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On Certain Kähler Quotients of Quaternionic Kähler Manifolds

V. Cortés[1,2], J. Louis[2,3], P. Smyth[4], H. Triendl[5]

[1] Department Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany. E-mail: cortes@math.uni-hamburg.de
[2] Zentrum für Mathematische Physik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany
[3] II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany. E-mail: jan.louis@desy.de
[4] Institut de Théorie des Phénomènes Physiques, EPFL, 1015 Lausanne, Switzerland. E-mail: paul.smyth@epfl.ch
[5] Institut de Physique Théorique, CEA Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette, France. E-mail: hagen.triendl@cea.fr

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Abstract: We prove that, given a certain isometric action of a two-dimensional Abelian group A on a quaternionic Kähler manifold M which preserves a submanifold $N \subset M$, the quotient $M' = N/A$ has a natural Kähler structure. We verify that the assumptions on the group action and on the submanifold $N \subset M$ are satisfied for a large class of examples obtained from the supergravity c-map. In particular, we find that all quaternionic Kähler manifolds M in the image of the c-map admit an integrable complex structure compatible with the quaternionic structure, such that $N \subset M$ is a complex submanifold. Finally, we discuss how the existence of the Kähler structure on M' is required by the consistency of spontaneous $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry breaking.

Introduction

Since the work of Galicki and Lawson [GL] it has been known that a quaternionic analogue of the well-known symplectic reduction exists. In fact, as shown in [ACDV], any isometric action of a Lie group G on a quaternionic Kähler manifold (M, g, Q) of nonzero scalar curvature gives rise to a $\mathfrak{g}^*$-valued section $P \in \Gamma(Q \otimes \mathfrak{g}^*)$ of the quaternionic structure $Q \subset \mathrm{End}\,(TM)$. P is called the moment map and by taking the quotient $\{P = 0\}/G$ one obtains a new quaternionic Kähler manifold, provided that the usual regularity assumptions are fulfilled.

In this paper, however, we are interested in constructing Kähler manifolds out of quaternionic Kähler manifolds. Such a procedure is needed in order to break supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in supersymmetric theories of gravity in four spacetime dimensions [FGP,L,LST1,LST2]. The reason is that quaternionic Kähler manifolds of negative scalar curvature occur as scalar manifolds of $\mathcal{N} = 2$ supergravity, whereas $\mathcal{N} = 1$ supergravity requires the scalar manifold to be Kähler. A natural but rather restrictive way to relate quaternionic Kähler manifolds to Kähler manifolds of lower dimension is to consider *Kähler submanifolds* $(N, g_N, J_N) \subset (M, g, Q)$, such that $g_N = g|_N$ and

$J_N \in \Gamma(Q|_N)$. It is shown in [AM] that the dimension of such a submanifold cannot exceed $2n$ if M has nonzero scalar curvature, where $\dim M = 4n$.

Our new idea is to drop the Kähler condition on (N, g_N, J_N) still maintaining the integrability of $J_N \in \Gamma(Q|_N)$. The final Kähler manifold M' is then obtained as an appropriate quotient $M' = N/A$ of N. To define the quotient we make use of two commuting Killing vector fields ξ_1, ξ_2, which generate a free proper isometric action of a two-dimensional Lie group A. The necessary technical assumptions on ξ_1, ξ_2 for our construction are formulated in terms of the corresponding moment maps $P_1, P_2 \in \Gamma(Q)$, see Theorem 5 and Corollary 1. The main result can be summarized as follows.

Theorem 1. *Let M be a quaternionic Kähler manifold of nonzero scalar curvature, $N \subset M$ a submanifold and ξ_1, ξ_2 Killing vector fields of M which satisfy the assumptions of Theorem 5 and Corollary 1. Then $M' = N/A$ carries an induced Kähler structure, where A is the transformation group generated by ξ_1, ξ_2.*

The main body of the article is devoted to the investigation of several classes of examples. As a first and simplest example we take $N = M = H^4_{\mathbb{R}} = H^1_{\mathbb{H}}$ the real hyperbolic four-space (which coincides with the quaternionic hyperbolic line) and obtain $M' = H^1_{\mathbb{C}}$. Then we study the quaternionic Kähler manifolds (M, g, Q) in the image of the c-map [CFG,FS]. These manifolds have negative scalar curvature and are associated with a (projective) special Kähler domain M_{sk}, the geometry of which can be encoded in a holomorphic prepotential $F(Z) = F(Z^1, \ldots, Z^n)$. As a first step in the study of the c-map examples we obtain the following general result, see Proposition 1 and 2.

Theorem 2. *Let (M, g, Q) be a quaternionic Kähler manifold in the image of the c-map. Then the quaternionic structure Q of M admits a global orthonormal frame (J_1, J_2, J_3) such that the almost complex structure $J_3 \in \Gamma(Q)$ is integrable. (M, J_3) is the total space of a holomorphic submersion $M \to M_{\mathrm{sk}}$ with all fibers biholomorphic to the domain $\mathbb{R}^{n+1} + iV \subset \mathbb{C}^{n+1}$, where* $\dim M = 4n$ *and*

$$V = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0 > \sum_{i=1}^{n-1} x_i^2 - x_n^2\}. \tag{0.1}$$

This should be contrasted with the situation for complete quaternionic Kähler manifolds of positive scalar curvature, which do not even admit an *almost* complex structure compatible[1] with the quaternionic structure [AMP]. Some interesting properties of the complex structure J_3 are described in Proposition 1 and Proposition 2. We then define a complex submanifold

$$N \subset (M, J_3),$$

see Proposition 4, associated with a choice of a null vector $v_0 \in TM_{\mathrm{ask}}$, where $M_{\mathrm{ask}} \to M_{\mathrm{sk}}$ is the affine special Kähler manifold associated with M_{sk}. The complex codimension of $N \subset M$ is $r + 1$, where r is the rank of a certain complex matrix (G_{AB}), which depends on the choice of v_0, see Eq. (3.16)[2] and the remark following it. The structure

[1] Note that the complex Grassmannians $Gr_2(\mathbb{C}^n)$ $(n \geq 3)$ do admit a complex structure which is even Kähler for the quaternionic Kähler metric but it does not belong to the quaternionic structure. It is known that these complex Grassmannians are the only complete quaternionic Kähler manifolds of positive scalar curvature which admit an almost complex structure [GMS].

[2] Note that $v_0 = \sum D^A \partial/\partial Z^A|_{Z_0} +$ c.c.

of the complex manifold N is described in Proposition 5. In particular, we find that N is always the total space of a holomorphic submersion

$$N \to M_{\mathrm{sk}}^{\wedge},$$

where $M_{\mathrm{sk}}^{\wedge} \subset M_{\mathrm{sk}}$ is a complex submanifold and the fibers are biholomorphic to $B_{\mathbb{C}}^{n-1} \times \mathbb{C}$. The inclusion $N \subset M$ maps the fibers of $N \to M_{\mathrm{sk}}^{\wedge}$ into the fibers of $M \to M_{\mathrm{sk}}$. Next we define two Killing vector fields ξ_1, ξ_2 on M, which depend on the choice of v_0. We show in Proposition 6 that they are tangent to $N \subset M$ and generate a holomorphic, free and proper action of the additive group $A = \mathbb{C}$ on N. We then have the following result, cf. Theorem 6.

Theorem 3. *The resulting quotient $M' = N/A$ is always the total space of a holomorphic submersion*

$$M' \to M_{\mathrm{sk}}^{\wedge},$$

where the fibers are isomorphic to the complex ball $B_{\mathbb{C}}^{n-1} \cong \mathbb{C}H^{n-1}$ with its standard complex hyperbolic metric of constant holomorphic sectional curvature -4. The projection $N \to M' = N/A$ maps the fibers of $N \to M_{\mathrm{sk}}^{\wedge}$ to the fibers of $M' \to M_{\mathrm{sk}}^{\wedge}$.

We also show that M' is complete if and only if the base manifold $M_{\mathrm{sk}}^{\wedge}$ is complete, see the Remark before Subsect. 3.2.1. Let us emphasize a subtle but crucial point in the construction. The fibers $M_p = \pi^{-1}(p)$ of $\pi : (M, g, J_3) \to M_{\mathrm{sk}}$ consist of a solvable Lie group G endowed with a family of left-invariant metrics $g_G(p)$ and left-invariant skew-symmetric complex structures $J_G(p)$:

$$(M_p, g|_{M_p}, J_3|_{M_p}) = (G, g_G(p), J_G(p)), \quad p \in M_{\mathrm{sk}}.$$

The group G is precisely the Iwasawa subgroup of $\mathrm{SU}(1, n+1)$, which is the group of holomorphic isometries of the complex hyperbolic space $\mathbb{C}H^{n+1} = \mathrm{U}(1, n+1)/(\mathrm{U}(1) \times \mathrm{U}(n+1)) = \mathrm{SU}(1, n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n+1))$. Since G acts simply transitively on $\mathbb{C}H^{n+1}$, we can identify the Kähler manifold $\mathbb{C}H^{n+1}$ with (G, g_{can}, J_{can}), where (g_{can}, J_{can}) is a left-invariant Kähler structure on G:

$$\mathbb{C}H^{n+1} = (G, g_{can}, J_{can}).$$

From the Riemannian point of view, the fibers $(M_p, g|_{M_p}) = (G, g_G(p))$ are as nice as possible. They are all isometric to $(G, g_{can}) \cong \mathbb{C}H^{n+1}$, although the metric $g_G(p)$ is never independent of $p \in M_{\mathrm{sk}}$. However, in view of the above discussion, the Hermitian manifold $(G, g_G(p), J_G(p))$ cannot be Kähler, since $2n + 2 = \dim G > \frac{1}{2} \dim M = 2n$. This means that $J_G(p)$ does not coincide with the canonical (parallel) complex structure $J_{can}(p)$ on $(G, g_G(p))H^{n+1}$, for which $(G, g_G(p), J_{can}(p)) \cong \mathbb{C}H^{n+1} = (G, g_{can}, J_{can})$. One can show that $(G, J_G(p))$ is not even biholomorphic[3] to $\mathbb{C}H^{n+1}$. This is related to the non-positivity of the quadratic form on the right-hand side of the inequality defining the complex domain $\mathbb{R}^{n+1} + iV \subset \mathbb{C}^{n+1} \cong (G, J_G(p))$, see (0.1). Summarizing, we have that

$$(G, g_G(p)) \cong (G, g_{can}) \cong \mathbb{C}H^{n+1} \quad \text{but} \quad (G, J_G(p)) \ncong (G, J_{can}) \cong \mathbb{C}H^{n+1}.$$

[3] A proof of this fact can be found in [CH], which includes the classification of skew-symmetric left-invariant complex structures on $(G, g_{can}) = \mathbb{C}H^{n+1}$.

It turns out that when passing to the quotient M', the fibers M'_p, $p \in M^\wedge_{sk}$, become all isometric *and* biholomorphic to $\mathbb{C}H^{n-1}$. In fact, we show that by considering the submanifold $N_p := N \cap M_p \subset M_p$, which is biholomorphic to $\mathbb{C}H^{n-1} \times \mathbb{C}$, and its quotient $M'_p = N_p/A$ we reduce the domain $M_p \cong \mathbb{R}^{n+1} + iV \ncong \mathbb{C}H^{n+1}$ to $M'_p \cong (\mathbb{R}^{n+1} + iV) \cap \mathbb{C}^{n-1} = \mathbb{R}^{n-1} + iV'$, where now

$$V' = \{(x_0, x_1, \ldots, x_{n-2}) \in \mathbb{R}^{n-1} | x_0 > \sum_{i=1}^{n-2} x_i^2\}$$

is defined by a positive definite quadratic form. Therefore, $\mathbb{R}^{n-1} + iV'$ is biholomorphic to $\mathbb{C}H^{n-1}$.

More detailed information is obtained in Sect. 3.2.1 and Sect. 3.2.2 when the prepotential is either quadratic or of the form $F = \frac{h(Z^1,\ldots,Z^{n-1})}{Z^n}$, where h is a homogeneous cubic polynomial with real coefficients. As usual in the physics literature, the latter class will be simply referred to as having *cubic prepotential*. It is particularly interesting for string theory compactifications and contains a wealth of homogeneous as well as inhomogeneous examples. We show that in the case of the cubic prepotential the dimension of M' can be as large as $\dim M - 8$ with $\dim M$ arbitrarily big. In the case of the quadratic prepotential the structure of M' is completely determined as follows, cf. Theorem 7.

Theorem 4. *The Kähler manifolds M' obtained from the above quotient construction applied to the quaternionic Kähler manifold $M = \frac{\mathrm{U}(2,n)}{\mathrm{U}(2)\times\mathrm{U}(n)} \to M_{sk} = H^{n-1}_{\mathbb{C}}$ are always isomorphic to $H^{n-1}_{\mathbb{C}} \times H^{n-1}_{\mathbb{C}}$, provided that $M^\wedge_{sk} \subset M_{sk}$ is complete. In this case, the holomorphic submersion $M' \to M^\wedge_{sk}$ is trivial and $M^\wedge_{sk} = M_{sk}$.*

So in this case, $\dim M' = \dim M - 4$.

The mathematical results obtained in this paper are motivated by the consistency of spontaneous $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry breaking [FGP,L,LST1,LST2] and in Sect. 4 we briefly discuss this relation. Quaternionic Kähler manifolds appear naturally in $\mathcal{N} = 2$ supergravity theories as part of the scalar field space. The Higgs mechanism responsible for the supersymmetry breaking requires two massive vector fields coupled to two Killing vector fields that fulfill the assumptions of Theorem 5. Furthermore, an $\mathcal{N} = 1$ effective action can be defined below the scale of supersymmetry breaking and is obtained by integrating out all massive degrees of freedom. Integrating out massive scalars corresponds to taking a submanifold $N \subset M$, while integrating out two massive vector fields corresponds to taking the quotient with respect to the two-dimensional Abelian Lie group A generated by the two Killing vectors, as specified in Theorem 1. Consistency with $\mathcal{N} = 1$ supersymmetry implies that the resulting scalar field space $M' = N/A$ should be Kähler.

1. Basic Results about Quaternionic Kähler Manifolds

In this section we recall some known facts about quaternionic Kähler manifolds, see e.g. [ACDV] for more details.

Definition 1. *A* **`quaternionic Kähler manifold`** *(M, g, Q) is a Riemannian manifold (M, g) which is endowed with a parallel skew-symmetric quaternionic structure $Q \subset \mathrm{End}\, TM$. If $\dim M = 4$ we require, in addition, that $Q \cdot R = 0$. (This condition is automatically satisfied if $\dim M > 4$.)*

Let $(J_\alpha)_{\alpha=1,2,3}$ be an orthonormal local frame of Q such that $J_3 = J_1 J_2$. Then

$$\nabla J_\alpha = -(\omega_\beta \otimes J_\gamma - \omega_\gamma \otimes J_\beta), \tag{1.1}$$

for some triplet of connection forms ω_α, where (α, β, γ) is always a cyclic permutation of $(1, 2, 3)$. These one-forms are related to the fundamental two-forms $\varphi_\alpha = g(\cdot, J_\alpha)$ by the following structure equations:

$$\nu \varphi_\alpha = d\omega_\alpha + \omega_\beta \wedge \omega_\gamma, \tag{1.2}$$

where $\nu = \frac{scal}{4n(n+2)}$ stands for the reduced scalar curvature, the quotient of the scalar curvature of (M, g) by that of $\mathbb{H}P^n$, with $4n = \dim M$. Quaternionic Kähler manifolds are Einstein; in particular, ν is a constant.

Now let ξ be a Killing vector field on a quaternionic Kähler manifold of nonzero scalar curvature, i.e. $\nu \neq 0$. Then Q is invariant under the flow of ξ, as well as under parallel transport. This implies that the endomorphism field $\nabla\xi$ is a section of the normaliser

$$N(Q) = Q \oplus Z(Q)$$

of Q in $\mathfrak{so}(TM)$. Here $Z(Q)$ stands for the centraliser of Q. Note that

$$N(Q)_p \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \quad \forall p \in M,$$

where $\mathfrak{sp}(n)$ is the Lie algebra of the compact symplectic group $\mathrm{Sp}(n)$, which is usually denoted by $\mathrm{USp}(2n)$ in the physics literature. Let us use

$$P := (\nabla\xi)^Q \in \Gamma(Q) \tag{1.3}$$

to denote the projection of $\nabla\xi$ onto Q. The section $P : M \to Q$ is called the `moment map` associated with ξ. Its covariant derivative is given by:

$$\nabla P = \frac{\nu}{2} \sum \varphi_\alpha(\cdot, \xi) \otimes J_\alpha. \tag{1.4}$$

For the last formula, see [ACDV] Prop. 2.

2. The New Quotient Construction

Theorem 5. *Let (M, g, Q) be a quaternionic Kähler manifold of nonzero scalar curvature, ξ_1, ξ_2 two Killing vector fields with corresponding moment maps $P_i \in \Gamma(Q)$, $i = 1, 2$, $N \subset M$ a submanifold and $\mathfrak{X}(N)$ the space of smooth vector fields on N such that:*

(i) $\xi_1|_N, \xi_2|_N \in \mathfrak{X}(N)$, $[\xi_1, \xi_2]|_N = 0$ *and* $|\xi_1| = |\xi_2| \neq 0$ *on* N,
(ii) $P_1 P_2|_N$ *is a section of* $Q|_N$ *which preserves* TN *and maps* $\xi_1|_N$ *to* $f\xi_2|_N$, *where* $f \in C^\infty(N)$ *is some nowhere vanishing function.*

Then the integrable distribution $\mathcal{D} \subset TN$ spanned by $\xi_1|_N, \xi_2|_N$ has an induced transversal Kähler structure (h, J). The complex structure J is induced by $I := \frac{1}{f} P_1 P_2|_N \in \Gamma(N, Q)$, which defines an integrable complex structure on N.

Remarks. 1) We will show below that for the quaternionic Kähler manifolds (M, g, Q) in the image of the c-map there exists a global orthonormal frame (J_1, J_2, J_3) of Q such that the almost complex structure J_3 is integrable. The above construction will then be applied to an appropriate complex submanifold N of (M, J_3).

2) The quaternionic Kähler manifolds in the image of the c-map include all the known homogeneous quaternionic Kähler manifolds of negative scalar curvature with the exception of the quaternionic hyperbolic spaces $H^n_{\mathbb{H}}$, $n \geq 2$.

Proof. Let us use $\mathcal{N} = \mathcal{D}^\perp \cong TN/\mathcal{D}$ to denote the Riemannian normal bundle of $\mathcal{D}$ in N. We then define the transversal metric $h \in \Gamma(S^2\mathcal{N}^*)$ as the restriction

$$h := g|_{\mathcal{N}\times\mathcal{N}}.$$

It follows from (i–ii) that $I := \frac{1}{f} P_1 P_2 \in \Gamma(N, Q)$ is an almost complex structure. We can choose an orthonormal local frame $(J_\alpha)_{\alpha=1,2,3}$ of Q such that $J_3 = I$ on N and $J_3 = J_1 J_2$. Since I preserves $\mathcal{D} \subset TN$, TN and therefore $\mathcal{N} = \mathcal{D}^\perp \subset TN$, we can define

$$J := I|_{\mathcal{N}} \in \Gamma(N, \text{End } \mathcal{N}).$$

Clearly, J_p is a skew-symmetric complex structure on the Euclidian vector space $(\mathcal{N}_p, h_p)$, for all $p \in M$. We claim that (h, J) defines a transversal Kähler structure for the foliation of N defined by the integral surfaces of the distribution $\mathcal{D}$. This means that (h, J) induces a Kähler structure on any submanifold $S \subset N$ transversal to $\mathcal{D}$ and that the Kähler structures on a pair of such submanifolds $S, S' \subset M$ intersecting the same leaves are related by the corresponding holonomy transformation of the foliation. To prove this it suffices to check that

$$\mathcal{L}_{\xi_i} g = 0, \tag{2.1}$$
$$\mathcal{L}_{\xi_i} I = 0, \tag{2.2}$$
$$[X, Y] \in \Gamma(T_I^{1,0}N), \quad \text{for all} \quad X, Y \in \Gamma(T_I^{1,0}N), \tag{2.3}$$
$$d\tilde{\varphi} = 0, \tag{2.4}$$

where $\tilde{\varphi}$ is the pull back of the fundamental form $\varphi = h(\cdot, J\cdot)$ to a two-form on N. Explicitly,

$$\tilde{\varphi}|_{\mathcal{D}\wedge TN} = 0, \quad \tilde{\varphi}|_{\wedge^2\mathcal{N}} = \varphi.$$

Equation (2.1) holds because the ξ_i are Killing fields. The Lie derivative $\mathcal{L}_{\xi_i} I$ of $I \in \Gamma(N, Q)$ is again a section of $Q|_N$, since any isometry of a quaternionic Kähler manifold of nonzero scalar curvature preserves the quaternionic structure Q. In order to prove (2.2), it thus suffices to check that:

$$(\mathcal{L}_{\xi_i} I)\xi_1 = \mathcal{L}_{\xi_i}(I\xi_1) - I\mathcal{L}_{\xi_i}\xi_1 = \mathcal{L}_{\xi_i}(\xi_2) = 0.$$

Equation (2.2) implies that on N we have

$$P_i = (\nabla\xi_i)^Q = (\nabla_{\xi_i} - \mathcal{L}_{\xi_i})^Q \equiv \frac{1}{2}\omega_1(\xi_i)J_1 + \frac{1}{2}\omega_2(\xi_i)J_2 \pmod{\mathbb{R}J_3}.$$

Combining this with the equation $P_1P_2|_N = fJ_3|_N$ we obtain that, on N,

$$P_i = \frac{1}{2}\omega_1(\xi_i)J_1 + \frac{1}{2}\omega_2(\xi_i)J_2, \tag{2.5}$$

where the vectors $v_i := (\omega_1(\xi_i), \omega_2(\xi_i)) \in \mathbb{R}^2$ satisfy

$$v_1 \perp v_2, \quad |v_1||v_2| = 4f \tag{2.6}$$

on N. This shows that the one-forms ω_1, ω_2 are pointwise linearly independent on N. In fact, their restrictions to $\mathcal{D}$ are linearly independent. Hence,

$$\mathcal{K} := \ker \omega_1|_N \cap \ker \omega_2|_N \subset TN$$

is a distribution complementary to $\mathcal{D}$. We will now show that $\mathcal{K} = \mathcal{N}$. To see this, we calculate the covariant derivative of P_i on N:

$$\begin{aligned}\nabla P_i &= \frac{1}{2}(\nabla\omega_1(\xi_i)) \otimes J_1 + \frac{1}{2}(\nabla\omega_2(\xi_i)) \otimes J_2 \\ &\quad -\frac{1}{2}\omega_1(\xi_i)(\omega_2 \otimes J_3 - \omega_3 \otimes J_2) - \frac{1}{2}\omega_2(\xi_i)(\omega_3 \otimes J_1 - \omega_1 \otimes J_3) \\ &\equiv -\frac{1}{2}\,(\omega_1(\xi_i)\omega_2 - \omega_2(\xi_i)\omega_1) \otimes J_3 \pmod{T^*M \otimes (\mathbb{R}J_1 \oplus \mathbb{R}J_2)}.\end{aligned}$$

Comparing with (1.4), we obtain

$$\nu\varphi_3(\xi_i, \cdot) = \omega_1(\xi_i)\omega_2 - \omega_2(\xi_i)\omega_1$$

along N. This implies that $\mathcal{K} = \mathcal{N}$. It also shows that the two-form $\nu\varphi_3 - \omega_1 \wedge \omega_2$ vanishes on $\mathcal{D} \wedge TM$ along N and coincides with $\nu\varphi_3$ on $\mathcal{N}$. This means that

$$\nu\tilde{\varphi} = (\nu\varphi_3 - \omega_1 \wedge \omega_2)|_N \overset{(1.2)}{=} d\omega_3|_N,$$

proving (2.4). It remains to check the integrability condition (2.3), which shows that I defines a complex structure on N. Equation (1.1) implies that $\nabla_X I = 0$, for all $X \in \mathcal{N} = \mathcal{K}$. Using the symmetry of the Levi-Civita connection and the fact that $I\mathcal{N} = \mathcal{N}$, we can easily check that

$$\begin{aligned}[X - iIX, Y - iIY] &= [X, Y] - [IX, IY] - i([X, IY] + [IX, Y]) \\ &= [X, Y] - [IX, IY] - iI([X, Y] - [IX, IY]),\end{aligned}$$

for all $X, Y \in \Gamma(\mathcal{N})$. It remains to calculate $[\xi_1 - i\xi_2, Y - iIY]$ for any $Y \in \mathfrak{X}(N)$:

$$[\xi_1 - i\xi_2, Y - iIY] \overset{(2.2)}{=} [\xi_1 - i\xi_2, Y] - iI[\xi_1 - i\xi_2, Y] \in T_I^{1,0}M.$$

This proves (2.3). □

Corollary 1. *If, in addition to the assumptions of Theorem 5, the vector fields $\xi_1|_N, \xi_2|_N$ generate a free and proper action of a two-dimensional Abelian Lie group A on the submanifold $N \subset M$, then the quotient $M' := N/A$ is a smooth manifold, which inherits a Kähler structure (h, J) from the transversal geometry of the integrable distribution $\mathcal{D}$. The projection $(N, g) \to (M', h)$ is a Riemannian submersion and a principal fiber bundle with structure group A. Moreover, $(N, I) \to (M', J)$ is holomorphic, where $I \in \Gamma(N, Q)$ is the (integrable) almost complex structure which maps $\xi_1|_N$ to $\xi_2|_N$. If, more generally, the proper action of A is only locally free with finite stabilisers, then (M', h, J) is a Kähler orbifold.*

3. Examples

3.1. Hyperbolic 4-space. As a first example, let us consider the four-dimensional hyperbolic space

$$M = H^4_{\mathbb{R}} = \frac{\mathrm{SO}_0(1,4)}{\mathrm{SO}(4)}.$$

The solvable Iwasawa subgroup L of $\mathrm{SO}_0(1,4) = \mathrm{Isom}_0(M)$ acts simply transitively on M and we can identify M with the group manifold L endowed with a left-invariant metric g of constant curvature -1. (M, g) is a quaternionic Kähler manifold with the quaternionic structure Q spanned by three left-invariant complex structures $J_\alpha, \alpha = 1, 2, 3$. The Lie algebra

$$\mathfrak{l} := \mathrm{Lie}\, L = \mathfrak{a} + \mathfrak{n}$$

is the orthogonal sum of a three-dimensional Abelian nilradical $\mathfrak{n} = \mathrm{span}\{X_\alpha = J_\alpha X_0 | \alpha = 1, 2, 3\}$ and a one-dimensional subalgebra $\mathfrak{a} = \mathbb{R}X_0$, where X_0 is a unit vector such that $ad_{X_0}|_{\mathfrak{n}} = \mathrm{Id}$. Decomposing the Levi-Civita connection,

$$\nabla_X Y = \frac{1}{2}\sum \omega_\alpha(X) J_\alpha Y + \bar{\nabla}_X Y, \quad X, Y \in \mathfrak{l},$$

such that $\bar{\nabla}_X J_\alpha = 0$, one can easily compute $\omega_\alpha = -X^*_\alpha$, where (X^*_a) is the dual basis of $\mathfrak{l}^*$.

Let us use $k_a, a = 0, 1, 2, 3$, to denote the (right-invariant) Killing vector field which coincides with the left-invariant vector field X_a at $e \in L$. A straightforward calculation shows that

$$k_0(p) = X_0(p) - \frac{e^{-x^0} - 1}{x^0}\sum x^\alpha X_\alpha(p),$$
$$k_\alpha = e^{-x^0} X_\alpha,$$

at $p = \exp(x) \in L = M$, where $x = \sum x^\alpha X_\alpha$. This allows us to compute the moment maps P_α of the three commuting Killing vector fields κ_α:

$$P_1 = (\nabla k_1)^Q = -(\mathcal{L}_{k_1})^Q + \nabla^Q_{k_1} = \frac{1}{2}\sum \omega_\alpha(k_1) J_1 = -\frac{1}{2} e^{-x^0} J_1,$$

since $(\mathcal{L}_k)^Q = 0$ for any right-invariant Killing vector field k and $\omega_\alpha = -X^*_\alpha$. Summarising, we have shown that

$$P_\alpha = -\frac{1}{2} e^{-x^0} J_\alpha, \tag{3.1}$$

in accordance with [FGP]. Thus we have

$$P_1 P_2 k_1 = f k_2, \quad 4f = |k_1|^2 = |k_2|^2 = e^{-2x^0} > 0, \tag{3.2}$$

and we can choose $\xi_i = k_i, i = 1, 2$, in agreement with conditions $(i–ii)$ in Theorem 5. The Killing vector fields k_1, k_2 generate the left-action of the normal subgroup $A_2 = \exp \mathfrak{a}_2 \subset L$, $\mathfrak{a}_2 = \mathrm{span}\{X_1, X_2\}$. Therefore, we can apply Theorem 5 and Corollary 1 to $N = M$. The quotient $M' = M/A_2$ is the complex hyperbolic line $M' \cong H^1_{\mathbb{C}}$, which again has constant curvature -1 and admits the simply transitive solvable group L/A_2 of holomorphic isometries.

3.2. Quaternionic Kähler manifolds in the image of the c-map. There is a class of quaternionic Kähler manifolds of negative scalar curvature of the form $M = M_{\rm sk} \times G$, where $M_{\rm sk}$ is a (projective) special Kähler manifold of dimension $2n-2$ and G is the solvable Iwasawa subgroup of $\mathrm{SU}(1, n+1)$, which is a semidirect product of a $(2n+1)$-dimensional Heisenberg group with $\mathbb{R}$. For simplicity we will assume from now on that $M_{\rm sk}$ admits a global system of special coordinates. Such manifolds are called (projective) `special Kähler domains`. Note that the quaternionic Kähler metric on M cannot be a product metric, since quaternionic Kähler manifolds are irreducible. The construction of these manifolds out of the special Kähler base is called the (supergravity) c-map [CFG, FS]. It has been recently shown that the quaternionic Kähler manifold M is complete if $M_{\rm sk}$ is complete [CMX]. As we will show, the class of quaternionic Kähler manifolds in the image of the c-map gives numerous examples for the quotient construction introduced in Theorem 5.

In the following we will briefly describe the construction of the c-map, see [CFG, FS, CMX] for more detailed information. Any (projective) special Kähler manifold $M_{\rm sk}$ can be realised as the base of a holomorphic $\mathbb{C}^*$-principal bundle $M_{\rm ask} \to M_{\rm sk}$. The total space $M_{\rm ask}$ has the structure of an affine special Kähler manifold, which admits special holomorphic local coordinates Z^A, $A = 1, \ldots, n$, such that the geometric data of $M_{\rm ask}$ are encoded in a holomorphic function $F(Z^1, \ldots, Z^n)$ called the `holomorphic prepotential`.[4] The functions $z^a = Z^a/Z^n$, $a = 1, \ldots, n-1$, induce local coordinates on $M_{\rm sk}$ and the Kähler potential $\mathcal{K}(z)$ of $M_{\rm sk}$ can be explicitly expressed in terms of the prepotential F. In fact, $\mathcal{K}(z) = K(z, 1)$, where

$$K(Z) = -\ln(2Z^A N_{AB} \bar{Z}^B), \quad N_{AB} = \mathrm{Im}\, F_{AB}, \tag{3.3}$$

where the subscripts on F denote derivatives with respect to Z^A e.g. $F_A = \partial F/\partial Z^A$. The solvable Lie group G admits a natural system of global coordinates[5] $(\phi, \tilde{\phi}, a^A, b_A)$, $A = 1, \ldots, n$. A basis for the right-invariant vector fields on G is given in these coordinates by

$$\begin{aligned}
k_\phi &= \frac{1}{2}\frac{\partial}{\partial\phi} - \tilde{\phi}\frac{\partial}{\partial\tilde{\phi}} - \frac{1}{2}a^A\frac{\partial}{\partial a^A} - \frac{1}{2}b_A\frac{\partial}{\partial b_A},\\
k_{\tilde{\phi}} &= -2\frac{\partial}{\partial\tilde{\phi}},\\
k_A &= \frac{\partial}{\partial a^A} + b_A\frac{\partial}{\partial\tilde{\phi}},\\
\tilde{k}^A &= \frac{\partial}{\partial b_A} - a^A\frac{\partial}{\partial\tilde{\phi}}.
\end{aligned} \tag{3.4}$$

These vector fields obey the commutation relations

$$\begin{aligned}
[k_\phi, k_{\tilde{\phi}}] &= k_{\tilde{\phi}}, & [k_\phi, k_A] &= \frac{1}{2}k_A,\\
[k_\phi, \tilde{k}^A] &= \frac{1}{2}\tilde{k}^A, & [k_A, \tilde{k}^B] &= \delta_A^B k_{\tilde{\phi}},
\end{aligned} \tag{3.5}$$

with all other commutators vanishing.

[4] Readers familiar with the supergravity literature might prefer to label the coordinates by $I = 0, 1, \ldots, n-1$, as is done from Sect. 3.2.2 onwards.

[5] In supergravity theories arising as effective theories of type II compactifications the scalar manifold $M_{\rm sk}$ is spanned by deformations of the metric and the Neveu-Schwarz B-field, while $(\phi, \tilde{\phi}, a^A, b_A)$ correspond to the dilaton, the axion and the 2n real Ramond-Ramond scalars, respectively.

Recall that a `quaternionic vielbein` on a quaternionic Kähler manifold (or, more generally, on an almost quaternionic Hermitian manifold) (M, g, Q) is a coframe which belongs to the $\mathrm{Sp}(n)\mathrm{Sp}(1)$-structure defined by (g, Q), cf. [ACDGV]. More explicitly, it is a system of complex-valued one-forms $\mathcal{U}^{\mathcal{A}m}$, $\mathcal{A} = 1, 2$, $m = 1, \dots, 2n$, such that the metric takes the form

$$g = \sum \epsilon_{\mathcal{A}\mathcal{B}}\epsilon_{lm}\mathcal{U}^{\mathcal{A}l} \otimes \mathcal{U}^{\mathcal{B}m}, \tag{3.6}$$

and such that the quaternionic structure Q on TM corresponds to the standard quaternionic structure on the first factor $\mathbb{C}^2$ of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$. Here $\epsilon = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$. Note that the metric and quaternionic structure are completely determined by specifying a quaternionic vielbein.

In [FS] it was proven that

$$\mathcal{U}^{\mathcal{A}m} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{u} & \bar{e} & -v & -E \\ \bar{v} & \bar{E} & u & e \end{pmatrix} \tag{3.7}$$

is a quaternionic vielbein of a quaternionic Kähler structure (g, Q) on a domain M if the one-forms $\mathcal{U}^{\mathcal{A}m}$ are defined by

$$\begin{aligned} u &= \mathrm{i}\, e^{K/2+\phi} Z^A (\mathrm{d}b_A - F_{AB}\mathrm{d}a^B), \\ v &= \frac{1}{2} e^{2\phi}\big[\mathrm{d}e^{-2\phi} - \mathrm{i}(\mathrm{d}\tilde{\phi} + b_A \mathrm{d}a^A - a^A \mathrm{d}b_A)\big], \\ E^b &= -\frac{\mathrm{i}}{2} e^{\phi - K/2} \Pi_A{}^b N^{AB} (\mathrm{d}b_B - \bar{F}_{BC}\mathrm{d}a^C), \\ e^b &= \Pi_A{}^b \mathrm{d}Z^A = e_a^b dz^a. \end{aligned} \tag{3.8}$$

Here (Z^A), $A = 1, \dots, n$, are the homogeneous coordinates of M_{sk}, which are functions on the affine special Kähler domain M_{ask},

$$\Pi_A{}^b = (e_a{}^b, -z^a e_a{}^b)$$

is defined using the vielbein $e_a{}^b$ on M_{sk}. In the above formulas one may simply put $(Z^A) = (z^a, 1)$ to obtain differential forms which are manifestly defined on $M = M_{\mathrm{sk}} \times G$, rather than horizontal $\mathbb{C}^*$-invariant forms on $M_{\mathrm{ask}} \times G \to M_{\mathrm{sk}} \times G$. It is shown in [CMX] that, although the prepotential F and the vielbeins are coordinate dependent, the quaternionic Kähler structure does not depend (up to isomorphism) on the choice of special coordinates.

Remark. It is also shown in [CMX] that a global quaternionic Kähler structure can be defined even if M_{sk} cannot be covered by a single system of special coordinates. In that case one has to replace $M = M_{\mathrm{sk}} \times G$ by the total space of a possibly nontrivial bundle over M_{sk}.

Using the quaternionic vielbein given in (3.7) we can define three almost complex structures J_α on M by

$$\mathcal{U}^{\mathcal{A}m} \circ J_\alpha = -\mathrm{i}(\sigma_\alpha)^{\mathcal{A}}_{\mathcal{B}}\mathcal{U}^{\mathcal{B}m}, \tag{3.9}$$

where $(\sigma_\alpha)^{\mathcal{A}}{}_{\mathcal{B}}$ are the $su(2)$ generators

$$(\sigma_1)^{\mathcal{A}}{}_{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (\sigma_2)^{\mathcal{A}}{}_{\mathcal{B}} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad (\sigma_3)^{\mathcal{A}}{}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $Q = \mathrm{span}\{J_\alpha | \alpha = 1, 2, 3\}$ is a skew-symmetric parallel quaternionic structure with respect to the quaternionic Kähler metric

$$g = u\bar{u} + v\bar{v} + \sum (e^b \bar{e}^b + E^b \bar{E}^b) \tag{3.10}$$

on M defined by (3.6). (Recall that $u\bar{u} = \bar{u}u = \frac{1}{2}(u \otimes \bar{u} + \bar{u} \otimes u)$.)

Proposition 1. *The almost complex structure J_3 is integrable for any quaternionic Kähler manifold $(M = M_{\mathrm{sk}} \times G, g, Q)$ in the image of the c-map. Moreover, the factors of the product $M_{\mathrm{sk}} \times G$ are complex submanifolds of the complex manifold (M, J_3). The restriction of J_3 to the first factor coincides (at any point of M) with the original complex structure J on the Kähler manifold M_{sk}, whereas the submanifold $G = \{p\} \times G \subset M_{\mathrm{sk}} \times G = M$ with the Hermitian structure induced by (g, J_3) is not Kähler. Nevertheless, the submanifold $G \subset M$ with its induced metric is isometric to the complex hyperbolic space $H^{n+1}_{\mathbb{C}}$ with the Kähler metric of constant holomorphic sectional curvature -4.*

Proof. According to (3.7) and (3.9), the one-forms $u, \bar{v}, e^b, \bar{E}^b$ constitute a basis for the space of $(1, 0)$-forms of J_3. We can compute their exterior derivative to be [FS]

$$\begin{aligned}
\mathrm{d}u &= \left(-\frac{1}{2}(v+\bar{v}) + \frac{\bar{Z}^A N_{AB} \mathrm{d}Z^B - Z^A N_{AB} \mathrm{d}\bar{Z}^B}{2\bar{Z}^A N_{AB} Z^B}\right) \wedge u - \bar{E} \wedge e, \\
\mathrm{d}\bar{v} &= -v \wedge \bar{v} + \bar{u} \wedge u - E \wedge \bar{E}, \\
\mathrm{d}e^a &= -\omega^a{}_b \wedge e^b, \\
\mathrm{d}\bar{E}^a &= \left(-\bar{\omega}^a{}_b - \frac{1}{2}(v+\bar{v})\delta^a_b + \frac{\bar{Z}^A N_{AB} \mathrm{d}Z^B - Z^A N_{AB} \mathrm{d}\bar{Z}^B}{2\bar{Z}^A N_{AB} Z^B}\delta^a_b\right) \wedge \bar{E}^b \\
&\quad + \bar{e} \wedge u + \frac{1}{4}(\bar{Z}^A N_{AB} Z^B)\Pi_{b\,A} N^{AB} N^{CD} \Pi^a_D E^b \wedge \mathrm{d}F_{BC},
\end{aligned}$$

where ω is the connection one-form of M_{sk} and the index b is lowered by means of the Kronecker symbol. Since there is no $(0, 2)$-form appearing on the right-hand side, J_3 is integrable by virtue of the Newlander-Nirenberg theorem. The two distributions tangent to the factors of the product manifold $M_{\mathrm{sk}} \times G$ are defined by $u = v = E^b = \bar{u} = \bar{v} = \bar{E}^b = 0$ and $e^b = \bar{e}^b = 0$, respectively. This shows that both distributions are J_3-invariant and, hence, that the leaves are complex submanifolds. The formula $e^b = e^b_a dz^a$ implies that the complex structures $J_3|_{M_{\mathrm{sk}}}$ and J coincide. It is known that a Kähler submanifold $S \subset M$ of a quaternionic Kähler manifold M, such that the complex structure of S is subordinate to the quaternionic structure, has at most dimension $\frac{1}{2}\dim M$ [AM]. Since $\dim G = 2n+2$, $2n = \frac{1}{2}\dim M$, $G \subset M$ cannot be a Kähler submanifold with the complex structure induced by J_3. Alternatively, one may check by a direct calculation that the fundamental two-form $\varphi_3 = g(\cdot, J_3)$ is not closed. For a proof of the last statement of the proposition see [CMX]. □

In the next proposition we give more detailed information about the complex structure J_3.

Proposition 2. *(i) The complex structure J_3 on the quaternionic Kähler manifold $M = M_{\mathrm{sk}} \times G$ is of the form $J_3 = J + J_G$, where J is the complex structure of the projective special Kähler domain M_{sk} and $(J_G(p))_{p\in M_{\mathrm{sk}}}$ is a smooth family of left-invariant complex structures $J_G(p)$ on G.*

(ii) The projection $\pi : M \to M_{\mathrm{sk}}$ is a holomorphic submersion with fibers $(G, J_G(p))$ biholomorphic to the domain

$$F(n+1) := \{(w^0, w_1, \ldots, w_n) \in \mathbb{C}^{n+1} \,|\, \mathrm{Re}\, w^0 > \sum_{A=1}^{n-1} (\mathrm{Im}\, w_A)^2 - (\mathrm{Im}\, w_n)^2\} \subset \mathbb{C}^{n+1},$$

for all $p \in M_{\mathrm{sk}}$. The total space (M, J_3) admits a fiber preserving open holomorphic embedding into the trivial holomorphic bundle $M_{\mathrm{sk}} \times \mathbb{C}^{n+1}$.

Proof. (i) By Proposition 1, the complex structure J_3 on $M = M_{\mathrm{sk}} \times G$ is the sum of the complex structure J on the base and a family of complex structures $J_G(p)$ on the fibers $\{p\} \times G \cong G$. To prove that $J_G(p)$ is left-invariant it suffices to check that the Lie derivative of the one-forms (3.8) with respect to the right-invariant vector fields (3.4) vanishes. That is a straightforward calculation.

(ii) We define a fiber preserving holomorphic embedding $\Psi : M \to M_{\mathrm{sk}} \times \mathbb{C}^{n+1}$ by $\Psi = (\pi, w^0, w_A)$, where

$$w^0 := e^{-2\phi} + i(\tilde{\phi} + a^A(b_A - F_{AB}a^B)), \quad w_A := b_A - F_{AB}a^B. \tag{3.11}$$

One can easily check that the functions w^0, w_A are J_3-holomorphic, cf. [LST2]. We claim that Ψ maps M biholomorphically onto the domain defined by the inequality

$$\mathrm{Re}\, w^0 > -N^{AB}\, \mathrm{Im}\, w_A\, \mathrm{Im}\, w_B. \tag{3.12}$$

In fact, for fixed $p \in M_{\mathrm{sk}}$, the linear map

$$\mathbb{R}^{2n} \ni (a^A, b_B) \mapsto (w_A) \in \mathbb{C}^n$$

is an isomorphism, whereas the variable $w^0 = e^{-2\phi} + i(\tilde{\phi} + a^A w_A)$ is constrained by the inequality $\mathrm{Re}\, w^0 > a^A\, \mathrm{Im}\, w_A$. Expressing (a^A) by (w_A) yields

$$a^A = -N^{AB}\, \mathrm{Im}\, w_B$$

and thus (3.12). For fixed $p \in M_{\mathrm{sk}}$ we can choose the special coordinates such that $(N^{AB}(p)) = \mathrm{diag}(-1, \ldots, -1, 1)$. This shows that π is a holomorphic submersion with fibers biholomorphic to $F(n+1)$. $\square$

Given the explicit form of the vielbein (3.7) the SU(2) connection ω^x reads [FS]

$$\begin{aligned} \omega^1 &= \mathrm{i}(\bar{u} - u), \qquad \omega^2 = u + \bar{u}, \\ \omega^3 &= \frac{\mathrm{i}}{2}(v - \bar{v}) - \mathrm{i}\, e^K \left(Z^A N_{AB} \mathrm{d}\bar{Z}^B - \bar{Z}^A N_{AB} \mathrm{d}Z^B \right). \end{aligned} \tag{3.13}$$

It can be checked that the natural action of G on $M = M_{\mathrm{sk}} \times G$ preserves the Ferrara-Sabharwal metric g [CMX]. The moment maps P_λ of the Killing vectors k_λ given in (3.4) take the following simple form:

$$P_\lambda = \frac{1}{2}\sum \omega_\alpha(k_\lambda)J_\alpha. \tag{3.14}$$

This follows from $\nabla k_\lambda = \nabla_{k_\lambda} - \mathcal{L}_{k_\lambda}$, since $\mathcal{L}_{k_\lambda} J_\alpha = 0$.[6]

In order to define the submanifold $N \subset M$ to which we will apply the quotient construction of Theorem 5, we choose constant complex vectors (C_A) and $(D^A) \neq 0$ and a constant $\tilde{C}$, where D^A obeys

$$\sum_{A,B=1}^{n} N_{AB}(Z_0)D^A\bar{D}^B = 0 \tag{3.15}$$

at some point $Z_0 = (Z_0^1, \dots, Z_0^n) \in M_{\mathrm{ask}}$. Here M_{ask} is identified with a domain in $\mathbb{C}^n$ by means of the special coordinates. Since the affine special Kähler metric $\sum N_{AB}(Z_0)dZ^A d\bar{Z}^B$ is indefinite, such a vector (D^A) does always exist. We will assume that the rank of the matrix

$$G_{AB}(Z) = \sum_C D^C F_{ABC}(Z) \tag{3.16}$$

is constant in a neighborhood of Z_0. Then, by restricting to that neighborhood, we can assume that the rank is constant on M_{ask}. That implies that the system of equations

$$\sum_A D^A F_{AB}(Z) = C_B := \sum_A D^A F_{AB}(Z_0) \tag{3.17}$$

defines a complex submanifold $M_{\mathrm{ask}}^\wedge \subset M_{\mathrm{ask}}$ of complex dimension $n - r$, where $r = \mathrm{rk}(G_{AB})$.

Proposition 3. *$r \leq n - 1$ and $M_{\mathrm{ask}}^\wedge$ fibers over a complex submanifold $M_{\mathrm{sk}}^\wedge \subset M_{\mathrm{sk}}$ of dimension $n-1-r$. In particular, M_{sk} is of dimension zero if (G_{AB}) has maximal rank.*

Proof. Since F_{AB} is homogeneous of degree zero, the vector (Z^A) is in the kernel of the matrix (G_{AB}), which implies $r \leq n - 1$. Due to the homogeneity of Eq. (3.17), $M_{\mathrm{ask}}^\wedge$ is a cone over a complex submanifold $M_{\mathrm{sk}}^\wedge \subset M_{\mathrm{sk}}$. □

Remark. More generally, for the smoothness of $M_{\mathrm{ask}}^\wedge$ it is sufficient to assume that the rank of (G_{AB}) is constant on a complex submanifold containing (a neighborhood of Z_0 in) the analytic set defined by (3.17).

Now we define a subset $N \subset M$ by the system

$$D^A F_{AB}(Z) = C_B, \qquad D^A(b_A - F_{AB}a^B) = \tilde{C}. \tag{3.18}$$

We claim that $N \subset M$ is submanifold of codimension $2r + 2$. More precisely, it is a subbundle of $M_{\mathrm{sk}}^\wedge \times G$ with fibers of codimension 2. To see this it suffices to recall that the first equation of (3.18) defines the submanifold $M_{\mathrm{sk}}^\wedge \subset M_{\mathrm{sk}}$ and to note that over points of $M_{\mathrm{sk}}^\wedge$ the second equation reduces to

$$D^A b_A - C_B a^B = \tilde{C},$$

[6] Note that the formula (3.14) differs by a factor 1/2 from that of [M], since our definition (1.3) of the moment map differs from that of [M] by the same factor. This can be easily checked with the help of formula (1.4).

which is a system of two real affine equations. The two real equations are independent if and only if the vector (D^A, C_B) is not a complex multiple of a real vector. The latter property follows from the fact that vector belongs to the tangent space $L = T_{dF(Z_0)}\mathcal{C}$ of the Lagrangian cone $\mathcal{C} = \{dF(Z)|Z \in M_{\text{ask}} \subset \mathbb{C}^n\} \subset T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$, which satisfies $L \cap \bar{L} = 0$, see [ACD].

Proposition 4. *Under the above assumptions, $N \subset M$ is a complex submanifold with respect to the complex structure J_3. More precisely, the homogeneous equation*

$$D^A b_A - C_A a^A = 0 \tag{3.19}$$

defines a subgroup $G^\wedge \subset G$ of codimension 2 and $N = M_{\text{sk}}^\wedge \times S$ is the product of the complex submanifold $M_{\text{sk}}^\wedge \subset M_{\text{sk}}$ and a submanifold $S \subset G$, which is a left-translate, $S = xG^\wedge$, of the subgroup $G^\wedge \subset G$ by an element $x \in G$ satisfying the inhomogeneous equation

$$D^A b_A - C_A a^A = \tilde{C}. \tag{3.20}$$

The fibers $\{p\} \times S \subset \{p\} \times G$, $p \in M_{\text{sk}}^\wedge$, are complex hypersurfaces with respect to the complex structure on $\{p\} \times G \subset M$ induced by J_3.

Proof. In order to prove that $N \subset M$ and the fibers of $N \to M_{\text{sk}}^\wedge$ are complex submanifolds, it suffices to show that the one-form

$$d(D^A(b_A - F_{AB}a^B)) = D^A(db_A - F_{AB}da^B) + D^A a^B dF_{AB}$$

is of type $(1, 0)$. This is obvious for the second term. In order to analyse the first term, we decompose

$$D^A = cZ^A + H^A,$$

where $\bar{H}^A N_{AB} Z^B = 0$ and $c \in \mathbb{C}$. Then

$$D^A(db_A - F_{AB}da^B) = -cie^{-K/2-\phi}u + c_b \bar{E}^b,$$

where the coefficients $c_b \in \mathbb{C}$ are determined by the equation $-\frac{\mathrm{i}\bar{c}_b}{2}e^{\phi-K/2}\Pi_A{}^b N^{AB} = \bar{H}^B$. This proves that $D^A(db_A - F_{AB}da^B)$ is of type $(1, 0)$.

To check that (3.19) defines a subgroup $G^\wedge \subset G$, we recall[7] [CMX] that in the coordinates $(\phi, \tilde{\phi}, a^A, b_B)$ the group multiplication in G is given by:

$$\begin{aligned}&(\phi, \tilde{\phi}, a, b) \cdot (\phi', \tilde{\phi}', a', b')\\ &\quad = (\phi + \phi', \tilde{\phi} + e^{-2\phi}\tilde{\phi}' + e^{-\phi}(a^A b'_A - a'^A b_A), a + e^{-\phi}a', b + e^{-\phi}b').\end{aligned} \tag{3.21}$$

From this formula we see that the set of solutions of (3.19) is closed under multiplication and contains the neutral element and the inverse

$$(\phi, \tilde{\phi}, a, b)^{-1} = (-\phi, -e^{2\phi}\tilde{\phi}, -e^{\phi}a, -e^{\phi}b)$$

of any element $(\phi, \tilde{\phi}, a, b)$ satisfying (3.19). Let $x \in G$ be any element satisfying (3.20). Using the multiplication law (3.21) we can easily check that $xG^\wedge$ is a subset of the solution space of (3.20), which we know is an affine subspace of $G \cong \mathbb{R}^{2n+2}$ of codimension 2. This proves that $xG^\wedge$ coincides with the set of solutions of (3.20), that is, with the fiber of $N \to M_{\text{sk}}^\wedge$. □

[7] Our additive variable ϕ is related to the corresponding variable λ in [CMX] by $\lambda = -2\phi$.

In the next proposition we give more detailed information about the complex submanifold $N \subset (M, J_3)$.

Proposition 5. *(i) The complex structure induced by J_3 on $N = M^{\wedge}_{\mathrm{sk}} \times S$ is of the form $J + J_S$, where J is the complex structure on $M^{\wedge}_{\mathrm{sk}}$ and $(J_S(p))_{p \in M^{\wedge}_{\mathrm{sk}}}$ is a smooth family of left-invariant complex structures on $S = xG^{\wedge} \cong G^{\wedge}$.*

(ii) The projection $\pi_N : N \to M^{\wedge}_{\mathrm{sk}}$ is a holomorphic submersion with fibers $(S, J_S(p))$ biholomorphic to $B^{n-1}_{\mathbb{C}} \times \mathbb{C}$, for all $p \in M^{\wedge}_{\mathrm{sk}}$. The total space (N, J_3) admits a fiber preserving open holomorphic embedding into the trivial holomorphic bundle $M^{\wedge}_{\mathrm{sk}} \times \mathbb{C}^n$.

Proof. (i) It follows from Proposition 2 and Proposition 4 that the complex structure of N is of the form $J + J_S$, where $J_S = J_G|_S$. Identifying $S = xG^{\wedge}$ with the group $G^{\wedge} \subset G$ by means of the left-translation with x^{-1}, we can consider J_S as a complex structure on the group $G^{\wedge}$. Then the left-invariance of J_S follows from that of J_G.

(ii) Using the fiber preserving open holomorphic embedding Ψ of $\pi : M \to M_{\mathrm{sk}}$ into $M_{\mathrm{sk}} \times \mathbb{C}^{n+1}$ defined in (3.11), we see that $\pi_N : N \to M^{\wedge}_{\mathrm{sk}}$ is embedded into $\pi|_{M^{\wedge}_{\mathrm{sk}}}$ by one complex affine equation $D^A w_A = \tilde{C}$, which reduces the trivial bundle $M^{\wedge}_{\mathrm{sk}} \times \mathbb{C}^{n+1} \subset M_{\mathrm{sk}} \times \mathbb{C}^{n+1}$ to a trivial bundle $\cong M^{\wedge}_{\mathrm{sk}} \times \mathbb{C}^n$ and the fiber $F(n+1)$ of π to $F'(n-1) \times \mathbb{C}$. In fact, for fixed $p \in M^{\wedge}_{\mathrm{sk}}$ we can choose special coordinates such that $(N^{AB}(p)) = \mathrm{diag}(-1, \ldots, -1, 1)$ and $D = (0, \ldots, 0, 1, 1)$. Then the fiber is defined by

$$\mathrm{Re}\, w^0 > \sum_{A=1}^{n-1} (\mathrm{Im}\, w_A)^2 - (\mathrm{Im}\, w_n)^2, \quad w_n + w_{n-1} = \tilde{C}.$$

Elimination of w_n yields the domain

$$\{(w^0, \ldots, w_{n-1}) \in \mathbb{C}^n \,|\, \mathrm{Re}\, w^0 - 2\, \mathrm{Im}\, \tilde{C}\, \mathrm{Im}\, w_{n-1} + (\mathrm{Im}\, \tilde{C})^2 > \sum_{A=1}^{n-2} (\mathrm{Im}\, w_A)^2\} \subset \mathbb{C}^n,$$

which is biholomorphic to $F'(n-1) \times \mathbb{C}$ by the affine transformation $(w^0, w_1, \ldots, w_{n-1}) \mapsto (w^0 - 2\, \mathrm{Im}\, \tilde{C}\, \mathrm{Im}\, w_{n-1} + (\mathrm{Im}\, \tilde{C})^2, w_1, \ldots, w_{n-1})$, where

$$F'(n-1) := \{(w^0, w_1, \ldots, w_{n-2}) \in \mathbb{C}^n \,|\, \mathrm{Re}\, w^0 > \sum_{A=1}^{n-2} (\mathrm{Im}\, w_A)^2\} \subset \mathbb{C}^{n-1}.$$

Now it suffices to note that $F'(n-1)$ is biholomorphic to the ball $B^{n-1}_{\mathbb{C}}$. □

Before we go on, let us summarize what we found so far by the following commutative diagram consisting of holomorphic fiber preserving embeddings:

$$\begin{array}{ccc} M = M_{\mathrm{sk}} \times G & \overset{\Psi}{\hookrightarrow} & M_{\mathrm{sk}} \times \mathbb{C}^{n+1} \\ \cup & & \cup \\ N = M^{\wedge}_{\mathrm{sk}} \times S & \overset{\Psi|_N}{\hookrightarrow} & M^{\wedge}_{\mathrm{sk}} \times \mathbb{C}^n, \end{array}$$

where the horizontal embeddings are open and the vertical ones are of complex codimension $r+1$. Recall that r is the complex codimension of $M^{\wedge}_{\mathrm{sk}} \subset M_{\mathrm{sk}}$, S is a left-translate of a subgroup $G^\wedge \subset G$ and $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ is an affine hyperplane (which is linear if $S = G^\wedge$). The fibers of $M \to M_{\mathrm{sk}}$ are biholomorphic to $F(n+1)$, whereas the fibers of $N \to M_{\mathrm{sk}}^\wedge$ are biholomorphic to $B_{\mathbb{C}}^{n-1} \times \mathbb{C}$.

Let us now define two Killing vectors ξ_i, $i = 1, 2$, on M by

$$\begin{aligned} \xi_1 &= \mathrm{Re}\, D^A k_A + \mathrm{Re}\, C_A \tilde{k}^A + \mathrm{Re}\, \tilde{C} k_{\tilde{\phi}}, \\ \xi_2 &= \mathrm{Im}\, D^A k_A + \mathrm{Im}\, C_A \tilde{k}^A + \mathrm{Im}\, \tilde{C} k_{\tilde{\phi}}. \end{aligned} \tag{3.22}$$

From (3.14), (3.13), (3.8) and (3.18) we see that both P_1 and P_2 lie in the plane spanned by J_1 and J_2. Therefore, we find $P_1 P_2 = f J_3$ for some function f. Furthermore, there is $J_3 k_1 = k_2$. Hence we can apply Theorem 5 and Corollary 1, provided that ξ_1, ξ_2 are tangent to N and generate a free and proper action. This is shown in the next proposition.

Proposition 6. *The vector fields ξ_1, ξ_2 generate a free and proper holomorphic action of a vector group $\mathbb{R}^2 \cong \mathbb{C}$ on the submanifold $N \subset M$. In the coordinates $(z^a, \phi, \tilde{\phi}, a^A, b_A)$ the action of $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ is given by $(z, \phi, \tilde{\phi}, a, b) \mapsto (z, \phi, \tilde{\phi}', a', b')$, where*

$$\begin{aligned} \tilde{\phi}' &= \tilde{\phi} - \lambda_1 \mathrm{Re}\, \tilde{C} - \lambda_2 \mathrm{Im}\, \tilde{C}, \\ a^{A\prime} &= a^A + \lambda_1 \mathrm{Re}\, D^A + \lambda_2 \mathrm{Im}\, D^A, \\ b_A' &= b_A + \lambda_1 \mathrm{Re}\, C_A + \lambda_2 \mathrm{Im}\, C_A. \end{aligned} \tag{3.23}$$

In the holomorphic coordinates (z^a, w^0, w_A) the action of $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ is given by $(z^a, w^0, w_A) \mapsto (z^a, \zeta^0, \zeta_A)$, where

$$\zeta^0 = w^0 + i\lambda \bar{D}^A w_A - i\lambda \overline{\tilde{C}} + i\frac{\lambda^2}{4} \bar{D}^A (\bar{C}_A - F_{AB} \bar{D}^B), \tag{3.24}$$

$$\zeta_A = w_A + \frac{\lambda}{2} (\bar{C}_A - F_{AB} \bar{D}^B). \tag{3.25}$$

Proof. First note that

$$\begin{aligned} \xi_1|_N &= \mathrm{Re}\, D^A \frac{\partial}{\partial a^A} + \mathrm{Re}\, C_A \frac{\partial}{\partial b_A} - \mathrm{Re}\, \tilde{C} \frac{\partial}{\partial \phi}, \\ \xi_2|_N &= \mathrm{Im}\, D^A \frac{\partial}{\partial a^A} + \mathrm{Im}\, C_A \frac{\partial}{\partial b_A} - \mathrm{Im}\, \tilde{C} \frac{\partial}{\partial \phi}. \end{aligned} \tag{3.26}$$

We can easily check that ξ_1, ξ_2 are tangent to $N = \{D^A b_A - C_A a^A = \tilde{C}\}$. In fact, this is a consequence of the two equations $D^A C_A - C_A D^A = 0$ and $D^A \bar{C}_A - C_A \bar{D}^A = -2i D^A N_{AB} \bar{D}^B = 0$. Let us use φ_j^t to denote the flow of the vector field ξ_j and put

$$\varphi^\lambda := \varphi_1^{\lambda_1} \circ \varphi_2^{\lambda_2}, \quad \lambda = \lambda_1 + i\lambda_2.$$

Then (3.26) shows that $\varphi^\lambda|_N$ is given by (3.23). We see that in these coordinates the action consists of translations along a plane. In particular, it is free and proper. Expressing φ^λ in holomorphic coordinates yields (3.24)–(3.25), which shows that the action $\mathbb{C} \times N \to N$ is holomorphic. $\square$

The Kähler manifold $M' = N/A$ constructed from Corollary 1 is of real dimension $4(n-1)-2r$, where r was the complex codimension of $M_{\mathrm{sk}}^{\wedge} \subset M_{\mathrm{sk}}$. Thus the minimal dimension of M' is $2(n-1)$, which is attained when the base manifold $M_{\mathrm{sk}}^{\wedge}$ is discrete. The maximal dimension $4(n-1)$ is attained, when $M_{\mathrm{sk}}^{\wedge} = M_{\mathrm{sk}}$.

Theorem 6. *Let (M', h) be the Kähler manifold obtained as above from the quotient construction of Corollary 1 applied to a quaternionic Kähler manifold $(M = M_{\mathrm{sk}} \times G, g)$ in the image of the c-map. Then M' is the total space of a holomorphic submersion over the complex submanifold $M_{\mathrm{sk}}^{\wedge} \subset M_{\mathrm{sk}}$ with fibers biholomorphic to $B_{\mathbb{C}}^{n-1}$. The metric of the fiber is given by*

$$h_{\mathrm{fib}} = \frac{1}{4}e^{4\phi}|\mathrm{d}x^0 + 2\,\mathrm{i}((\mathrm{Im}\,x)_a\delta^{ab})\mathrm{d}x_b|^2 + \frac{1}{2}e^{2\phi}\mathrm{d}\bar{x}_a\delta^{ab}\mathrm{d}x_b, \tag{3.27}$$

with respect to some global system of holomorphic coordinates $(x^0, x_1, \ldots, x_{n-2})$ on the fiber. As a consequence, the fiber is isometric (but not biholomorphic, unless $n \le 2$) to $H_{\mathbb{C}}^{n-1}$ with its metric of constant holomorphic sectional curvature -4.

Proof. Since the action on N generated by ξ_1 and ξ_2 is holomorphic, see Proposition 6, and preserves the fibers of the holomorphic submersion $\pi_N : N \to M_{\mathrm{sk}}^{\wedge}$, we have an induced holomorphic submersion $\pi' : M' \to M_{\mathrm{sk}}^{\wedge}$. We know already (see the proof of Proposition 5) that $\pi_N : N \to M_{\mathrm{sk}}^{\wedge}$ is holomorphically embedded into $\pi|_{M_{\mathrm{sk}}^{\wedge}} : \pi^{-1}(M_{\mathrm{sk}}^{\wedge}) \to M_{\mathrm{sk}}^{\wedge}$ with fibers of complex codimension one and that $\pi^{-1}(M_{\mathrm{sk}}^{\wedge})$ is an open subset of the trivial bundle $M_{\mathrm{sk}}^{\wedge} \times \mathbb{C}^{n+1}$. The fiber $S = xG^{\wedge}$ of π_N is the intersection of the fiber $G \cong F(n+1)$ of $\pi|_{M_{\mathrm{sk}}^{\wedge}}$ with the complex affine hyperplane defined by the equation $D^A w_A = \tilde{C}$ in the holomorphic fiber coordinates (w^0, w_A). Let V^A be any vector such that $V^A(\bar{C}_A - F_{AB}\bar{D}^B) \neq 0$ holds on some neighborhood $U \subset M_{\mathrm{sk}}^{\wedge}$. Such a vector exists, since $\bar{C}_A - F_{AB}\bar{D}^B = (\bar{F}_{AB} - F_{AB})\bar{D}^B = -2iN_{AB}\bar{D}^B$ and $D \neq 0$. Consider the subgroup $G' \subset G$ defined by the homogeneous equations $D^A w_A = V^A w_A = 0$. One can check that G' is isomorphic to the Iwasawa subgroup of $\mathrm{SU}(1, n-1)$. The reason is that the canonical symplectic form ω on $\mathbb{R}^{2n}$ is nondegenerate on the real subspace Π' of $\mathbb{R}^{2n}$ which corresponds to the complex subspace of $\mathbb{C}^n$ defined by $D^A w_A = V^A w_A = 0$ under the isomorphism $(a^A, b_B) \mapsto (w_A)$. In fact, Π' is complementary in $\Pi^{\perp,\omega}$ to the plane $\Pi \subset \Pi^{\perp,\omega} \subset \mathbb{R}^{2n}$ spanned by the real and imaginary part of the complex vector (D^A, C_B). The plane Π is precisely the kernel of ω on $\Pi^{\perp,\omega}$. (Note that for the same reason $G^{\wedge}$ is not isomorphic to the Iwasawa subgroup of $\mathrm{SU}(1, n)$.) The complex submanifold $S' := xG' \subset S = xG^{\wedge}$ intersects all the orbits of the vector group A generated by the two Killing vector fields ξ_1 and ξ_2 transversally and exactly in one point, as follows from (3.25). Therefore, it is biholomorphic to the quotient $A \backslash S$, which is the fiber of the holomorphic submersion $\pi' : M' \to M_{\mathrm{sk}}^{\wedge}$. This proves that the fiber is biholomorphic to G' endowed with a left-invariant complex structure $J' = J'(p)$, $p \in M_{\mathrm{sk}}^{\wedge}$. Using the fact that $G^{\wedge}$ and, hence, $G' \subset G^{\wedge}$ normalizes A in G, one can show that the fiber metric corresponds to a left-invariant metric $g' = g'(p)$ on G'. Since $N_{AB}dw_A d\bar{w}_B < 0$ on $\{D^A w_A = V^A w_A\} \cong \mathbb{C}^{n-2} \subset \mathbb{C}^n$ we get that $(G', J', g') \cong \mathbb{C}H^{n-1}$.

In order to make the above argument more explicit, let us compute the Kähler metric of the fiber of $M' \to M_{\mathrm{sk}}^{\wedge}$ in holomorphic coordinates and show that it is indeed the complex hyperbolic metric of constant holomorphic sectional curvature -4. The metric of M is given by (3.10). Let us recall that (Π_A^b) is the matrix which represents the projection $TM_{\mathrm{ask}} \to TM_{\mathrm{sk}}$ with respect to the special holomorphic coordinate frame on M_{ask} and a unitary frame on M_{sk}. By the definition of the projective special Kähler metric, we have

$$\frac{N_{AB}}{ZN\bar{Z}} = -\delta_{ab}\Pi^a_A\Pi^b_B + \frac{N_{AC}\bar{Z}^C N_{BD}Z^D}{(ZN\bar{Z})^2}, \tag{3.28}$$

where $ZN\bar{Z} = \sum Z^A N_{AB}\bar{Z}^B = \frac{e^{-K}}{2}$, cf. (3.3). Multiplying (3.28) with the inverse matrix of the left-hand side yields

$$\delta^B_A = -\frac{1}{2}e^{-K}\Pi_{Ab}\bar{\Pi}^b_C N^{-1\,CB} + 2e^K N_{AC}\bar{Z}^C Z^B. \tag{3.29}$$

Restricting to the fiber over a point $p \in M^{\wedge}_{\text{sk}}$ and using the identity (3.29) we find

$$\begin{aligned} g_{\text{fib}} = \mathrm{d}\phi^2 + \frac{1}{4}e^{4\phi}|\mathrm{d}\tilde{\phi} + b_A\mathrm{d}a^A - a^A\mathrm{d}b_A|^2 - \frac{1}{2}e^{2\phi}(\mathrm{d}b_A - \bar{F}_{AC}\mathrm{d}a^C)N^{-1\,AB} \\ \times(\mathrm{d}b_B - F_{BD}\mathrm{d}a^D) + 2e^{K+2\phi}|Z^A(\mathrm{d}b_A - F_{AB}\mathrm{d}a^B)|^2, \end{aligned}$$

which is the canonical metric of $F(n+1)$. Using the coordinates (3.11) the fiber metric $g_{\text{fib}} := g|_{\pi^{-1}(p)}$ takes the following form:

$$\begin{aligned} g_{\text{fib}} = \frac{1}{4}e^{4\phi}|\mathrm{d}w^0 - 2\,\mathrm{i}(\mathrm{Im}\,w)_A N^{-1\,AB}\mathrm{d}w_B|^2 - \frac{1}{2}e^{2\phi}\mathrm{d}\bar{w}_A N^{-1\,AB}\mathrm{d}w_B \\ +2e^{K+2\phi}|Z^A\mathrm{d}w_A|^2. \end{aligned} \tag{3.30}$$

The metric $h_{\text{fib}} := h|_{M'_p}$ of the fiber $M'_p := (\pi')^{-1}(p)$ of $\pi' : M' \to M^{\wedge}_{\text{sk}}$ is obtained by first restricting g_{fib} to the submanifold $N_p := \pi_N^{-1}(p) \subset \pi^{-1}(p)$ defined by $D^A w_A = \tilde{C}$ and then taking the quotient by the isometric $\mathbb{R}^2$-action generated by the Killing vector fields ξ_1 and ξ_2. These vector fields can be combined in the holomorphic vector field

$$k := \xi_2 + i\xi_1 = -i(\xi_1 - i\xi_2) = \bar{D}^A N_{AB}\frac{\partial}{\partial w_B} - 2\,\mathrm{i}\,\bar{D}^A(\mathrm{Im}\,w)_A\frac{\partial}{\partial w^0},$$

see (3.24)–(3.25). Since the quotient map $\tau : N \to M' = N/A$ is a Riemannian submersion, as is its restriction $\tau_p : N_p \to M'_p$, the metric h_{fib} on $M'_p \cong F(n-1)$ is determined by the degenerate symmetric tensor field

$$(\tau_p)^* h_{\text{fib}} = \tilde{g}_{\text{fib}} - \frac{\tilde{g}_{\text{fib}}(k,\cdot)\tilde{g}_{\text{fib}}(\bar{k},\cdot) + \tilde{g}_{\text{fib}}(\bar{k},\cdot)\tilde{g}_{\text{fib}}(k,\cdot)}{\tilde{g}_{\text{fib}}(k,\bar{k})},$$

where $\tilde{g}_{\text{fib}} = g|_{N_p} = g_{\text{fib}}|_{N_p}$. Since $\sum Z^A N_{AB}\bar{Z}^B = \frac{e^{-K}}{2} > 0$ and therefore $\sum Z^A N_{AB}\bar{D}^B \neq 0$, we see as above that the equivalence classes $[w^0, w_A]$ corresponding to the holomorphic $\mathbb{C}$-action generated by k each contain exactly one representative which fulfills

$$Z^A w_A = 0.$$

Recall that the index A runs from 1 to n. In particular, the $n+1$ holomorphic fiber coordinates are $(w^0, w_1, \ldots, w_n)$. By a linear change of special coordinates (Z^A), if necessary, we can assume that at our base point p we have $Z^1 = 1$. Because (Z^A) always has positive norm and (D^A) is null, we know that $D^a \neq D^1 Z^a$ for some $a \in \{2, \ldots, n\}$, let us say for $a = n$. Therefore, we can find coordinates $\{x^0, x_a\}$, $a = 2, \dots, n-1$, for the fiber of $M' \to M^{\wedge}_{\mathrm{sk}}$ as follows. We put $\alpha = 1/(D^n - D^1 Z^n)$ and observe that the map

$$
\begin{aligned}
(x^0, x_a) &\mapsto (w^0, w_A)\\
&= \left(x^0, \alpha((Z^n D^a - Z^a D^n)x_a - Z^n \tilde{C}), x_a, \alpha(\tilde{C} - (D^a - D^1 Z^a)x_a)\right),
\end{aligned}
\tag{3.31}
$$

is an affine isomorphism from $\mathbb{C}^{n-1}$ onto the affine subspace $E \subset \mathbb{C}^{n+1}$ defined by $D^A w_A = \tilde{C}$ and $Z^A w_A = 0$. Therefore, it induces a biholomorphic map from an open subset of $\mathbb{C}^{n-1}$ onto $M'_p = N_p/A \cong E \cap N_p$. On the complex hypersurface $\mathcal{H} := E \cap N_p \subset N_p$ (defined by $Z^A w_A = 0$) we have

$$Z^A \mathrm{d}w_A = 0, \qquad D^A \mathrm{d}w_A = 0.$$

From this one computes

$$g(k, \cdot)|_{\mathcal{H}} = \tilde{g}_{\mathrm{fib}}(k, \cdot)|_{\mathcal{H}} = 0,$$

and therefore concludes that the projection $N_p \to M'_p$ restricts to a biholomorphic isometry $\mathcal{H} \to M'_p$. Using this isomorphism, the metric h_{fib} of M'_p is identified with the metric $g_{\mathcal{H}} = g|_{\mathcal{H}} = g_{\mathrm{fib}}|_H$ of the hypersurface $\mathcal{H} \subset N_p$, which is

$$g_{\mathcal{H}} = \frac{1}{4} e^{4\phi} |\mathrm{d}x^0 - 2\,\mathrm{i}(\tilde{N}_0^b + (\mathrm{Re}\, x)_a \tilde{N}_1^{ab} + (\mathrm{Im}\, x)_a \tilde{N}_2^{ab})\mathrm{d}x_b|^2 - \frac{1}{2} e^{2\phi} \mathrm{d}\bar{x}_a \tilde{N}^{ab} \mathrm{d}x_b,$$

where

$$
\begin{aligned}
\tilde{N}^{ab} = {} & N^{-1\,ab} + \bar{\alpha}(\bar{Z}^n \bar{D}^a - \bar{Z}^a \bar{D}^n) N^{-1\,1b} + \alpha N^{-1\,a1}(Z^n D^b - Z^b D^n)\\
& + |\alpha|^2 (\bar{Z}^n \bar{D}^a - \bar{Z}^a \bar{D}^n) N^{-1\,11} (Z^n D^b - Z^b D^n) - \bar{\alpha}(\bar{D}^a - \bar{D}^1 \bar{Z}^a) N^{-1\,nb}\\
& - \alpha N^{-1\,an}(D^b - D^1 Z^b) + |\alpha|^2 (\bar{D}^a - \bar{D}^1 \bar{Z}^a) N^{-1\,nn} (D^b - D^1 Z^b)\\
& - |\alpha|^2 (\bar{Z}^n \bar{D}^a - \bar{Z}^a \bar{D}^n) N^{-1\,1n} (D^b - D^1 Z^b)\\
& - |\alpha|^2 (\bar{D}^a - \bar{D}^1 \bar{Z}^a) N^{-1\,n1} (Z^n D^b - Z^b D^n)
\end{aligned}
$$

is Hermitian and negative definite and the other coefficients are given by

$$
\begin{aligned}
\tilde{N}_0^a = {} & \mathrm{Im}(\alpha \tilde{C})\big((N^{-1\,na} - Z^n N^{-1\,1a}) + \alpha(N^{-1\,n1} - Z^n N^{-1\,11})(Z^n D^a - Z^a D^n)\\
& + \alpha(N^{-1\,1n} - N^{-1\,nn})(D^a - D^1 Z^a)\big),\\
\tilde{N}_1^{ab} = {} & \mathrm{Im}(\alpha(Z^n D^a - Z^a D^n))(N^{-1\,1b} + \alpha N^{-1\,11}(Z^n D^b - Z^b D^n)\\
& - \alpha N^{-1\,1n}(D^b - D^1 Z^b)) - \mathrm{Im}(\alpha(D^a - D^1 Z^a))\\
& \times (N^{-1\,nb} + \alpha N^{-1\,n1}(Z^n D^b - Z^b D^n) - \alpha N^{-1\,nn}(D^b - D^1 Z^b)),\\
\tilde{N}_2^{ab} = {} & \mathrm{Re}(\alpha(Z^n D^a - Z^a D^n))(N^{-1\,1b} + \alpha N^{-1\,11}(Z^n D^b - Z^b D^n)\\
& - \alpha N^{-1\,1n}(D^b - D^1 Z^b)) + (N^{-1\,ab} + \alpha N^{-1\,a1}(Z^n D^b - Z^b D^n)\\
& - \alpha N^{-1\,an}(D^b - D^1 Z^b)) - \mathrm{Re}(\alpha(D^a - D^1 Z^a))\\
& \times (N^{-1\,nb} + \alpha N^{-1\,n1}(Z^n D^b - Z^b D^n) - \alpha N^{-1\,nn}(D^b - D^1 Z^b)).
\end{aligned}
$$

By a linear change of holomorphic coordinates we can assume that $\tilde{N}^{ab} = -\delta^{ab}$. Finally, by changing the coordinate x^0 into $x^0 - 2i \tilde{N}_0^a x_a - 2i x_a \tilde{N}_1^{ab} x_b$ we obtain the form

$$g_{\mathcal{H}} = \frac{1}{4}e^{4\phi}|\mathrm{d}x^0 - 2\,\mathrm{i}((\mathrm{Im}\,x)_a M^{ab})\mathrm{d}x_b|^2 + \frac{1}{2}e^{2\phi}\mathrm{d}\bar{x}_a\delta^{ab}\mathrm{d}x_b,$$

where $M^{ab} = \tilde{N}_2^{ab} - i\tilde{N}_1^{ab} = \tilde{N}^{ab} = -\delta^{ab}$. Note that this metric has the same form as the fiber of $M \to M_{\mathrm{sk}}$, which we already know has constant holomorphic sectional curvature. To compare the metrics it suffices to put $N_{AB} = -\eta_{AB}$ (the Minkowski scalar product) and $(Z^A) = (1, 0, \ldots, 0)$ in (3.30), which yields

$$g_{\mathrm{fib}} = \frac{1}{4}e^{4\phi}|\mathrm{d}w^0 - 2\,\mathrm{i}((\mathrm{Im}\,w)_A\eta^{AB})\mathrm{d}w_B|^2 + \frac{1}{2}e^{2\phi}\mathrm{d}\bar{w}_A\delta^{AB}\mathrm{d}w_B.$$

Changing the coordinate w_1 to $\bar{w}_1$ brings this metric to the more standard form (3.27), but in $n+1$ instead of $n-1$ complex dimensions. □

Remark. The above proof shows that the quotient Kähler manifold M' can be described as follows. As a smooth manifold,

$$M' = M_{\mathrm{sk}}^{\wedge} \times G',$$

where G' is the Iwasawa subgroup of $\mathrm{SU}(1, n-1)$. The Kähler structure $(J_{M'}, g_{M'})$ of M' is of the form

$$J_{M'} = J_{M_{\mathrm{sk}}^{\wedge}} + J', \quad g_{M'} = g_{M_{\mathrm{sk}}^{\wedge}} + g',$$

where $(J'(p), g'(p))_{p \in M_{\mathrm{sk}}^{\wedge}}$ is a family of left-invariant Kähler structures on G' such that $(G', J'(p), g'(p))$ is isomorphic to $\mathbb{C}H^{n-1}$ with its standard Kähler structure for all p. Applying Theorem 2 of [CMX], this shows, in particular, that M' is complete if the submanifold $M_{\mathrm{sk}}^{\wedge} \subset M_{\mathrm{sk}}$ is complete.

We shall now consider some explicit examples of the new quotient construction applied to quaternionic Kähler manifolds in the image of the c-map.

3.2.1. Quadratic prepotential. Let us first analyze the case of a quadratic prepotential F, i.e. $F(Z^1, \ldots, Z^n)$ is a quadratic polynomial such that the real symmetric matrix $N_{AB} = \mathrm{Im}\,F_{AB}$ is of signature $(1, n-1)$. The corresponding $4n$-dimensional quaternionic Kähler manifold is the Hermitian symmetric space

$$M = \frac{\mathrm{U}(2, n)}{\mathrm{U}(2) \times \mathrm{U}(n)}.$$

Proposition 7. *In the case of quadratic prepotential, the holomorphic submersion $\pi : M \to M_{\mathrm{sk}} = H_{\mathbb{C}}^{n-1}$ of Proposition 2 is a trivial holomorphic fiber bundle and (M, J_3) is biholomorphic to $H_{\mathbb{C}}^{n-1} \times F(n+1)$.*

Proof. Since F_{AB} is constant, the fiber preserving open embedding $\Psi : M \to M_{\mathrm{sk}} \times \mathbb{C}^{n+1}$ defined in (3.11) is a biholomorphic isomorphism onto its image $M_{\mathrm{sk}} \times F(n+1)$. □

In this case, the first condition in (3.18) is automatically satisfied at every point of M as soon as it is satisfied at one point. Hence, N is of dimension $4n-2$ and M' is of dimension $4n-4$.

Proposition 8. *In the case of quadratic prepotential, the holomorphic submersion $\pi_N : N \to M_{\mathrm{sk}}^{\wedge} = M_{\mathrm{sk}}$ of Proposition 5 is a trivial holomorphic fiber bundle and the complex submanifold $N \subset (M, J_3)$ is biholomorphic to $M_{\mathrm{sk}} \times \mathbb{C} \times F'(n-1) = H_{\mathbb{C}}^{n-1} \times \mathbb{C} \times F'(n-1)$, for any choice of null vector $(D^A) \in \mathbb{C}^{1,n-1}$ and any $\tilde{C} \in \mathbb{C}$.*

Proof. Since N_{AB} is now constant, it follows immediately from the proof of Proposition 5 that the submanifold $N \subset M \cong M_{\mathrm{sk}} \times \mathbb{C}^{n+1}$ is biholomorphic to $M_{\mathrm{sk}} \times \mathbb{C} \times F'(n-1)$. □

Theorem 7. *The Kähler manifolds M' obtained from the quotient construction of Corollary 1 applied to the quaternionic Kähler manifold $M = \frac{\mathrm{U}(2,n)}{\mathrm{U}(2)\times\mathrm{U}(n)}$ are isomorphic to*

$$H_{\mathbb{C}}^{n-1} \times H_{\mathbb{C}}^{n-1},$$

for any choice of null vector $(D^A) \in \mathbb{C}^{1,n-1}$ and any $\tilde{C} \in \mathbb{C}$.

Proof. The holomorphic submersion $M' \to M_{\mathrm{sk}}^{\wedge}$ of Theorem 6 is, in this case, a trivial holomorphic fiber bundle over $M_{\mathrm{sk}}^{\wedge} = M_{\mathrm{sk}} = H_{\mathbb{C}}^{n-1}$. This follows from the proof of Theorem 6, since the constructions are now independent of $p \in M_{\mathrm{sk}}$. For the same reason, the metric is the product of the metric on the base and the metric on the fiber. □

3.2.2. Cubic prepotential. Now let us turn to the case of a cubic prepotential, i.e.

$$F = \frac{1}{6} d_{ijk} \frac{Z^i Z^j Z^k}{Z^0}, \tag{3.32}$$

where the lower case indices run from 1 to $n-1$. Note that, from now on, the special coordinates Z^I run from Z^0 to Z^{n-1}. Putting $z^i = Z^i/Z^0$, the first equation in (3.18) turns into

$$C_I = \begin{pmatrix} d_{ijk}(\frac{1}{3}D^0 z^i - \frac{1}{2}D^i) z^j z^k \\ -d_{ijk}(\frac{1}{2}D^0 z^j - D^j) z^k \end{pmatrix}, \tag{3.33}$$

which defines a Kähler submanifold $M_{\mathrm{sk}}^{\wedge}$ of M_{sk} under our general assumptions on the rank of the matrix (3.16), see the Remark after Proposition 3. By means of the coordinates $z^1, \dots, z^{n-1}$ we will identify M_{sk} with an open subset of $\mathbb{C}^{n-1}$.

Proposition 9. *Let $z_0 \in M_{\mathrm{sk}} \subset \mathbb{C}^{n-1}$ be a solution of Eq. (3.33) and $U \subset M_{\mathrm{sk}}$ an open neighborhood of z_0. Suppose that the rank of the matrix*

$$m_{ij} := d_{ijk}(D^k - D^0 z^k) \tag{3.34}$$

is constant on U. Then $M_{\mathrm{sk}}^{\wedge} \subset M_{\mathrm{sk}}$ is a complex submanifold of complex codimension $r = \mathrm{rk}(m_{ij})$. (More generally, it suffices to assume that the rank of (m_{ij}) is constant on a complex submanifold containing the algebraic subset of $U \subset \mathbb{C}^{n-1}$ defined by (3.33).)

Proof. The Jacobi matrix of the map $z \mapsto D^J F_{IJ}|_{Z=(1,z)}$ is given by

$$\begin{pmatrix} -m_{jk} z^k \\ m_{ij} \end{pmatrix}. \tag{3.35}$$

Since the first row is a linear combination of the other rows, the rank of that matrix coincides with the rank of (m_{ij}). □

Remark. Note that, as in the case of general prepotential, given a null vector (D^I) at $Z = (1, z) \in M_{\mathrm{ask}}$ we can define (C_I) such that (3.33) holds at z. Therefore, we can always assume that $M_{\mathrm{sk}}^{\wedge} \neq \emptyset$. For generic (d_{ijk}), (D^I) and z the rank of m_{ij} is maximal and so $\dim M_{\mathrm{sk}}^{\wedge} = 0$. Let us also keep in mind the trivial fact that for any $z_0 = (z_0^i) \in M_{\mathrm{sk}}$ there is always a nonzero vector (D^I) which satisfies (3.15) at $Z_0 = (1, z_0)$. The set of all such vectors (the null cone without its origin) is a $\mathbb{C}^*$-invariant real hypersurface of $T_{Z_0} M_{\mathrm{ask}} = \mathbb{C}^n$. Finally, let us point out that the constant vector (D^I) defining the submanifold $M_{\mathrm{sk}}^{\wedge}$ is a null vector not only at Z_0 but at any point Z of $M_{\mathrm{ask}}^{\wedge}$, since

$$\bar{D}^I N_{IJ}(Z) D^J = \frac{1}{2i}(\bar{D}^I C_I - D^I \bar{C}_I) = \bar{D}^I N_{IJ}(Z_0) D^J = 0.$$

The following proposition can be used in explicit examples to obtain an upper bound on the dimension of $M_{\mathrm{sk}}^{\wedge}$, which is defined by (3.33).

Proposition 10. *Let z_0 be any point of $M_{\mathrm{sk}}^{\wedge}$. A necessary condition for a vector $\alpha = \alpha^i \partial_{z^i} \in T_{z_0} M_{\mathrm{sk}}$ to be tangent to $M_{\mathrm{sk}}^{\wedge}$ is to satisfy the following equations:*

$$d_{ijk}(D^j - D^0 z_0^j)\alpha^k = 0 \tag{3.36}$$

and

$$d_{ijk}\alpha^i \alpha^j \alpha^k = 0. \tag{3.37}$$

Proof. Consider a complex analytic curve $\tau \mapsto z(\tau) = (z^i(\tau))$ in $M_{\mathrm{sk}}^{\wedge}$ through $z_0 = (z_0^i)$:

$$z^i(\tau) = z_0^i + \tau\alpha^i + \tau^2\beta^i + \tau^3\gamma^i + \cdots .$$

Then the last $n - 1$ equations of (3.33) are satisfied up to cubic order in τ if and only if:

$$0 = d_{ijk}(D^j - D^0 z_0^j)\alpha^k = 0, \tag{3.38}$$

$$0 = d_{ijk}(D^j - D^0 z_0^j)\beta^k - \frac{1}{2} D^0 d_{ijk}\alpha^j \alpha^k \quad \text{and} \tag{3.39}$$

$$0 = d_{ijk}(D^j - D^0 z_0^j)\gamma^k - D^0 d_{ijk}\alpha^j \beta^k. \tag{3.40}$$

The first equation already gives (3.36). Considering the τ^3-component of the first equation of (3.33) we also obtain

$$-d_{ijk} z_0^i (D^j - D^0 z_0^j)\gamma^k + (2D^0 z_0^i - D^i) d_{ijk}\alpha^j \beta^k + \frac{1}{3} D^0 d_{ijk}\alpha^i \alpha^j \alpha^k = 0. \tag{3.41}$$

Inserting (3.39)–(3.40) into (3.41), we find

$$d_{ijk}\alpha^i \alpha^j \alpha^k = 0.$$

□

Remark. Note that one can always find D^I s.t.

$$d_{ijk}(D^j - D^0 z_0^j)\alpha^k = 0$$

is not fulfilled for any $\alpha \neq 0$. On the other hand, depending on the particular form of d_{ijk}, one can adjust D^I in order to obtain examples for which $\dim N$ is large. We will discuss such examples in the remainder of this paper.

A low-dimensional example. We shall now see that a simple low-dimensional example with a one-dimensional manifold $M^{\wedge}_{\rm sk}$ is provided by the STU model with two coordinates fixed. The corresponding quaternionic Kähler manifold is the symmetric space

$$M = \frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(4)\times \mathrm{SO}(4)},$$

which is the c-map image of the special Kähler manifold

$$M_{\rm sk} = \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)^3.$$

Choosing appropriate inhomogeneous coordinates $z = (z^1 = S, z^2 = T, z^3 = U)$, the prepotential (3.32) is determined by

$$F(z^0 = 1, z) = STU.$$

Equation (3.33) defining the submanifold $M^{\wedge}_{\rm sk} \subset M_{\rm sk}$ now reads

$$2D^0 STU - D^S TU - D^T SU - D^U ST = C_0, \tag{3.42}$$
$$(D^0 T - D^T)(D^0 U - D^U) = D^T D^U - D^0 C_S, \tag{3.43}$$
$$(D^0 S - D^S)(D^0 U - D^U) = D^S D^U - D^0 C_T, \tag{3.44}$$
$$(D^0 S - D^S)(D^0 T - D^T) = D^S D^T - D^0 C_U, \tag{3.45}$$

where $(D^S, D^T, D^U) = (D^1, D^2, D^3)$ and we are assuming that $D^0 \neq 0$. From the last three equations we already see that two of the three coordinates, say S and T, must be fixed to the values $\langle S\rangle := \frac{D^S}{D^0}$ and $\langle T\rangle := \frac{D^T}{D^0}$ in order to keep the third coordinate, here U, free. Note that this is not possible for arbitrary choices of D^S and D^T since the coordinates have to satisfy $\sum_{I,J=0}^{3} N_{IJ} z^I \bar{z}^J > 0$. Therefore, we will assume that $(1, \langle S\rangle, \langle T\rangle)$ can be extended to a vector $(1, \langle S\rangle, \langle T\rangle, \langle U\rangle)$ spanning a complex line which is positive definite with respect to the pseudo-Hermitian metric (N_{IJ}). We will call such vectors *time-like*. One can check that all the above equations are solved for

$$C_S = D^U\langle T\rangle, \qquad C_T = D^U\langle S\rangle, \qquad C_U = D^0\langle S\rangle\langle T\rangle, \qquad C_0 = -D^U\langle S\rangle\langle T\rangle,$$

with U remaining arbitrary. Therefore, the coordinate U parameterises $M^{\wedge}_{\rm sk}$. It is straightforward to check that for any choice of $D^0 \neq 0$, D^S and D^T as above, the null condition (3.15) can be satisfied by appropriately choosing D^U. This ensures $D^0 U - D^U \neq 0$ on $M^{\wedge}_{\rm sk}$. The latter inequality implies that the matrix (m_{ij}) of Proposition 9 has rank two, which again proves that $M^{\wedge}_{\rm sk} \subset M_{\rm sk}$ is a one-dimensional complex submanifold. The resulting Kähler manifold M' has complex dimension 4.

We can also consider the quantum STU model, where the prepotential is given by

$$F(1, z) = STU + \frac{1}{3}T^3,$$

and the corresponding 6-dimensional special Kähler manifold $M_{\rm sk}$ admits a 4-dimensional group of automorphisms, which acts freely on $M_{\rm sk}$, as follows from [CMX], Example 3 in Sect. 4.2.

Again we can try to fix the values of one or two of the variables and use the remaining ones as parameters. In this case, Proposition 10 immediately implies that T cannot belong to the remaining parameters. Comparing with Eqs. (3.42)–(3.45) for the STU model, we see that only the conditions (3.42) and (3.44) are modified by the extra term in the prepotential. The new version of (3.44) reads

$$(D^0 S - D^S)(D^0 U - D^U) + (D^0 T - D^T)^2 = D^S D^U + (D^T)^2 - D^0 C_T,$$

which together with (3.43) and (3.45) implies that T must be fixed to some value $\langle T\rangle$. If $\langle T\rangle = \frac{D^T}{D^0}$, then also $S = \langle S\rangle = \frac{D^S}{D^0}$ and we come back to the solution for the STU model, now with

$$\begin{aligned} C_S &= D^U\langle T\rangle, \qquad C_T = D^U\langle S\rangle + D^T\langle T\rangle,\\ C_U &= D^0\langle S\rangle\langle T\rangle, \qquad C_0 = -D^U\langle S\rangle\langle T\rangle - \frac{1}{3}D^T\langle T\rangle^2. \end{aligned}$$

Again, U parameterises $M^{\wedge}_{\text{sk}}$ and we find again a 4-dimensional Kähler manifold M' as in the STU model. If the constant $\langle T\rangle$ is chosen to be real, then the term $\frac{1}{3}T^3$ in the prepotential will not contribute to the metric of M' and so we get the same Kähler metric as for the unperturbed STU model. Otherwise, the metric will change by a conformal factor of the form $\frac{e^{-2K_0}}{e^{-2K}} = (\frac{e^{-K_0}}{e^{-K_0}+c})^2$, where $c = \frac{8}{3}(\text{Im}\langle T\rangle)^3$ and K_0 is the Kähler potential of the unperturbed STU model.

High-dimensional examples. We can construct examples $M^{\wedge}_{\text{sk}}$ with high dimension by extending the example above to the manifold

$$M = \frac{\text{SO}_0(4,n)}{\text{SO}(4)\times\text{SO}(n)}, \quad n \geq 4, \tag{3.46}$$

which is the c-map image of

$$M_{\text{sk}} = ST[2,n-2] := \frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}_0(2,n-2)}{\text{SO}(2)\times\text{SO}(n-2)}.$$

The latter has complex dimension $n-1$. By appropriately choosing inhomogeneous coordinates the prepotential becomes

$$F(1,z) = STU + Sy^\ell y^m \delta_{\ell m}, \tag{3.47}$$

where now $z = (S,T,U,y)$, $y = (y^\ell)$ and $\ell, m = 1,\ldots,n-4$. For this prepotential we find from (3.33),

$$\begin{aligned} D^0 C_0 &= D^0(D^0 S - D^S)TU + D^0 S(D^0 T - D^T)U - D^0 D^U ST - SD^\ell\delta_{\ell m}D^m\\ &\quad + D^0(D^0 S - D^S)y^\ell\delta_{\ell m}y^m + S(D^0 y^\ell - D^\ell)\delta_{\ell m}(D^0 y^m - D^m), &(3.48)\\ D^0 C_S &= D^\ell\delta_{\ell m}D^m + D^T D^U - (D^0 T - D^T)(D^0 U - D^U)\\ &\quad -(D^0 y^\ell - D^\ell)\delta_{\ell m}(D^0 y^m - D^m), &(3.49)\\ D^0 C_T &= -(D^0 S - D^S)(D^0 U - D^U) + D^S D^U, &(3.50)\\ D^0 C_U &= -(D^0 S - D^S)(D^0 T - D^T) + D^S D^T, &(3.51)\\ D^0 C_\ell &= -2(D^0 S - D^S)\delta_{\ell m}(D^0 y^m - D^m) + 2D^S\delta_{\ell m}D^m. &(3.52) \end{aligned}$$

From the $n = 4$ example we expect that at least two directions should be fixed to a constant. Indeed, we have to at least fix S to the value $\langle S\rangle = \frac{D^S}{D^0}$ in order to solve

Eqs. (3.50)–(3.52). From (3.49) we then get an additional quadratic equation in the remaining coordinates that can be solved by choosing

$$T = \frac{D^T}{D^0} - \frac{\delta_{\ell m}(D^0 y^\ell - 2D^\ell)y^m}{D^0 U - D^U}$$

and (D^U, D^ℓ) as usual such that (3.15) holds at some base point $Z_0 = (1, z_0)$ and $D^0 U - D^U \neq 0$ on M_{sk} (one may have to replace M_{sk} by a neighborhood of z_0 for the latter). The solution for C_I is given by

$$C_S = \frac{D^T D^U}{D^0}, \qquad C_T = D^U \langle S \rangle, \qquad C_U = \langle S \rangle D^T,$$
$$C_\ell = 2\langle S \rangle \delta_{\ell m} D^m, \qquad C_0 = -\frac{D^T D^U}{D^0}\langle S \rangle.$$

Hence, the dimension of $M_{\mathrm{sk}}^\wedge$ is $2(n-3)$. The manifold M' therefore has dimension $4(n-2)$, which is eight smaller than the dimension of M. Note that the dimension of the submanifold $M_{\mathrm{sk}}^\wedge \subset M_{\mathrm{sk}}$ is only so high because we are fixing the direction S, which appears in both parts of the direct product manifold M_{sk}, i.e. in both terms in (3.47). This is already suggested by Proposition 10, which implies that in each monomial of $\sum_{i=1}^{n-1} d_{ijk} z^i z^j z^k$ at least one variable must be fixed. Also, it is known that the only special Kähler manifolds which are decomposable as a product are the symmetric spaces $ST[2, \ell]$, $\ell \geq 1$, [FVP]. The next step is to study special Kähler manifolds that are not symmetric and, hence, are not decomposable.

Examples of homogeneous manifolds. Let us now discuss the case of a homogeneous quaternionic manifold of negative scalar curvature that is not necessarily symmetric. These manifolds have been classified (under certain assumptions) in [A,Ce,dWV,C]. One simple class is the one that is in the image of the $c \circ r$ map[8] of the hyperbolic spaces

$$H_{\mathbb{R}}^{n-2} = \frac{\mathrm{SO}_0(n-2,1)}{\mathrm{SO}(n-2)}, \quad n \geq 3,$$

which is defined by the holomorphic prepotential

$$F(1, z) = S(ST - x^\ell \delta_{\ell m} x^m),$$

where $z = (S, T, x)$, $x = (x^\ell)$ and the indices ℓ, m run from 1 to $n-3$. Thus, the corresponding special Kähler manifold M_{sk} is still of complex dimension $n-1$. It is known that the corresponding quaternionic Kähler manifold M can be presented as a solvable Lie group $\mathcal{T}(p)$, $p = n-3$, of rank 3 with a left invariant quaternionic Kähler structure [C]. The only symmetric space in this series is $\mathcal{T}(0) = \frac{\mathrm{SO}_0(3,4)}{\mathrm{SO}(3)\times\mathrm{SO}(4)}$. We will consider the case $p \geq 1$.

Inserting this prepotential into (3.33) gives

$$C_0 = 2(D^0 S - D^S)ST - D^T S^2 - S(D^0 x^\ell - 2D^\ell)\delta_{\ell m} x^m - (D^0 S - D^S)x^\ell \delta_{\ell m} x^m, \tag{3.53}$$

$$C_S = -(D^0 S - 2D^S)T - (D^0 T - 2D^T)S + (D^0 x^\ell - 2D^\ell)\delta_{\ell m} x^m, \tag{3.54}$$

$$C_T = -(D^0 S - 2D^S)S, \tag{3.55}$$

$$C_\ell = (D^0 S - 2D^S)\delta_{\ell m} x^m + \delta_{\ell m}(D^0 x^m - 2D^m)S. \tag{3.56}$$

[8] The r-map is a construction of special Kähler manifolds, which was introduced by de Wit and Van Proeyen in [dWV]. See [CMX] for a recent discussion of some of its mathematical properties.

From (3.55) we see that S is always fixed, i.e. locally constant on $M^{\wedge}_{\mathrm{sk}}$. If S is fixed to some value $\langle S\rangle$ such that $D^0\langle S\rangle - D^S \neq 0$ one can conclude from (3.56) and (3.54) that T and x^m are also fixed. If $D^0\langle S\rangle - D^S = 0$, we find that (3.56) does not fix any further coordinates but only determines the value of C_ℓ. In contrast, (3.54) reads

$$D^\ell \delta_{\ell m} D^m + D^0 C_S - 2D^0 D^T \langle S\rangle = (D^0 x^\ell - D^\ell)\delta_{\ell m}(D^0 x^m - D^m),$$

which is a quadratic equation and fixes one of the complex degrees of freedom (which we will simply call moduli). Therefore, the minimal number of fixed moduli is two in the base space and two in the fiber. Now let us turn to the cubic equation (3.53). By using $D^0\langle S\rangle - D^S = 0$, we can write it as

$$D^0 C_0 + D^0 D^T \langle S\rangle^2 + \langle S\rangle D^\ell \delta_{\ell m} D^m = \langle S\rangle (D^0 x^\ell - D^\ell)\delta_{\ell m}(D^0 x^m - D^m),$$

which reduces to the above quadratic equation if C_0 is chosen properly. This shows that we can construct examples such that the final Kähler manifold M' has complex dimension $2n - 4$.

Now let us turn to a second series of homogeneous quaternionic Kähler manifolds $\mathcal{W}(p, q)$, which is a generalization of (3.46) and has the prepotential

$$F(1, z) = F(1, S, T, U, x, y) = STU + Sy^\ell \delta_{\ell m} y^m + T x^a \delta_{ab} x^b,$$

where $x = (x^\ell) \in \mathbb{R}^p$, $y = (y^a) \in \mathbb{R}^q$. The Alekseevsky spaces $\mathcal{W}(p, q)$ are of dimension $4n = 4(p + q + 4)$ and are symmetric only if $p = 0$ or $q = 0$. We will consider the case $p, q \geq 1$. Equation (3.33) now reads

$$\begin{aligned}
D^0 C_0 = {} & D^0(D^0 S - D^S)TU + D^0 S(D^0 T - D^T)U - D^0 D^U ST - SD^\ell \delta_{\ell m} D^m \\
& + D^0(D^0 S - D^S) y^\ell \delta_{\ell m} y^m + S(D^0 y^\ell - D^\ell)\delta_{\ell m}(D^0 y^m - D^m) \\
& + D^0(D^0 T - D^T) x^a \delta_{ab} x^b + T(D^0 x^a - D^a)\delta_{ab}(D^0 x^b - D^b) \\
& - T D^a \delta_{ab} D^b,
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
D^0 C_S = {} & D^\ell \delta_{\ell m} D^m + D^T D^U - (D^0 T - D^T)(D^0 U - D^U) \\
& - (D^0 y^\ell - D^\ell)\delta_{\ell m}(D^0 y^m - D^m),
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
D^0 C_T = {} & D^a \delta_{ab} D^b + D^S D^U - (D^0 S - D^S)(D^0 U - D^U) \\
& - (D^0 x^a - D^a)\delta_{ab}(D^0 x^b - D^b),
\end{aligned} \tag{3.59}$$

$$D^0 C_U = -(D^0 S - D^S)(D^0 T - D^T) + D^S D^T, \tag{3.60}$$

$$D^0 C_\ell = -2(D^0 S - D^S)\delta_{\ell m}(D^0 y^m - D^m) + 2D^S \delta_{\ell m} D^m, \tag{3.61}$$

$$D^0 C_a = -2(D^0 T - D^T)\delta_{ab}(D^0 x^b - D^b) + 2D^T \delta_{ab} D^b. \tag{3.62}$$

We see from (3.61) that the y^m can only be free if S is fixed to the value $\langle S\rangle = \frac{D^S}{D^0}$. However, from (3.59) we see that then the x^a must fulfill a quadratic equation. Similarly, if one does not want to fix all the moduli x^a, one must fix $\langle T\rangle = \frac{D^T}{D^0}$, cf. (3.62), and (3.58) gives one quadratic equation for the y^ℓ.[9] Let us now turn to the cubic equation (3.57). We see that for $\langle S\rangle = \frac{D^S}{D^0}$ and $\langle T\rangle = \frac{D^T}{D^0}$, this equation reduces to the quadratic equations encountered before, giving no new constraint on the remaining moduli. Therefore, four

[9] Note that the alternative of fixing e.g. the y^ℓ to $y^\ell = \frac{D^\ell}{D^0}$ reduces the set of equations to those for the case (3.46), with the same set of solutions. In that case only the fiber dimension differs from the M' obtained for (3.46).

moduli in the base space are fixed, which together with the two fiber directions make six fixed moduli. The dimension of the resulting Kähler manifold M' is thus eight smaller than that of the quaternionic manifold M.

General homogeneous manifolds. Finally, let us discuss the case of a general homogeneous space with cubic prepotential.[10] The prepotential for the general cubic case is given by [dWV]

$$F = h^1[(h^2)^2 - h^\mu \delta_{\mu\nu} h^\nu] - h^2 h^\ell \delta_{\ell m} h^m + h^\mu \gamma_{\mu\ell m} h^\ell h^m.$$

Here, the index μ labels $q+1$ fields while ℓ labels r fields that form representations of the $(q+1)$-dimensional Clifford algebra. Accordingly, the matrices γ_μ fulfill the Clifford algebra. The special Kähler base of M is therefore parameterised $3+q+r$ complex scalars. Thus the dimension of M is $4(4+q+r)$.

The analysis of possible dimensions of the Kähler quotient M' is done analogously to the examples discussed above. Inserting the above prepotential into (3.33), one finds

$$\begin{aligned}
D^0 C_0 &= 2D^0 h^1 h^2 (D^0 h^2 - D^2) - D^0 D^1 ((h^2)^2 - h^\mu \delta_{\mu\nu} h^\nu) \\
&\quad -2D^0 h^1 h^\mu \delta_{\mu\nu} (D^0 h^\nu - D^\nu) \\
&\quad -D^0 (D^0 h^2 - D^2) h^\ell \delta_{\ell m} h^m - h^2 (D^0 h^\ell - D^\ell) \delta_{\ell m} (D^0 h^m - D^m) \\
&\quad +D^0 \gamma_{\mu\ell m} (D^0 h^\mu - D^\mu) h^\ell h^m + \gamma_{\mu\ell m} h^\mu (D^0 h^\ell - D^\ell)(D^0 h^m - D^m) \\
&\quad +h^2 D^\ell \delta_{\ell m} D^m - h^\mu \gamma_{\mu\ell m} D^\ell D^m, \qquad (3.63) \\
D^0 C_1 &= (D^2)^2 - D^\mu \delta_{\mu\nu} D^\nu - (D^0 h^2 - D^2)^2 \\
&\quad +(D^0 h^\mu - D^\mu) \delta_{\mu\nu} (D^0 h^\nu - D^\nu), \qquad (3.64) \\
D^0 C_2 &= 2D^1 D^2 - D^\ell \delta_{\ell m} D^m - 2(D^0 h^1 - D^1)(D^0 h^2 - D^2) \\
&\quad +(D^0 x^\ell - D^\ell) \delta_{\ell m} (D^0 x^m - D^m), \qquad (3.65) \\
D^0 C_\mu &= \gamma_{\mu\ell m} D^\ell 2D^m - 2D^1 \delta_{\mu\nu} D^\nu + 2(D^0 h^1 - D^1) \delta_{\mu\nu} (D^0 h^\nu - D^\nu) \\
&\quad -(D^0 h^\ell - D^\ell) \gamma_{\mu\ell m} (D^0 h^m - D^m), \qquad (3.66) \\
D^0 C_\ell &= 2[(D^0 h^2 - D^2)\delta_{\ell m} - (D^0 h^\mu - D^\mu)\gamma_{\mu\ell m}](D^0 h^m - D^m) \\
&\quad -2D^2 \delta_{\ell m} D^m + 2D^\mu \gamma_{\mu\ell m} D^m. \qquad (3.67)
\end{aligned}$$

From (3.67) we see that the only h^ℓ that can stay massless are those in the kernel of the matrix

$$M_{\ell m}(h^2, h^\mu) = [(D^0 h^2 - D^2)\delta_{\ell m} + (D^0 h^\mu - D^\mu)\gamma_{\mu\ell m}].$$

On the other hand, a direction in the (h^2, h^μ)-plane can only remain unfixed if $D^0 h^\ell - D^\ell = 0$ holds for at least some of the scalars h^ℓ. In general, the minimal set of fixed scalars consists of just (h^2, h^μ). If we fix these scalars to $\langle h^2 \rangle = \frac{D^2}{D^0}$ and $\langle h^\mu \rangle = \frac{D^\mu}{D^0}$, then (3.64) and (3.67) are fulfilled for all values of the h^ℓ. Furthermore, we find $q+1$ quadratic equations for the h^ℓ from (3.65) and (3.66), which also solve (3.63). In total,

[10] The case of a quadratic prepotential has been discussed above.

this gives $2q+2$ fixed (complex) directions, leading to a Kähler quotient M' of (complex) dimension $2r+4$.[11]

4. Kähler Quotients and Spontaneous Partial Supersymmetry Breaking

The construction of the Kähler quotient of quaternionic-Kähler manifolds presented in Sect. 2 first arose in the physics literature in the derivation of the low-energy effective action of spontaneous $\mathcal{N}=2$ to $\mathcal{N}=1$ supersymmetry breaking in supergravity [L,LST1,LST2]. Let us close this paper by linking the mathematical analysis of the previous sections to the physical perspective of refs. [L,LST1,LST2].

The spectrum of $\mathcal{N}=2$ supergravity includes the gravitational multiplet together with n_{v} vector- and n_{h} hypermultiplets. Each hypermultiplet contains four real scalars which together span a $4n_{\mathrm{h}}$-dimensional field space M that is constrained by $\mathcal{N}=2$ supersymmetry to be quaternionic-Kähler. A necessary condition for a maximally-symmetric solution of the $\mathcal{N}=2$ supergravity field equations to preserve only $\mathcal{N}=1$ supersymmetry is that two isometries of the quaternionic-Kähler manifold are gauged [FGP,LST1]. The Higgs mechanism then makes the corresponding two vector fields massive, with the charged scalars providing the longitudinal degrees of freedom. Consistency with $\mathcal{N}=1$ supersymmetry demands that these isometries satisfy the assumed properties of Theorem 5.

In order to derive an effective action valid below the scale of supersymmetry breaking $m_{3/2}$ one needs to integrate out all fields with masses of order $m_{3/2}$. Integrating out massive scalar fields corresponds to taking a submanifold $N\subset M$, while integrating out the two massive vector fields corresponds to taking the quotient with respect to the two-dimensional Abelian Lie group A generated by the two Killing vectors, as specified in Theorem 1. The two charged scalars act as Goldstone bosons and are removed from the scalar field space by the quotient construction described in Theorem 5. Consistency with $\mathcal{N}=1$ supersymmetry implies that the resulting scalar field space $M'=N/A$ should be Kähler.

For generic quaternionic-Kähler manifolds M the precise identification of massive versus massless fields or, in other words the identification of the submanifold N, is difficult. However, for the case of special quaternionic-Kähler manifolds, i.e. manifolds in the image of the c-map, N is determined by (3.18), which we repeat here for convenience:

$$D^A F_{AB}(Z)=C_B, \qquad D^A(b_A-F_{AB}a^B)=\tilde{C}. \tag{4.1}$$

These equations give $2r_{\mathrm{F}}+2$ real conditions, where $r_{\mathrm{F}}=\operatorname{rank}(F_{ABC}D^C)$. From this one can read off the dimension of the submanifold N to be $4n_{\mathrm{h}}-2(r_{\mathrm{F}}+1)$.

The dimension of the quotient M' is two less than that of the submanifold N_{h}. Therefore, the specific dimensions of the quotient M'_{h} is model-dependent and depends on the number of hypermultiplet scalars which remain massless, i.e. on the dimension of N. The maximal rank of $F_{ABC}D^C$ is $n_{\mathrm{h}}-1$ due to $F_{ABC}X^A=0$, therefore for generic F and D^A the dimension of N is $2n_{\mathrm{h}}$, cf. Proposition 3. In other words, generically all moduli in the special Kähler base of the special quaternionic-Kähler manifold are fixed. However, only two of the axionic scalars in the G-fiber are fixed. For special choices of

[11] Alternatively, one could choose to fix h^1 and all h^ℓ, with one additional constraint coming from (3.64), resulting in a Kähler quotient M' of complex dimension $4+2q+r$. Depending on q and r, this might be larger or smaller than $2r+4$. Since r must be a multiple of the dimension of the fundamental representation of the q-dimensional Clifford algebra, generically r will be much larger than q. Note that there is also the possibility of fixing some h^ℓ and some (h^2,h^μ), which we do not discuss any further here.

the prepotential F and fine-tuned D^A one can increase the dimension of M', as discussed in detail in Sect. 3.

$\mathcal{N} = 2$ gauged supergravities in four dimensions appear in the low-energy limit of compactifications of string theory on Calabi-Yau and, more generally, $SU(3) \times SU(3)$-structure manifolds. In all these theories the quaternionic-Kähler manifold are of the special form described in [CFG, FS]. In the limit of large volume the holomorphic prepotential simplifies to become cubic. In Sect. 3.2.2 we analysed a large class of special quaternionic-Kähler manifolds with cubic prepotentials, including the examples of general homogeneous manifolds classified in [dWV] and the inhomogeneous quantum STU model. We found that it is possible to obtain both high- and low-dimensional moduli spaces, with the latter being generic. From the perspective of string theory compactifications, the fact that we generically find low-dimensional moduli spaces is particularly attractive as it suggests that moduli stabilisation can be easily implemented.

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