.: An invariance principle for reversible Markov processes. Applications to random motions in random environments. J. Stat. Phys. **55**, 787–855 (1989)
8. Dudley, R.M.: Real analysis and probability. Belmont, CA: Wadsworth 1989
9. Friedman, A.: Partial differential equations of parabolic type. Englewood Cliff, NJ: Prentice Hall 1964
10. Friedman, A.: Stochastic differential equations and applications vol. 1. New York: Academic Press 1975
11. Karatzas, I., Shreve, S.: Brownian motion and stochastic calculus. Berlin: Springer 1988
12. Kipnis, C., Varadhan, S.R.S.: A central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. Commun. Math. Phys. **104**, 1–19 (1986)
13. Kozlov, S.M.: The method of averaging and walks in inhomogeneous environments. Russ. Math. Surv. **40**, 73–145 (1985)
14. Kunnemann, R.: The diffusion limit for reversible jump processes in $\mathbb{Z}^d$ with ergodic bond conductivities. Commun. Math. Phys. **90**, 27–68 (1983)
15. Lawler, G.F.: Weak convergence of a random walk in a random environment. Commun. Math. Phys. **87**, 81–87 (1982)

16. Mallat, S.: A wavelet tour of signal processing, 2nd edn. San Diego: Academic Press 1999
17. Molchanov, S.A.: Lectures on random media. Lect. Notes Math., vol. 1581, pp. 242–411. Berlin: Springer 1994
18. Olla, S.: Homogenization of diffusion processes in random fields. Palaiseau: École Polytechnique 1994
19. Olla, S.: Central limit theorems for tagged particles and for diffusions in random environment. In: Milieux Aléatoires. Panoramas et Synthèses, Numéro 12. Société Mathématique de France 2001
20. Osada, H.: Homogenization of diffusion processes with random stationary coefficients. In: Probability Theory and Mathematical Statistics (Tbilissi 1982). Lect. Notes Math., vol. 1021, pp. 507–517. Berlin: Springer 1983
21. Papanicolaou, G., Varadhan, S.R.S.: Boundary value problems with rapidly oscillating random coefficients. In: Random Fields. ed. by J. Fritz, D. Szasz. Janyos Bolyai Series, pp. 835–873. Amsterdam: North-Holland 1981
22. Papanicolaou, G., Varadhan, S.R.S.: Diffusion with random coefficients. Statistics and probability: Essays in honor of C.R. Rao, ed. by G. Kallianpur, P.R. Krishnajah, J.K. Gosh, pp. 547–552. Amsterdam: North Holland 1982
23. Revuz, D., Yor, M.: Continuous martingales and Brownian motion, 3rd edn. New York: Springer 1998
24. Shaked, M., Shanthikumar, J.G.: Stochastic orders and their applications. Boston: Academic Press 1994
25. Sidoravicius, V., Sznitman, A.S.: Quenched invariance principles for walks on clusters of percolation or among random conductances. Probab. Theor. Relat. Fields **129**, 219–244 (2004)
26. Stroock, D.W., Varadhan, S.R.S.: Multidimensional diffusion processes. Berlin: Springer 1979
27. Sznitman, A.S.: Topics in random walk in random environment. Notes of course at School and Conference on Probability Theory, May 2002, pp. 203–266. ICTP Lecture Series, Trieste 2004
28. Sznitman, A.S.: Brownian motion, obstacles and random media. Berlin: Springer 1998
29. Sznitman, A.S.: On a class of transient random walks in random environment. Ann. Probab. **29**, 723–764 (2001)
30. Sznitman, A.S.: On new examples of ballistic random walks in random environment. Ann. Probab. **31**, 285–322 (2003)
31. Yurinsky, V.V.: Average of an elliptic boundary problem with random coefficients. Sib. Math. J. **21**, 470–482 (1980)
32. Zeitouni, O.: Random walks in random environment. Lect. Notes Math., vol. 1837, pp. 190–312. Berlin: Springer 2004