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Calculus of Variations

# Large solutions for biharmonic maps in four dimensions

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**Abstract** We seek critical points of the Hessian energy functional $E_\Omega(u) = \int_\Omega |\Delta u|^2 dx$, where $\Omega = \mathbb{R}^4$ or $\Omega$ is the unit disk $B$ in $\mathbb{R}^4$ and $u : \Omega \to S^4$. We show that $E_{\mathbb{R}^4}$ has a critical point which is not homotopic to the constant map. Moreover, we prove that, for certain prescribed boundary data on $\partial B$, $E_B$ achieves its infimum in at least two distinct homotopy classes of maps from $B$ into $S^4$.

## 1 Introduction

Let $\Omega \subset \mathbb{R}^4$ be a smooth domain, $S^4$ the unit sphere in $\mathbb{R}^5$ and consider

$$\mathcal{D}^{l,p}(\Omega, S^4) := \{u \in L^\infty(\Omega, S^4) : \nabla^l u \in L^p(\Omega, \mathbb{R}^{5\cdot 4^l})\}$$

equipped with the semi-norm $\|u\|_{\mathcal{D}^{l,p}(\Omega,S^4)} := \sum_{|\alpha|=l} \int_\Omega |\frac{\partial^\alpha u}{\partial x^\alpha}|^p dx$. Observe that the spaces $\mathcal{D}^{l,p}$ locally coincide with the usual Sobolev spaces $W^{l,p}$. Furthermore, we define the Hessian energy (or biharmonic energy)

$$E_\Omega(u) := \int_\Omega |\Delta u|^2 dx.$$

A map $u \in \mathcal{D}^{2,2}(\Omega, S^4)$ is (*weakly extrinsically*) *biharmonic* if $u$ is a critical point of the Hessian energy functional with respect to compactly supported variations on $S^4$, that is, if for all $\xi \in C_0^\infty(\Omega, S^4)$ we have

$$\frac{d}{dt}|_{t=0} E_\Omega(\pi(u + t\xi)) = 0,$$

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where $\pi$ denotes the nearest point projection onto $S^4$. Computing the corresponding Euler–Lagrange equation, we see that $u \in \mathcal{D}^{2,2}(\Omega, S^4)$ is biharmonic iff $u$ verifies

$$\Delta^2 u = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\nabla u \nabla \Delta u)u \tag{1}$$

in the sense of distributions. We define $u$ to be *minimizing* if $u$ minimizes the Hessian energy among all maps $v \in \mathcal{D}^{2,2}(\Omega, S^4)$ satisfying $u - v \in W_0^{2,2}$. Hence, minimizing maps are biharmonic. Chang et al. proved in [7] the smoothness of biharmonic maps defined from a 4-dimensional domain to a sphere. For recent improvements, we refer to Strzelecki [32], Wang [33] and Lamm and Rivière [15]. Thus, biharmonic maps verify (1) pointwise.

The simplest examples of biharmonic maps $\mathbb{R}^4 \to S^4$ are the constant maps. Our first aim here is to show the existence of biharmonic maps $u : \mathbb{R}^4 \to S^4$ having a non-trivial topological degree. As in [4] we introduce

$$Q_\Omega(u) = \frac{1}{\mathcal{H}^4(S^4)} \int_\Omega J_4(\nabla u) dx$$

for $u \in \mathcal{D}^{2,2}(\Omega, S^4) \subset \mathcal{D}^{1,4}(\Omega, S^4)$, where $J_4(\nabla u) := \det(u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_4})$ is the Jacobian determinant in 4 dimensions. When $\Omega = \mathbb{R}^4$, we observe that, due to Corollary 3.2, $Q_{\mathbb{R}^4}(u) \in \mathbb{Z}$. Thus, for $u \in \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$, the topological degree $deg(u) := Q_{\mathbb{R}^4}(u)$ is well defined. For $k \in \mathbb{Z}$, we then consider the non-empty homotopy class

$$\Xi^k := \{u \in \mathcal{D}^{2,2}(\mathbb{R}^4, S^4) : deg(u) = k\}.$$

The existence of non-constant biharmonic maps $\mathbb{R}^4 \to S^4$ now follows from

**Theorem 1.1** *For $k \in \{-1, 1\}$, there exists a smooth map $u \in \Xi^k$ such that*

$$16\mathcal{H}^4(S^4) < E_{\mathbb{R}^4}(u) = \mathcal{I} := \inf_{v \in \Xi^k} E_{\mathbb{R}^4}(v) \leq 24\mathcal{H}^4(S^4).$$

By symmetry of the Jacobian determinant and of the Hessian energy, it is sufficient to prove Theorem 1.1 in the case $k = 1$. Moreover, we have

$$\mathcal{I} = \inf_{u \in \Xi^1} E(u) = \inf_{u \in \Xi^{-1}} E(u).$$

Our second problem focuses on the existence of non-minimizing biharmonic maps from the unit ball $B \subset \mathbb{R}^4$ into $S^4$. We consider $\gamma \in W^{2,2}(B, S^4)$ and define

$$\Xi_\gamma := \{u \in W^{2,2}(B, S^4) : u - \gamma \in W_0^{2,2}\}.$$

As $\Xi_\gamma$ is a complete metric space and $E_B$ is weakly lower semi-continuous with respect to the $\mathcal{D}^{2,2}$-topology, there exists some $\underline{u} \in \Xi_\gamma$ such that

$$E_B(\underline{u}) = \inf_{u \in \Xi_\gamma} E_B(u).$$

In view of Corollary 3.3, for $u \in \Xi_\gamma$ we have

$$Q_B(u) - Q_B(\underline{u}) \in \mathbb{Z}.$$

For $k \in \mathbb{Z}$, define the homotopy class

$$\Xi_\gamma^k := \{u \in \Xi_\gamma : Q_B(u) - Q_B(\underline{u}) = k\}.$$

It is not hard to see that $\Xi_\gamma^k$ is non-empty for any $k$.

Fix a smooth map $u^* \in \Xi^1$ with $E_{\mathbb{R}^4}(u^*) = \mathcal{I}$ as given, for instance, by Theorem 1.1. For $R > 0$, let $\gamma(x) := u^*(Rx)$, $x \in B$. For $R > 0$ sufficiently small, the above minimizer $\underline{u} \in \Xi_\gamma$ of $E_B$ coincides with $\gamma$. We fix such an $R > 0$. We show

**Theorem 1.2** $\inf_{\Xi^1_\gamma} E_B$ *is achieved.*

This gives the following

**Corollary 1.3** *There exist* (*at least*) *two distinct critical points in* $\Xi_\gamma$ *of* $E_B$.

In view of the results of Brezis-Coron [4] and Jost [11] for harmonic maps, the special choice of $\gamma$ may seem unnecessary. However, the present problem is of fourth order. Carrying over the method of sphere-attaching from the second order case therefore becomes very delicate. In particular, geometric considerations require that the gradients of the absolute minimizer $\underline{u}$ of $E_B$ in $\Xi_\gamma$ and $u^*$ are almost identical at some point. If $\gamma(x) := u^*(Rx)$ (for $R$ sufficiently small), this condition is trivially satisfied. However, at this moment, we have no general criteria to guarantee this condition. See Lemma 5.1 and Remark 5.1 for further details.

In the proof of both theorems, we have to deal with the problem that the classes $\Xi^k$, respectively $\Xi^k_\gamma$, are not closed in the weak $\mathcal{D}^{2,2}$-topology. If we take arbitrary minimizing sequences for $E_{\mathbb{R}^4}$ (resp. $E_B$) in $\Xi^k$ (resp. $\Xi^k_\gamma$), we may encounter the phenomena of concentration and vanishing at infinity as introduced by Lions in [17] and [18]. Therefore, we have to choose our minimizing sequences carefully in order to assure compactness in the limit.

Following the scheme presented in [17] and [18], we show that every minimizing sequence $(u_k)_{k\in\mathbb{N}}$ in $\Xi^1$ for $E_{\mathbb{R}^4}(\cdot)$ converges, up to translations and rescalings, to a map $u$ in $\Xi^1$. Our proof strongly relies on uniform estimates for $\int_{\mathbb{R}^4} |J_4(\nabla u_k)|dx$ for a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ of prescribed topological degree $d = 1$, that allow us to show that the degree is conserved in the limit $k \to \infty$.

In order to prove Theorem 1.2, we show the existence of a map $u \in \Xi^1_\gamma$, verifying $E_B(u) < E_B(\underline{u}) + \mathcal{I} - \delta$ for some $\delta > 0$. For a minimizing sequence $(u_k)_{k\in\mathbb{N}} \subset \Xi^1_\gamma$ s.t. $E_B(u_k) < E_B(\underline{u}) + \mathcal{I} - \delta$, we then can exclude bubbling and conclude that the limiting map $u$ belongs to $\Xi^1_\gamma$.

Similar constructions first appear in Wente's "*sphere attaching lemma*" in [36, Theorem 3.5] in the context of surfaces of prescribed constant mean curvature. Further results relying on this technique can be found in Steffen [26], Steffen [27], Wente [37], Struwe [29], Steffen [28], Brezis and Coron [3], Brezis and Coron [5], Struwe [30] and Struwe [31] in the context of surfaces of prescribed constant mean curvature, and in Brezis and Coron [4], Jost [11], Giaquinta et al. [8], Soyeur [25], Hardt and Lin [9], Qing [22], Kuwert [13], Rivière [23] and Weitkamp [34] in the case of harmonic maps, and in Kusner [12], Bauer-Kuwert [2] for Willmore surfaces.

The paper is organized as follows. In Sect. 2 we compare our results with the respective results in the theory of harmonic maps. In Sect. 3.1, we assign a topological degree to the Sobolev maps in $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$. In Sect. 3.2, we describe the behaviour of the volume functional under weak $\mathcal{D}^{2,2}$-convergence. In Sect. 3.3, we prove a concentration compactness lemma stating that the minimizing sequences for the biharmonic energy locally converge in $\mathcal{D}^{2,2}$ except on a countable set of points. Sect. 3.4 is devoted to some gluing constructions allowing us to isolate possible concentration points of the minimizing sequences. In Sect. 4 we prove Theorem 1.1; in Sect. 5, finally, we present the proof of Theorem 1.2.

In what follows, we denote $u = (u^1, u^2, u^3, u^4, u^5)$ and we tacitly sum over repeated indices (unless otherwise stated). We let $B_r(x)$ be the ball of radius $r > 0$ centered at $x \in \mathbb{R}^4$, and define $B_r := B_r(0)$, $B := B_1$ and $A(r,q) := B_{(1+q)r} \setminus B_r$. Moreover, $\nabla_M$ is the usual gradient on the Riemannian manifold $M$. We let $\nabla := \nabla_{\mathbb{R}^4}$ and $E(\cdot) := E_{\mathbb{R}^4}(\cdot)$.

## 2 Short review of the Dirichlet problem for harmonic maps

For a smooth domain $\Omega \subset \mathbb{R}^2$, consider for $u \in W^{1,2}(\Omega, S^2)$ the Dirichlet energy $D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$. The (*weak*) *harmonic maps* from $\Omega$ into $S^2$ are the critical points of $D_\Omega(u)$ with respect to compactly supported variations on $S^2$. The resulting Euler–Lagrange equation states that $u \in W^{1,2}(\Omega, S^2)$ is harmonic iff $u$ verifies

$$\Delta u = -|\nabla u|^2 u \tag{2}$$

in the sense of distributions. We define $u$ to be *D-minimizing* if $u$ minimizes the Dirichlet energy among all maps $v \in W^{1,2}(\Omega, S^2)$ satisfying $u - v \in W_0^{1,2}$. Thus, D-minimizing maps are harmonic. F. Hélein [10] proved that harmonic maps defined on a 2-dimensional domain are smooth. Therefore, harmonic maps verify (2) pointwise.

Here again, the simplest examples of harmonic maps are the constant maps. A non-trivial harmonic map is given by the inverse of the stereographic projection $\kappa$. Indeed, observe that

$$|\nabla u|^2 \geq 2J_2(\nabla u), \tag{3}$$

where $J_2(\nabla u) := \det(u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2})$ is the Jacobian determinant in 2 dimensions, and consider $X := \{u \in W^{1,2}(\mathbb{R}^2, S^2) : \int_{\mathbb{R}^2} J_2(\nabla u) dx = \mathcal{H}^2(S^2)\}$. It follows from (3) that $D_{\mathbb{R}^2}(v) \geq 2\mathcal{H}^2(S^2)$ for $v \in X$, and equality holds for $\kappa \in X$ (and all maps being conformally equivalent to $\kappa$). Thus, $\kappa$ minimizes the Dirichlet energy in the homotopy class $X$ of $W^{1,2}$- maps of topological degree 1. Hence, $\kappa$ is harmonic.

Consider now the space $W_\gamma^{1,2}(B, S^2) := \{v \in W^{1,2}(B, S^2) : v = \gamma \text{ on } \partial B\}$, where $B$ denotes the unit disk in $\mathbb{R}^2$ and $\gamma \in W^{1,2}(B, S^2)$. It is generally impossible to determine explicit harmonic maps in $W^{1,2}(B, S^2)$. However, as $W_\gamma^{1,2}(B, S^2)$ is a complete metric space and $D_B(\cdot)$ is weakly lower semi-continuous with respect to the $W^{1,2}$-topology, there exists a D-minimizing map $\underline{u} \in W_\gamma^{1,2}(B, S^2)$. Brezis and Coron [4], and Jost [11] proved independently that $D_\Omega$ also attains its infimum in (at least one) homotopy class different from $[\underline{u}]$ when $\gamma \not\equiv cte$. Conversely, if $\gamma \equiv cte.$, Lemaire [16] proved that the constant maps are the only harmonic maps. See Giaquinta et al. [8], Kuwert [13], Qing [22], Soyeur [25] and Weitkamp [34] for further results on the Dirichlet problem for harmonic maps in two dimensions.

## 3 Prerequisites

### 3.1 Topological degree of Sobolev maps

In this subsection, we show that we can define a topological degree for maps in the Sobolev class $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$.

In view of the embedding $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4) \hookrightarrow \mathcal{D}^{1,4}(\mathbb{R}^4, S^4)$, it is sufficient to show the following density result for maps in $\mathcal{D}^{1,4}(\mathbb{R}^4, S^4)$. This is a direct generalization of the density result of Schoen and Uhlenbeck [24] for maps in $H^1$. See also Brezis and Coron [4].

**Theorem 3.1** (Density) *For $u \in \mathcal{D}^{1,4}(\mathbb{R}^4, S^4)$, there exists a sequence of maps $(u_k)_{k\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^4, S^4) \cap \mathcal{D}^{1,4}(\mathbb{R}^4, S^4)$ and a sequence of radii $r_k \to \infty$, as $k \to \infty$, such that*

$$u_k|_{\mathbb{R}^4\setminus B_{r_k}} \equiv cte.$$

*and*

$$u_k \to u \text{ in } \mathcal{D}^{1,4}.$$

*Proof* Let $\sigma : S^4 \to \mathbb{R}^4$ be the stereographic projection, which maps the south pole into 0. We set $v(p) = u(\sigma(p))$ for $p \in S^4$. We define

$$\mathcal{D}^{1,4}(S^4, S^4) := \{f \in L^4(S^4, S^4) : \|f\|_{\mathcal{D}^{1,4}(S^4,S^4)} < \infty\},$$

with

$$\|f\|^4_{\mathcal{D}^{1,4}(S^4,S^4)} := \int_{S^4} |\nabla_{S^4} f|^4 dvol_{S^4}.$$

Observe that $\|v\|_{\mathcal{D}^{1,4}(S^4,S^4)} = \|u\|_{\mathcal{D}^{1,4}(\mathbb{R}^4,S^4)} < \infty$. Hence, $v \in \mathcal{D}^{1,4}(S^4, S^4)$. Consider

$$\mathbb{B}_{\frac{1}{k}}(p) := \left\{q \in S^4 : |q - p| < \frac{1}{k}\right\}$$

and

$$\overline{v}_k(p) := \fint_{\mathbb{B}_{\frac{1}{k}}(p)} v(q) dvol_{S^4}(q).$$

Thus, we have

$$\overline{v}_k \in \mathcal{C}(S^4, \mathbb{R}^5) \cap \mathcal{D}^{1,4}(S^4, \mathbb{R}^5)$$

and

$$\overline{v}_k \to v \text{ in } \mathcal{D}^{1,4}(S^4, \mathbb{R}^5).$$

Now we define, for $\phi_k \in \mathcal{C}_0^\infty(\mathbb{B}_{\frac{2}{k}}(N))$ ($N$ north pole), with $0 \le \phi_k \le 1$ and $\phi_k|_{\mathbb{B}_{\frac{1}{k}}(N)} \equiv 1$,

$$v_k(p) := \overline{v}_k(p) + \phi_k(p)\left(\overline{v}_k(N) - \overline{v}_k(p)\right).$$

We estimate using Poincaré's inequality

$$
\begin{aligned}
dist(v_k(p), S^4) &\leq \fint_{\mathbb{B}_{\frac{1}{k}}(p)} |v(q) - \overline{v}_k(p)| dvol_{S^4}(q) + |\overline{v}_k(p) - v_k(p)| \\
&\leq Ck \left( \int_{\mathbb{B}_{\frac{1}{k}}(p)} |\nabla_{S^4} v|^2 \right)^{\frac{1}{2}} + |\phi_k(p)| \fint_{\mathbb{B}_{\frac{1}{k}}(p)} |\overline{v}_k(N) - v(q)| dvol_{S^4}(q) \\
&\leq Ck \left( \int_{\mathbb{B}_{\frac{1}{k}}(p)} |\nabla_{S^4} v|^2 \right)^{\frac{1}{2}} + C \fint_{\mathbb{B}_{\frac{3}{k}}(N)} |\overline{v}_k(N) - v(q)| dvol_{S^4}(q) \\
&\leq C \left( \int_{\mathbb{B}_{\frac{1}{k}}(p)} |\nabla_{S^4} v|^4 \right)^{\frac{1}{4}} + C \left( \int_{\mathbb{B}_{\frac{3}{k}}(N)} |\nabla_{S^4} v|^4 \right)^{\frac{1}{4}}. \qquad (4)
\end{aligned}
$$

As the right hand side decreases monotonically to zero as $k \to \infty$, Dini's theorem implies uniform convergence. Thus, for $k$ sufficiently large, we can project $v_k(p)$ onto $S^4$ and assume that $v_k \in \mathcal{C}(S^4, S^4)$. Moreover, $v_k \to v$ in $\mathcal{D}^{1,4}$ and $v_k$ is constant in a neighborhood of the north pole $N$.

The sequence $u_k := v_k(\sigma^{-1}(x))$ has now all the required properties. □

*Remark 3.1* The approximating sequences $u_k$ can be chosen in the class of smooth maps after mollification with a smooth kernel.

It follows immediately

**Corollary 3.2** *For $u \in \mathcal{D}^{1,4}(\mathbb{R}^4, S^4)$, we have*

$$\int_{\mathbb{R}^4} J_4(\nabla u) dx = k \mathcal{H}^4(S^4)$$

*with $k \in \mathbb{Z}$.*

Moreover, we get

**Corollary 3.3** *For $u, v \in \mathcal{D}^{1,4}(B, S^4)$ with $u = v$ on $\partial B$, we have*

$$\int_B (J_4(\nabla u) - J_4(\nabla v)) dx = k \mathcal{H}^4(S^4)$$

*with $k \in \mathbb{Z}$.*

*Proof* We define $w : \mathbb{R}^4 \to S^4$ as

$$w(x) := \begin{cases} u(x) & \text{for } x \in B \\ v(\frac{x}{|x|^2}) & \text{for } x \in \mathbb{R}^4 \setminus B. \end{cases}$$

As $u, v \in \mathcal{D}^{1,4}(B, S^4)$, we verify that $w \in \mathcal{D}^{1,4}(\mathbb{R}^4, S^4)$ and

$$\int_B (J_4(\nabla u) - J_4(\nabla v))dx = \int_{\mathbb{R}^4} J_4(\nabla w)dx.$$

Thus, the result follows from Corollary 3.2. □

3.2 Weak continuity properties of the volume functional

In this subsection we prove a generalization of Wente's weak continuity property of the volume functional

$$V(u) = \int_\Omega J_4(\nabla u)dx$$

in [35, Sect. III]. More precisely, the following properties for the weak $\mathcal{D}^{2,2}$-topology hold.

**Proposition 3.4**

1. *Let $u_k \rightharpoonup u$ in $\mathcal{D}^{2,2}(B_R, S^4)$. Then, we have for a.e.* $0 < r < R$

$$\int_{B_r} J_4(\nabla u_k)dx = \int_{B_r} J_4\left(\nabla(u_k - u)\right)dx + \int_{B_r} J_4(\nabla u)dx + o(1), \tag{5}$$

   *where $o(1) \longrightarrow 0$ as $k \to \infty$.*
   *We call r a good radius if r verifies* (5).
2. *For $u_k \rightharpoonup u$ in $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$, it holds*

$$\int_{\mathbb{R}^4} J_4(\nabla u_k)dx = \int_{\mathbb{R}^4} J_4\left(\nabla(u_k - u)\right)dx + \int_{\mathbb{R}^4} J_4(\nabla u)dx + o(1).$$

3. *For $u_k \rightharpoonup u$ in $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ and $\xi \in C_0^\infty(\mathbb{R}^4, \mathbb{R}_{>0})$, it holds*

$$\int_{\mathbb{R}^4} \xi |J_4(\nabla u_k) - J_4(\nabla u)|dx \le \int_{\mathbb{R}^4} \xi \left|det\left(u_k, \frac{\partial v_k}{\partial x_1}, \frac{\partial v_k}{\partial x_2}, \frac{\partial v_k}{\partial x_3}, \frac{\partial v_k}{\partial x_4}\right)\right| dx + o(1)$$

   *with $v_k := u_k - u$.*

In order to prove Proposition 3.4, we need the following

**Lemma 3.5** *For $\Omega = B_r$ $(r > 0)$ or $\Omega = \mathbb{R}^4$, let $e_k \rightharpoonup e$, $f_k \rightharpoonup f$, $g_k \rightharpoonup g$, $h_k \rightharpoonup 0$ in $\mathcal{D}^{2,2}(\Omega, S^4)$ as $k \to \infty$, and $w \in \mathcal{D}^{2,2}(\Omega, S^4)$. Then, as $k \to \infty$, we have that*

$$\int_\Omega \left|det\left(e_k, \frac{\partial f_k}{\partial x_1}, \frac{\partial g_k}{\partial x_2}, \frac{\partial h_k}{\partial x_3}, \frac{\partial w}{\partial x_4}\right)\right| dx \longrightarrow 0.$$

*Proof* We consider, for $\epsilon > 0$ and $\Omega = B_r$ (compact case), $w_\epsilon$ smooth such that $\|w - w_\epsilon\|_{\mathcal{D}^{2,2}} \longrightarrow 0$ as $\epsilon \to 0$. It follows from the multilinearity of the determinant, Hölder's inequality and the continuity of the embedding $W^{2,2} \hookrightarrow W^{1,4}$ (Sobolev's embedding theorem)

$$\begin{aligned}
&\int_{B_r} \left| det\left(e_k, \frac{\partial f_k}{\partial x_1}, \frac{\partial g_k}{\partial x_2}, \frac{\partial h_k}{\partial x_3}, \frac{\partial w}{\partial x_4}\right)\right| dx \\
&\quad \le \int_{B_r} \left| det\left(e_k, \frac{\partial f_k}{\partial x_1}, \frac{\partial g_k}{\partial x_2}, \frac{\partial h_k}{\partial x_3}, \frac{\partial (w - w_\epsilon)}{\partial x_4}\right)\right| dx \\
&\qquad + \int_{B_r} \left| det\left(e_k, \frac{\partial f_k}{\partial x_1}, \frac{\partial g_k}{\partial x_2}, \frac{\partial h_k}{\partial x_3}, \frac{\partial w_\epsilon}{\partial x_4}\right)\right| dx \\
&\quad \le \|e_k\|_{L^\infty(B_r)} \|\nabla f_k\|_{L^4(B_r)} \|\nabla g_k\|_{L^4(B_r)} \|\nabla h_k\|_{L^4(B_r)} \|\nabla (w - w_\epsilon)\|_{L^4(B_r)} \\
&\qquad + \|e_k\|_{L^\infty(B_r)} \|\nabla f_k\|_{L^4(B_r)} \|\nabla g_k\|_{L^4(B_r)} \|\nabla h_k\|_{L^2(B_r)} \|\nabla w_\epsilon\|_{L^\infty(B_r)} \\
&\quad \le C\|w - w_\epsilon\|_{\mathcal{D}^{2,2}(B_r)} + K\|\nabla h_k\|_{L^2(B_r)},
\end{aligned}$$

where $C$ is independent of $k$ and $\epsilon$, and $K$ is independent of $k$.

The embedding $\mathcal{D}^{2,2}(B_r, S^4) = W^{2,2}(B_r, S^4) \hookrightarrow W^{1,2}(B_r, S^4)$ $(B_r \subset \mathbb{R}^4)$ is compact by Rellich's theorem. Thus, the second term converges to zero as $k$ tends to infinity. Then, we let $\epsilon$ tend to zero and the claim follows for $\Omega = B_r$ (compact case).

For $\Omega = \mathbb{R}^4$, we choose $r_l \to \infty$ $(l \to \infty)$ and estimate with Hölder's inequality

$$\begin{aligned}
&\int_{\mathbb{R}^4} \left| det\left(e_k, \frac{\partial f_k}{\partial x_1}, \frac{\partial g_k}{\partial x_2}, \frac{\partial h_k}{\partial x_3}, \frac{\partial w}{\partial x_4}\right)\right| dx \\
&\quad \le \int_{B_{r_l}} \left| det\left(e_k, \frac{\partial f_k}{\partial x_1}, \frac{\partial g_k}{\partial x_2}, \frac{\partial h_k}{\partial x_3}, \frac{\partial w}{\partial x_4}\right)\right| dx + C\left(\int_{\mathbb{R}^4 \setminus B_{r_l}} |\nabla w|^4 dx\right)^{\frac{1}{4}},
\end{aligned}$$

where $C$ is independent of $k$ and $l$. First, we let $k \to \infty$ and the first term converges to zero (as in the compact case $\Omega = B_r$). Then, we let $l$ tend to infinity and the second term vanishes due to the absolute continuity property of the Lebesgue integral. □

Now we are able to prove Proposition 3.4. Define $v_k := u_k - u$. Using the multilinearity of the determinant we have

$$J_4(\nabla u_k) - J_4(\nabla u) = \sum_{j=0}^{4} A_j + J_4(\nabla v_k),$$

where $A_j$ is a sum of terms where $j$ derivatives of $v_k$ (and $4-j$ derivatives of $u$) appear (e.g. $\det(u, \frac{\partial u}{\partial x_1}, \frac{\partial v_k}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial v_k}{\partial x_4})$ appears in $A_2$; $\det(v_k, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial v_k}{\partial x_3}, \frac{\partial u}{\partial x_4})$ appears in $A_1$).

For $j = 1, 2, 3$, and $\Omega = B_r$ $(r > 0)$ or $\Omega = \mathbb{R}^4$, it holds

$$\|A_j\|_{L^1(\Omega)} \longrightarrow 0,$$

as $k \to \infty$. This follows directly from Lemma 3.5. Moreover, we observe that

$$|A_0| = \left| det\left(v_k, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}\right)\right| \le C|\nabla u|^4 \in L^1.$$

Thus, applying Lebesgue's dominate convergence theorem implies that $A_0 \to 0$ in $L^1(\Omega)$. Since $A_4 + J_4(\nabla v_k) = det\left(u_k, \frac{\partial v_k}{\partial x_1}, \frac{\partial v_k}{\partial x_2}, \frac{\partial v_k}{\partial x_3}, \frac{\partial v_k}{\partial x_4}\right)$, this completes the proof of the third affirmation.

It remains to consider $A_4$ in order to complete the proof of the two first affirmations. We rewrite using integrations by parts

$$\int_{B_r} A_4 dx = \int_{B_r} det\left(u, \frac{\partial v_k}{\partial x_1}, \frac{\partial v_k}{\partial x_2}, \frac{\partial v_k}{\partial x_3}, \frac{\partial v_k}{\partial x_4}\right) dx$$
$$= \sum_{\substack{\alpha,\beta,\gamma,\delta,\epsilon \\ i,j,m,s}} \int_{\partial B_r} (-1)^s u^\alpha v_k^\beta \frac{\partial v_k^\gamma}{\partial x_i} \frac{\partial v_k^\delta}{\partial x_j} \frac{\partial v_k^\epsilon}{\partial x_m} d\sigma$$
$$+ \sum_{\substack{\alpha,\beta,\gamma,\delta,\epsilon \\ i,j,m,n,s}} \int_{B_r} (-1)^s v_k^\alpha \frac{\partial u^\beta}{\partial x_i} \frac{\partial v_k^\gamma}{\partial x_j} \frac{\partial v_k^\delta}{\partial x_m} \frac{\partial v_k^\epsilon}{\partial x_n} dx \tag{6}$$

Using Fubini's theorem, we have for a.e. $r$ that $(v_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,2}(\partial B_r, S^4)$. By the compactness of the embedding $W^{2,2}(\partial B_r, S^4) \hookrightarrow W^{1,3}(\partial B_r, S^4)$, the first term converges to zero, as $k$ tends to infinity. The second term may be treated as in the proof of Lemma 3.5. This completes the proof of the first affirmation.

It remains to consider the case $\Omega = \mathbb{R}^4$. Using Theorem 3.1, we may assume that $u_k \equiv cte.$ and $u \equiv cte. + o_{\mathcal{D}^{2,2}}$ on $\mathbb{R}^4 \setminus B_{R_k}$ for some sequence of radii $R_k \to \infty$ as $k \to \infty$, and $o_{\mathcal{D}^{2,2}} \to 0$ in $\mathcal{D}^{2,2}$. Thus, Eq. (6) implies

$$\left| \int_{\mathbb{R}^4} det\left(u, \frac{\partial v_k}{\partial x_1}, \frac{\partial v_k}{\partial x_2}, \frac{\partial v_k}{\partial x_3}, \frac{\partial v_k}{\partial x_4}\right) dx \right| \leq \sum_{\substack{\alpha,\beta,\gamma,\delta,\epsilon \\ i,j,m,n,s}} \left| \int_{\mathbb{R}^4} (-1)^s v_k^\alpha \frac{\partial u^\beta}{\partial x_i} \frac{\partial v_k^\gamma}{\partial x_j} \frac{\partial v_k^\delta}{\partial x_m} \frac{\partial v_k^\epsilon}{\partial x_n} dx \right| + o(1),$$

where $o(1) \to 0$ as $k \to \infty$. The second affirmation follows now as in the proof of Lemma 3.5 and we are done.

### 3.3 Concentration compactness lemma

**Lemma 3.6** (Concentration compactness) *Suppose $u_k \rightharpoonup u \in \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ and $\mu_k = |\nabla^2 u_k|^2 dx \rightharpoonup \mu$, $\nu_k = J_4(\nabla u_k)dx \rightharpoonup \nu$ weakly in the sense of measures, where $\mu$ and $\nu$ are bounded signed measures on $\mathbb{R}^4$.*
*Then, we have:*
*There exists some at most countable set $J$, a family $\{x^{(j)} \in J\}$ of distinct points on $\mathbb{R}^4$, and a family $\{\nu^{(j)}, j \in J\}$ of non-zero real numbers such that*

$$\nu = J_4(\nabla u)dx + \sum_{j\in J} \nu^{(j)} \delta_{x^{(j)}},$$

*where $\delta_x$ is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^4$.*

*Proof* We follow the scheme of [18]. Let $v_k := u_k - u \in \mathcal{D}^{2,2}(\mathbb{R}^4, \mathbb{R}^5)$. Then we have that $v_k \rightharpoonup 0$ weakly in $\mathcal{D}^{2,2}$. Moreover, we define $\lambda_k := |\nabla^2 v_k|^2 dx$, $\omega_k := \nu_k - J_4(\nabla u)dx$, $\omega_k^+$ the positive part of $\omega_k$, $\omega_k^-$ the negative part of $\omega_k$ and $\overline{\omega}_k := \omega_k^+ + \omega_k^-$ the total variation.

We may assume that $\lambda_k \rightharpoonup \lambda$, while $\omega_k \rightharpoonup \omega = \nu - J_4(\nabla u)dx$, $\omega_k^+ \rightharpoonup \omega^+$, $\omega_k^- \rightharpoonup \omega^-$ and $\overline{\omega}_k \rightharpoonup \overline{\omega} = \omega^+ + \omega^-$ weakly in the sense of measures, where $\lambda, \omega^+, \omega^-, \overline{\omega} \geq 0$.

For $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^4, \mathbb{R})$ with Proposition 3.4, we have

$$\begin{aligned}\int_{\mathbb{R}^4} \xi^4 d\overline{\omega} &= \lim_{k\to\infty} \int_{\mathbb{R}^4} \xi^4 d\overline{\omega}_k \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^4} \xi^4 |J_4(\nabla u_k) - J_4(\nabla u)| dx \\ &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^4} \xi^4 \left| det \left( u_k, \frac{\partial v_k}{\partial x_1}, \frac{\partial v_k}{\partial x_2}, \frac{\partial v_k}{\partial x_3}, \frac{\partial v_k}{\partial x_4} \right) \right| dx.\end{aligned}$$

As $\xi$ has compact support, Rellich's theorem implies that lower order terms vanish as $k \to \infty$ and with Nirenberg's interpolation inequality in [21, p. 11], we get

$$\begin{aligned}\int_{\mathbb{R}^4} \xi^4 d\overline{\omega} &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^4} \left| det \left( u_k, \frac{\partial (\xi v_k)}{\partial x_1}, \frac{\partial (\xi v_k)}{\partial x_2}, \frac{(\xi \partial v_k)}{\partial x_3}, \frac{\partial (\xi v_k)}{\partial x_4} \right) \right| dx \qquad (7) \\ &\leq C \liminf_{k\to\infty} \int_{\mathbb{R}^4} |\nabla(\xi v_k)|^4 dx \\ &\leq C \liminf_{k\to\infty} \left( \int_{\mathbb{R}^4} |\nabla^2(\xi v_k)|^2 dx \right)^2 \\ &\leq C \liminf_{k\to\infty} \left( \int_{\mathbb{R}^4} \xi^2 |\nabla^2 v_k|^2 dx \right)^2 \\ &\leq C \left( \int_{\mathbb{R}^4} \xi^2 d\lambda \right)^2. \qquad (8)\end{aligned}$$

Let $\{x^{(j)}; j \in J\}$ be the atoms of the measure $\overline{\omega}$ and decompose $\overline{\omega} = \overline{\omega}_0 + \sum_{j\in J} \mu^{(j)} \delta_{x^{(j)}}$, with $\mu^{(j)} > 0$ and $\overline{\omega}_0$ free of atoms.

Since $\int_{\mathbb{R}^4} d\overline{\omega} < \infty$, $J$ is an at most countable set. Now, for any open set $\Omega \subset \mathbb{R}^4$, by (7) with $\xi = \xi_k \in \mathcal{C}_0^\infty(\Omega)$ converging to the characteristic function of $\Omega$ as $k \to \infty$, we have

$$\int_\Omega d\overline{\omega} \leq C \left( \int_\Omega d\lambda \right)^2. \qquad (9)$$

In particular, $\overline{\omega}$ is absolutely continuous with respect to $\lambda$ and by the Radon–Nikodym theorem there exists $f \in L^1(\mathbb{R}^4, \lambda)$ such that $d\overline{\omega} = f d\lambda$, $\lambda$-almost everywhere.

Moreover, for $\lambda$-almost every $x \in \mathbb{R}^4$, we have

$$f(x) = \lim_{\rho\to 0} \frac{\int_{B_\rho(x)} d\overline{\omega}}{\int_{B_\rho(x)} d\lambda}.$$

But then, using (9), if $x$ is not an atom of $\lambda$,

$$f(x) \leq \lim_{\rho\to 0}\left(\frac{C\left(\int_{B_\rho(x)} d\lambda\right)^2}{\int_{B_\rho(x)} d\lambda}\right) = C\lim_{\rho\to 0}\int_{B_\rho(x)} d\lambda = 0,$$

$\lambda$-almost everywhere. Since $\lambda$ has only countably many atoms and $\overline{\omega}_0$ has no atoms, this implies that $\overline{\omega}_0 = 0$.

We conclude that

$$\nu = J_4(\nabla u)dx + \sum_{j\in J}\nu^{(j)}\delta_{x^{(j)}},$$

with $\nu^{(j)} \neq 0$. This completes the proof. □

### 3.4 Gluing lemmas

**Lemma 3.7** *Suppose $u, v \in W^{2,2}(S^3, S^4)$. For $\epsilon > 0$, there exists $w \in W^{2,2}(S^3\times[0,\epsilon], \mathbb{R}^5)$ such that $w$ agrees with $u$ in a neighborhood of $S^3\times\{0\}$, $w$ agrees with $v$ in a neighborhood of $S^3 \times \{\epsilon\}$,*

$$\int_{S^3\times[0,\epsilon]} |\nabla^2_{S^3\times[0,\epsilon]} w|^2 \leq C\epsilon\int_{S^3}\left(|\nabla^2_{S^3}u|^2 + |\nabla^2_{S^3}v|^2\right)$$
$$+C\epsilon^{-1}\int_{S^3}|\nabla_{S^3}(u-v)|^2 + C\epsilon^{-3}\int_{S^3}|u-v|^2,$$

*and*

$$dist(w, S^4) \leq C\|u-v\|_{L^\infty(S^3)}$$

*almost everywhere on $S^3 \times [0,\epsilon]$.*

*Proof* Choose $\psi \in \mathcal{C}^\infty([0,\epsilon])$ with $0 \leq \psi \leq 1$, $\psi \equiv 1$ in a neighborhood of 0, $\psi \equiv 0$ in a neighborhood of $\epsilon$, $|\psi'| \leq C\epsilon^{-1}$ and $|\psi''| \leq C\epsilon^{-2}$. We define

$$w(x,s) := v(x) + \psi(s)(u(x) - v(x))$$

for $(x,s) \in S^3 \times [0,\epsilon]$. We estimate

$$|\nabla^2_{S^3\times[0,\epsilon]}w|^2 \leq C\left(|\nabla^2 u|^2 + |\nabla^2 v|^2 + \epsilon^{-2}|\nabla(u-v)|^2 + \epsilon^{-4}|u-v|^2\right).$$

Integration over $S^3 \times [0,\epsilon]$ yields the energy estimate.

As $v(S^3) \subset S^4$, we have for a.e. $(x,s) \in S^3 \times [0,\epsilon]$

$$dist(w(x,s), S^4) \leq \|u-v\|_{L^\infty(S^3)}.$$

□

**Lemma 3.8** *Let $u_k \rightharpoonup u$ in $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ as $k \to \infty$. After passing to a subsequence, there exists, for a.e. $\rho > 0$, a sequence of positive real numbers $\epsilon_k \to 0$ $(k \to \infty)$ and a sequence of maps $(v_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ such that*

$$v_k = \begin{cases} u_k & \textit{for } x \in B_\rho \\ u & \textit{for } x \in \mathbb{R}^4 \setminus B_{(1+\epsilon_k)\rho}\end{cases}$$

*and*

$$\int_{A(\rho,\epsilon_k)} |\nabla^2 v_k|^2 dx \longrightarrow 0,$$

*as* $k \to \infty$.

*Proof* As $u_k$ is bounded in $\mathcal{D}^{2,2}$, Lemma 3.9 below applied to $f_k(r) := \int_{\partial B_r} |\nabla^2 u_k|^2 dx$ implies that, after passing to a subsequence, we have for a.e. $\rho > 0$ and every sequence of positive real numbers $\gamma_k \to 0$

$$\int_{A(\rho,\gamma_k)} |\nabla^2 u_k|^2 dx \longrightarrow 0, \tag{10}$$

as $k \to \infty$. Henceforth, we consider a fixed $\rho > 0$ verifying (10) and $\epsilon_k \to 0$ ($k \to \infty$) to be fixed later. Fubini's theorem implies that there is a set of $\sigma_k \in (\rho, (1+\frac{\epsilon_k}{2})\rho)$ of positive measure, such that for $p \in \{2, \frac{7}{2}\}$, we have

$$\begin{aligned} \epsilon_k \sigma_k \int_{\partial B_{\sigma_k}} \left(|\nabla^2 u_k|^2 + |\nabla^2 u|^2\right) &\le C \int_{A(\rho,\epsilon_k)} \left(|\nabla^2 u_k|^2 + |\nabla^2 u|^2\right) \\ \epsilon_k \sigma_k \int_{\partial B_{\sigma_k}} |\nabla(u_k - u)|^p &\le C \int_{A(\rho,\epsilon_k)} |\nabla(u_k - u)|^p \\ \epsilon_k \sigma_k \int_{\partial B_{\sigma_k}} |u_k - u|^2 &\le C \int_{A(\rho,\epsilon_k)} |u_k - u|^2. \end{aligned} \tag{11}$$

Now we can apply Lemma 3.7 (with $\frac{\epsilon_k}{4}$ instead of $\epsilon_k$) to the functions $\overline{u}_k(x) := u_k(\sigma_k x)$ and $\overline{u}(x) := u(\sigma_k x)$, thus giving $\overline{v}_k$ on $S^3 \times [0, \frac{\epsilon_k}{4}]$ with $\overline{v}_k = \overline{u}_k$ in a neighborhood of $S^3 \times \{0\}$, $\overline{v}_k = \overline{u}$ in a neighborhood of $S^3 \times \{\frac{\epsilon_k}{4}\}$ and verifying

$$\begin{aligned} &\int_{S^3 \times [0,\frac{\epsilon_k}{4}]} |\nabla^2_{S^3 \times [0,\frac{\epsilon_k}{4}]} \overline{v}_k|^2 \\ &\le C\epsilon_k \int_{S^3} \left(|\nabla^2_{S^3} \overline{u}_k|^2 + |\nabla^2_{S^3} \overline{u}|^2\right) + \frac{C}{\epsilon_k} \int_{S^3} |\nabla_{S^3}(\overline{u}_k - \overline{u})|^2 + \frac{C}{\epsilon_k^3} \int_{S^3} |\overline{u}_k - \overline{u}|^2 \\ &\le C\epsilon_k \sigma_k \int_{\partial B_{\sigma_k}} \left(|\nabla^2 u_k|^2 + |\nabla^2 u|^2\right) + \frac{C}{\epsilon_k \sigma_k} \int_{\partial B_{\sigma_k}} |\nabla(u_k - u)|^2 + \frac{C}{(\epsilon_k \sigma_k)^3} \int_{\partial B_{\sigma_k}} |u_k - u|^2 \\ &\le C \int_{A(\rho,\epsilon_k)} \left(|\nabla^2 u_k|^2 + |\nabla^2 u|^2\right) + \frac{C}{\sigma_k^2 \epsilon_k^2} \int_{B_{2\rho}} |\nabla(u_k - u)|^2 + \frac{C}{\sigma_k^4 \epsilon_k^4} \int_{B_{2\rho}} |u_k - u|^2 \end{aligned} \tag{12}$$

by (11), and almost everywhere on $S^3 \times [0, \frac{\epsilon_k}{4}]$

$$dist(\overline{v}_k, S^4) \leq C\|u_k - u\|_{L^\infty(S^3)} \leq C\left(\int\limits_{S^3} |\nabla(u_k - u)|^{\frac{7}{2}}\right)^{\frac{2}{7}}$$

$$\leq C\left(\sigma_k^{\frac{1}{2}} \int\limits_{\partial B_{\sigma_k}} |\nabla(u_k - u)|^{\frac{7}{2}}\right)^{\frac{2}{7}}$$

$$\leq C\left(\sigma_k^{-\frac{1}{2}} \epsilon_k^{-1} \int\limits_{B_{2\rho}} |\nabla(u_k - u)|^{\frac{7}{2}}\right)^{\frac{2}{7}}. \tag{13}$$

by Sobolev's inequality and (11). As $u_k \to u$ in $W^{1,\frac{7}{2}}_{loc}$ (up to a subsequence), we may choose $\epsilon_k$ tending sufficiently slowly to zero such that the two last terms in (12) and the right hand side in (13) converge to zero as $k \to \infty$. The first term on the right hand side in (12) converges to zero due to (10). This implies that

$$\int\limits_{S^3\times[0,\frac{\epsilon_k}{4}]} |\nabla^2_{S^3\times[0,\frac{\epsilon_k}{4}]} \overline{v}_k|^2 = o(1) \tag{14}$$

and almost everywhere on $S^3 \times [0, \frac{\epsilon_k}{4}]$

$$dist(\overline{v}_k, S^4) = o(1),$$

where $o(1) \to 0$ as $k \to \infty$. Let $\Pi$ be the nearest point projection onto $S^4$. Then, for $k$ sufficiently large, $\Pi \circ \overline{v} : S^3 \times [0, \frac{\epsilon_k}{4}] \to S^4$ is well-defined and smooth. So finally, after passing to a subsequence, we can define a suitable sequence of maps $v_k$ by

$$v_k(x) := \begin{cases} u_k(x) & \text{for } |x| \leq \sigma_k \\ \Pi\left(\overline{v}_k(\frac{x}{|x|}, \frac{|x|}{\sigma_k} - 1)\right) & \text{for } \sigma_k \leq |x| \leq (1 + \frac{\epsilon_k}{4})\sigma_k \\ u(\psi(|x|)\frac{x}{|x|}) & \text{for } (1 + \frac{\epsilon_k}{4})\sigma_k \leq |x| \leq (1 + \frac{\epsilon_k}{2})\sigma_k \\ u(x) & \text{for } |x| \geq (1 + \frac{\epsilon_k}{2})\sigma_k, \end{cases}$$

where $\psi(t)$ is a $\mathcal{C}^2(\mathbb{R})$ function with the properties $\psi((1+\frac{\epsilon_k}{4})\sigma_k) = \sigma_k$, $\psi((1+\frac{\epsilon_k}{2})\sigma_k) = (1 + \frac{\epsilon_k}{2})\sigma_k$, $\psi'((1 + \frac{\epsilon_k}{4})\sigma_k) = 0$ and $\psi'((1 + \frac{\epsilon_k}{2})\sigma_k) = 1$ . In view of (14), it is now easy to conclude. □

In order to complete the proof of Lemma 3.8, we have to show the following

**Lemma 3.9** *Let $f_k \geq 0$ be a bounded sequence in $L^1(\mathbb{R})$. After passing to a subsequence, we have for a.e. $\rho > 0$ and every sequence of positive real numbers $\gamma_k \to 0$*

$$\int\limits_{\rho}^{\rho+\gamma_k} f_k dx \longrightarrow 0,$$

*as $k \to \infty$.*

*Proof* Define $F_k(y) := \int_{-\infty}^{y} f_k dx$. Observe that $F_k \in W^{1,1}(\mathbb{R}) \hookrightarrow \mathcal{C}^0(\mathbb{R})$ increases monotonically and $|F_k| \leq C$. Thus, after passing to a subsequence, $F_k \to F$ a.e. and $F$ increases monotonically. It follows that the set $sing(F)$, where $F$ is discontinuous, is countable. Hence, for any $\rho \notin sing(F)$ and any sequence $\alpha_l \to 0$, as $l \to \infty$, we have $F(\rho + \alpha_l) \to F(\rho)$, as $l \to \infty$.

Now fix $\rho \notin sing(F)$ and consider sequences of positive real numbers $\beta_m \overset{m\to\infty}{\longrightarrow} 0$ and $\gamma_k \overset{k\to\infty}{\longrightarrow} 0$ such that for any fixed $m \in \mathbb{N}$, as $k \to \infty$, we have $F_k(\rho \pm \beta_m) \overset{k\to\infty}{\longrightarrow} F(\rho \pm \beta_m)$. Then, for $k \geq k_0(m)$ we have

$$\begin{aligned} 0 \leq F_k(\rho + \gamma_k) - F_k(\rho) &\leq F_k(\rho + \beta_m) - F_k(\rho - \beta_m) \\ &= F(\rho + \beta_m) - F(\rho - \beta_m) + o(1), \end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$. The claim follows as we first let $k \to \infty$ and then also pass to the limit $m \to \infty$. □

A direct consequence of Lemma 3.8 is the following

**Lemma 3.10** *Let $u_k \rightharpoonup u$ in $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ as $k \to \infty$. After passing to a subsequence, there exists a sequence of positive real numbers $\epsilon_k \to 0$ $(k \to \infty)$ and a sequence of maps $(w_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ such that*

$$w_k = \begin{cases} u_k & \text{for } x \in B_{\epsilon_k} \\ u & \text{for } x \in \mathbb{R}^4 \setminus B_{(1+\epsilon_k)\epsilon_k} \end{cases}$$

*and*

$$\int_{A(\epsilon_k,\epsilon_k)} |\nabla^2 w_k|^2 dx \longrightarrow 0,$$

*as $k \to \infty$.*

*Remark 3.2* Lemma 3.8 and 3.10 may be viewed as a variant of Luckhaus' lemma (see [19] and [20]) for second derivatives in the critical Sobolev dimension.

**Lemma 3.11** *Let $u \in \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$. For every sequence of positive real numbers $\epsilon_k \to 0$ $(k \to \infty)$, there exists a sequence of maps $(w_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ such that*

$$w_k = \begin{cases} u & \text{for } x \in B_{\epsilon_k} \\ \pi(u_{B_{\epsilon_k}}) & \text{for } x \in \mathbb{R}^4 \setminus B_{2\epsilon_k} \end{cases}$$

*and*

$$\int_{B_{2\epsilon_k} \setminus B_{\epsilon_k}} |\nabla^2 w_k|^2 dx \longrightarrow 0,$$

*as $k \to \infty$, $\pi$ is the nearest point projection onto $S^4$ and $u_{B_{\epsilon_k}} := \fint_{B_{\epsilon_k}} u dx$.*

*Proof* Choose $\psi \in \mathcal{C}_0^\infty([0, 2\epsilon_k])$ with $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $[0, \epsilon_k]$, $\psi \equiv 0$ in a neighborhood of $2\epsilon_k$, $|\psi'| \leq C\epsilon_k^{-1}$ and $|\psi''| \leq C\epsilon_k^{-2}$. We define

$$w_k(x) := u_{B_{\epsilon_k}} + \psi(|x|)(u(x) - u_{B_{\epsilon_k}}),$$

and estimate with Poincaré's and Sobolev's inequality

$$\int_{B_{2\epsilon_k}} |\nabla^2 w_k|^2 dx \le C \int_{B_{2\epsilon_k}} \left(|\nabla^2 u|^2 + \epsilon_k^{-2}|\nabla u|^2 + \epsilon_k^{-4}|u - u_{B_{\epsilon_k}}|^2\right) dx$$
$$\le C \int_{B_{2\epsilon_k}} |\nabla^2 u|^2 dx + C \left( \int_{B_{2\epsilon_k}} |\nabla u|^4 dx \right)^{\frac{1}{2}}$$
$$\le C \int_{B_{2\epsilon_k}} |\nabla^2 u|^2 dx = o(1),$$

where $o(1) \to 0$ as $k \to \infty$. Moreover, we have for a.e. $x \in B_{2\epsilon_k}$

$$dist(w_k(x), S^4) \le \fint_{B_{\epsilon_k}(x)} |u(y) - u_{B_{\epsilon_k}}|dy + |\psi(|x|)||u(x) - u_{B_{\epsilon_k}}|. \tag{15}$$

As in (4) Poincaré's and Sobolev's inequality imply that the first term in (15) is bounded by

$$C\fint_{B_{\epsilon_k}(x)} |u(y) - u_{B_{\epsilon_k}}|dy \le C \left( \int_{B_{3\epsilon_k}} |\nabla u|^4 dy \right)^{\frac{1}{4}} \le C \left( \int_{B_{3\epsilon_k}} |\nabla^2 u|^2 dy \right)^{\frac{1}{2}}. \tag{16}$$

For the second term in (15), we define

$$u_k(x) := \fint_{B_{\epsilon_k}(x)} u(y)dy.$$

As $u_k \to u$ in $L^2$, it follows, after passing to a subsequence, that $u_k$ converges to $u$ pointwise a.e. Thus, we estimate

$$|\psi(|x|)||u(x) - u_{B_{\epsilon_k}}| \le \fint_{B_{\epsilon_k}(x)} |u(y) - u_{B_{\epsilon_k}}|dy + |u_k(x) - u(x)|. \tag{17}$$

The last term in (17) converges to zero a.e., as $k \to \infty$. Therefore, the inequalities (15), (16) and (17) imply that $dist(w_k(x), S^4)$ converges to zero, as $k \to \infty$, for a.e. $x \in B_{2\epsilon_k}$. Thus, for $k$ sufficiently large, we can project $w_k$ onto $S^4$ and assume that $w_k \in W^{2,2}(\mathbb{R}^4, S^4)$ and $w_k = \pi(u_{B_{\epsilon_k}})$ in a neighborhood of $\partial B_{2\epsilon_k}$. This completes the proof. □

A direct consequence of Lemma 3.10 and Lemma 3.11 is

**Lemma 3.12** *Let $u_k \rightharpoonup u$ in $\mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ as $k \to \infty$. After passing to a subsequence, there exists a sequence of positive real numbers $\epsilon_k \to 0$ $(k \to \infty)$ and a sequence of maps $(w_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ defined as*

$$w_k := \begin{cases} u_k & \text{for } x \in B_{\epsilon_k} \\ \pi(u_{B_{\epsilon_k}}) & \text{for } x \in \mathbb{R}^4 \setminus B_{4\epsilon_k} \end{cases}$$

*and*

$$\int_{B_{4\epsilon_k}\setminus B_{\epsilon_k}} |\nabla^2 w_k|^2 dx \longrightarrow 0,$$

*as* $k \to \infty$, $\pi$ *is the nearest point projection onto* $S^4$ *and* $u_{B_{\epsilon_k}} := \fint_{B_{\epsilon_k}} u dx$.

## 4 Large solutions on $\mathbb{R}^4$

In this section, we give a proof of Theorem 1.1. As mentioned in the introduction, it is sufficient to show that $\inf_{\Xi^1} E$ is achieved. Therefore, let $(u_k)_{k\in\mathbb{N}} \subset \Xi^1$ be a minimizing sequence for $E(\cdot)$. We deduce from $|u_k|^2 = 1$ that $\Delta u_k^\alpha u_k^\alpha = -|\nabla u_k|^2$. Combining this with Lemma 5.7 in the appendix and integrating by parts we obtain

$$\begin{aligned} 16\mathcal{H}^4(S^4) &\le 16 \int_{\mathbb{R}^4} |J_4(\nabla u_k)| dx \le \int_{\mathbb{R}^4} |\nabla u_k|^4 dx \\ &= \int_{\mathbb{R}^4} |u_k^\alpha \Delta u_k^\alpha|^2 dx \le \int_{\mathbb{R}^4} |\Delta u_k|^2 dx = \int_{\mathbb{R}^4} |\nabla^2 u_k|^2 dx \le C, \end{aligned} \tag{18}$$

where $C < \infty$ is independent of $k$.

Now, as $E(\cdot)$ is invariant under rescalings and translations, there are sequences $(r_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $(\overline{x}_k)_{k\in\mathbb{N}} \subset \mathbb{R}^4$, such that $\overline{u}_k$, defined as

$$\overline{u}_k(x) := u_k\left(\frac{x - \overline{x}_k}{r_k}\right) \subset \Xi^1,$$

is a minimizing sequence for $E(\cdot)$ verifying (18) and

$$\int_{B_1(0)} J_4(\nabla \overline{u}_k) dx = \sup_{x_0 \in \mathbb{R}^4} \int_{B_1(x_0)} J_4(\nabla \overline{u}_k) dx = \frac{1}{4}\mathcal{H}^4(S^4), \tag{19}$$

for all $k \in \mathbb{N}$. Replacing $u_k$ by $\overline{u}_k$, if necessary, henceforth we may assume $\overline{x}_k = 0$ and $r_k = 1$.

Consider now the families of measures $\mu_k := |\nabla^2 u_k|^2 dx$, $\nu_k := J_4(\nabla u_k) dx$, $\nu_k^+$ the positive part of $\nu_k$, $\nu_k^-$ the negative part of $\nu_k$ and $\overline{\nu}_k := \nu_k^+ + \nu_k^- = |J_4(\nabla u_k)| dx$. We have

$$\begin{aligned} \int_{\mathbb{R}^4} d\nu_k &= \mathcal{H}^4(S^4), \\ \int_{\mathbb{R}^4} d\overline{\nu}_k &= (1 + \overline{e}_k)\mathcal{H}^4(S^4), \\ \int_{\mathbb{R}^4} d\nu_k^+ &= (1 + e_k)\mathcal{H}^4(S^4), \\ \int_{\mathbb{R}^4} d\nu_k^- &= e_k \mathcal{H}^4(S^4), \end{aligned} \tag{20}$$

with

$$\bar{e}_k = 2e_k \geq 0.$$

Let $\sigma$ denote the stereographic projection from $S^4$ to $\mathbb{R}^4$. From the appendix 5.2, we get that $\sigma^{-1} \in \Xi^1$ is biharmonic with $E(\sigma^{-1}) = 24\mathcal{H}^4(S^4)$. If $\sigma^{-1}$ is minimizing, we are done and $\min_{u\in\Xi^1} E(u) = 24\mathcal{H}^4(S^4)$. Otherwise, it follows from (18), that there exists $\delta > 0$ independent of $k$ such that

$$0 \leq \bar{e}_k \leq \frac{1}{2} - 2\delta$$

for $k$ sufficiently large. Thus, we get with (20), that there exists $\delta > 0$ independent of $k$ such that

$$0 \leq \int\limits_{\mathbb{R}^4} d\nu_k^- \leq \frac{1}{4}\mathcal{H}^4(S^4) - \delta \tag{21}$$

for $k$ sufficiently large. This implies with (19), that for all $r \geq 1$ and $k$ sufficiently large

$$0 < \delta \leq \int\limits_{B_r(0)} d\nu_k \leq \frac{5}{4}\mathcal{H}^4(S^4).$$

Hence, we get, after passing to a subsequence, that

$$\begin{aligned} u_k &\rightharpoonup u \quad && \text{weakly in } \mathcal{D}^{2,2}(\mathbb{R}^4, S^4),\\ \mu_k &\rightharpoonup \mu \quad && \text{weakly in the sense of measures},\\ \nu_k &\rightharpoonup \nu \quad && \text{weakly in the sense of measures}, \end{aligned}$$

with

$$0 < \int\limits_{\mathbb{R}^4} d\nu \leq \liminf_{k\to\infty} \int\limits_{\mathbb{R}^4} d\nu_k^+ \leq \frac{5}{4}\mathcal{H}^4(S^4) < 2\mathcal{H}^4(S^4). \tag{22}$$

From Lemma 3.6, we deduce that

$$\nu = J_4(\nabla u)dx + \sum_{j\in J} \nu^{(j)}\delta_{x^{(j)}} \tag{23}$$

for certain points $x^{(j)} \in \mathbb{R}^4$ $(j \in J)$, $\nu^{(j)} \in \mathbb{R} \setminus \{0\}$ and $J$ a countable set.

We prove in Lemma 4.1 below that $J = \emptyset$. This implies with (22), (23) and Corollary 3.2 that

$$\int\limits_{\mathbb{R}^4} J_4(\nabla u)dx = \mathcal{H}^4(S^4).$$

Thus, $u \in \Xi^1$ and $E(u) = \min_{\Xi^1} E$. The smoothness of $u$ follows from [7].

Now we want to show

$$16\mathcal{H}^4(S^4) < \int\limits_{\mathbb{R}^4} |\Delta u|^2 dx \leq 24\mathcal{H}^4(S^4). \tag{24}$$

Let $v$ be the minimizer of $E$ in $\Xi^1$. Then,

$$16\mathcal{H}^4(S^4) \leq 16 \int\limits_{\mathbb{R}^4} |J_4(\nabla v)|dx \leq \int\limits_{\mathbb{R}^4} |\nabla v|^4 dx \leq \int\limits_{\mathbb{R}^4} |\Delta v|^2 dx \leq 24\mathcal{H}^4(S^4). \tag{25}$$

The second inequality is achieved iff $v$ is conformal, i.e. $v = \sigma^{-1} \circ c$, where $c : \mathbb{R}^4 \to \mathbb{R}^4$ is conformal. Thus, c belongs to the Möbius group, generated by the translations, rescalings and the inversion at the unit sphere. Since $\sigma^{-1}$ is not harmonic, then also no conformal map $v \in \Xi^1$ is harmonic. However, the third inequality is achieved iff $v$ is harmonic. Thus, (24) has to hold.

In order to complete the proof of Theorem 1.1, we need to show

**Lemma 4.1** *J is empty.*

*Proof* Suppose $J \neq \emptyset$. Choose $l \in J$ and set $x^{(l)} = 0$ (translation). Applying Proposition 3.4 and Lemma 3.9, we may choose a good radius $0 < \rho < 1$ such that for $J(\rho) := \{j \in J : x^{(j)} \in B_\rho\}$ it holds

$$\sum_{j \in J(\rho)\setminus\{l\}} \nu^{(j)} < \left|\nu^{(l)}\right|, \tag{26}$$

(We set $\sum_{j\in\emptyset} \nu^{(j)} := 0$.) and such that for all sequences of positive real numbers $\gamma_k \to 0$ as $k \to \infty$, we have

$$\int\limits_{A(\rho,\gamma_k)} |\nabla^2 u_k|^2 dx \longrightarrow 0 \tag{27}$$

as $k \to \infty$. Moreover, we may assume

$$\left|\int\limits_{B_\rho} J_4(\nabla u) dx\right| \le \int\limits_{B_\rho} |J_4(\nabla u)| dx \le \frac{1}{16}\int\limits_{B_\rho} |\nabla u|^4 dx \le \frac{1}{16}\int\limits_{B_\rho} |\Delta u|^2 dx \le \frac{1}{4}\mathcal{H}^4(S^4), \tag{28}$$

and that Lemma 3.8 is valid for $\rho$, $u_k$ and $u$. Hence, we get $\epsilon_k \to 0$ $(k \to \infty)$ and maps $(v_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ such that

$$v_k := \begin{cases} u_k & \text{for } x \in B_\rho \\ u & \text{for } x \in \mathbb{R}^4 \setminus B_{(1+\epsilon_k)\rho} \end{cases}$$

and

$$\int\limits_{A(\rho,\epsilon_k)} |\nabla v_k|^4 dx \le C \int\limits_{A(\rho,\epsilon_k)} |\nabla^2 v_k|^2 dx \longrightarrow 0 \tag{29}$$

as $k \to \infty$. As $v_k \rightharpoonup u$ in $\mathcal{D}^{2,2}$, we get from Proposition 3.4

$$\begin{aligned}\int\limits_{\mathbb{R}^4} J_4\left(\nabla(v_k - u)\right) dx &= \int\limits_{\mathbb{R}^4} J_4(\nabla v_k) dx - \int\limits_{\mathbb{R}^4} J_4(\nabla u) dx + o(1) \\ &= (d_k - d)\mathcal{H}^4(S^4) + o(1),\end{aligned} \tag{30}$$

where $d_k$ is the topological degree of $v_k$ and $d$ the degree of $u$. Notice that, according to Corollary 3.2, $d_k - d \in \mathbb{Z}$.

We get from (19), (21), (23), (28) and (29) that

$$\begin{aligned}
\left|\int_{\mathbb{R}^4} J_4\left(\nabla(v_k-u)\right)dx\right| &= \left|\lim_{k\to\infty}\int_{B_\rho} J_4\left(\nabla(u_k-u)\right)dx\right| \\
&= \left|\lim_{k\to\infty}\int_{B_\rho} J_4(\nabla u_k)dx - \int_{B_\rho} J_4(\nabla u)dx\right| + o(1) \\
&= \left|\sum_{j\in J(\rho)} \nu^{(j)}\right| + o(1) \\
&= \left|\int_{B_\rho} d\nu - \int_{B_\rho} J_4(\nabla u)dx\right| + o(1) \\
&\leq \int_{B_\rho} d\overline{\nu}_k + \frac{1}{4}\mathcal{H}^4(S^4) + o(1) \\
&\leq \int_{B_1} d\nu_k + 2\int_{\mathbb{R}^4} d\nu_k^- + \frac{1}{4}\mathcal{H}^4(S^4) + o(1) \\
&< \mathcal{H}^4(S^4),
\end{aligned} \tag{31}$$

for $k$ sufficiently large. Thus,

$$\int_{\mathbb{R}^4} J_4(\nabla(v_k-u))dx = 0.$$

This implies with (31) that

$$\sum_{j\in J(\rho)} \nu^{(j)} = 0.$$

This is a contradiction to (26). Thus, $J=\emptyset$. □

## 5 Large solutions on the unit disk

### 5.1 Energy gain

Let $\tau_z$ be the inversion at the unit sphere centered in $z$ given by $\tau_z : \mathbb{R}^4 \cup \{\infty\} \to \mathbb{R}^4 \cup \{\infty\}$, $\tau_z(x) := \frac{x-z}{|x-z|^2} + z$ for $x \notin \{z,\infty\}$, $\tau_z(z) := \infty$ and $\tau_z(\infty) := z$. We abbreviate $\tau := \tau_0$.

We consider for $f \in \mathcal{C}^\infty(B, S^4)$ and $y \in B$ the sets

$$\mathcal{E} := \mathcal{D}^{2,2}(\mathbb{R}^4, S^4) \cap \mathcal{C}^\infty(\mathbb{R}^4, S^4)$$

and

$$\mathcal{F}_{f,y} := \{u \in \mathcal{E} : u \circ \tau_y \in \mathcal{E},\ (u \circ \tau_y)(y) = f(y) \quad \text{and} \quad \nabla(u \circ \tau_y)(y) = \lambda \nabla f(y) \\ \text{for some } \ \lambda > 0\}.$$

In general, for $u \in \mathcal{E}$, we do not have that $u \circ \tau_y \in \mathcal{E}$. However, Lemma 5.6 below states that for $\gamma$, as defined in the introduction, $\mathcal{F}_{\gamma,y}$ is non-empty.

We have the following energy comparison lemma.

**Lemma 5.1** *Let $v \in W^{2,2}(B, S^4) \cap \mathcal{C}^\infty(B, S^4)$ be such that $\nabla v(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^4$, and consider $w \in \mathcal{F}_{v,x_0}$. There exists $u \in W^{2,2}(B, S^4)$ such that $u - v \in W_0^{2,2}$,*

$$\int\limits_B (J_4(\nabla u) - J_4(\nabla v))\, dx = \int\limits_{\mathbb{R}^4} J_4(\nabla w) dx \tag{32}$$

*and*

$$\int\limits_B |\Delta u|^2 dx < \int\limits_B |\Delta v|^2 dx + \int\limits_{\mathbb{R}^4} |\Delta w|^2 dx. \tag{33}$$

*Proof* Let $B_r(x_0) \subset B$. We may assume, after performing a translation and dilation that $x_0 = 0$ and $r = 1$. Since $w \in \mathcal{F}_{v,0}$, the inversion at the unit sphere $\tau$ gives the map $y := w \circ \tau \in \mathcal{E}$ with $\nabla y(0) = \lambda \nabla v(0)$ for some $\lambda > 0$. Thus, $w = y \circ \tau$. We may assume, due to Lemma 5.2 below, that $y(0) = v(0) = N = (0, 0, 0, 0, 1)$, $y_i(0)$ is orthogonal to $y_j(0)$ and $v_i(0)$ is orthogonal to $v_j(0)$ for $i \neq j$. Indeed, we replace $v$ by $R_1 \circ v \circ R_2$ and $w$ by $R_1 \circ w \circ R_2$, with suitable $R_1 \in SO(5)$ and $R_2 \in SO(4)$. Furthermore, we have $y_i^5(0) = 0 = v_i^5(0)$ for $1 \leq i \leq 4$.

We consider on $\mathbb{R}^4 \setminus \{0\}$ the spherical coordinates $\theta_1$, $\theta_2$ and $\theta_3$, which are related to the usual Euclidean coordinates by $x^i = r\eta^i$ $(1 \leq i \leq 4)$, where

$$\begin{aligned} \eta^1 &= \cos\theta_1 \\ \eta^2 &= \sin\theta_1 \cos\theta_2 \\ \eta^3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ \eta^4 &= \sin\theta_1 \sin\theta_2 \sin\theta_3. \end{aligned}$$

We define for $\epsilon$ sufficiently small

$$u(x) = u_\epsilon(x) := \begin{cases} v(x) & \text{for } x \in B \setminus B_{2\epsilon} \\ \overline{u}(x) & \text{for } x \in B_{2\epsilon} \setminus B_\epsilon \\ \overline{w}(x) & \text{for } x \in B_\epsilon, \end{cases}$$

where $\overline{w}(x) := w(l^{-1}\epsilon^{-2}x)$, $l$ is a constant to be fixed later. $\overline{u}$ is defined as

$$\begin{aligned} \overline{u}^\alpha(r, \eta) &= a^\alpha r^3 + b^\alpha r^2 + c^\alpha r + d^\alpha \text{ for } 1 \leq \alpha \leq 4, \\ \overline{u}^5(r, \eta) &= \sqrt{1 - \sum_{1 \leq \alpha \leq 4} \left(\overline{u}^\alpha(r, \eta)\right)^2}, \end{aligned} \tag{34}$$

where $a^\alpha, b^\alpha, c^\alpha$ and $d^\alpha$ depend only on $\theta, \phi, \psi$, and are such that

$$\begin{aligned} \overline{u}^\alpha = \overline{w}^\alpha,\ \frac{\partial \overline{u}^\alpha}{\partial r} &= \frac{\partial \overline{w}^\alpha}{\partial r} \qquad \text{on } \partial B_\epsilon, \\ \overline{u}^\alpha = v^\alpha,\ \frac{\partial \overline{u}^\alpha}{\partial r} &= \frac{\partial v^\alpha}{\partial r} \qquad \text{on } \partial B_{2\epsilon}. \end{aligned} \tag{35}$$

We clearly have $u \in W^{2,2}(B, S^4)$. Now we define $z(x) := w(l^{-1}\tau(x)) \in \mathcal{E}$, i.e. $w(x) = z(l^{-1}\tau(x))$, $\overline{w}(x) = z(\epsilon^2\tau(x))$ and $z(x) = y(lx)$. Thus, $\overline{w}$ agrees with $z$ on $\partial B_\epsilon$ and the boundary conditions on $\partial B_\epsilon$ read

$$\overline{u}^\alpha(x) = z^\alpha(x) \text{ and } \frac{\partial \overline{u}^\alpha}{\partial r}(x) = -\frac{\partial z^\alpha}{\partial r}(x). \tag{36}$$

We consider the Taylor expansions for $z$ and $v$ for any fixed $l$

$$z^\alpha(x) = z^\alpha(0) + z_i^\alpha(0)x^i + o(x) \text{ and } v^\alpha(x) = v^\alpha(0) + v_i^\alpha(0)x^i + o(x), \tag{37}$$

their partial derivatives

$$z_i^\alpha(x) = z_i^\alpha(0) + o(1) \text{ and } v_i^\alpha(x) = v_i^\alpha(0) + o(1), \tag{38}$$

their radial derivatives

$$\frac{\partial z^\alpha}{\partial r}(x) = z_i^\alpha(0)\eta^i + o(1) \text{ and } \frac{\partial v^\alpha}{\partial r}(x) = v_i^\alpha(0)\eta^i + o(1), \tag{39}$$

and their Laplacians

$$\Delta z^\alpha(x) = \Delta z^\alpha(0) + o(1) \text{ and } \Delta v^\alpha(x) = \Delta v^\alpha(0) + o(1). \tag{40}$$

Combinig (34), (35), (36), (37) and (39) with $y(0) = v(0) = N = (0,0,0,0,1)$ gives, for $1 \le \alpha \le 4$, the following system of linear equations for $a^\alpha$, $b^\alpha$, $c^\alpha$ and $d^\alpha$

$$\begin{cases} a^\alpha\epsilon^3 + b^\alpha\epsilon^2 + c^\alpha\epsilon + d^\alpha &= z_i^\alpha(0)\eta^i\epsilon + o(\epsilon) \\ 3a^\alpha\epsilon^2 + 2b^\alpha\epsilon + c^\alpha &= -z_i^\alpha(0)\eta^i + o(1) \\ 8a^\alpha\epsilon^3 + 4b^\alpha\epsilon^2 + 2c^\alpha\epsilon + d^\alpha &= 2v_i^\alpha(0)\eta^i\epsilon + o(\epsilon) \\ 12a^\alpha\epsilon^2 + 4b^\alpha\epsilon + c^\alpha &= v_i^\alpha(0)\eta^i + o(1). \end{cases} \tag{41}$$

We verify that the solution to (41) satisfies

$$\begin{aligned} a^\alpha &= a_i^\alpha\eta^i + o(\epsilon^{-2}), & a_i^\alpha &:= (z_i^\alpha(0) - 3v_i^\alpha(0))\epsilon^{-2} \\ b^\alpha &= b_i^\alpha\eta^i + o(\epsilon^{-1}), & b_i^\alpha &:= (-4z_i^\alpha(0) + 14v_i^\alpha(0))\epsilon^{-1} \\ c^\alpha &= c_i^\alpha\eta^i + o(1), & c_i^\alpha &:= 4z_i^\alpha(0) - 19v_i^\alpha(0) \\ d^\alpha &= d_i^\alpha\eta^i + o(\epsilon), & d_i^\alpha &:= 8v_i^\alpha(0)\epsilon. \end{aligned} \tag{42}$$

Exactly in the same way, we verify, for $1 \le j \le 3$, that

$$\begin{aligned} \tfrac{\partial a^\alpha}{\partial\theta_j} &= a_i^\alpha \tfrac{\partial\eta^i}{\partial\theta_j} + o(\epsilon^{-2}), \\ \tfrac{\partial b^\alpha}{\partial\theta_j} &= b_i^\alpha \tfrac{\partial\eta^i}{\partial\theta_j} + o(\epsilon^{-1}), \\ \tfrac{\partial c^\alpha}{\partial\theta_j} &= c_i^\alpha \tfrac{\partial\eta^i}{\partial\theta_j} + o(1), \\ \tfrac{\partial d^\alpha}{\partial\theta_j} &= d_i^\alpha \tfrac{\partial\eta^i}{\partial\theta_j} + o(\epsilon). \end{aligned} \tag{43}$$

It holds for the spherical harmonics $\eta^i$ $(1 \le i \le 4)$

$$(\Delta_{S^3} + 3)\eta^i = 0.$$

Thus, we verify as before

$$\begin{aligned} \Delta_{S^3}a^\alpha &= a_i^\alpha\Delta_{S^3}\eta^i + o(\epsilon^{-2}) = -3a_i^\alpha\eta^i + o(\epsilon^{-2}), \\ \Delta_{S^3}b^\alpha &= b_i^\alpha\Delta_{S^3}\eta^i + o(\epsilon^{-1}) = -3b_i^\alpha\eta^i + o(\epsilon^{-1}), \\ \Delta_{S^3}c^\alpha &= c_i^\alpha\Delta_{S^3}\eta^i + o(1) = -3c_i^\alpha\eta^i + o(1), \\ \Delta_{S^3}d^\alpha &= d_i^\alpha\Delta_{S^3}\eta^i + o(\epsilon) = -3d_i^\alpha\eta^i + o(\epsilon). \end{aligned} \tag{44}$$

We are now able to compute the biharmonic energy expense $E_a$ of $u$ on the annulus $B_{2\epsilon} \setminus B_\epsilon$, given by

$$E_a = \sum_{\alpha=1}^{5} \int_{B_{2\epsilon}\setminus B_\epsilon} |\Delta \overline{u}^\alpha|^2 dx.$$

We consider the Laplacian in spherical coordinates

$$\Delta = r^{-3}\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} + r^{-2}\Delta_{S^3}. \tag{45}$$

For $1 \leq \alpha \leq 4$, this gives with (34) and (44) on $B_{2\epsilon} \setminus B_\epsilon$

$$r^{-2}\Delta_{S^3}\overline{u}^\alpha = -3(a^\alpha r + b^\alpha + c^\alpha r^{-1} + d^\alpha r^{-2}) + o(\epsilon^{-1}). \tag{46}$$

Furthermore, we compute

$$r^{-3}\frac{\partial}{\partial r} r^3 \frac{\partial \overline{u}^\alpha}{\partial r} = 15a^\alpha r + 8b^\alpha + 3c^\alpha r^{-1}. \tag{47}$$

Thus, inserting (46) and (47) in (45) yields

$$\Delta \overline{u}^\alpha = (12a_i^\alpha r + 5b_i^\alpha - 3d_i^\alpha r^{-2})\eta^i + o(\epsilon^{-1}). \tag{48}$$

Moreover, we have

$$\int_{S^3} \eta^i \eta^j dvol_{S^3} = \frac{\pi^2}{2}\delta_{ij}. \tag{49}$$

Hence, we compute for $1 \leq \alpha \leq 4$ (here we do not sum over $\alpha$)

$$\begin{aligned}\frac{2}{\pi^2}\int_{S^3} |\Delta \overline{u}^\alpha|^2 dvol_{S^3} &= 144|a_i^\alpha|^2 r^2 + 25|b_i^\alpha|^2 + 9|d_i^\alpha|^2 r^{-4} \\ &\quad +120 a_i^\alpha b_i^\alpha r - 72 a_i^\alpha d_i^\alpha r^{-1} - 30 b_i^\alpha d_i^\alpha r^{-2} + o(\epsilon^{-2}),\end{aligned}$$

and with (42)

$$\begin{aligned}&\frac{2}{\pi^2}\int_{B_{2\epsilon}\setminus B_\epsilon} |\Delta \overline{u}^\alpha|^2 dx \\ &= 1512|a_i^\alpha|^2\epsilon^6 + \frac{375}{4}|b_i^\alpha|^2\epsilon^4 + 9\ln 2|d_i^\alpha|^2 + 744 a_i^\alpha b_i^\alpha \epsilon^5 - 168 a_i^\alpha d_i^\alpha \epsilon^3 - 45 b_i^\alpha d_i^\alpha \epsilon^2 + o(\epsilon^2) \\ &= \left(36(z_i^\alpha(0))^2 - 132 v_i^\alpha(0) z_i^\alpha(0) + (576\ln 2 - 273)(v_i^\alpha(0))^2\right)\epsilon^2 + o(\epsilon^2). \qquad (50)\end{aligned}$$

Now we show that

$$\int_{B_{2\epsilon}\setminus B_\epsilon} |\Delta \overline{u}^5|^2 dx = o(\epsilon^2). \tag{51}$$

As $\overline{u}^5 = \sqrt{1 - \sum_{\alpha=1}^4 \left(\overline{u}^\alpha\right)^2}$, we compute

$$\begin{aligned} \overline{u}_i^5 &= -\frac{\sum_{\alpha=1}^4 \overline{u}^\alpha \overline{u}_i^\alpha}{\sqrt{1 - \sum_{\alpha=1}^4 \left(\overline{u}^\alpha\right)^2}} \\ \Delta \overline{u}^5 &= -\frac{\sum_{\alpha=1}^4 (|\nabla \overline{u}^\alpha|^2 + \overline{u}^\alpha \Delta \overline{u}^\alpha)}{\sqrt{1 - \sum_{\alpha=1}^4 \left(\overline{u}^\alpha\right)^2}} + \frac{(\sum_{\alpha=1}^4 \overline{u}^\alpha \overline{u}_i^\alpha)^2}{\left(1 - \sum_{\alpha=1}^4 \left(\overline{u}^\alpha\right)^2\right)^{\frac{3}{2}}}. \end{aligned} \tag{52}$$

From (34), (42), (43) and (48), we infer that

$$\begin{aligned} |\overline{u}^\alpha| &= |a^\alpha r^3 + b^\alpha r^2 + c^\alpha r + d^\alpha| \le C(\epsilon^{-2} r^3 + \epsilon^{-1} r^2 + r + \epsilon) \le C\epsilon, \\ \left|\frac{\partial \overline{u}^\alpha}{\partial r}\right| &= |3a^\alpha r^2 + 2b^\alpha r + c^\alpha| \le C(\epsilon^{-2} r^2 + \epsilon^{-1} r + 1) \le C, \\ \left|\frac{\partial \overline{u}^\alpha}{\partial \theta_1}\right| &= \left|\frac{\partial a^\alpha}{\partial \theta_1} r^3 + \frac{\partial b^\alpha}{\partial \theta_1} r^2 + \frac{\partial c^\alpha}{\partial \theta_1} r + \frac{\partial d^\alpha}{\partial \theta_1}\right| \le C(\epsilon^{-2} r^3 + \epsilon^{-1} r^2 + r + \epsilon) \le C\epsilon, \\ \left|\frac{\partial \overline{u}^\alpha}{\partial \theta_2}\right| &\le \cdots \le C\epsilon \sin\theta_1, \\ \left|\frac{\partial \overline{u}^\alpha}{\partial \theta_3}\right| &\le \cdots \le C\epsilon \sin\theta_1 \sin\theta_2, \end{aligned}$$

and

$$|\Delta \overline{u}^\alpha| = |(12 a_i^\alpha r + 5 b_i^\alpha - 3 d_i^\alpha r^{-2})\eta^i| + o(\epsilon^{-1}) \le C(\epsilon^{-2} r + \epsilon^{-1} + \epsilon r^{-2}) \le C\epsilon^{-1}$$

on $B_{2\epsilon} \setminus B_\epsilon$. Consequently, we have

$$\sum_{\alpha=1}^4 \left(\overline{u}^\alpha\right)^2 \le C\epsilon^2$$

and

$$|\nabla \overline{u}^\alpha|^2 = \left|\frac{\partial \overline{u}^\alpha}{\partial r}\right|^2 + r^{-2}\left(\left|\frac{\partial \overline{u}^\alpha}{\partial \theta_1}\right|^2 + \frac{1}{\sin^2\theta_1}\left|\frac{\partial \overline{u}^\alpha}{\partial \theta_2}\right|^2 + \frac{1}{\sin^2\theta_1 \sin^2\theta_2}\left|\frac{\partial \overline{u}^\alpha}{\partial \theta_3}\right|^2\right) \le C.$$

Introducing these estimates in (52) gives

$$|\Delta \overline{u}^5|^2 \le C,$$

and

$$\int_{B_{2\epsilon} \setminus B_\epsilon} \left|\Delta \overline{u}^5\right|^2 dx \le C\epsilon^4.$$

This completes the proof of (51). Combining (50) and (51) gives

$$E_a = \pi^2 \left(18 (z_i^\alpha(0))^2 - 66 v_i^\alpha(0) z_i^\alpha(0) + \left(288 \ln 2 - \frac{273}{2}\right)(v_i^\alpha(0))^2\right)\epsilon^2 + o(\epsilon^2). \tag{53}$$

(Since $y_i^5(0) = 0 = v_i^5(0)$ for $1 \le i \le 4$, we sum over all possible values of $\alpha$.)

The biharmonic energy expense $E_o$ of $u$ on $B \setminus B_{2\epsilon}$ is

$$E_o \leq \int\limits_B |\Delta v|^2 dx. \tag{54}$$

On $B_\epsilon$, the biharmonic energy of $u$ equals

$$E_i = \int\limits_{B_\epsilon} |\Delta \overline{w}|^2 dx.$$

Recalling $\overline{w}(x) = z(\epsilon^2 \tau(x))$, we compute with $\overline{z}(x) := \overline{w}(\epsilon^2 x) = z(\tau(x))$

$$\int\limits_{\mathbb{R}^4 \setminus B_\epsilon} |\Delta \overline{w}|^2 dx = \int\limits_{\mathbb{R}^4 \setminus B_{\epsilon^{-1}}} |\Delta \overline{z}|^2 dx = \int\limits_{B_\epsilon} |\Delta z^\alpha - 4 z_i^\alpha \frac{x^i}{|x|^2}|^2 dx.$$

From the Taylor expansion (38) and (40) on $B_\epsilon$, we deduce

$$|\Delta z^\alpha - 4 z_i^\alpha \frac{x^i}{|x|^2}|^2 = 16 z_i^\alpha(0) z_j^\alpha(0) \eta^i \eta^j r^{-2} + o(\epsilon^{-2}).$$

It follows with (49)

$$\int\limits_{\mathbb{R}^4 \setminus B_\epsilon} |\Delta \overline{w}|^2 dx = 4\pi^2 (z_i^\alpha(0))^2 \epsilon^2 + o(\epsilon^2),$$

and consequently

$$E_i = \int\limits_{\mathbb{R}^4} |\Delta w|^2 dx - 4\pi^2 (z_i^\alpha(0))^2 \epsilon^2 + o(\epsilon^2). \tag{55}$$

Combining (53), (54) and (55) with $z_i(0) = l y_i(0)$ and $y_i(0) = \lambda v_i(0)$ gives the total energy $E_B(u) = E_o + E_a + E_i$ of $u$

$$\begin{aligned} E_B(u) \leq & \int\limits_B |\Delta v|^2 dx + \int\limits_{\mathbb{R}^4} |\Delta w|^2 dx \\ & + \pi^2 \left( 14 (y_i^\alpha(0))^2 l^2 - 66 v_i^\alpha(0) y_i^\alpha(0) l + \left( 288 \ln 2 - \frac{273}{2} \right) (v_i^\alpha(0))^2 \right) \epsilon^2 + o(\epsilon^2) \\ = & \int\limits_B |\Delta v|^2 dx + \int\limits_{\mathbb{R}^4} |\Delta w|^2 dx \\ & + \pi^2 \left( 14 \lambda^2 l^2 - 66 \lambda l + \left( 288 \ln 2 - \frac{273}{2} \right) \right) |\nabla v(0)|^2 \epsilon^2 + o(\epsilon^2). \end{aligned}$$

As $\nabla v(0) \neq 0$, we may choose an adequate $l$ and $\epsilon_0 > 0$ sufficiently small such that (33) holds for $0 < \epsilon < \epsilon_0$. We take for example $l = \frac{2}{\lambda}$.

We show now that $u$ satisfies (32). We have

$$\int\limits_{B \setminus B_{2\epsilon}} J_4(\nabla u) dx = \int\limits_B J_4(\nabla v) dx + o(1). \tag{56}$$

From (53), we infer

$$\int_{B_{2\epsilon}\setminus B_\epsilon} J_4(\nabla u)dx \leq \int_{B_{2\epsilon}\setminus B_\epsilon} |\Delta u|^2 dx = o(1). \tag{57}$$

As the volume functional is conformally invariant, we get

$$\begin{aligned}\int_{B_\epsilon} J_4(\nabla u)dx &= \int_{\mathbb{R}^4} J_4(\nabla \overline{w})dx - \int_{\mathbb{R}^4\setminus B_\epsilon} J_4(\nabla \overline{w})dx \\ &= \int_{\mathbb{R}^4} J_4(\nabla w)dx - \int_{B_\epsilon} J_4(\nabla z)dx \\ &= \int_{\mathbb{R}^4} J_4(\nabla w)dx + o(1).\end{aligned} \tag{58}$$

Combining (56)–(58) gives

$$\int_B (J_4(\nabla u) - J_4(\nabla v))\, dx = \int_{\mathbb{R}^4} J_4(\nabla w)dx + o(1).$$

For $0 < \epsilon < \epsilon_0$ sufficiently small, Corollary 3.3 gives (32). □

*Remark 5.1* At first sight the assumptions on the gradient of $v$ and $w$ in Lemma 5.1 may seem to be unnecessarily restrictive conditions. Moreover, it would seem that the choice of the interpolant could be improved by replacing $\overline{u}^\alpha$ for $1 \leq \alpha \leq 4$ in (34) with the biharmonic function satisfying $\Delta^2 \overline{u}^\alpha = 0$ and the boundary conditions (35), or that the ratio of the inner and outer radii of the interpolating annulus could be modified. However, neither the insertion of the biharmonic functions nor a different choice of the radii of the annulus lead to a significally better energy gain that would allow to weaken the assumptions. Therefore, in order to circumvent more ponderous notations and involved computations, we have chosen the present approach.

Furthermore, from a geometric point of view, it seems quite natural, that the gradients of $v$ and $w$ should be almost identical (up to rescalings) in order to control the biharmonic energy of the interpolant.

In order to complete the proof of Lemma 5.1, we need to show

**Lemma 5.2** *Let* $f \in \mathcal{C}^\infty(B_r, S^4)$, $0 < r \leq \infty$. *There exists rotations* $R_1 \in SO(5)$ *and* $R_2 \in SO(4)$, *such that* $g := R_1 \circ f \circ R_2 \in \mathcal{C}^\infty(B_r, S^4)$ *verifies*

1. $g(0) = N$,
2. $\frac{\partial g}{\partial x^i}(0)$ *is orthogonal to* $\frac{\partial g}{\partial x^j}(0)$ *for* $i \neq j$,
3. $E_{B_r}(f) = E_{B_r}(g)$.

*Proof* It is clear that there exists a rotation $R_1 \in SO(5)$ s.t. $h = R_1 \circ f$ verifies condition 1. $dh(0)$ may be viewed as a 4x5-matrix. If we define $g(p) := h(R_2 p)$, we verify

$$dg(p) = dh(p) \cdot R_2,$$

where $\cdot$ is the usual matrix multiplication. Hence,

$$(dg(0))^T \cdot dg(0) = R_2^T \cdot (dh(0))^T \cdot dh(0) \cdot R_2.$$

As $(dh(0))^T \cdot dh(0) \in \mathrm{Symm}(4)$, the theorem of principal axes gives a rotation $R_2 \in SO(4)$, s.t. $(dg(0))^T \cdot dg(0) \in \mathrm{Diag}(4)$. Thus, $g$ verifies condition 2. The last condition follows from the invariance of the biharmonic energy under rotations. □

We have furthermore the following

**Theorem 5.3** (Unique Continuation) *Let $u, \overline{u}$ be two biharmonic maps in $\mathcal{C}^\infty(\Omega, S^4)$. If they agree to infinitely high order at some point, then $u = \overline{u}$ everywhere on $\Omega$.*

*Proof* Any smooth biharmonic map $u \in \mathcal{C}^\infty(\Omega, S^4)$ verifies the Euler–Lagrange equation associated to the Hessian energy functional

$$\Delta^2 u = -\left(|\Delta u|^2 + 2|\nabla^2 u|^2 + 4\nabla u \nabla \Delta u\right) u.$$

Defining the new variables $v = \nabla u$, $w = \Delta u$, it follows that any biharmonic map satisfies the elliptic second order equation

$$\Delta y = F(u, \nabla u, \nabla v, w, \nabla w), \tag{59}$$

with

$$y = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \text{ and } F = \begin{pmatrix} w \\ \nabla w \\ -(w^2 + 2|\nabla v|^2 + 4\nabla u \nabla w)u \end{pmatrix}. \tag{60}$$

Consider now two biharmonic maps $u, \overline{u} \in \mathcal{C}^\infty(\Omega, S^4)$ with the corresponding new variables $v, \overline{v}, w, \overline{w}, y$ and $\overline{y}$. Moreover, we define $z := y - \overline{y}$. As $u, \overline{u}$ and all their derivatives are locally bounded in $\Omega$, it follows from (59) and (60)

$$|\Delta z^\beta| \leq C \left\{ \sum_{i,\alpha} |z_i^\alpha| + \sum_\alpha |z^\alpha| \right\}$$

on every open and bounded $U \subset \Omega$. As $z^\beta$ vanish to infinitely high order at some point, Aronszajn's generalization of Carleman's unique continuation theorem in [1] implies that $z = 0$ on $U$. The result follows from the connectedness of $\Omega$. □

**Corollary 5.4** *If a biharmonic map $u \in \mathcal{C}^\infty(\Omega, S^4)$ is constant on some open set $U \subset \Omega$, then $u$ is constant on $\Omega$.*

**Corollary 5.5** *The minimizers of $E(\cdot)$ in $\Xi^1$ have non-vanishing gradient almost everywhere.*

Now let $u^*$ and $\gamma$ be defined as in the introduction. Then, we have the following

**Lemma 5.6** *For $y \in B$, there exists $u \in \mathcal{F}_{\gamma,y} \cap \Xi^1$ s.t. $E(u) = \mathcal{I}$.*

*Proof* We assume, after possible translations, that $y = 0$. Define

$$u := u^* \circ \sigma_N \circ \sigma_S^{-1} : \overline{\mathbb{R}^4} \setminus \{0\} \longrightarrow S^4,$$

where $\overline{\mathbb{R}^4} := \mathbb{R}^4 \cup \infty$, $\sigma_N$ and $\sigma_S$ are the stereographic projections s.t. $\sigma_N(N) = \infty$, $\sigma_N(S) = 0$, $\sigma_S(S) = \infty$ and $\sigma_S(N) = 0$. Observe that $u = u^* \circ \tau$, $(u \circ \tau)(0) = u^*(0)$,

$\nabla(u \circ \tau)(0) = \nabla u^*(0)$ and deg($u$)= 1. We show now that $u$ has a removable singularity at 0. Indeed, we infer from [6] (or direct computation)

$$E(u) = \int_{\mathbb{R}^4} |\Delta u|^2 dx = \int_{S^4} |\Delta_{S^4}(u^* \circ \sigma_N)|^2 dvol_{S^4} + 2\int_{S^4} |\nabla_{S^4}(u^* \circ \sigma_N)|^2 dvol_{S^4}$$
$$= \int_{\mathbb{R}^4} |\Delta u^*|^2 dx = \mathcal{I}.$$

Thus, $u$ verifies the biharmonic equation on $\mathbb{R}^4 \setminus \{0\}$. We show that $u$ verifies the biharmonic equation on $\mathbb{R}^4$. Consider $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^4)$, and smooth cut-off functions $0 \le \eta_\epsilon \le 1$ satisfying $\eta_\epsilon(x) \equiv 0$ for $|x| \le \epsilon$, $\eta_\epsilon(x) \equiv 1$ for $|x| \ge 2\epsilon$, $|\nabla \eta_\epsilon| \le C\epsilon^{-1}$ and $|\nabla^2 \eta_\epsilon| \le C\epsilon^{-2}$. Inserting $\phi = \eta_\epsilon \psi \in \mathcal{C}_0^\infty(\mathbb{R}^4 \setminus \{0\})$ in the biharmonic equation (1) gives

$$\int_{\mathbb{R}^4} (\Delta u + |\nabla u|^2 u)\Delta(\eta_\epsilon \psi) + 2\int_{\mathbb{R}^4} (|\nabla u|^2 \nabla u - < \Delta u, \nabla u > u)\nabla(\eta_\epsilon \psi)$$
$$- \int_{\mathbb{R}^4} (|\Delta u|^2 u + |\nabla u|^2 \Delta u)\eta_\epsilon \psi = 0 \tag{61}$$

We compute with $a := \Delta u + |\nabla u|^2 u$ and $b := |\nabla u|^2 \nabla u - < \Delta u, \nabla u > u$ that

$$\int_{\mathbb{R}^4} a\Delta(\eta_\epsilon \psi) = \int_{\mathbb{R}^4} a\Delta\psi\eta_\epsilon + 2\int_{\mathbb{R}^4} a\nabla\psi\nabla\eta_\epsilon + \int_{\mathbb{R}^4} a\psi\Delta\eta_\epsilon \tag{62}$$

and

$$\int_{\mathbb{R}^4} b\nabla(\eta_\epsilon \psi) = \int_{\mathbb{R}^4} b\nabla\psi\eta_\epsilon + \int_{\mathbb{R}^4} b\psi\nabla\eta_\epsilon. \tag{63}$$

Recalling that $\eta_\epsilon$ is constant outside the annulus $A_\epsilon := B_{2\epsilon} \setminus B_\epsilon$, $|u| = 1$ and $|\nabla u|^4 \le |\Delta u|^2$, we estimate using Hölder's inequality

$$2\int_{\mathbb{R}^4} a\nabla\psi\nabla\eta_\epsilon \le C\left(\int_{A_\epsilon} |\Delta u|^2\right)^{\frac{1}{2}} \left(\int_{A_\epsilon} |\nabla\eta_\epsilon|^2\right)^{\frac{1}{2}} \le C\epsilon \xrightarrow{\epsilon\to 0} 0, \tag{64}$$

$$\int_{\mathbb{R}^4} a\psi\Delta\eta_\epsilon \le C\left(\int_{A_\epsilon} |\Delta u|^2\right)^{\frac{1}{2}} \left(\int_{A_\epsilon} |\Delta\eta_\epsilon|^2\right)^{\frac{1}{2}} \le C\left(\int_{A_\epsilon} |\Delta u|^2\right)^{\frac{1}{2}} \xrightarrow{\epsilon\to 0} 0 \tag{65}$$

and

$$\int_{\mathbb{R}^4} b\psi\nabla\eta_\epsilon \le \left(\int_{A_\epsilon} |\Delta u|^2\right)^{\frac{1}{2}} \left(\int_{A_\epsilon} |\nabla u|^4\right)^{\frac{1}{4}} \left(\int_{A_\epsilon} |\nabla\eta_\epsilon|^4\right)^{\frac{1}{4}} \le C\left(\int_{A_\epsilon} |\Delta u|^2\right)^{\frac{3}{4}} \xrightarrow{\epsilon\to 0} 0. \tag{66}$$

Combining (61)-(66) gives, as $\epsilon \to 0$, that

$$\int_{\mathbb{R}^4} (\Delta u + |\nabla u|^2 u)\Delta\psi + 2\int_{\mathbb{R}^4} (|\nabla u|^2 \nabla u - < \Delta u, \nabla u > u)\nabla\psi$$
$$- \int_{\mathbb{R}^4} (|\Delta u|^2 u + |\nabla u|^2 \Delta u)\psi = 0$$

for every test function $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^4)$. Hence, $u$ is a weak extrinsic biharmonic map on $\mathbb{R}^4$. Thus, we may assume that $u$ is smooth on $\mathbb{R}^4$. This completes the proof. Similar arguments can be found in the proof of [14, Theorem 2.2.]. □

## 5.2 Proof of Theorem 1.2

Now we are able to prove Theorem 1.2. Let $u^*$, $\gamma$ and $R$ be defined as in the introduction. Recall that $\underline{u} = \gamma$. From Corollary 5.5, we infer the existence of a point $x_0 \in B$, s.t. $\nabla\underline{u}(x_0) \neq 0$. Furthermore, Lemma 5.6 gives $u^* \in \mathcal{F}_{\underline{u},x_0} \cap \Xi^1$ such that $E(u^*) = \mathcal{I}$. Applying Lemma 5.1 to $u^*$ and $\underline{u}$ yields $\delta > 0$ and $f \in \Xi^1_\gamma$ such that

$$E_B(f) < E_B(\underline{u}) + \mathcal{I} - \delta.$$

Thus, let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence for $E_B$ in $\Xi^1_\gamma$ such that

$$E_B(u_k) < E_B(\underline{u}) + \mathcal{I} - \delta. \tag{67}$$

Consider $\mu_k := |\nabla^2 u_k|dx$, $\nu_k := J_4(\nabla u_k)dx$, $\nu_k^+$ the positive part of $\nu_k$ and $\nu_k^-$ the negative part of $\nu_k$. We may extract a subsequence, such that

$$\begin{aligned} u_k &\rightharpoonup u && \text{weakly in } \mathcal{D}^{2,2}(B, S^4),\\ \mu_k &\rightharpoonup \mu && \text{weakly in the sense of measures},\\ \nu_k &\rightharpoonup \nu && \text{weakly in the sense of measures}, \end{aligned}$$

and

$$\int_B d\nu - \int_B J_4(\nabla\underline{u})dx = \mathcal{H}^4(S^4). \tag{68}$$

We deduce from Lemma 3.6 (after extension of $u_k, u, \mu_k, \mu, \nu_k$ and $\nu$ to $\mathbb{R}^4$), that

$$\nu = J_4(\nabla u)dx + \sum_{j\in J} \nu^{(j)}\delta_{x^{(j)}} \tag{69}$$

for certain points $x^{(j)} \in B$ $(j \in J)$, $\nu^{(j)} \in \mathbb{R}\setminus\{0\}$ and $J$ a countable set.
Suppose $J \neq \emptyset$. Choose $l \in J$. After performing a translation $t$, we may assume $x^{(l)} = 0$. Henceforth, we set $B := t(B)$. Let $\pi$ be the nearest point projection onto $S^4$ and define for $r > 0$

$$u_{B_r} := \fint_{B_r} u dx.$$

Applying Lemma 3.12, we get maps $(w_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$ s.t.

$$w_k = \begin{cases} u_k & \text{for } x \in B_{\epsilon_k}\\ \pi(u_{B_{\epsilon_k}}) & \text{for } x \in \mathbb{R}^4 \setminus B_{4\epsilon_k} \end{cases}$$

and

$$\int_{B_{4\epsilon_k}\backslash B_{\epsilon_k}} |\nabla^2 w_k|^2 dx \longrightarrow 0,$$

as $k \to \infty$. Thus,

$$E(w_k) = E_{B_{\epsilon_k}}(u_k) + o(1),$$

with $o(1) \to 0$ as $k \to \infty$. For $A \subset \mathbb{R}^4$ open set, we consider the characteristic function $\chi_A : \mathbb{R}^4 \to \{0,1\}$. Then, $\chi_{B\backslash B_{\epsilon_k}} \Delta u_k$ converges weakly to $\Delta u$ in $L^2$ and $E_B(u) \leq E_{B\backslash B_{\epsilon_k}}(u_k)$. Hence, we infer from (67) and $E_B(\underline{u}) \leq E_B(u)$

$$\begin{aligned} E(w_k) &\leq E_B(u_k) - E_B(u) + o(1) \\ &< \mathcal{I} - \delta + o(1) \leq 24\mathcal{H}^4(S^4). \end{aligned} \tag{70}$$

On the other hand, as $(w_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{2,2}(\mathbb{R}^4, S^4)$, we have

$$\int_{\mathbb{R}^4} J_4(\nabla w_k) dx = d_k \mathcal{H}^4(S^4),$$

with $d_k \in \mathbb{Z}$. Moreover,

$$\int_{\mathbb{R}^4} J_4(\nabla w_k) dx = \int_{B_{\epsilon_k}} J_4(\nabla u_k) dx + o(1) = \nu^{(l)} + o(1) \neq 0.$$

Thus, $|d_k| > 0$ for $k$ sufficiently large. Furthermore, as $|w_k|^2 = 1$ (i.e. $\Delta w_k^\alpha w_k^\alpha = -|\nabla w_k|^2$), we have with Lemma 5.7

$$24\mathcal{H}^4(S^4) \geq E(w_k) = \int_{\mathbb{R}^4} |\Delta w_k|^2 dx \geq \int_{\mathbb{R}^4} |\nabla w_k|^4 dx \geq 16 \int_{\mathbb{R}^4} |J_4(\nabla w_k)| dx \geq 16|d_k|\mathcal{H}^4(S^4).$$

In particular, $|deg(w_k)| = 1$, Theorem 1.1 implies $E(w_k) \geq \mathcal{I}$. This is a contradiction to (70). Hence, $J = \emptyset$. This implies with (68) and (69) that $u \in \Xi^1_\gamma$, and we are done.

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## Appendix A: An inequality

**Lemma 5.7** *Consider $a, b, c, d \in \mathbb{R}$. Then, it holds*

$$16abcd \leq \left(a^2 + b^2 + c^2 + d^2\right)^2.$$

*Proof* Applying $2xy \leq x^2 + y^2$ gives

$$\begin{aligned} &4abcd \leq 2a^2b^2 + 2c^2d^2, \quad 4abcd \leq 2a^2c^2 + 2b^2d^2, \\ &4abcd \leq 2a^2d^2 + 2b^2c^2, \quad 4abcd \leq a^4 + b^4 + c^4 + d^4. \end{aligned}$$

Adding these inequalities establishes the desired inequality. □

## Appendix B: Inverse of the stereographic projection

Here, we give some explicit computations concerning the inverse of the stereographic projection $\kappa := \sigma^{-1}$. $\kappa$ is explicitly given (for example) as follows. For $1 \le i \le 4$, we define

$$\begin{aligned} \kappa : \mathbb{R}^4 &\to S^4 \\ x &\mapsto \left( \tfrac{2}{1+|x|^2} x^i, \tfrac{1-|x|^2}{1+|x|^2} \right). \end{aligned}$$

We compute for $1 \le i, j \le 4$

$$\begin{aligned} \kappa^i_j &= \tfrac{2\delta_{ij}(1+|x|^2) - 4x^i x^j}{(1+|x|^2)^2}, & \kappa^5_j &= -\tfrac{4x^j}{(1+|x|^2)^2}, \\ \Delta\kappa^i &= -\tfrac{24x^i}{(1+|x|^2)^2} + \tfrac{16x^i|x|^2}{(1+|x|^2)^3}, & \Delta\kappa^5 &= -\tfrac{16}{(1+|x|^2)^3}, \\ \Delta^2\kappa^i &= \tfrac{768x^i}{(1+|x|^2)^5}, & \Delta^2\kappa^5 &= \tfrac{384(1-|x|^2)}{(1+|x|^2)^5}. \end{aligned}$$

We deduce that

$$\Delta^2\kappa = \frac{384}{(1+|x|^2)^4}\kappa,$$

i.e. $\kappa$ is biharmonic.

Moreover, we compute

$$|\Delta\kappa|^2 = \frac{64(4+|x|^2)}{(1+|x|^2)^4}.$$

It follows with $\mathcal{H}^3(S^3) = 2\pi^2$ and $\mathcal{H}^4(S^4) = \frac{8}{3}\pi^2$, that

$$\begin{aligned} E(\kappa) = \int_{\mathbb{R}^4} |\Delta\kappa|^2 dx &= 64\mathcal{H}^3(S^3) \int_0^\infty \frac{4+r^2}{(1+r^2)^4} r^3 dr \\ &= 64\pi^2 \left( \frac{1}{(1+r^2)^3} - \frac{1}{(1+r^2)^2} - \frac{1}{1+r} \right)\Bigg|_0^\infty \\ &= 64\pi^2 = 24\mathcal{H}^4(S^4). \end{aligned}$$

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