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Stable multilevel splittings of boundary edge element spaces

Ralf Hiptmair · Shipeng Mao

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Abstract We establish the stability of nodal multilevel decompositions of lowest-order conforming boundary element subspaces of the trace space $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ on boundaries of triangulated Lipschitz polyhedra. The decompositions are based on nested triangular meshes created by uniform refinement and the stability bounds are uniform in the number of refinement levels.

The main tool is the general theory of P. Oswald (*Interface preconditioners and multilevel extension operators*, in Proc. 11th Intern. Conf. on Domain Decomposition Methods, London, 1998, pp. 96–103) that teaches, when stability of decompositions of boundary element spaces with respect to trace norms can be inferred from corresponding stability results for finite element spaces. $\boldsymbol{H}(\mathbf{curl}, \Omega)$-stable discrete extension operators are instrumental in this.

Stable multilevel decompositions immediately spawn subspace correction preconditioners whose performance will not degrade on very fine surface meshes. Thus, the results of this article demonstrate how to construct optimal iterative solvers for the linear systems of equations arising from the Galerkin edge element discretization of boundary integral equations for eddy current problems.

Keywords Trace spaces · Boundary element methods · Edge elements · Multilevel preconditioning

Mathematics Subject Classification (2000) 65N12 · 65N15 · 65N30

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R. Hiptmair (✉) · S. Mao
SAM, ETH Zürich, 8092 Zürich, Switzerland
e-mail: hiptmair@sam.math.ethz.ch

S. Mao
e-mail: shipeng.mao@sam.math.ethz.ch

1 Introduction

The pioneering work of J. Xu [63, 64] revealed how simple stability properties of decompositions of Galerkin trial and test spaces for symmetric positive definite variational problems translate into good properties of induced subspace correction preconditioners and iterative solvers. This paved the way for a comprehensive theoretical understanding of multigrid methods and multilevel preconditioners for low-order finite element discretizations of symmetric positive definite elliptic variational problems. In this context it is crucial to show that stability of multilevel splittings holds *uniformly* with respect to the local and global resolution of the finite element space.

This was first accomplished for $H^1(\Omega)$-conforming linear Lagrangian finite elements on quasi-uniform hierarchies of meshes [9, 11, 12, 42, 61]. Later the results were extended to sequences of meshes created by adaptive mesh refinement, see [21, 62, 66] and [43, Sect. 4.2.2]. The developments for $\boldsymbol{H}(\mathbf{curl}, \Omega)$-elliptic variational problems and their discretization by means of edge elements followed a similar path: uniform stability was established in the case of regular refinement [5, 23, 30, 33, 51] and then sequences of locally refined meshes were tackled successfully [19, 37, 65].

Symmetric positive definite variational problems are also common in the variational formulation of boundary integral equations (BIE) of the first kind [20], [48, Sect. 3.4]. Therefore, stable decompositions of boundary element (BEM) spaces immediately yield good subspace correction preconditioners for the linear systems we obtain from the Galerkin BEM discretization of those BIEs. Preconditioning for BEM has attracted considerable attention recently, since modern matrix compression techniques for discrete BIE entail the use of iterative solvers, whose efficiency often hinges on powerful preconditioners.

Plenty of stability results for the piecewise polynomial subspaces of the classical trace spaces $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ associated with scalar 2nd-order elliptic boundary value problems have been found. In [18, 56] stability proofs are given for closed curves, in [2, 44] for surfaces and adaptive refinement. These results were extended to the p and hp version of BEM in [25, 27, 28, 53, 54, 57] and to screen problems in [24, 58]. Related techniques are substructuring techniques [1, 26] and multilevel wavelet preconditioners for BEM [49, 59, 60].

Scant attention was paid to 1st-kind boundary integral equations set in the (tangential) trace space $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ of $\boldsymbol{H}(\mathbf{curl}, \Omega)$, see [15–17] for the relevant trace theorems. These BIE (for complex-valued surface fields) occur in the context of eddy current simulations in computational electromagnetism, see, e.g., [35, 36]. Their low-order Galerkin discretizations naturally rely on surface edge elements [7], also known as RWG boundary elements [46]. On fine surface meshes one ends up with poorly conditioned linear systems of equations, which may be tackled by the strategy of operator preconditioning, see [34]. This requires fast solvers for discrete $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$-elliptic variational BIE. The multilevel preconditioners proposed in this paper can be used for this purpose.

Hitherto no multilevel stability theory has been developed for surface edge element spaces and only a few ideas in the direction of multilevel preconditioning have been floated [3, 4]. It is the goal of this paper to fill the gap and show the uniform stability of multilevel splitting of edge BEM subspaces of $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ on hierarchies of nested triangular surface meshes created by regular refinement. Our key idea is to take the cue from the analysis of BIE in trace spaces, detach oneself from the boundary, whisk estimates to a finite element setting in the volume and harness their mature multilevel theory. This is made possible by the general theory of [45, Sect. 1], which we are going to review in Sect. 2. In a sense, with more than ten years delay we follow up on the final remark in [45] that "it is intriguing to look at the consequences of our approach in connection with multilevel splittings for **H**(div) and **H**(**curl**)".

Admittedly, our domain centered approach fails to be "intrinsic to Γ". However, we believe it still covers sufficiently general situations as far as BIE on closed surfaces are concerned. Screen problems are outside its scope, but it is possible to extend our technique, see Rem. 6.5.

The results of this paper are not completely satisfactory in one respect: reliance on an inverse inequality in one estimate precludes the treatment of multilevel splittings arising from hierarchies of locally refined surface meshes.

A recurring motive in this article is the perspective to view the trace space $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ as a member of a family of spaces, which is suggested by the following commuting diagram

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl}, \Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div}, \Omega) \\ {\scriptstyle |\partial\Omega \text{ (point trace)}}\downarrow & & {\scriptstyle \cdot\times\boldsymbol{n}|\partial\Omega \text{ (tangential trace)}}\downarrow & & {\scriptstyle \cdot\cdot\boldsymbol{n}|\partial\Omega \text{ (normal trace)}}\downarrow \\ H^{\frac{1}{2}}(\Gamma) & \xrightarrow{\mathbf{curl}_\Gamma} & \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) & \xrightarrow{\mathrm{div}_\Gamma} & H^{-\frac{1}{2}}(\Gamma). \end{array} \tag{1.1}$$

Here, Γ stands for the boundary of the Lipschitz domain Ω with exterior unit normal vectorfield $\boldsymbol{n}$. A subscript Γ designates surface differential operators. The diagram (1.1) is natural, once the function spaces are identified as a Hilbert complex corresponding to the deRham complex of differential forms, and the various trace operators are recognized as incarnations of the trace of differential forms [6, Sect. 2].

It is an important observation that (1.1) carries over to the discrete setting of finite element spaces and boundary element spaces, because both can be viewed as spaces of discrete differential forms built upon triangulations, see [6, 10, 29]. To elaborate this let us equip Ω with a tetrahedral finite element mesh Ω_h. On it we consider the finite element spaces

- $\mathcal{S}_1(\Omega_h) \subset H^1(\Omega)$ of piecewise linear continuous Lagrangian finite element functions (3D Whitney 0-forms),
- $\boldsymbol{\mathcal{ND}}_1(\Omega_h) \subset \boldsymbol{H}(\mathbf{curl}, \Omega)$ of Nedelec's first family of edge elements [41] (3D Whitney 1-forms),
- $\boldsymbol{\mathcal{RT}}_0(\Omega_h) \subset \boldsymbol{H}(\mathrm{div}, \Omega)$ of 3D div-conforming finite elements [41] (3D Whitney 2-forms).

Taking the restriction of Ω_h to $\Gamma := \partial\Omega$ furnishes a triangular mesh of Γ, which supports the boundary element spaces

- $\mathcal{S}_1(\Gamma_h) \subset H^{\frac{1}{2}}(\Gamma)$ of piecewise linear continuous boundary elements (2D Whitney 0-forms),

– $\boldsymbol{\mathcal{RT}}_0(\Gamma_h) \subset \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ of surface Raviart-Thomas vector fields [47] (2D Whitney 1-forms),
– $\mathcal{Q}_0(\Gamma_h) \subset H^{-\frac{1}{2}}(\Gamma)$ of piecewise constant boundary element functions (2D Whitney 2-forms).

More details will be given below in Sect. 3. Then straightforward computations establish the commuting relationships, a discrete counterpart of (1.1)

$$\begin{array}{ccccc} \mathcal{S}_1(\Omega_h) & \xrightarrow{\mathbf{grad}} & \boldsymbol{\mathcal{ND}}_1(\Omega_h) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{RT}}_0(\Omega_h) \\ {\scriptstyle |\partial\Omega \atop \text{(point trace)}}\big\downarrow & & {\scriptstyle \cdot\times\boldsymbol{n}|\partial\Omega \atop \text{(tangential trace)}}\big\downarrow & & {\scriptstyle \cdot\cdot\boldsymbol{n}|\partial\Omega \atop \text{(normal trace)}}\big\downarrow \\ \mathcal{S}_1(\Gamma_h) & \xrightarrow{\mathbf{curl}_\Gamma} & \boldsymbol{\mathcal{RT}}_0(\Gamma_h) & \xrightarrow{\mathrm{div}_\Gamma} & \mathcal{Q}_0(\Gamma_h). \end{array} \tag{1.2}$$

Our approach will make heavy use of these relationships throughout.

We point out that from (1.2) it is immediate that all relevant lowest-order conforming boundary element spaces arise from taking the traces of finite element spaces. This observation was what initially made us try and connect stability estimates for boundary element spaces with analogous results for finite elements in the volume.

We also point out that extending the results to surface edge elements of a *fixed* higher polynomial degree is a mere technicality that we forgo in order to keep the presentation simple. Of course, in this case most constants will depend on the polynomial degree and no useful results for p-version BEM can be expected.

2 Abstract theory

In this section we revisit the theory presented in [45] in a slightly simplified form. For the sake of completeness we give most results with proofs, which closely follow Oswald's original work.

On a real Hilbert space V we consider the variational problem

$$\text{find} \quad u \in V: \quad a(u, v) = f(v) \quad \forall v \in V, \tag{2.1}$$

where $f \in V'$ is a bounded linear functional, and $a(\cdot,\cdot)$ a continuous, V-elliptic bilinear form with associated operator $\mathsf{A}: V \mapsto V'$. It supplies an inner product on V and the "energy norm" $\|v\|_A^2 := a(v, v)$. An additive subspace correction preconditioner $\mathsf{M}: V' \mapsto V$ for (2.1) is induced by the (not necessarily direct) splitting

$$V = \sum_{i=0}^{L} V_i, \quad V_i \text{ is a closed subspace of } V, \tag{2.2}$$

and defined through [43, Sect. 4.1], [55, Sect. 2.1]

$$\mathsf{MA} = \sum_{i=0}^{L} \mathsf{P}_i. \tag{2.3}$$

Here, we have written P_i for the $a(\cdot,\cdot)$-orthogonal projections $\mathsf{P}_i : V \mapsto V_i$ onto V_i, $i = 0, \dots, L$.

Following [43, Sect. 4.1] we introduce a norm on V by

$$|||v|||_A^2 = \inf\left\{\sum_{i=0}^{L} \|v_i\|_A^2;\ v_i \in V_i, v = \sum_{i=0}^{L} v_i\right\}, \quad \forall v \in V. \tag{2.4}$$

It allows the concise statement of the following fundamental result [43, Thm. 16]

Theorem 2.1 *Let* A *and* M *defined as above, then for any* $v \in V$

$$a((\mathsf{MA})^{-1} v, v) = |||v|||_A^2. \tag{2.5}$$

If there exist two constants λ *and* Λ *such that*

$$\lambda \|v\|_A^2 \le |||v|||_A^2 \le \Lambda \|v\|_A^2, \tag{2.6}$$

then we have the following estimate for the spectral condition number

$$\kappa(\mathsf{MA}) \le \frac{\Lambda}{\lambda}. \tag{2.7}$$

Next, we consider a pair of Hilbert spaces V, X connected by a linear *surjective* operator $\mathsf{T} : V \mapsto X$, $X = \mathsf{T}(V)$. Further, let $d(\cdot,\cdot)$ be a bounded, symmetric, and X-elliptic bilinear form with associated operator $\mathsf{D} : X \mapsto X'$. For the related norm on X we write $\|\cdot\|_D$.

Let V be split according to (2.2), which induces a splitting of X by

$$X = \sum_{i=0}^{L} X_i, \quad X_i := \mathsf{T}(V_i). \tag{2.8}$$

Assume that (2.6) holds for the decomposition of V. The question is, under what conditions we can infer an analogous estimate for (2.8). The answer is given by P. Oswald in [45] and he identifies the following sufficient conditions:

Assumption STO The operator $\mathsf{T} : V \longrightarrow X$ is bounded

$$\|\mathsf{T}\, v\|_D \le C_0 \|v\|_A, \quad \forall v \in V \tag{STO}$$

with constant $C_0 > 0$.

Assumption USEO There exist bounded (extension) operators $\mathsf{E} : X \longrightarrow V$ and $\mathsf{E}_i : X_i \longrightarrow V_i$, $0 \le i \le L$, *uniformly with respect to the choice of subspace index* i, such that, with $C_1, C_2 > 0$,

$$\mathsf{T} \circ \mathsf{E} = \mathsf{Id} \text{ on } X, \qquad \|\mathsf{E}\xi\|_A \le C_1 \|\xi\|_D, \quad \forall \xi \in X, \tag{USEO.1}$$

$$\xi_i = \mathsf{T}(\mathsf{E}_i \xi_i), \qquad \|\mathsf{E}_i \xi_i\|_A \le C_2 \|\xi_i\|_D, \quad \forall \xi_i \in X_i, \forall i = 0, \dots, L. \tag{USEO.2}$$

Remark 2.1 There may be some subspaces $V_i \subset V$ such that $X_i = \mathsf{T}(V_i) = \{0\}$. We still keep them to simplify notations. Obviously, the only choice of E_i for such i is the null operator such that $E_i(X_i) = \{0\}$, too.

Remark 2.2 **STO** and **USEO** are abbreviations of ***s**table **t**race **o**perator* and ***u**niformly **s**table **e**xtension **o**perator*, respectively.

Now, we are in a position to state and prove a key abstract result, see [45, Thm. 1].

Theorem 2.2 *Under the assumptions **STO** and **USEO** we have the norm equivalence*

$$\frac{\lambda}{C_0^2 C_2^2} \|\xi\|_D^2 \leq |||\xi|||_D^2 \leq \Lambda C_0^2 C_1^2 \|\xi\|_D^2, \quad \forall \xi \in X, \tag{2.9}$$

where

$$|||\xi|||_D^2 := \inf\left\{\sum_{i=0}^{L} \|\xi_i\|_D^2 ;\ \xi_i \in X_i,\ \xi = \sum_{i=0}^{L} \xi_i\right\}, \quad \xi \in X, \tag{2.10}$$

and λ *and* Λ *are the constants from* (2.6).

Proof Let us first prove the upper bound in (2.9). Pick any $\xi \in X$. According to the assumption **USEO**, we have

$$\xi = \mathsf{T}(\mathsf{E}\xi), \quad \|\mathsf{E}\xi\|_A \leq C_1 \|\xi\|_D. \tag{2.11}$$

By (2.2) there exist $w_i, 0 \leq i \leq L$ such that $\mathsf{E}\xi = \sum_{i=0}^{L} w_i$. Furthermore, we can assume this decomposition realizes the $|||\cdot|||_A$-norm of $\mathsf{E}\xi$ up to a given $\epsilon > 0$, i.e.,

$$|||\mathsf{E}\xi|||_A^2 = \sum_{i=0}^{L} \|w_i\|_A^2 - \epsilon. \tag{2.12}$$

Thus we get a decomposition of ξ by

$$\xi = \sum_{i=0}^{L} \xi_i, \quad \xi_i = \mathsf{T}w_i \in X_i,\ 0 \leq i \leq L. \tag{2.13}$$

From this we conclude

$$\begin{aligned} |||\xi|||_D^2 &\overset{\inf}{\leq} \sum_{i=0}^{L} \|\xi_i\|_D^2 \overset{(2.13)}{=} \sum_{i=0}^{L} \|\mathsf{T}w_i\|_D^2 \overset{(\text{STO})}{\leq} C_0^2 \sum_{i=0}^{L} \|w_i\|_A^2 \\ &\overset{(2.12)}{=} C_0^2 (|||\mathsf{E}\xi|||_A^2 + \epsilon) \overset{(2.6)}{\leq} \Lambda C_0^2 \|\mathsf{E}\xi\|_A^2 + C_0^2 \epsilon \overset{(\text{USEO.1})}{\leq} \Lambda C_0^2 C_1^2 \|\xi\|_D^2 + C_0^2 \epsilon. \end{aligned}$$

Since this holds for all $\epsilon > 0$, the proof of the upper bound is done.

Next, we prove the lower bound in (2.9). For any $\xi \in X$ and a decomposition $\xi = \sum_{i=0}^{L} \xi_i$ with $\xi_i \in X_i$, $0 \le i \le L$, from the assumption **USEO** again, we know that

$$\mathsf{E}_i \xi_i \in V_i, \quad \xi_i = \mathsf{T}(\mathsf{E}_i \xi_i), \qquad \|\mathsf{E}_i \xi_i\|_A \le C_2 \|\xi_i\|_D . \tag{2.14}$$

Then we can assemble the estimates into

$$\begin{aligned}
\sum_{i=0}^{L} \|\xi_i\|_D^2 &\overset{(2.14)}{\geq} \frac{1}{C_2^2} \sum_{i=0}^{L} \|\mathsf{E}_i \xi_i\|_A^2 \overset{\inf}{\geq} \frac{1}{C_2^2} \left|\!\left|\!\left| \sum_{i=0}^{L} \mathsf{E}_i \xi_i \right|\!\right|\!\right|_A^2 \overset{(2.6)}{\geq} \frac{\lambda}{C_2^2} \left\| \sum_{i=0}^{L} E_i \xi_i \right\|_A^2 \\
&\overset{(\mathrm{STO})}{\geq} \frac{\lambda}{C_2^2 C_0^2} \left\| \mathsf{T}\left(\sum_{i=0}^{L} \mathsf{E}_i \xi_i \right) \right\|_D^2 = \frac{\lambda}{C_2^2 C_0^2} \left\| \sum_{i=0}^{L} \mathsf{T}(\mathsf{E}_i \xi_i) \right\|_D^2 \\
&\overset{(2.14)}{=} \frac{\lambda}{C_2^2 C_0^2} \left\| \sum_{i=0}^{L} \xi_i \right\|_D^2 = \frac{\lambda}{C_2^2 C_0^2} \|\xi\|_D^2 .
\end{aligned}$$

□

Combining Thm. 2.1 and Thm. 2.2 we conclude the following condition number estimate from Assumptions **STO** and **USEO**

$$\kappa(\mathsf{M}^D \mathsf{D}) \le C_0^4 C_1^2 C_2^2 \frac{\Lambda}{\lambda}. \tag{2.15}$$

Here, we wrote $\mathsf{M}^D : X' \mapsto X$ for the subspace correction preconditioner induced by the splitting (2.8) in the same way as M emerged from (2.2).

For the concrete application to multilevel preconditioning in boundary element spaces the ingredients of the abstract theory will be given the following meanings:

- The space V will stand for a conforming finite element space built on a volume mesh and suitable for the Galerkin discretization of the s.p.d. variational problem (2.1).
- The estimate (2.6) will express the stability of some (multilevel) decomposition of the finite element space.
- The operator T will be the trace operator associated with the energy norm.
- Its range X is a boundary element space contained in the natural trace space.

In the following two sections we provide details with emphasis on $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming edge elements and the corresponding $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$-conforming boundary elements.

3 Discrete spaces

As introduced in Sect. 1, let Ω_h be a tetrahedral finite element mesh of the bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$. As above, write Γ_h for the triangular surface mesh of

$\Gamma := \partial\Omega$ arising through restricting Ω_h to Γ. Next, we briefly review the definitions of the standard *finite element spaces*, see [31, Sect. 3.2] for more details.

The space $\mathcal{S}_1(\Omega_h)$ of piecewise linear Lagrangian finite element functions reads

$$\mathcal{S}_1(\Omega_h) := \left\{ v \in H^1(\Omega) : \begin{array}{l} v_{|T}(\boldsymbol{x}) = \boldsymbol{a}_T \cdot \boldsymbol{x} + \alpha_T \\ \text{with } \boldsymbol{a}_T \in \mathbb{R}^3,\ \alpha_T \in \mathbb{R} \end{array}, \forall T \in \Omega_h \right\}. \tag{3.1}$$

We adopt the notation $\mathcal{V}(\Omega_h)$ for the set of vertices of Ω_h and recall the standard Lagrange basis $\{b_{\boldsymbol{p}}\}_{\boldsymbol{p}\in\mathcal{V}(\Omega_h)}$ of $\mathcal{S}_1(\Omega_h)$ consisting of locally supported "tent functions" defined through

$$b_{\boldsymbol{p}} \in \mathcal{S}_1(\Omega_h), \quad b_{\boldsymbol{p}}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} = \boldsymbol{p}, \\ 0, & \text{if } \boldsymbol{x} \in \mathcal{V}(\Omega_h) \setminus \{\boldsymbol{p}\}, \end{cases} \quad \boldsymbol{p} \in \mathcal{V}(\Omega_h). \tag{3.2}$$

When restricted to a single tetrahedron $T \in \Omega_h$, each basis function agrees with one local barycentric coordinate function λ_i, $i = 1, \ldots, 4$.

The edge element space $\boldsymbol{\mathcal{ND}}_1(\Omega_h)$ is given by

$$\boldsymbol{\mathcal{ND}}_1(\Omega_h) := \left\{ \mathbf{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \begin{array}{l} \mathbf{v}_{|T}(\boldsymbol{x}) = \boldsymbol{a}_T \times \boldsymbol{x} + \boldsymbol{b}_T \\ \text{with } \boldsymbol{a}_T \in \mathbb{R}^3,\ \boldsymbol{b}_T \in \mathbb{R}^3 \end{array}, \forall T \in \Omega_h \right\}. \tag{3.3}$$

For each edge we fix a direction (orientation). The local basis functions $\mathbf{b}_e$, $e \in \mathcal{E}(\Omega_h)$, of $\boldsymbol{\mathcal{ND}}_1(\Omega_h)$ are associated with the edges of Ω_h (edge set $\mathcal{E}(\Omega_h)$). We scale them such that for the path integrals

$$\int_s \mathbf{b}_e \cdot \mathrm{d}\mathbf{s} = \begin{cases} 1, & \text{if } s = e, \\ 0, & \text{if } s \in \mathcal{E}(\Omega_h) \setminus \{e\}, \end{cases} \quad e \in \mathcal{E}(\Omega_h). \tag{3.4}$$

Remember the local representation

$$\mathbf{b}_{e\,|T} = \lambda_j \, \mathbf{grad}\, \lambda_i - \lambda_i \, \mathbf{grad}\, \lambda_j, \tag{3.5}$$

when e is the edge connecting the vertices i and j, $\{i, j\} \subset \{1, \ldots, 4\}$, of the tetrahedron T.

We will also need the space $\boldsymbol{\mathcal{RT}}_0(\Omega_h)$ of face element functions

$$\boldsymbol{\mathcal{RT}}_0(\Omega_h) := \left\{ \mathbf{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) : \begin{array}{l} \mathbf{v}_{|T}(\boldsymbol{x}) = \alpha_T \boldsymbol{x} + \boldsymbol{b}_T \\ \text{with } \alpha_T \in \mathbb{R},\ \boldsymbol{b}_T \in \mathbb{R}^3 \end{array}, \forall T \in \Omega_h \right\}. \tag{3.6}$$

Writing $\mathcal{F}(\Omega_h)$ for the set of faces of Ω_h, there is a canonical basis $\{\mathbf{b}_F\}_{F\in\mathcal{F}(\Omega_h)}$ of $\boldsymbol{\mathcal{RT}}_0(\Omega_h)$ consisting of locally supported functions of $\boldsymbol{\mathcal{RT}}_0(\Omega_h)$ fixed by

$$\int_S \mathbf{b}_F \cdot \mathrm{d}\mathbf{F} = \begin{cases} 1, & \text{if } S = F, \\ 0, & \text{if } S \in \mathcal{F}(\Omega_h) \setminus \{F\}, \end{cases} \quad F \in \mathcal{F}(\Omega_h), \tag{3.7}$$

where each face is endowed with a fixed crossing direction (orientation). When restricted to a tetrahedron the basis functions can be written as

$$\mathbf{b}_{F\,|T} = \lambda_i \,\mathbf{grad}\,\lambda_j \times \mathbf{grad}\,\lambda_k + \lambda_j \,\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_i \\ + \lambda_j \,\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_i, \tag{3.8}$$

where the face F is spanned by the vertices i, j and k, $i, j, k \in \{1, \dots, 4\}$.

The last and simplest finite element space is the space $\mathcal{Q}_0(\Omega_h)$ of piecewise constant functions on Ω_h, equipped with the standard basis $\{b_T\}_{T\in\Omega_h}$ of characteristic functions of the elements of Ω_h scaled such that $\int_K b_K \,\mathrm{d}\boldsymbol{x} = 1$.

Throughout, these finite element spaces will be endowed with the norm of the underlying Sobolev spaces $H^1(\Omega)$, $\boldsymbol{H}(\mathbf{curl}, \Omega)$, $\boldsymbol{H}(\mathrm{div}, \Omega)$ and $L^2(\Omega)$, respectively.

The degrees of freedom dual to the bases introduced above can be extended to functionals on smooth functions and vector fields, respectively. Thus, they define canonical interpolation operators Π_X, $X \in \{\mathrm{S}, \mathrm{Nd}, \mathrm{RT}, \mathrm{Q}\}$ (with ranges $\mathcal{S}_1(\Omega_h)$, $\mathcal{ND}_1(\Omega_h)$, $\mathcal{RT}_0(\Omega_h)$, $\mathcal{Q}_0(\Omega_h)$, respectively), which enjoy a fundamental *commuting diagram property*, see, e.g., [31, Sect. 3.2] or [6, Sect. 5.2],

$$\begin{array}{ccccccc} C^\infty(\Omega) & \xrightarrow{\mathbf{grad}} & (C^\infty(\Omega))^3 & \xrightarrow{\mathbf{curl}} & (C^\infty(\Omega))^3 & \xrightarrow{\mathrm{div}} & C^\infty(\Omega) \\ \Pi_{\mathrm{S}}\downarrow & & \Pi_{\mathrm{Nd}}\downarrow & & \Pi_{\mathrm{RT}}\downarrow & & \Pi_{\mathrm{Q}}\downarrow \\ \mathcal{S}_1(\Omega_h) & \xrightarrow{\mathbf{grad}} & \mathcal{ND}_1(\Omega_h) & \xrightarrow{\mathbf{curl}} & \mathcal{RT}_0(\Omega_h) & \xrightarrow{\mathrm{div}} & \mathcal{Q}_0(\Omega_h). \end{array} \tag{3.9}$$

We learn from [6, Sect. 5.5], [31, Sect. 3.2] that the top and bottom sequences in (1.2) are *exact*, provided that Ω has trivial topology, that is, the co-homology of a ball. For the sake of lucidity, we will largely forgo the discussion of general topologies and make the following assumption. Remark 6.1 will briefly indicate how to deal with more general situations.

Assumption 3.1 The domain Ω is connected with vanishing first ("no tunnels") and second ("no cavities") Betti numbers.

As hinted in the Introduction, the relevant *boundary element spaces* on Γ_h can be generated by taking suitable traces of finite element functions. Hence, we first recall the continuous trace operators, see [17, 39], the pointwise trace T_p, the tangential trace $\mathsf{T}_{\boldsymbol{t}}$, and the normal component trace $\mathsf{T}_{\boldsymbol{n}}$,

$$\begin{cases} \mathsf{T}_p : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\Gamma), \\ \mathsf{T}_p v(\boldsymbol{x}) := v(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma,\ \forall v \in C^\infty(\overline{\Omega}), \end{cases} \tag{3.10}$$

$$\begin{cases} \mathsf{T}_{\boldsymbol{t}} : \boldsymbol{H}(\mathbf{curl}, \Omega) \longrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \\ \mathsf{T}_{\boldsymbol{t}} \mathbf{v}(\boldsymbol{x}) := \mathbf{v}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma,\ \forall \mathbf{v} \in (C^\infty(\overline{\Omega}))^3, \end{cases} \tag{3.11}$$

$$\begin{cases} \mathsf{T}_{\boldsymbol{n}} : \boldsymbol{H}(\mathrm{div}, \Omega) \longrightarrow H^{-\frac{1}{2}}(\Gamma), \\ \mathsf{T}_{\boldsymbol{n}}\mathbf{v}(\boldsymbol{x}) := \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma, \ \forall \mathbf{v} \in (C^\infty(\overline{\Omega}))^3. \end{cases} \tag{3.12}$$

Then we define the boundary element spaces

$$\mathcal{S}_1(\Gamma_h) := \mathsf{T}_p(\mathcal{S}_1(\Omega_h)) \subset H^{\frac{1}{2}}(\Gamma), \tag{3.13}$$

$$\boldsymbol{\mathcal{RT}}_0(\Gamma_h) := \mathsf{T}_{\boldsymbol{t}}(\boldsymbol{\mathcal{ND}}_1(\Omega_h)) \subset \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \tag{3.14}$$

$$\mathcal{Q}_0(\Gamma_h) := \mathsf{T}_{\boldsymbol{n}}(\boldsymbol{\mathcal{RT}}_0(\Omega_h)) \subset H^{-\frac{1}{2}}(\Gamma), \tag{3.15}$$

where $\mathcal{Q}_0(\Gamma_h)$ is the space of piecewise constant discontinuous functions on Γ_h. For all these spaces canonical bases $\{\beta_{\boldsymbol{p}}\}_{\boldsymbol{p}\in\mathcal{V}(\Gamma_h)} \subset \mathcal{S}_1(\Gamma_h)$, $\{\boldsymbol{\beta}_e\}_{e\in\mathcal{E}(\Gamma_h)} \subset \boldsymbol{\mathcal{RT}}_0(\Gamma_h)$, and $\{\beta_F\}_{F\in\Gamma_h} \subset \mathcal{Q}_0(\Gamma_h)$ can be obtained by merely taking the appropriate traces of those finite element bases functions belonging to vertices, edges, or faces, respectively, contained in Γ:

$$\beta_{\boldsymbol{p}} := \mathsf{T}_p(b_{\boldsymbol{p}}) \quad \text{for } \boldsymbol{p} \in \mathcal{V}(\Omega_h) \cap \Gamma, \tag{3.16}$$

$$\boldsymbol{\beta}_e := \mathsf{T}_{\boldsymbol{t}}(\mathbf{b}_e) \quad \text{for } e \in \mathcal{E}(\Omega_h),\ e \subset \Gamma, \tag{3.17}$$

$$\beta_F := \mathsf{T}_{\boldsymbol{n}}(\mathbf{b}_F) \quad \text{for } F \in \mathcal{F}(\Omega_h),\ F \subset \Gamma. \tag{3.18}$$

Eventually, Ass. 3.1 also ensures that we have an *exact* discrete DeRham sequence formed by the boundary element spaces:

$$\{1\} \longrightarrow \mathcal{S}_1(\Gamma_h) \xrightarrow{\mathbf{curl}_\Gamma} \boldsymbol{\mathcal{RT}}_0(\Gamma_h) \xrightarrow{\mathrm{div}_\Gamma} \mathcal{Q}_0(\Gamma_h) \longrightarrow \{0\}, \tag{3.19}$$

where we refer to [17] for the definition of the vector valued surface rotation $\mathbf{curl}_\Gamma$ and the surface divergence div_Γ.

4 Multilevel decompositions

Now we specify the particular setting required for the envisaged multilevel preconditioners and their analysis.

Assumption 4.1 For some $L \in \mathbb{N}$, $\Gamma_L := \Gamma_h$ is the finest surface mesh in a sequence $\Gamma_0 \prec \Gamma_1 \prec \cdots \prec \Gamma_L$ of *nested* triangular meshes, for which Γ_0 still resolves the faces of Γ.

Here, $\Gamma_{i-1} \prec \Gamma_i$ expresses the nestedness of two meshes in the sense that each closed cell of Γ_{i-1} is the union of closed cells of Γ_i. In order to link boundary elements and finite elements, we have to take for granted that the hierarchy of surface meshes fits a corresponding *auxiliary hierarchy of volume meshes*.

Assumption 4.2 The surface meshes Γ_i are to be the restrictions to Γ of the members of a sequence of nested tetrahedral meshes $\Omega_0 \prec \Omega_1 \prec \cdots \prec \Omega_L$: $\Gamma_i := \Omega_{i\,|\Gamma}$.

We point out that this assumption is not unduly restrictive; every "reasonable" triangular surface mesh can certainly be extended to a tetrahedral mesh of Ω. In particular, surface meshes created by (local) refinement like bisection strategies can be obtained as restrictions to Γ of suitably (locally) refined tetrahedral meshes.

As usual, we rule out severely distorted elements in both sequences of meshes:

Assumption 4.3 Both sequences $(\Omega_l)_{l=0}^L$, $(\Gamma_l)_{l=0}^L$ of meshes are *uniformly shape-regular*, that is,

$$\exists C_s > 0: \quad \max_{T\in\Gamma_l} \frac{h_T}{\rho_T}, \ \max_{K\in\Omega_l} \frac{h_K}{\rho_K} \le C_s \quad \forall l = 0, \dots, L. \tag{4.1}$$

Here we adopted the conventional notation h_T, h_K for the diameter of a mesh cell (element), and ρ_T, ρ_K for the radius of the largest inscribed circle. For technical reasons, which will become clear in Sect. 5, we demand that all elements of a mesh have "about the same size"

Assumption 4.4 The sequence $(\Omega_l)_{l=0}^L$ of meshes is *quasi-uniform*, that is,

$$\exists C_u > 0: \quad C_u^{-1} \le \frac{\max_{K\in\Omega_l} h_K}{\min_{K\in\Omega_l} h_K} \le C_u \quad \forall l = 0, \dots, L. \tag{4.2}$$

It goes without saying that a quasi-uniformity condition like (4.2) is also satisfied by $(\Gamma_l)_{l=0}^L$.

An easy way to generate sequences of meshes complying with Ass. 4.1–4.4 is the global regular refinement of a coarse tetrahedral mesh Ω_0 of Ω; each tetrahedron is successively split into eight smaller according to the rules put forth in [38]. For the surface meshes this amounts to splitting each triangle into four congruent triangles of half the size.

On all of the meshes Ω_i and Γ_i, $i = 0, \dots, L$, we can define the finite element and boundary element spaces introduced in the previous section. The index l for the sequences of meshes may be dubbed the *level*. The level as a superscript will tag the standard (canonical) basis functions of a finite element or boundary element space built on a mesh on a certain level.

The theory of (local) multilevel preconditioning for edge elements developed in [30, 33, 37] and [65, Sect. 5] suggests the following multilevel decomposition

$$\mathcal{ND}_1(\Omega_h) = \underbrace{\mathcal{ND}_1(\Omega_0)}_{=:\mathbf{V}_0} + \sum_{l=1}^{L} \Big\{ \sum_{e\in\mathcal{E}_l} \underbrace{\mathrm{Span}(\mathbf{b}_e^l)}_{=:\mathbf{V}_e^l} + \sum_{p\in\mathcal{V}_l} \underbrace{\mathrm{Span}(\mathbf{grad}\, b_p^l)}_{=:\mathbf{V}_p^l} \Big\}. \tag{4.3}$$

The inner product of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ will provide the s.p.d. bilinear form a on $\mathcal{ND}_1(\Omega_h)$. Thus, for the concrete splitting (4.3) the induced "multilevel norm" on $\mathcal{ND}_1(\Omega_h)$ in analogy to (2.10) is

$$
\begin{aligned}
&|||\mathbf{v}_h|||^2 \\
&:= \inf \left\{ \begin{array}{l} \|\mathbf{v}_0\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \sum_{l=1}^{L} \left\{ \sum_{e\in\mathcal{E}_l} \left\|\mathbf{v}_e^l\right\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \sum_{p\in\mathcal{V}_l} \left\|\mathbf{v}_{\boldsymbol{p}}^l\right\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)} \right\}, \\ \mathbf{v}_0 + \sum_{l=1}^{L} \left\{ \sum_{e\in\mathcal{E}_l} \mathbf{v}_e^l + \sum_{p\in\mathcal{V}_l} \mathbf{v}_{\boldsymbol{p}}^l \right\} = \mathbf{v}_h, \quad \begin{array}{l} \mathbf{v}_e^l \in \mathbf{V}_e^l,\ \mathbf{v}_{\boldsymbol{p}}^l \in \mathbf{V}_{\boldsymbol{p}}^l, \\ \mathbf{v}_0 \in \mathbf{V}_0. \end{array} \end{array} \right\}
\end{aligned}
\tag{4.4}
$$

The next key result addresses the uniform stability of the local nodal multilevel splitting (4.3), that is, the norm equivalence (2.6) for this concrete case.

Theorem 4.1 *There are constants* $0 < \lambda \leq \Lambda$ *depending only on* Ω *and* C_s *such that*

$$
\lambda \|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq |||\mathbf{v}_h||| \leq \Lambda \|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \quad \forall \mathbf{v}_h \in \mathcal{ND}_1(\Omega_h). \tag{4.5}
$$

Proofs of this theorem in the setting of this article can be found in [30, 33] and even more general situations (local refinement) in [19, 37, 65].

5 Stable extensions

Now we tackle the key assumption **USEO** of the abstract theory of Sect. 2 for the finite element space $\mathcal{ND}_1(\Omega_h)$, the associated surface edge element space $\mathcal{RT}_0(\Gamma_h)$ and the splitting (4.3). First, we have to find a discrete extension operator $\mathsf{E} : \mathcal{RT}_0(\Gamma_h) \mapsto \mathcal{ND}_1(\Omega_h)$, uniformly bounded with respect to the norms of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Secondly, we have to show that the mappings $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \to \boldsymbol{H}(\mathbf{curl}, \Omega)$ that take an edge basis function of $\mathcal{RT}_0(\Gamma_l)$ to the one of $\mathcal{ND}_1(\Omega_l)$ associated with the same edge enjoy a norm bound independent of the basis function and the level. A similar result is needed for "tent functions" in $\mathcal{S}_1(\Gamma_h)$ in order to deal with the **curl**-free terms in the splitting (4.3).

5.1 Discrete extension in $\boldsymbol{H}(\mathbf{curl}, \Omega)$

We focus on the meshes Ω_h and $\Gamma_h := \Omega_{h|\Gamma}$ and designate by h their common "meshwidth", that is, the diameter of the largest element of Ω_h. To begin with, we recall the construction of bounded discrete extension operators $\mathcal{S}_1(\Gamma_h) \to \mathcal{S}_1(\Omega_h)$.

Lemma 5.1 *There exists an extension operator* $\mathsf{E}^0 : \mathcal{S}_1(\Gamma_h) \mapsto \mathcal{S}_1(\Omega_h)$ *such that for any* $\psi_h \in \mathcal{S}_1(\Gamma_h)$ *we have*

$$
\mathsf{T}_p(\mathsf{E}^0\psi_h) = \psi_h \tag{5.1}
$$

and

$$
\|\mathsf{E}^0\psi_h\|_{H^1(\Omega)} \leq C\, \|\psi_h\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{5.2}
$$

where $C > 0$ *depends only on* Ω *and the shape-regularity*[1] *of the triangulation* Ω_h.

[1] The phrase that a constant "depends on shape-regularity" means that this constant may be a function of C_s from (4.1).

Proof For any $\psi_h \in \mathcal{S}_1(\Gamma_h) \subset H^{\frac{1}{2}}(\Gamma)$, we consider its $H^1(\Omega)$-extension defined as the solution $\phi \in H^1(\Omega)$ of the auxiliary boundary value problem

$$\begin{cases} -\triangle\phi + \phi = 0, & \text{in } \Omega, \\ \mathsf{T}_p\phi = \psi_h, & \text{on } \Gamma, \end{cases} \tag{5.3}$$

which satisfies the obvious estimate

$$\|\phi\|_{H^1(\Omega)} \le C\,\|\psi_h\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{5.4}$$

where the constant $C > 0$ only depends on the domain Ω.

Let $\mathsf{Q}_h : H^1(\Omega) \to \mathcal{S}_1(\Omega_h)$ be the so-called Scott-Zhang type *quasi-interpolation operator*, which is continuous and preserves boundary values in $\mathcal{S}_1(\Gamma_h)$, see [52]. Thus, if we define

$$\mathsf{E}^0\psi_h := \mathsf{Q}_h\phi, \quad \forall\psi_h \in \mathcal{S}_1(\Gamma_h), \tag{5.5}$$

by [52, Thm. 3.1] there exists a constant $C > 0$ depending only on the shape-regularity of the mesh, such that

$$\|\mathsf{E}^0\psi_h\|_{H^1(\Omega)} = \|\mathsf{Q}_h\phi\|_{H^1(\Omega)} \le C\|\phi\|_{H^1(\Omega)} \overset{(5.4)}{\le} C\,\|\psi_h\|_{H^{\frac{1}{2}}(\Gamma)}.$$

The preservation of boundary values follows from [52, Thm. 2.1]. □

Unfortunately, this recipe fails for $\boldsymbol{H}(\mathbf{curl}, \Omega)$, because tangential traces of functions in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ may not even belong to $L^2(\Gamma)$. A construction of a quasi-interpolation operator based on volume integrals was pursued in [50]. Yet, this quasi-interpolation onto $\boldsymbol{\mathcal{ND}}_1(\Omega_h)$ does not preserve non-homogeneous boundary values. Thus, we take a completely different tack exploiting the connections between different spaces depicted in (1.1), (1.2).

We start with an auxiliary elliptic lifting theorem:

Lemma 5.2 *There exists a $\epsilon_\Omega \in (0, \frac{1}{2})$ depending solely on the geometry of Ω, such that for any $\epsilon \in [-\frac{1}{2}, \epsilon_\Omega)$ and $\mu \in H^\epsilon(\Gamma)$, $\int_\Gamma \mu\, dS = 0$, we can find a vector field*

$$\mathbf{w} \in \boldsymbol{H}(\mathrm{div}, \Omega), \quad \mathrm{div}\,\mathbf{w} = 0, \quad \mathsf{T}_{\boldsymbol{n}}\mathbf{w} = \mu, \tag{5.6}$$

which enjoys the stability

$$\|\mathbf{w}\|_{\boldsymbol{H}^{\frac{1}{2}+\epsilon}(\Omega)} \le C\,\|\mu\|_{H^\epsilon(\Gamma)}, \tag{5.7}$$

with $C > 0$ depending only on Ω and ϵ.

Proof Thanks to [22, Cor. 23.5], there is $\epsilon_\Omega \in]0, \frac{1}{2})$ such that for $\epsilon \in [-\frac{1}{2}, \epsilon_\Omega)$ the solution of the homogeneous Neumann problem

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \mu, & \text{on } \Gamma, \end{cases} \tag{5.8}$$

belongs to $H^{\frac{3}{2}+\epsilon}(\Omega)$ and satisfies

$$\|u\|_{H^{\frac{3}{2}+\epsilon}(\Omega)} \leq C \|\mu\|_{H^\epsilon(\Gamma)},$$

where all constants depend on Ω only. Setting $\mathbf{w} := \mathbf{grad}\, u$ gives us the desired vector field. □

The following interpolation error estimate for the canonical interpolation onto $\mathcal{RT}_0(\Omega_h)$ is well-known, see Theorem 5.25 of [40] and [31, Thm. 3.16].

Lemma 5.3 *For any* $\epsilon \in (0, \frac{1}{2}]$, *the canonical interpolation operator* $\Pi_{\mathrm{RT}} : (C^\infty(\Omega))^3 \mapsto \mathcal{RT}_0(\Omega_h)$ *can be extended to a continuous mapping* $\Pi_{\mathrm{RT}} : (H^{\frac{1}{2}+\epsilon}(\Omega))^3 \mapsto \mathcal{RT}_0(\Omega_h)$ *and satisfies*

$$\|\boldsymbol{u} - \Pi_{RT}\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq C\, h^{\frac{1}{2}+\epsilon} \|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}+\epsilon}(\Omega)}, \tag{5.9}$$

with $C > 0$ *depending only on* Ω, ϵ, *and the shape-regularity of the mesh* Ω_h.

The following lemma furnishes an *inverse inequality* for piecewise constant boundary element functions.

Lemma 5.4 *For any* $\epsilon \in (0, \frac{1}{2})$, *we have*

$$\|\mu_h\|_{H^\epsilon(\Gamma)} \leq C\, h^{-(\frac{1}{2}+\epsilon)} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall \mu_h \in \mathcal{Q}_0(\Gamma_h), \tag{5.10}$$

with $C > 0$ *depending on the shape-regularity* and quasi-uniformity[2] *of* Γ_h.

Proof Without further notice, all constants in this proof may depend on ϵ, Γ, the shape-regularity and quasi-uniformity of Γ_h.

(i) From [13, Appendix] we learn that $\mathcal{Q}_0(\Gamma_h) \subset H^\epsilon(\Gamma)$ for all $0 \leq \epsilon < \frac{1}{2}$ with

$$\|\mu_h\|_{H^\epsilon(\Gamma)} \leq C h^{-\epsilon} \|\mu_h\|_{L^2(\Gamma)} \quad \forall \mu_h \in \mathcal{Q}_0(\Gamma_h). \tag{5.11}$$

(ii) Let $\beta \in H^1(\Gamma)$ denote the sum of cubic bubble functions (products of barycentric coordinate functions) associated with the triangles of Γ_h. Clearly, we have

[2] The phrase that a constant "depends on quasi-uniformity" means that this constant may be a function of C_u from (4.2).

$\|\beta\|_{H^1(\Gamma)} \leq Ch^{-1} \|\beta\|_{L^2(\Gamma)}$. For any $\mu_h \in \mathcal{Q}_0(\Gamma_h)$ this implies

$$\|\mu_h\|_{L^2(\Gamma)} \leq C \frac{\int_\Gamma \mu_h \beta \, \mathrm{d}S}{\|\beta\|_{L^2(\Gamma)}} \leq Ch^{-1} \frac{\int_\Gamma \mu_h \beta \, \mathrm{d}S}{\|\beta\|_{H^1(\Gamma)}} \leq Ch^{-1} \|\mu_h\|_{H^{-1}(\Gamma)}. \tag{5.12}$$

(iii) From (5.12) we obtain by interpolation between the Sobolev spaces $L^2(\Gamma)$ and $H^{-1}(\Gamma)$

$$\|\mu_h\|_{L^2(\Gamma)} \leq Ch^{-\frac{1}{2}} \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \mu_h \in \mathcal{Q}_0(\Gamma_h). \tag{5.13}$$

Combined with (5.11) this gives the assertion of the lemma. □

The idea for the construction of the stable discrete extension operator $\mathsf{E}: \mathcal{RT}_0(\Gamma_h) \mapsto \mathcal{ND}_1(\Omega_h)$ is the following: given $\boldsymbol{\xi}_h \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, first find a divergence-free extension of $\mathrm{div}_\Gamma \boldsymbol{\xi}_h \in \mathcal{Q}_0(\Gamma_h)$ and determine a suitable vector potential. This is done in the next lemma. Then extend a div_Γ-free remainder using a discrete surface scalar potential, see the proof of Thm. 5.1 below. For the sake of brevity, we set

$$\mathcal{Q}_{0,0}(\Gamma_h) := \left\{ \mu_h \in \mathcal{Q}_0(\Gamma_h), \int_\Gamma \mu_h \, \mathrm{d}S = 0 \right\}.$$

Lemma 5.5 *There exists a discrete extension operator* $\mathsf{E}^1 : \mathcal{Q}_{0,0}(\Gamma_h) \mapsto \mathcal{ND}_1(\Omega_h)$ *such that*

$$\mathsf{T}_{\boldsymbol{n}}(\mathbf{curl}(\mathsf{E}^1 \mu_h)) = \mu_h \quad \forall \mu_h \in \mathcal{Q}_{0,0}(\Gamma_h), \tag{5.14}$$

and

$$\left\| \mathsf{E}^1 \mu_h \right\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)} \leq C \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}, \tag{5.15}$$

with $C > 0$ depending only on Ω, and the shape-regularity and quasi-uniformity of Γ_h and Ω_h.

Proof Fix an $\epsilon \in (0, \epsilon_\Omega]$ with ϵ_Ω from Lemma 5.2. Lemma 5.4 teaches that $\mu_h \in \mathcal{Q}_{0,0}(\Gamma_h)$ actually belongs to $H^\epsilon(\Gamma)$. Write $\mathbf{w} \in \boldsymbol{H}(\mathrm{div}, \Omega) \cap (H^{\frac{1}{2}+\epsilon}(\Omega))^3$ for the divergence-free extension of μ_h according to Lemma 5.2.

Then we combine the preceding lemmas,

$$\|\mathbf{w} - \Pi_{\mathrm{RT}} \mathbf{w}\|_{\mathbf{L}^2(\Omega)} \overset{\text{Lemma 5.3}}{\leq} C h^{\frac{1}{2}+\epsilon} \|\mathbf{w}\|_{\mathbf{H}^{\frac{1}{2}+\epsilon}(\Omega)} \overset{\text{Lemma 5.2}}{\leq} C h^{\frac{1}{2}+\epsilon} \|\mu_h\|_{H^\epsilon(\Gamma)}$$
$$\overset{\text{Lemma 5.4}}{\leq} C \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)},$$

which implies that

$$\|\Pi_{\mathrm{RT}} \mathbf{w}\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{w} - \Pi_{RT} \mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} \leq C \|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{5.16}$$

Here, the estimate for $\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}$ is immediate from the variational formulation of the Neumann problem (5.8).

On the other hand, since $\operatorname{div}\mathbf{w}=0$, by the commuting diagram property (3.9) for the canonical interpolation operators, we know that $\operatorname{div}\Pi_{\mathrm{RT}}\mathbf{w}=0$. Thanks to Ass. 3.1, this implies that there exists $\mathbf{v}_h \in \mathcal{ND}_1(\Omega_h)$ such that

$$\mathbf{curl}\,\mathbf{v}_h = \Pi_{\mathrm{RT}}\mathbf{w}, \quad \|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\|\Pi_{\mathrm{RT}}\mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \tag{5.17}$$

It can be confirmed easily that $\mathsf{T}_{\boldsymbol{n}}(\mathbf{curl}\,\mathbf{v}_h)=\mathsf{T}_{\boldsymbol{n}}(\Pi_{\mathrm{RT}}\mathbf{w})=\mathsf{T}_{\boldsymbol{n}}\mathbf{w}=\mu_h$ on Γ. Then, if we define $\mathsf{E}^1\mu_h := \mathbf{v}_h$, (5.15) follows from (5.16). □

From [15] recall that the norm of $\boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma)$ is defined as

$$\|\boldsymbol{\psi}\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma)} := \|\boldsymbol{\psi}\|^2_{\boldsymbol{H}_\parallel^{-\frac{1}{2}}(\Gamma)} + \|\operatorname{div}_\Gamma \boldsymbol{\psi}\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \quad \boldsymbol{\psi}\in \boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma), \tag{5.18}$$

where $\boldsymbol{H}_\parallel^{\frac{1}{2}}(\Gamma)$ is the tangential trace space of $(H^1(\Omega))^3$ with dual $\boldsymbol{H}_\parallel^{-\frac{1}{2}}(\Gamma)$.

Theorem 5.1 *There exists a discrete extension operator* $\mathsf{E}:\mathcal{RT}_0(\Gamma_h)\mapsto\mathcal{ND}_1(\Omega_h)$ *such that for any* $\boldsymbol{\xi}_h\in\mathcal{RT}_0(\Gamma_h)$ *we have*

$$\mathsf{T}_{\boldsymbol{t}}(\mathsf{E}\boldsymbol{\xi}_h)=\boldsymbol{\xi}_h \tag{5.19}$$

and

$$\left\|\mathsf{E}\boldsymbol{\xi}_h\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C_{\mathsf{E}}\,\left\|\boldsymbol{\xi}_h\right\|_{\boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma)} \tag{5.20}$$

with $C_{\mathsf{E}}>0$ *depending only on* Ω *and the shape-regularity and quasi-uniformity of the mesh.*

Proof Given $\boldsymbol{\xi}_h\in\mathcal{RT}_0(\Gamma_h)$, we know that $\operatorname{div}_\Gamma\boldsymbol{\xi}_h\in\mathcal{Q}_{0,0}(\Gamma_h)$. Then by Lemma 5.5, we have a discrete extension $\mathsf{E}^1(\operatorname{div}_\Gamma\boldsymbol{\xi}_h)\in\mathcal{ND}_1(\Omega_h)$ with

$$\left\|\mathsf{E}^1(\operatorname{div}_\Gamma\boldsymbol{\xi}_h)\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\,\left\|\operatorname{div}_\Gamma\boldsymbol{\xi}_h\right\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{5.21}$$

However, we cannot expect such an extension to preserve the boundary values as required by (5.19). A closer inspection of $\boldsymbol{\xi}_h-\mathsf{T}_{\boldsymbol{t}}(\mathsf{E}^1(\operatorname{div}_\Gamma\boldsymbol{\xi}_h))$ reveals that

$$\operatorname{div}_\Gamma(\boldsymbol{\xi}_h-\mathsf{T}_{\boldsymbol{t}}(\mathsf{E}^1(\operatorname{div}_\Gamma\boldsymbol{\xi}_h)))=\operatorname{div}_\Gamma\boldsymbol{\xi}_h-\mathsf{T}_{\boldsymbol{n}}(\mathbf{curl}\,\mathsf{E}^1(\operatorname{div}_\Gamma\boldsymbol{\xi}_h))=0,$$

which, thanks to Ass. 3.1 and the discrete exact sequence (3.19), means that there exists a function $\psi_h\in\mathcal{S}_1(\Gamma_h)$ with $\int_\Gamma\psi_h\,\mathrm{d}S=0$, such that

$$\mathbf{curl}_\Gamma\,\psi_h=\boldsymbol{\xi}_h-\mathsf{T}_{\boldsymbol{t}}(\mathsf{E}^1(\operatorname{div}_\Gamma\boldsymbol{\xi}_h)). \tag{5.22}$$

Since $\mathbf{curl}_\Gamma : \{\psi \in H^{\frac{1}{2}}(\Gamma),\ \int_\Gamma \psi \,\mathrm{d}S = 0\} \mapsto \boldsymbol{H}_\parallel^{-\frac{1}{2}}(\Gamma)$ is injective with closed range [17], we can estimate

$$\begin{aligned}
\|\psi_h\|_{H^{\frac{1}{2}}(\Gamma)} &\leq C \,\|\mathbf{curl}_\Gamma\, \psi_h\|_{\boldsymbol{H}_\parallel^{-\frac{1}{2}}(\Gamma)} = C \,\left\|\boldsymbol{\xi}_h - \mathsf{T}_{\boldsymbol{t}}(\mathsf{E}^1(\mathrm{div}_\Gamma\, \boldsymbol{\xi}_h))\right\|_{\boldsymbol{H}_\parallel^{-\frac{1}{2}}(\Gamma)} \\
&\leq C \left(\left\|\boldsymbol{\xi}_h\right\|_{\boldsymbol{H}_\parallel^{-\frac{1}{2}}(\Gamma)} + \left\|\mathsf{E}^1(\mathrm{div}_\Gamma\, \boldsymbol{\xi}_h)\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\right) \\
&\overset{(5.21)}{\leq} C \,\left\|\boldsymbol{\xi}_h\right\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} .
\end{aligned}$$

By Lemma 5.1, we have a discrete extension $\mathsf{E}^0\psi_h \in \mathcal{S}_1(\Omega_h)$ with

$$\left\|\mathsf{E}^0\psi_h\right\|_{H^1(\Omega)} \leq C \,\|\psi_h\|_{H^{\frac{1}{2}}(\Gamma)} .$$

Then we can define

$$\mathsf{E}\boldsymbol{\xi}_h := \mathsf{E}^1(\mathrm{div}_\Gamma\, \boldsymbol{\xi}_h) + \mathbf{grad}\,\mathsf{E}^0(\psi_h)$$

and it is immediate from (1.2) that E satisfies (5.19) and (5.20). □

5.2 Local extension of basis functions

Lemma 5.6 *For any $\boldsymbol{p} \in \mathcal{V}(\Gamma_h)$ the nodal basis functions $b_{\boldsymbol{p}} \in \mathcal{S}_1(\Omega_h)$ and $\beta_{\boldsymbol{p}} \in \mathcal{S}_1(\Gamma_h)$ linked by (3.16) satisfy*

$$\left\|b_{\boldsymbol{p}}\right\|_{H^1(\Gamma)} \leq C_p \left\|\beta_{\boldsymbol{p}}\right\|_{H^{\frac{1}{2}}(\Gamma)} ,$$

with a constant $C_p > 0$ that depends on the shape regularity constant C_s from (4.1) only.

Proof Write $h_{\boldsymbol{p}}$ for the largest diameter of elements abutting $\boldsymbol{p} \in \mathcal{V}(\Gamma_h)$. Thanks to shape regularity we can resort to simple scaling arguments and local inverse inequalities to confirm

$$\begin{aligned}
\left\|b_{\boldsymbol{p}}\right\|_{H^1(\Omega)} \leq C \left|b_{\boldsymbol{p}}\right|_{H^1(\Omega)} \leq C h_{\boldsymbol{p}}^{\frac{1}{2}}, \\
C \leq \left\|\beta_{\boldsymbol{p}}\right\|_{H^1(\Gamma)} \leq C h_{\boldsymbol{p}}^{-\frac{1}{2}} \left\|\beta_{\boldsymbol{p}}\right\|_{H^{\frac{1}{2}}(\Gamma)} ,
\end{aligned}$$

with $C > 0$ depending only on shape-regularity. □

Lemma 5.7 *For any edge $e \in \mathcal{E}(\Omega_h)$,which is located on the boundary, the canonical basis functions $\mathbf{b}_e$ and $\boldsymbol{\beta}_e$ of $\mathcal{ND}_1(\Omega_h)$ and $\mathcal{RT}_0(\Gamma_h)$, respectively, complying with (3.17), satisfy*

$$\|\mathbf{b}_e\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C_e \left\|\boldsymbol{\beta}_e\right\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} , \tag{5.23}$$

with $C_e > 0$ depending only on shape regularity, that is, on C_s from (4.1).

Proof By scaling argument and simple calculations, we have for any edge $e \in \mathcal{E}(\Gamma_h)$ (with length h_e)

$$\|\mathbf{b}_e\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\|\,\mathbf{curl}\,\mathbf{b}_e\|_{\mathbf{L}^2(\Omega)} \leq C h_e^{-\frac{1}{2}},$$

with $C > 0$ depending only on shape-regularity. Next, we use the inverse inequality (5.13) locally to see

$$C h_e^{-1} \leq \left\|\mathrm{div}_\Gamma\, \boldsymbol{\beta}_e\right\|_{L^2(\Gamma)} \leq C h_e^{-\frac{1}{2}} \left\|\mathrm{div}_\Gamma\, \boldsymbol{\beta}_e\right\|_{H^{-\frac{1}{2}}(\Gamma)}. \qquad \square$$

6 Proof of uniform stability

Now we are in a position to apply the abstract theory of Sect. 2 with

- $V = \mathcal{ND}_1(\Omega_h) \subset \boldsymbol{H}(\mathbf{curl}, \Omega)$, and a given by the $\boldsymbol{H}(\mathbf{curl}, \Omega)$-inner product,
- $X = \mathcal{RT}_0(\Gamma_h) \subset \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, and d agreeing with the inner product of the trace space $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$,
- T as the continuous and surjective trace operator $\mathsf{T}_{\boldsymbol{t}} : \boldsymbol{H}(\mathbf{curl}, \Omega) \rightarrow \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$,
- $\mathsf{E} : X \rightarrow V$ provided by the discrete extension operator introduced in Thm. 5.1.

Recalling (3.17), the splitting of $\mathcal{RT}_0(\Gamma_h) = \mathsf{T}_{\boldsymbol{t}}\mathcal{ND}_1(\Omega_h)$ induced by the nodal multilevel decomposition (4.3) of the edge finite element space according to (2.8) is straightforward:

$$\mathcal{RT}_0(\Gamma_h) = \underbrace{\mathcal{RT}_0(\Gamma_0)}_{=:\mathbf{X}_0} + \sum_{l=1}^{L}\Bigg\{\sum_{e\in\mathcal{E}(\Gamma_l)} \underbrace{\mathrm{Span}(\boldsymbol{\beta}_e^l)}_{=:\mathbf{X}_e^l} + \sum_{\boldsymbol{p}\in\mathcal{V}(\Gamma_l)} \underbrace{\mathrm{Span}(\mathbf{curl}_\Gamma\, \beta_{\boldsymbol{p}}^l)}_{=:\mathbf{X}_{\boldsymbol{p}}^l}\Bigg\}. \tag{6.1}$$

The splitting (4.3) is the concrete counterpart of (2.2), where the spaces $\mathbf{X}_0$, $\mathbf{X}_e^l$, and $\mathbf{X}_{\boldsymbol{p}}^l$ correspond to the X_i's of (2.8). It remains to fix the subspace extension operators E_i from Ass. **USEO**. In concrete terms, we search for extension operators $\mathsf{E}_0 : \mathbf{X}_0 \rightarrow \mathbf{V}_0$, $\mathsf{E}_e^l : \mathbf{V}_e^l \rightarrow \mathbf{X}_e^l$, and $\mathsf{E}_{\boldsymbol{p}}^l : \mathbf{X}_{\boldsymbol{p}}^l \rightarrow \mathbf{V}_{\boldsymbol{p}}^l$, where the spaces involved are defined in (4.3) and (6.1).

The operators E_e^l and $\mathsf{E}_{\boldsymbol{p}}^l$ act between one-dimensional spaces, which leaves little freedom. In light of (3.17) and (3.16) we set

$$\mathsf{E}_e^l(\alpha\boldsymbol{\beta}_e^l) := \alpha\mathbf{b}_e^l, \quad \alpha \in \mathbb{R},\; e \in \mathcal{E}(\Gamma_l), \tag{6.2}$$

$$\mathsf{E}_{\boldsymbol{p}}^l(\alpha\,\mathbf{curl}_\Gamma\, \beta_{\boldsymbol{p}}^l) := \alpha\,\mathbf{grad}\, b_{\boldsymbol{p}}^l, \quad \alpha \in \mathbb{R},\; \boldsymbol{p} \in \mathcal{V}(\Gamma_l). \tag{6.3}$$

For $\mathsf{E}_0 : \mathcal{RT}_0(\Gamma_0) \mapsto \mathcal{ND}_1(\Omega_0)$ we employ the discrete extension operator from Sect. 5.1 on the pair of coarsest meshes and denote its stability constant from (5.20) by C_{E_0}.

Theorem 6.1 *Under Ass. 3.1–4.4 the nodal multilevel decomposition (6.1) of the surface edge element space $\mathcal{RT}_0(\Gamma_h)$ is uniformly stable in the sense that there exist constants $0 < \lambda_\Gamma \leq \Lambda_\Gamma$ that depend only on Ω, the shape regularity measure C_s from (4.1) and the quasi-uniformity measure C_u from (4.2) such that*

$$\lambda_\Gamma \left\| \boldsymbol{\xi}_h \right\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \leq |||\boldsymbol{\xi}_h|||_\Gamma \leq \Lambda_\Gamma \left\| \boldsymbol{\xi}_h \right\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \quad \forall \boldsymbol{\xi}_h \in \mathcal{RT}_0(\Gamma_h),$$

where $|||\cdot|||_\Gamma$ is the multilevel norm induced by the splitting (6.1), cf. (2.10).

Proof We have to verify the assumptions of the abstract Thm. 2.2. Ass. **STO** is clear by the continuity of the trace operator $\mathsf{T}_{\boldsymbol{t}} : \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, see [17]. The extension operator $\mathsf{E} : \mathcal{RT}_0(\Gamma_h) \mapsto \mathcal{ND}_1(\Omega_h)$ enjoys the properties (USEO.1) with C_{E}, see Theorem 5.1. Similarly, the extension operator E_0 : $\mathcal{RT}_0(\Gamma_0) \mapsto \mathcal{ND}_1(\Omega_0)$ enjoys the properties (USEO.2) with C_{E_0}. Further, the simple extension operators E_e^l and $\mathsf{E}_{\boldsymbol{p}}^l$ discussed above clearly comply with (USEO.2) thanks to Lemmas 5.6 and 5.7. □

Remark 6.1 In fact, we can dispense with the simplifying assumption Ass. 3.1 on the topology of Ω with some extra technical effort. Topological obstructions can interfere with the existence of potentials. Such potentials are used twice in the construction of the discrete extention operator E.

Firstly, we need a vector potential in the proof of Lemma 5.5. Yet we notice that the Lemma will be invoked only for $\mu_h \in \mathcal{Q}_0(\Gamma_h)$ with vanishing mean on all connected components of Γ. This guarantees the existence of a vector potential for $\mathbf{w}$. Hence, Lemma 5.5 does not hinge on Ass. 3.1.

The discrete scalar surface potential ψ_h for $\boldsymbol{\zeta}_h$ occurring in the proof of Theorem 5.1 may not exist, if the first Betti number of Ω does not vanish. However, by adding a suitable weighted sum of discrete co-homology surface vector fields $\in \mathcal{RT}_0(\Gamma_0)$ on the coarsest surface mesh, we can ensure the existence of a discrete scalar potential. The discrete co-homology surface vector fields can be extended to functions $\in \mathcal{ND}_1(\Omega_0)$ in a rather arbitrary fashion, because all this is done on the coarsest level and a dependence of the constants on Ω_0 is acceptable. With this new twist, the construction of E works for general Ω and Thm. 5.1 still holds with an extra dependence of the constants on Ω_0.

Remark 6.2 We could not find a discrete extension operator E, for which stability can be proved without resorting to a global inverse inequality. Such inverse inequalities invariable entail an assumption on the quasi-uniformity of the sequence of meshes, see our Ass. 4.4. Except for the analysis of the extension operator, all other aspects of the theory developed in this article carry over to shape-regular families of *locally refined* meshes.

Remark 6.3 The technique adopted in this paper can also be used to tackle the stability of multilevel decompositions of $\mathcal{S}_1(\Gamma_h)$ and $\mathcal{Q}_0(\Gamma_h)$ in the trace spaces $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, respectively. We do not dwell on this, because the results have already been found by other means [2].

Remark 6.4 The approach of this article also works for discrete traces spaces defined on *skeletons*, that is, the union of boundaries of adjacent polyhedra, provided that counterparts of Ass. 3.1 through Ass. 4.4 are satisfied, where the skeleton mesh has to be extended to finite element volume meshes in all polyhedra.

Remark 6.5 Our theory can be extended to *screen problems* (see [14] for a profound discussion) under the following geometric assumption:

> Let Γ be an orientable two-dimensional manifold with boundary homeomorphic to a disc, for which there is a Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ such that Γ is the union of a *some* of its faces.

Then we may consider variational problems set in the space $\boldsymbol{H}_{00}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, which is the space of distributions on Γ for which extension by zero on $\partial\Omega \setminus \Gamma$ provides elements of $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \partial\Omega)$ [15, Sect. 2.3].

Via extension by zero to $\partial\Omega \setminus \Gamma$ our approach can be applied, the role of the V_i being played by spaces of edge finite element functions with vanishing tangential components on $\partial\Omega \setminus \Gamma$. For them the stability of multilevel decompositions analogous to (4.3) has been established in [37].

The remaining prerequisites of the abstract theory of Sect. 2 are available in the present screen setting, too. For instance, for the construction of a global discrete extension operator that respects zero boundary conditions we may exactly follow the policy of Sect. 5.1 after extension by zero, exploiting the fact that both face interpolation onto $\mathcal{RT}_0(\Omega_h)$ and discrete potentials respect zero boundary conditions on $\partial\Omega \setminus \Gamma$.

7 Implementation and numerical test

The parallel subspace correction preconditioner defined by (6.1) can efficiently be implemented in the spirit of multigrid methods, also called multilevel diagonal scaling in this context, see [30, Sect. 6]. We give an algebraic description close to what has to be coded actually.

Write $\mathbf{D}_l$, $l = 0, \dots, L$, for the Galerkin matrix of the inner product $d(\cdot,\cdot)$ of $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ with respect to the standard basis $\{\boldsymbol{\beta}_e^l\}_{e\in\mathcal{E}(\Gamma_l)}$ of $\mathcal{RT}_0(\Gamma_h)$. These $\sharp\mathcal{E}(\Gamma_l) \times \sharp\mathcal{E}(\Gamma_l)$-matrices will be dense, generically. Note that $\mathbf{D}_L$ is the matrix of the linear system to be solved.

Next, write $\mathbf{P}_l$, $l = 1, \dots, L$, for the so-called *prolongation matrices* of size $\sharp\mathcal{E}(\Gamma_l) \times \sharp\mathcal{E}(\Gamma_{l-1})$. Each entry corresponds to a pair of edges (e, e'), $e \in \mathcal{E}(\Gamma_l)$, $e' \in \mathcal{E}(\Gamma_{l-1})$, and is defined by the refinement relation

$$\boldsymbol{\beta}_{e'}^{l-1} = \sum_{e\in\mathcal{E}(\Gamma_l)} (\mathbf{P}_l)_{e,e'} \boldsymbol{\beta}_e^l, \quad e' \in \mathcal{E}(\Gamma_{l-1}). \tag{7.1}$$

Hence, these matrices will be sparse with at most eight non-zero entries per column.

Eventually, we need the discrete surface curl matrices $\mathbf{L}_l$ of size $\sharp\mathcal{E}(\Gamma_l) \times \sharp\mathcal{V}(\Gamma_l)$, which agree with the oriented edge-vertex incidence matrices of the mesh Γ_l [31, Sect. 3.1], and can be defined by

$$\mathbf{curl}_{\Gamma}\, \beta^l_{\boldsymbol{p}} = \sum_{e \in \mathcal{E}(\Gamma_l)} (\mathbf{L}_l)_{e,\boldsymbol{p}} \boldsymbol{\beta}^l_e, \quad \boldsymbol{p} \in \mathcal{V}(\Gamma_l). \tag{7.2}$$

They have entries $\in \{-1, 0, 1\}$ and exactly two non-zero elements per row. Note that through the formula $\mathbf{S}_l := \mathbf{L}_l^T \mathbf{D}_l \mathbf{L}_l$ we can obtain the dense matrix $\mathbf{S}_l \in \mathbb{R}^{\sharp\mathcal{V}(\Gamma_l),\sharp\mathcal{V}(\Gamma_l)}$ with entries $d(\mathbf{curl}_{\Gamma}\, \beta_{\boldsymbol{p}}, \mathbf{curl}_{\Gamma}\, \beta_{\boldsymbol{p}'})$, $\boldsymbol{p}, \boldsymbol{p}' \in \mathcal{V}(\Gamma_l)$.

Algorithm 1 Algorithmic realization of the parallel subspace correction preconditioner based on (6.1). The meanings of the matrices are explained in the text.

function $\mathsf{psc}(l \in \{1, \ldots, L\}, \mathbf{r} \in \mathbb{R}^{\sharp\mathcal{E}(\Gamma_l)})$
if $l = 0$ **then**
  return $\mathbf{D}_0^{-1}\mathbf{r}$;
else
  $\mathbf{c} \in \mathbb{R}^{\sharp\mathcal{E}(\Gamma_l)}$: $\quad c_e := \dfrac{r_e}{(\mathbf{D}_l)_{e,e}}, \quad e \in \mathcal{E}(\Gamma_l)$;
  $\boldsymbol{\rho} = \mathbf{L}_l^T \mathbf{r}$;
  $\boldsymbol{\gamma} \in \mathbb{R}^{\sharp\mathcal{V}(\Gamma_l)}$: $\quad \gamma_{\boldsymbol{p}} := \dfrac{\rho_{\boldsymbol{p}}}{(\mathbf{S}_l)_{\boldsymbol{p},\boldsymbol{p}}}, \quad \boldsymbol{p} \in \mathcal{V}(\Gamma_l)$;
  $\mathbf{c}_H := \mathsf{psc}(l-1, \mathbf{P}_l^T \mathbf{r})$;
  return $\mathbf{c} + \mathbf{L}_l \boldsymbol{\gamma} + \mathbf{P}_l \mathbf{c}_H$;
end if

The recursive implementation of the parallel subspace correction preconditioner is detailed in Alg. 1: the action of the preconditioning operator $\mathsf{M} : \boldsymbol{\mathcal{RT}}_0(\Gamma_h)' \mapsto \boldsymbol{\mathcal{RT}}_0(\Gamma_h)$ on a linear functional r passed as its coefficient vector $\mathbf{r}$ with respect to the standard dual basis of $\boldsymbol{\mathcal{RT}}_0(\Gamma_h)$ is realized as $\mathsf{M}r = \mathsf{psc}(L, \mathbf{r})$, where the result $\mathsf{M}r \in \boldsymbol{\mathcal{ND}}_1(\Omega_h)$ is returned in the form of the coefficient vector of its standard basis representation. The function psc can take the place of the preconditioning operator in a preconditioned iterative algorithm.

From Thm. 6.1 and Thm. 2.1 we immediately infer that the preconditioner will perform independently on the depth of refinement.

Corollary 7.1 *Under Ass. 4.1–4.4, the spectral condition number of the linear operator* $\mathbf{x} \mapsto \mathsf{psc}(L, \mathbf{D}_l \mathbf{x})$ *on* $\mathbb{R}^{\sharp\mathcal{E}(\Gamma_h)}$ *depends only on* Ω*, the shape-regularity measure* C_s *from* (4.1)*, and the quasi-uniformity constant* C_u *from* (4.2).

Moreover, the computational cost of executing psc except for the inversion of $\mathbf{D}_0$ is proportional to

$$\mathrm{cost}(\mathsf{psc}) \sim \sum_{l=1}^{L} \{\sharp\mathcal{E}(\Gamma_l) + \sharp\mathcal{V}(\Gamma_l)\}. \tag{7.3}$$

Due to the geometric increase of $\sharp\mathcal{E}(\Gamma_l)$ and $\sharp\mathcal{V}(\Gamma_l)$ with the level l, $\mathrm{cost}(\mathsf{psc})$ will be bounded by a small multiple of $\sharp\mathcal{E}(\Gamma_h)$. Also notice that only the diagonals of $\mathbf{D}_l$ and

Table 1 Numerical results. The matrix $\mathbf{B}_L$ stands for the preconditioning operator implemented in Alg. 1

Level L	0	1	2	3	4
$\dim \mathcal{RT}_0(\Gamma_h)$	18	72	288	1152	4608
$\kappa(\mathbf{D}_L)$	8.0406	23.7349	57.5309	221.7030	880.8022
$\kappa(\mathbf{B}_L\mathbf{D}_L)$	1	3.5299	5.8101	7.3791	8.2471

$\mathbf{S}_l$ will be needed. This implies *optimal complexity* of psc, that is, a computational effort proportional to the number of components of the argument vector $\mathbf{r}$.

The theoretical estimates are asymptotic in nature and feature unknown constants. Thus, we report a numerical experiment that allow to assess the actual performance of the multilevel preconditioner of Alg. 1. We use the equivalent definition of the $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$-inner product according to

$$d(\boldsymbol{\psi}, \boldsymbol{\mu}) := (\mathcal{V}\,\mathrm{div}_\Gamma\,\boldsymbol{\psi}, \mathrm{div}_\Gamma\,\boldsymbol{\mu})_{L^2(\Gamma)} + (\boldsymbol{\mathcal{A}}\boldsymbol{\psi}, \boldsymbol{\mu})_{L^2(\Gamma)},$$

where $\mathcal{V}$ and $\boldsymbol{\mathcal{A}}$, respectively, stand for the scalar and vectorial single layer potential integral operator belonging to $-\Delta$ in $\mathbb{R}^3$, see [32, Sect. 5–6].

We picked the cube $\Omega =]-1,1[^3$ and built the coarsest surface mesh Γ_0 by splitting each of its square faces into two triangles along the diagonal. Then, the Γ_i are obtained by splitting each triangle of Γ_{i-1} into four congruent triangles of half the size.

Table 1 lists the condition numbers without and with multilevel preconditioning for different system sizes.[3] It clearly conveys the efficacy of the preconditioner, though the condition numbers slightly increase with L, an effect, also observed in the case of multilevel preconditioners for discretized elliptic PDEs, see [8, Sect. 5, Rem. 2].

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[3] Computations done in MATLAB using dense matrices and the `cond()` command. For want of matrix compression no further refinement was possible.

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