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RESEARCH ARTICLE

Quantum gravitational bremsstrahlung: massless versus massive gravity

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Abstract The massive spin-2 quantum gauge theory previously developed is applied to calculate gravitational bremsstrahlung. It is shown that this theory is unique and free from defects. In particular, there is no strong coupling if the graviton mass becomes small. The cross sections go over smoothly into the ones of the massless theory in the limit of vanishing graviton mass. The massless cross sections are calculated for the full tensor theory.

1 Introduction and summary

Gravity with massive gravitons got much interest in recent years in view of the evidence for dark energy. However, various massive gravity theories suffer from serious defects. There may be a discontinuity if the graviton mass m_0 goes to zero (VDVZ discontinuity), a violation of Lorentz invariance or fields with wrong sign of the kinetic term in the Lagrangian ([1–4] and references given therein). It is misleading to call those fields "ghosts" because this name is occupied by the Fermi fields with integer spin which are essential in gauge theories (Faddeev—Popov ghosts). The reason for the failure is that those theories have been set up *without taking the gauge structure into account*. The early work by Boulware and Deser [5] can be criticized for the same reason. One should never forget that gravity is a gauge theory side by side with non-abelian gauge theories [6]. This is true for both, massless and massive gravity.

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In fact, D.R. Grigore and one of us (G.S.) have constructed the massive spin-2 gauge theory on Minkowski background according to the laws of perturbative quantum gauge invariance [7]. The result was that this theory is essentially unique and its classical limit agrees (in the pure graviton sector) with general relativity with a cosmological constant $\Lambda = m_0^2/2$. The consistency has been verified beyond linear gravity up to the quartic couplings of the graviton. A proof to all orders has been given in the massless case [8]. There is no doubt that massive gravity considered as spin-2 quantum gauge theory is mathematically consistent. There remains the question whether it is also consistent with nature for the very small graviton mass $m_0 = \sqrt{2\Lambda}$.

It has been shown in [7] that this theory has no discontinuity for $m_0 \to 0$ as far as the graviton propagator is concerned. Of course it is fully Lorentz invariant. Another question is whether the additional degrees of freedom of the massive graviton give rise to a low strong coupling scale as in the theories ([4]) mentioned above. This question is studied here for gravitational bremsstrahlung because this process might be close to experimental observations.

The origin of the so-called strong coupling is the longitudinal graviton mode of the form [see [4] Eq. (6)]

$$e_{\mu\nu} \sim \frac{k_\mu k_\nu}{m_0^2} + \cdots \tag{1.1}$$

Since $m_0 = \sqrt{2\Lambda}$ is very small, the scattering amplitudes involving this mode get large. But this disaster is not real because the longitudinal (massive) graviton states do not belong to the physical Hilbert space $\mathcal{H}_{\text{phys}}$. As well known in a gauge theory, $\mathcal{H}_{\text{phys}}$ is defined by means of the nilpotent gauge charge Q ($Q^2 = 0$) in the following, two possible forms [6]

$$\mathcal{H}_{\text{phys}} = \text{Ker}\, Q/\text{Ran}\, Q \tag{1.2}$$

$$\mathcal{H}_{\text{phys}} = \text{Ker}(QQ^+ + Q^+Q). \tag{1.3}$$

The second representation (1.3) is best suited for the construction of the physical states in momentum space. For massive gravity, Q is given in terms of the asymptotic free quantum fields by

$$Q \stackrel{\text{def}}{=} \int\limits_{x^0=t} d^3x \left[\partial_\nu h^{\mu\nu}(x) + m_0 v^\mu(x)\right] \overleftrightarrow{\partial}_0 u_\mu(x). \tag{1.4}$$

Here $h^{\mu\nu}$ is the symmetric tensor field, v^μ a Bose vector field and u_μ is the fermiomic ghost field. All fields satisfy Klein–Gordon equations with mass m_0, further properties are discussed in the next section.

To study massive gravity in the neighborhood of $m_0 = 0$, we choose a Lorentz frame where the graviton momentum is $k_\mu = (\omega, 0, 0, k_3)$, $\omega^2 = k_3^2 + m_0^2$. Then in Sect. 3, we determine the kernel of the selfadjoint operator in (1.3) by calculating its eigenfunctions for eigenvalue 0. The kernel consists of 6 modes. Two modes agree exactly with the two transverse graviton states of the massless theory. The other four modes have contributions from the emission operators of the vector field v^μ. In the limit $m_0 \to 0$, only the latter survive. For this reason, we call v^μ the vector-graviton field. Note that no longitudinal mode contributes to the physical subspace in this representation. Since the vector-graviton does not couple to ordinary matter in a gauge invariant way, the cross-sections for bremsstrahlung go over smoothly into the massless ones in the limit $m_0 \to 0$.

In the last section, we calculate the differential cross-section and the total radiated energy of quantum gravitational bremsstrahlung in the lowest order of perturbation theory for $m_0 = 0$. It seems that this has not been done before for the full tensor theory. In particular, we consider the case of an ultrarelativistic particle being scattered by a heavy mass M through a small angle. Under these assumptions, our results agree with classical calculations.

2 Field content and gauge structure

The basic free asymptotic fields of massive gravity are the symmetric tensor field $h^{\mu\nu}(x)$ with arbitrary trace, the fermionic ghost $u^\mu(x)$ and anti-ghost $\tilde{u}^\mu(x)$ fields and the bosonic vector-graviton field $v^\mu(x)$. They all satisfy the massive Klein–Gordon equations

$$(\Box + m_0^2)h^{\mu\nu} = 0 = (\Box + m_0^2)u^\mu \cdots \tag{2.1}$$

and are quantized as follows [7]

$$\left[h^{\alpha\beta}(x), h^{\mu\nu}(y)\right] = -ib^{\alpha\beta\mu\nu} D_{m_0}(x-y) \tag{2.2}$$

with

$$b^{\alpha\beta\mu\nu} = \frac{1}{2}\left(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}\right) \tag{2.3}$$

$$\left\{u^\mu(x), \tilde{u}^\nu(y)\right\} = i\eta^{\mu\nu} D_{m_0}(x-y) \tag{2.4}$$

$$\left[v^\mu(x),\ v^\nu(y)\right] = \frac{1}{2}\eta^{\mu\nu} D_{m_0}(x-y) \tag{2.5}$$

and zero otherwise. Here, $D_{m_0}(x)$ is the Jordan–Pauli distribution with mass m_0 and $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ the Minkowski tensor. Note that the vector-graviton field v^μ is essential to render the gauge charge Q nilpotent:

$$\begin{aligned} Q^2 = \frac{1}{2}\{Q, Q\} &= \frac{1}{2}\int d^3x(\partial_\nu h^{\mu\nu} + mv^\mu)\left\{\overleftrightarrow{\partial}_0 u_\mu, Q\right\} \\ &\quad -\frac{1}{2}\int d^3x\left[\partial_\nu h^{\mu\nu} + mv^\mu, Q\right]\overleftrightarrow{\partial}_0 u_\mu \\ &= 0. \end{aligned}$$

The factor 1/2 in (2.5) is due to the convention (2.3).

The commutators or anticommutators with Q, respectively, define the gauge variations of the asymptotic fields. The commutation relations immediately yield

$$d_Q h^{\mu\nu} \stackrel{\text{def}}{=} [Q, h^{\mu\nu}] = -\frac{i}{2}\big(\partial^\nu u^\mu + \partial^\mu u^\nu - \eta^{\mu\nu}\partial_\alpha u^\alpha\big) \tag{2.6}$$

$$d_Q u^\mu \stackrel{\text{def}}{=} \{Q, u\} = 0 \tag{2.7}$$

$$d_Q \tilde{u}^\mu \stackrel{\text{def}}{=} \{Q, \tilde{u}^\mu\} = i\big(\partial_\nu h^{\mu\nu} - m_0 v^\mu\big) \tag{2.8}$$

$$d_Q v^\mu \stackrel{\text{def}}{=} [Q, v^\mu] = -\frac{i}{2} m_0 u^\mu. \tag{2.9}$$

Note that if (2.6) is considered as an equation between classical fields, this is just the infinitesimal version of the diffeomorphic coordinate transformations of general relativity.

Now we are able to formulate gauge invariance of the S-matrix. Let

$$S(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n\, T_n(x_1, \ldots, x_n)\, g(x_1) \cdots g(x_n) \tag{2.10}$$

be the perturbative S-matrix where $g(x)$ is a Schwartz test function which switches the interaction. The time-ordered products T_n are expressed in terms of free fields defined above. Then $d_Q T_n$ is well defined and gauge invariance of $S(g)$ is defined perturbatively as follows:

First order gauge invariance:

$$d_Q T_1(x) = i \frac{\partial}{\partial x^\mu} T^\mu_{1/1}(x) \tag{2.11}$$

nth order gauge invariance:

$$d_Q T_n = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T\big\{T_1(x_1) \ldots T^\mu_{1/1}(x_l) \cdot \ldots T_1(x_n)\big\} \tag{2.12}$$

where the time-ordered products must be appropriately normalized. It is a very nice feature that the conditions (2.11) and (2.12) for $n = 2$ determine the theory essentially uniquely (that means up to divergence and coboundary couplings). No classical Lagrangian is needed. It was shown in [7] that in case of the massive spin-2 theory we have:

$$\begin{aligned} T_1 = {} & h^{\mu\nu}\partial_\mu h \partial_\nu h - 2h^{\mu\nu}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} - 4h_{\mu\nu}\partial_\alpha h^{\mu\beta}\partial_\beta h^{\nu\alpha} - 2h_{\mu\nu}\partial_\alpha h^{\mu\nu}\partial^\alpha h \\ & + 4h_{\mu\nu}\partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta_\nu + 4\partial_\mu h^{\mu\nu} u^\alpha \partial_\alpha \tilde{u}_\nu - 4h^{\mu\nu}\partial_\mu u^\alpha \partial_\nu \tilde{u}_\alpha + 4h^{\mu\nu}\partial_\mu v^\alpha \partial_\nu v_\alpha \\ & - 4m_0 \partial_\mu v_\nu u^\mu \tilde{u}^\nu + m_0^2\Big(-\frac{4}{3}h^{\mu\nu}h_{\mu\alpha}h^\alpha_\nu + h^{\mu\nu}h_{\mu\nu}h - \frac{1}{6}h^3\Big) \end{aligned} \tag{2.13}$$

up to a coupling constant, where $h = h^\mu_\mu$ is the trace; all products are normally ordered. As mentioned in the introduction, this result is in agreement with the expansion of the Einstein–Hilbert Lagrangian with a cosmological constant [7].

For bremsstrahlung we need the gravity-matter couplings. We represent the matter by a non-hermitian scalar field φ of mass m. Its coupling to $h_{\mu\nu}$ can be derived from gauge invariance as in the massless case ([6], Sect. 5.9):

$$T^m(x) = i\kappa\Big(h^{\mu\nu}\partial_\mu\varphi^+\partial_\nu\varphi - \frac{m^2}{2}h\varphi^+\varphi\Big). \tag{2.14}$$

But in the massive case, there may also be a coupling between φ and the vector-graviton field v^μ of the form

$$T_1 = a_1\partial_\mu v^\mu\varphi^+\varphi + a_2 v^\mu\partial_\mu\varphi^+\varphi + a_3 v^\mu\varphi^+\varphi. \tag{2.15}$$

In order to fulfill first order gauge invariance (2.11), $d_Q T_1$ must be a divergence. Using $d_Q\varphi = 0$ and (2.9) we obtain

$$d_Q T_1 = -\frac{i}{2}m_0 a_1\partial_\mu v^\mu\varphi^+\varphi - \frac{i}{2}m_0(a_2 u^\mu\partial_\mu\varphi^+\varphi + a_3 u^\mu\varphi^+\partial_\mu\varphi).$$

This must be a divergence

$$= -\frac{i}{2}m_0\partial_\mu(b_1 u^\mu\varphi^+\varphi) = -\frac{i}{2}b_1(\partial_\mu u^\mu\varphi^+\varphi + u^\mu\partial_\nu\varphi^+\varphi + u^\mu\varphi^+\partial_\mu\varphi).$$

Then it follows that $b_1 = a_1 = a_2 = a_3$ and

$$T_1 = a_1\partial_\mu(v^\mu\varphi^+\varphi).$$

Since such a divergence coupling is physically irrelevant, there exists no non-trivial gauge invariant coupling between v^μ and φ.

Finally, the interaction of the φ-particle with an external classical potential is described by (2.14) where $h^{\mu\nu}$ is substituted by $h^{\mu\nu}_{\rm ext}$. In the Newtonian limit, which we will consider later, we have

$$h^{00} = -2\phi_N \tag{2.16}$$

where ϕ_N is the Newtonian potential.

3 The physical Hilbert space for massive gravitons

To get a concrete Hilbert space representation, we must express the various fields by means of emission and absorption operators. We follow the discussion of the massless case in [6] as close as possible. We decompose $h^{\alpha\beta}$ into its traceless part and the trace h

$$h^{\alpha\beta}(x) = H^{\alpha\beta}(x) + \frac{1}{4}\eta^{\alpha\beta}h(x). \tag{3.1}$$

From (2.2) we obtain the following commutation relations

$$\big[h(x), h(y)\big] = 4i D_{m_0}(x-y) \tag{3.2}$$

$$\big[H^{\alpha\beta}(x), H^{\mu\nu}(y)\big] = -i t^{\alpha\beta\mu\nu} D_{m_0}(x-y) \tag{3.3}$$

with

$$t^{\alpha\beta\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}\big(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}\big) = t^{\mu\nu\alpha\beta} \tag{3.4}$$

and

$$\big[H^{\alpha\beta}(x), h(y)\big] = 0. \tag{3.5}$$

We claim that the fields in (3.2) and (3.3) can be represented as follows:

$$H^{\alpha\beta}(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega}} \Big(a_{\alpha\beta}(\vec{k})e^{-ikx} + \eta^{\alpha\alpha}\eta^{\beta\beta} a^+_{\alpha\beta}(\vec{k})e^{ikx}\Big) \tag{3.6}$$

where $a_{\alpha\beta} = a_{\beta\alpha}$ is symmetric and satisfies the commutation relation

$$\big[a_{\alpha\beta}(\vec{k}), a^+_{\mu\nu}(\vec{k}')\big] = \eta^{\alpha\alpha}\eta^{\beta\beta} t^{\alpha\beta\mu\nu}\delta(\vec{k}-\vec{k}'). \tag{3.7}$$

The trace part is given by

$$h(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega}} \Big(d(\vec{k})e^{-ikx} - d^+(\vec{k})e^{ikx}\Big) \tag{3.8}$$

with

$$\big[d(\vec{k}), d^+(\vec{k}')\big] = 4\delta(\vec{k}-\vec{k}'). \tag{3.9}$$

Since the right-hand side is positive, the h-sector of the Fock space can be constructed in the usual way by applying products of d^+'s to the vacuum.

The situation is not so simple in the H-sector because the righthand side of (3.7) is not a diagonal matrix. We perform a linear transformation of the diagonal operators $a_{\alpha\alpha}$ and $a^+_{\alpha\alpha}$ in such a way that the new operators are usual annihilation and creation operators

$$\big[\tilde{a}_{\alpha\alpha}(\vec{k}), \tilde{a}^+_{\beta\beta}(\vec{k}')\big] = \delta_{\alpha\beta}\delta(\vec{k}-\vec{k}'). \tag{3.10}$$

The following transformation does the job:

$$\begin{aligned}
a_{00} &= \frac{1}{2}\big(\tilde{a}_{11} + \tilde{a}_{22} + \tilde{a}_{33}\big)\\
a_{11} &= \frac{1}{2}\big(-\tilde{a}_{11} + \tilde{a}_{22} + \tilde{a}_{33}\big)\\
a_{22} &= \frac{1}{2}\big(\tilde{a}_{11} - \tilde{a}_{22} + \tilde{a}_{33}\big)\\
a_{33} &= \frac{1}{2}\big(\tilde{a}_{11} + \tilde{a}_{22} - \tilde{a}_{33}\big).
\end{aligned} \tag{3.11}$$

We note that $\tilde{a}_{00}$ does not appear because one pair of absorption and emission operators is superfluous due to the trace condition $H^\alpha{}_\alpha = 0$. In fact, from (3.11) we see

$$\sum_{j=1}^{3} a_{jj} = a_{00}. \tag{3.12}$$

The Fock representation can now be constructed as usual by means of $\tilde{a}_{11}^+, \tilde{a}_{22}^+, \tilde{a}_{33}^+$ and $a_{\alpha\beta}^+$ with $\alpha \neq \beta$.

The other fields have the following representation in terms of emission and absorption operators:

$$u^\mu(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2E_k}} \left(c_2^\mu(\vec{k}) e^{-ipx} - \eta^{\mu\mu} c_1^\mu(\vec{k})^+ e^{ipx} \right) \tag{3.13}$$

$$\tilde{u}^\mu(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2E_k}} \left(-c_1^\mu(\vec{k}) e^{-ipx} - \eta^{\mu\mu} c_2^\mu(\vec{k})^+ e^{ipx} \right) \tag{3.14}$$

$$v^\mu(x) = (2\pi)^{-3/2} \int \frac{d^3k}{2\sqrt{E_k}} \left(b^\mu(\vec{k}) e^{-ipx} - \eta^{\mu\mu} b^\mu(\vec{k})^+ e^{ipx} \right) \tag{3.15}$$

with the following (anti)commutation relations

$$\left\{ c_j^\mu(\vec{k}), c_l^\nu(\vec{k}')^+ \right\} = \delta_{jl} \delta_\nu^\mu \delta^3(\vec{k} - \vec{k}') \tag{3.16}$$

$$\left[b^\mu(\vec{k}), b^\nu(\vec{k})^+ \right] = \delta_\nu^\mu \delta^3(\vec{k} - \vec{k}'). \tag{3.17}$$

Then the gauge charge Q (1.4) can be written in momentum space as follows

$$Q = \int d^3k \left(A^\alpha(\vec{k})^+ c_2^\gamma(\vec{k}) - B^\alpha(\vec{k}) c_1^\gamma(\vec{k})^+ \right) \eta_{\alpha\gamma} \tag{3.18}$$

where

$$A^\alpha = \eta^{\alpha\alpha} a^{\alpha\beta}(\vec{k}) k^\beta - \frac{k^\alpha}{4} d(\vec{k}) - i m_1 \eta^{\alpha\alpha} b^\alpha \tag{3.19}$$

$$B^\alpha = \left(a^{\alpha\beta}(\vec{k}) k_\beta + \frac{k^\alpha}{4} d(\vec{k}) + i m_1 b^\alpha \right) \eta^{\alpha\alpha}. \tag{3.20}$$

The adjoint is given by

$$Q^+ = \int d^3k \left(c_2^\delta(\vec{k})^+ A^\beta(\vec{k}) - c_1^\delta(\vec{k}) B^\beta(\vec{k})^+ \right) \eta_{\delta\beta} \tag{3.21}$$

where

$$m_1 = \frac{m_0}{\sqrt{2}}. \tag{3.22}$$

According to (1.3), the physical Hilbert space is the kernel of the selfadjoint operator

$$\{Q, Q^+\} = \int d^3k\, d^3k' \Big(A^\alpha(\vec{k})^+ A^\beta(\vec{k}')\{c_2^\gamma(\vec{k}), c_2^\delta(\vec{k}')^+\}$$
$$+B^\beta(\vec{k}')^+ B^\alpha(\vec{k})\{c_1^\delta(\vec{k}'), c_1^\gamma(\vec{k})^+\} + c_2^\delta(\vec{k}')^+ c_2^\gamma(\vec{k})\big[A^\beta(\vec{k}'), A^\alpha(\vec{k})^+\big]$$
$$+c_1^\gamma(\vec{k})^+ c_1^\delta(\vec{k}')\big[B^\alpha(\vec{k}), B^\beta(\vec{k}')^+\big]\Big)\eta_{\alpha\gamma}\eta_{\beta\delta}. \tag{3.23}$$

We restrict to the graviton sector because the ghost sector is unphysical.

$$\{Q, Q^+\}|_{\text{graviton}} = \int d^3k \sum_{\alpha=0}^{3} \Big(A^{\alpha+}A^\alpha + B^{\alpha+}B^\alpha\Big). \tag{3.24}$$

It is convenient to introduce time-like and space-like components:

$$\begin{aligned}
A^0 &= k_0\big(a^{00} - a_\parallel^0 - \frac{d}{4} - \frac{im_1}{k_0}b^0\big)\\
A^j &= k_0\big(-a^{0j} + a_\parallel^j - \frac{k^j}{k_0}\frac{d}{4} + \frac{im_1}{k_0}b^j\big)\\
B^0 &= k_0\big(a^{00} + a_\parallel^0 + \frac{d}{4} + \frac{im_1}{k_0}b^0\big)\\
B^j &= k_0\big(-a^{0j} - a_\parallel^j - \frac{k^j}{k_0}\frac{d}{4} - \frac{im_1}{k_0}b^j\big)
\end{aligned} \tag{3.25}$$

where

$$a_\parallel^\mu = \frac{k_j}{k_0}a^{\mu j}. \tag{3.26}$$

We choose a Lorentz frame where $k_\mu = (\omega, 0, 0, k_3)$. Then we get for the integrand in (3.24)

$$\begin{aligned}
\sum_{\alpha=0}^{3}\Big(A^{\alpha+}A^\alpha + B^{\alpha+}B^\alpha\Big) = 2\omega^2 \Big\{ & a^{00+}a^{00} + \frac{k_3^2}{\omega^2}a^{03+}a^{03} + a^{01+}a^{01} + a^{02+}a^{02}\\
& + a^{03+}a^{03} + \frac{k_3^2}{\omega^2}\Big[a^{13+}a^{13} + a^{23+}a^{23} + a^{33+}a^{33}\Big]\\
& + \frac{1}{16}\big(1 + \frac{k_3^2}{\omega^2}\big)d^+d + \frac{im_1}{4\omega}d^+b^0 - \frac{im_1}{4\omega}b^{0+}d\\
& + \frac{im_1k_3}{\omega^2}a^{03+}b^0 - \frac{im_1k_3}{\omega^2}b^{0+}a^{03} + \frac{im_1k_3}{\omega^2}\Big[a^{13+}b^1\\
& + a^{23+}b^2 + a^{33+}b^3 - b^{1+}a^{13} - b^{2+}a^{23} - b^{3+}a^{33}\Big]\\
& + \frac{m_1^2}{\omega^2}\Big[b^{0+}b^0 + \sum_{j=1}^{3}b^{j+}b^j\Big]\Big\}.
\end{aligned} \tag{3.27}$$

Since a^{12+} does not appear herein, the states $a^{12+}\Omega$, where Ω is the Fock vacuum, certainly belong to the kernel of (3.24) and, hence, are in the physical subspace. We have still to substitute the diagonal operators $a^{\mu\mu}$ by means of (3.11) by the operators $\tilde{a}^{jj}$ which generate the Fock states. Then the quadratic form (3.27) can be represented in matrix notation A^+XA where A^+ stands for the emission operators

$$A^+ = (\tilde{a}^+_{11}, \tilde{a}^+_{22}, \tilde{a}^+_{33}, \sqrt{2}a^+_{01}, \sqrt{2}a^+_{02}, \sqrt{2}a^+_{03}, \sqrt{2}a^+_{13}, \sqrt{2}a^+_{23}, 1/2d^+, b^+_0, b^+_1, b^+_2, b^+_3) \tag{3.28}$$

where the numerical factors are necessary to get the states correctly normalized due to (3.7), (3.9), (3.10) and (3.17). According to (3.27), X is the hermitian 13×13 matrix

$$X = \begin{pmatrix}
z & z & m_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & im_1k_3 \\
z & z & m_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & im_1k_3 \\
m_1^2 & m_1^2 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -im_1k_3 \\
0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2z & 0 & 0 & 0 & \sqrt{2}im_1k_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & k_3^2 & 0 & 0 & 0 & \sqrt{2}im_1k_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & k_3^2 & 0 & 0 & 0 & \sqrt{2}im_1k_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & im_1\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{2}im_1k_3 & 0 & 0 & -im_1\omega & 2m_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}im_1k_3 & 0 & 0 & 0 & 2m_1^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}im_1k_3 & 0 & 0 & 0 & 2m_1^2 & 0 \\
-im_1k_3 & -im_1k_3 & im_1k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2m_1^2
\end{pmatrix} \tag{3.29}$$

where

$$z = \frac{1}{2}(\omega^2 + k_3^2). \tag{3.30}$$

The eigenvalues of this matrix are:

$$\begin{aligned}\lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0\\ \lambda_6 &= \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \omega^2 = 2m_1^2 + k_3^2\\ \lambda_{12} &= \lambda_{13} = 3m_1^2 + 2k_3^2.\end{aligned} \tag{3.31}$$

The 5 eigenvectors with eigenvalue 0 determine the kernel. They can be easily calculated from the matrix (3.29). Together with the previously found state $a_{12}^+\Omega$ we have 6 physical massive graviton modes:

$$\psi_1 = \sqrt{2}a_{12}^+\Omega \tag{3.32}$$

$$\psi_2 = c_2\left(\frac{im_1}{k_3}(\tilde{a}_{11} - \tilde{a}_{33}) + b_3\right)^+\Omega \qquad c_2 = \left(1 + \frac{2m_1^2}{k_3^2}\right)^{-1/2} \tag{3.33}$$

$$\psi_3 = c_3\left(\frac{2im_1}{k_3}a_{23} + b_2\right)^+\Omega \qquad c_3 = \left(1 + \frac{2m_1^2}{k_3^2}\right)^{-1/2} \tag{3.34}$$

$$\psi_4 = c_4\left(\frac{2im_1}{k_3}a_{13} + b_1\right)^+\Omega \qquad c_4 = \left(1 + \frac{2m_1^2}{k_3^2}\right)^{-1/2} \tag{3.35}$$

$$\begin{aligned}\psi_5 &= c_5\left(\frac{im_1}{2(m_1^2+k_3^2)}\left(2k_3a_{03} + \sqrt{2m_1^2+k_3^2}\,d\right) + b_0\right)^+\Omega\\ c_5 &= \left(1 + m_1^2\,\frac{3k_3^2+4m_1^2}{2(m_1^2+k_3^2)^2}\right)^{-1/2}\end{aligned} \tag{3.36}$$

$$\psi_6 = \frac{1}{\sqrt{2}}(\tilde{a}_{11} - \tilde{a}_{22})^+\Omega. \tag{3.37}$$

All physical states are normalized due to (3.7), (3.9), (3.10) and (3.17).

The states ψ_1 and ψ_6 agree exactly with the two transverse physical modes of the massless graviton. The other four massive physical graviton modes converge in the massless limit to the four vector-graviton states $b_\mu^+\Omega$, $\mu = 0, 1, 2, 3$. We have seen in the last section that the latter do not couple to matter in a gauge invariant way. This is the reason why the scattering amplitudes for bremsstrahlung go over smoothly to the massless ones in the limit $m_0 \to 0$. We will return to this in the next section.

4 Quantum gravitational bremsstrahlung

We consider a particle of mass m, described by a non-hermitian scalar field $\varphi(x)$, which is scattered by a fixed central gravitational potential induced by a mass M. The φ-particle looses energy-momentum and emits a (massive or massless) graviton. The interaction with the classical central potential ϕ will be treated according to the external field approximation where in the Newtonian limit, ϕ becomes the Newtonian potential ϕ_N. In lowest order, the process is described by four second-order Feynman diagrams

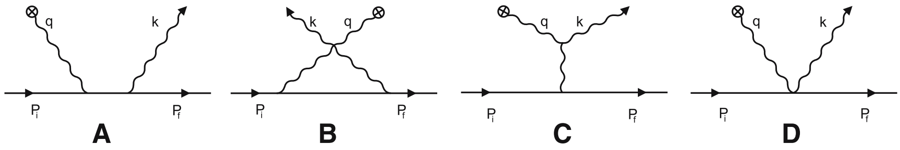

Fig. 1 Lowest order Feynman diagrams

according to (Fig. 1) where $p_i^\mu = (E_i, \vec{p}_i)$, $p_f^\mu = (E_f, \vec{p}_f)$ are the four-momenta of the initial and final particle, $k^\mu = (\omega, \vec{k})$ is the momentum of the bremsstrahl graviton, with $k_\mu k^\mu = m_0^2$. We will assume the mass m of the particle to be much smaller than the central mass $M \gg m$ and the scattering takes place under a small angle $\Theta' \ll m/E_i$ which in the classical picture corresponds to a large impact parameter ϱ. Then we are able to use the first Born approximation for the external field ϕ_N. Since at the end, we want to compare our results with the classical theory, we consider the case of ultra-relativistic particle energy ($m \ll E_i$) and low frequency ($\omega \ll E_i$). This limit is known as the only overlapping region of the Born and quasiclassical approximation.

According to our findings in the last section, we consider the S-matrix element for each massive graviton mode $\psi_a, a = 1, \ldots, 6$ in the final state separately. Again, we choose the coordinate system such that the outgoing graviton has momentum $k_\mu = (\omega, 0, 0, k_3)$. In the scalar product $(\phi_f, S\phi_i)$, where ϕ_i, ϕ_f is the initial, final state, respectively, only the emission operators $a_{\mu\nu}^+$ and d^+ in the final bremsstrahl graviton state are to be contracted with the coupling term (2.14) because there is no coupling to v^μ. In $\psi_2, \ldots, \psi_5$, these contributions are of order $O(m_0)$. Consequently, the matrix elements of these final states are of order $O(m_0)$, too, compared to the contributions $O(1)$ of ψ_1 and ψ_6. This proves that in the massless limit $m_0 \to 0$, the cross sections converge to the massless ones.

From now on, we consider bremsstrahlung of massless gravitons only. The S-matrix element has the form

$$(\phi_f^{\varrho\sigma}, S_A\phi_i) = \delta(E_f + E_k - E_i) M_A^{\varrho\sigma} \tag{4.1}$$

for each Feynman diagram A,B,C,D. Note that we have only energy conservation. Momentum is not conserved because the central big mass M absorbs a tiny amount of momentum, but its repulsion is neglected. The external Newtonian potential is in momentum space given by

$$\phi_N(\vec{q}) = -\sqrt{\frac{2}{\pi}} \frac{G_N M m}{\vec{q}^2} \tag{4.2}$$

where G_N is Newton's constant. For a general final graviton state $a^{\varrho\sigma +}\Omega$, we then find the following matrix element for diagram A:

$$M_A^{\varrho\sigma} = \frac{i\kappa^2 \pi^{-5/2}}{8\sqrt{E_i E_f \omega}} \frac{G_N M m}{\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^2} \cdot \frac{\left(E_i(E_f + \omega) - \frac{m^2}{2}\right)}{m^2 - (p_f + k)^2 - i0} p_{f\mu}(p_f + k)_\nu\, t^{\varrho\sigma\mu\nu} \tag{4.3}$$

where the coupling constant κ is given in terms of Newton's constant by ([6])

$$\kappa^2 = 32\pi G_N. \tag{4.4}$$

The result for diagram B is very similar. The final matrix elements are

$$\begin{aligned} M^{\varrho\sigma}_{A,B} = &\frac{i\kappa^2\pi^{-5/2}}{8\sqrt{E_i E_f \omega}} \frac{G_N M m}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} \frac{E_{i,f}(E_{f,i} \pm \omega) - \frac{m^2}{2})}{m^2 - (p_{f,i} \pm k)^2 - i0} \\ &\times \left(\frac{1}{2}\left[p^\varrho_{f,i}(p_{f,i} \pm k)^\sigma + p^\sigma_{f,i}(p_{f,i} \pm k)^\varrho\right] - \frac{1}{4} p_{f,i}(p_{f,i} \pm k)\,\eta^{\varrho\sigma} \right) \end{aligned} \tag{4.5}$$

where the upper sign and first subscript f is for diagram A and the other for B. For diagram C we obtain

$$\begin{aligned} M^{\varrho\sigma}_C = &\frac{i\kappa^2\pi^{-5/2}}{8\sqrt{E_i E_f \omega}} \frac{G_N M m}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} \left(-\frac{\omega(E_f - E_i)}{2(p_f - p_i)^2 - i0} \left[p^\varrho_f p^\sigma_i + p^\varrho_i p^\sigma_f - 2 p_f p_i\,\eta^{\varrho\sigma}\right] \right. \\ &- \frac{1}{2(p_f - p_i)^2 - i0} \left[(p_i - p_f)^\sigma \eta^{0\varrho} + (p_i - p_f)^\varrho \eta^{0\sigma} - \frac{1}{2}(E_i - E_f)\eta^{\varrho\sigma} \right] \\ &\left. \times \left[(E_f p_i + E_i p_f)k - \omega p_f p_i + m^2\omega \right] \right) \end{aligned} \tag{4.6}$$

and diagram D gives

$$M^{\varrho\sigma}_D = \frac{i\kappa^2 m^2 \pi^{-5/2}}{32\sqrt{E_i E_f \omega}} \frac{G_N M m}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} \left(\eta^{\varrho 0}\eta^{\sigma 0} - \frac{1}{4}\eta^{\varrho\sigma} \right). \tag{4.7}$$

By estimating the contributions from the 4 diagrams, we find that C is of order $O(\omega^2/m^2)$ and D of order $O(\omega m^2/E_i^3)$ compared to $O(1)$ for A and B. Therefore, restricting ourselves to the low frequency region $\omega \ll E_i$, we neglect the contributions from C and D. The two massless physical helicity states can be represented by the complex polarization "tensor" $\varepsilon^{\varrho\sigma}_\pm(\vec{k})$ introduced by Weinberg [9]. We choose a coordinate system where the direction of the initial momentum $\vec{p}_i$ is along the z-axis and the scattering takes place in the xz-plane. Let Θ be the angle between $\vec{p}_i$ and $\vec{k}$, Θ_{fk} the angle between $\vec{p}_f$ and $\vec{k}$, and Θ' is the scattering angle of the particle. Then the total matrix element becomes (under the same assumptions as [10])

$$M_\pm \approx \frac{2iG_N^2 M m}{\pi^{3/2}\omega^{3/2}E_i^2\Theta'^2} \varepsilon^{\varrho\sigma *}_\pm(\vec{k}) \left(\frac{p_{i\varrho}(p_i - k)_\sigma}{1 - v\cos\Theta} - \frac{p_{f\varrho}(p_f + k)_\sigma}{1 - v\cos\Theta_{fk}} \right) \tag{4.8}$$

where $v = |\vec{p}_i|/E_i$ and the star is complex conjugation. By squaring, summing over the two polarizations, and integrating over the spherical angles of the emitted graviton, we find the cross section

$$\frac{d\sigma}{d\omega d\Theta'} \approx \frac{64\pi G_N^4 M^2 m^2 E_i^2}{3\omega\Theta'}. \tag{4.9}$$

In order to make contact to the classical theory, we use the relation between the scattering angle Θ' and the impact parameter ϱ in the presence of a Newtonian field. In the ultra-relativistic case, it has the form $\Theta' \approx 4G_N M/\varrho$ and we use the definition of the energy cross section

$$\frac{d\mathcal{E}}{d\omega} = \frac{\omega}{2\pi\varrho} \frac{d\sigma}{d\varrho d\omega}. \tag{4.10}$$

With $\gamma = E_i/m$ we arrive at

$$\frac{d\mathcal{E}}{d\omega} \approx \frac{32 G_N^4 M^2 m^4 \gamma^2}{3\varrho^2}. \tag{4.11}$$

In order to obtain the total radiated energy

$$\mathcal{E} = \int_0^{\omega_{th}} \frac{d\mathcal{E}}{d\omega} d\omega \tag{4.12}$$

we must have an estimate for the threshold frequency ω_{th} of the low-frequency region in the case of gravitational interaction. In order to compare with classical results, we insert the classical value ([11]) $\omega_{th} \approx \gamma/\varrho$ in (4.12), and find the total energy radiated in a single encounter

$$\mathcal{E}^{\text{tot}} \approx \frac{32 G_N^4 M^2 m^4 \gamma^3}{3\varrho^3}. \tag{4.13}$$

Our result (4.11) agrees with the result obtained by Galtsov et al. [10] up to a factor of two. This factor is due to our two polarizations, because Galtsov et al. have only treated scalar bremsstrahlung. There have been studies on classical and semi-classical bremsstrahlung in the sixties and seventies ([11] and references given therein). The classical picture mainly consists of two massive colliding bodies, deflecting their trajectories and therefore emitting gravitational waves. All those classical calculations arrived at the total radiated energy $\mathcal{E} \sim \gamma^3/\varrho^3$ in agreement with our result (4.13).

One important point to mention is the range of validity of the results obtained from the Born approximation in quantum theory. Since the external field is expanded in powers of $G_N Mm$, we must have $G_N Mm \ll 1$. According to the discussion in [11] there are no astrophysical objects satisfying this condition. Therefore, for the astrophysical situation some resummation procedure is needed.

One might think that the massless limit of our massive gravity always agrees with general relativity. This is not the case. A counter-example is graviton-graviton scattering. In this process, the vector graviton states contribute. This is a consequence of the coupling term $4h^{\mu\nu}\partial_\mu v^\alpha \partial_\nu v_\alpha$ in (2.13) which survives in the limit $m_0 \to 0$!

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