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The "Butterfly Effect" in a Porous Slab

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Abstract The present Note is a comment on the article "Small and moderate Prandtl number chaotic convection in porous media in the presence of feedback control" by M. N. Mahmud and I. Hashim, published recently online in Transport in Porous Media, doi:10.1007/s11242-009-9511-1. We show that the only effect of the *feedback control* considered by Mahmud and Hashim is a simple rescaling of the initial conditions of the *Vadasz–Olek model* which describes the chaotic convection in a porous layer. Thus, the "suppression or enhancement of chaotic convection" found by Mahmud and Hashim as a result of the feedback control, in fact is a manifestation of the famous *Butterfly Effect*, i.e., a consequence of the high sensitivity of chaotic dynamics with respect to the initial conditions. We also point out in this Note that several results reported by Mahmud and Hashim are in error.

Keywords Chaotic convection · Porous layer · Lorenz model · Limit cycles · Strange attractors

In a recent paper by Mahmud and Hashim (2009) (hereinafter referred to as MH2009), the feedback control strategy of Bau (1999) has been implemented in the Vadasz–Olek model which describes the chaotic convection in a porous layer under various physical situations (see Vadasz and Olek 1998, 1999, 2000, as well as Vadasz 1999, 2003). The governing equations obtained by MH2009 for the scaled amplitudes of the stream function and temperature field are equivalent to the famous model equations of the atmospheric turbulence proposed by Lorenz (1963) and, for vanishing value of the feedback control parameter K, reduce to the governing equations of the Vadasz–Olek model. Accordingly, the declared aim of MH2009 was to perform a systematic study of the effect of control parameter K on the chaotic dynamics of the Vadasz–Olek model. Bearing in mind this background, the present Note sets the following three objectives:

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(i) To show that the feedback control considered by MH2009 in fact is equivalent to a simple rescaling of the initial conditions of the Vadasz–Olek model and thus has no influence on the stability features (critical Rayleigh numbers) of this model;
(ii) To explain why the "suppression or enhancement of chaotic convection" interpreted by MH2009 as a result of the feedback control is nothing else than a direct manifestation of the well-known *Butterfly Effect*; and
(iii) To point out that several results reported by MH2009 are in error.

(i) The governing equations obtained by MH2009 for the feedback-controlled Vadasz–Olek model read

$$\begin{aligned}\dot{X} &= \alpha\,[(1+K)\,Y - X]\,,\\ \dot{Y} &= \frac{R}{1+K}X - Y - (R-1)\,XZ,\\ \dot{Z} &= 4\gamma\,(XY - Z)\end{aligned} \tag{1}$$

and are subject to some prescribed initial conditions

$$X(0) = X_0, \quad Y(0) = Y_0, \quad Z(0) = Z_0, \tag{2}$$

All the quantities involved in the above equations are dimensionless. The variable X is related to the stream function amplitude and the variables Y and Z to the temperature amplitudes of the Vadasz–Olek model, respectively. The parameter α is related to the time derivative term in the Darcy equation, γ to the front aspect ratio of the porous layer and R is a scaled Rayleigh number. The dot denotes differentiation with respect to the dimensionless time t.

As already mentioned by MH2009, Eq. 1 describe a Lorenz-type dynamical system which for $K = 0$ reduce to the basic equations of the Vadasz–Olek model. Due to this latter aspect, there appears that the Vadasz–Olek model is merely a special case of the feedback-controlled dynamical system described by Eq. 1, so that for $K \neq 0$ new features of the critical dynamics could be expected. This appearance, however, is deceiving since Eq. 1 in fact is nothing more than a rescaled form of the governing equations of the Vadasz–Olek model. Indeed, changing from (X, Y, Z) to the rescaled variables $(\hat{X}, \hat{Y}, \hat{Z})$ defined by equation

$$\hat{X} = X, \quad \hat{Y} = (1+K)Y, \quad \hat{Z} = (1+K)Z \tag{3}$$

Equation 1 reduces for any K to Eqs.19–21 of Vadasz and Olek (1998),

$$\begin{aligned}\dot{\hat{X}} &= \alpha(\hat{Y} - \hat{X}),\\ \dot{\hat{Y}} &= R\hat{X} - \hat{Y} - (R-1)\hat{X}\hat{Z},\\ \dot{\hat{Z}} &= 4\gamma(\hat{X}\hat{Y} - \hat{Z})\end{aligned} \tag{4}$$

Therefore, what actually gets changed due to the feedback control is not the basic structure of the Vadasz–Olek equations but only the pertinent initial conditions which, according to Eqs. 3 and 2, become

$$\hat{X}(0) = X_0, \quad \hat{Y}(0) = (1+K)Y_0, \quad \hat{Z}(0) = (1+K)Z_0 \tag{5}$$

This result implies that the "suppression or enhancement of chaotic convection" found by MH2009 as a consequence of the feedback control is solely the effect of the changed initial conditions.

(ii) The changed initial conditions can actually lead to an essentially changed trajectory of the representative point (X, Y, Z) on the chaotic attractor (*strange attractor*) of a dynamical system since, as it is well known, the chaotic dynamics depends on the external perturbations very sensitively. The reason is that in the chaotic regime of a dynamical system the perturbations grow exponentially, and thus the distance between two initially neighboring trajectories increases exponentially with time. This phenomenon is known under the metaphoric name *Butterfly Effect,* indicating that in a chaotic dynamical state of the atmosphere, a tiny flap of a butterfly is sufficient to make the evolution unpredictable and thus the weather forecast impossible. On the other hand, the Lorenz-type systems being deterministic systems, their state $\{X(t), Y(t), Z(t)\}$ at any given instant t plays the role of "initial conditions" for their subsequent state at an infinitesimally neighboring time instant. Accordingly, the chaotic dynamics depends sensitively also on the changes in its very first initial conditions, i.e., on the changes in the starting conditions at $t = 0$. This is precisely what happens in the feedback control considered by MH2009.

(iii) Our above conclusion on the effect of the feedback control differs from the interpretation of MH2009 essentially. Indeed, the interpretation of MH2009 is based on their Eqs. 27 and 28 which are

$$\lambda^3 + (4\gamma + \alpha + 1)\,\lambda^2 + \left[4\gamma\,(\alpha + R) + \alpha K\,(R - 1)\right]\lambda + 8\gamma\alpha\,(K + 1)\,(R - 1) = 0 \tag{6}$$

and

$$R_{c2} = \frac{K\alpha\,(1 + \alpha - 4\gamma) - 4\gamma\alpha\,(3 + \alpha + 4\gamma)}{K\alpha\,(1 + \alpha - 4\gamma) + 4\gamma\,(1 - \alpha + 4\gamma)} \tag{7}$$

respectively. These equations are called by MH2009 "extensions" of the respective Eqs. 23 and 24 of Vadasz and Olek (1998). According to the above Eqs. 6 and 7, the eigenvalues λ of the stability matrix, as well as the critical Rayleigh number R_{c2} which marks the transition to the chaotic evolution of the system, depend on the feedback control parameter K essentially. This would actually imply that the effect of K is a change of the stability limits of the stationary points (corresponding to the uniform convection), as well as of the limit cycles (corresponding to the convection rolls). However, the crucial point is that although Eqs. 6 and 7 reduce for $K = 0$ to the correct results (23) and (24) of Vadasz and Olek (1998), for $K \neq 0$ they are simply faulty.

That the presence of K in Eqs. 6 and 7 must be an error, can be realized without any calculation. Indeed, as argued above, with the aid of the scaling transformations (3) the parameter K can be transferred from the governing equations (1) into the initial conditions. The result is the original Vadasz–Olek model (4), accompanied by the rescaled initial conditions (5). Bearing in mind that the stability limits of the Lorenz-type dynamical systems cannot depend on the initial conditions but only on the intrinsic parameters of the system (which in this case are α, γ, and R), we immediately see that Eqs. 6 and 7 cannot be correct. Moreover, according to our scaling argument, these equations must be replaced also for $K \neq 0$ by the respective equations (23) and (24) of Vadasz and Olek (1998) which, as mentioned above, coincide with the $K = 0$ case of Eqs. 6 and 7 and read

$$\lambda^3 + (4\gamma + \alpha + 1)\,\lambda^2 + 4\gamma\,(\alpha + R)\,\lambda + 8\gamma\alpha\,(R - 1) = 0 \tag{8}$$

and

$$R_{c2} = \frac{\alpha\,(3 + \alpha + 4\gamma)}{\alpha - 1 - 4\gamma} \tag{9}$$

respectively.

Although the above reasoning is reliable, in order to be more convincing, we also prove by explicit calculations that for $K \neq 0$, Eqs. 6 and 7 of MH2009 are in error.

The standard procedure to arrive to the eigenvalue equations of the stationary points (X_i, Y_i, Z_i), $i = 1, 2, 3$ of a Lorenz-type dynamical system is to linearize the governing equations around the states (X_i, Y_i, Z_i) by setting

$$X = X_i + \xi e^{\lambda t}, \quad Y = Y_i + \eta e^{\lambda t}, \quad Z = Z_i + \zeta e^{\lambda t} \tag{10}$$

where ξ, η, and ζ are small perturbations. It is easy to show that in case of Eq. 1, the matrix of the linearized system (i.e., the "stability matrix" M) has the form

$$M = \begin{pmatrix} \lambda + \alpha & -\alpha\,(1 + K) & 0 \\ (R - 1)\, Z_i - \frac{R}{1+K} & \lambda + 1 & (R - 1)\, X_i \\ -4\gamma Y_i & -4\gamma X_i & \lambda + 4\gamma \end{pmatrix} \tag{11}$$

The corresponding eigenvalue equations are then obtained as $\det(M) = 0$.

The three stationary points of the system of Eqs. 1 are

$$(X_1, Y_1, Z_1) = (0, 0, 0)\,, \quad \left(X_{2,3}, Y_{2,3}, Z_{2,3}\right) = \left(\pm 1, \pm\frac{1}{1+K}, \frac{1}{1+K}\right) \tag{12}$$

For the point $(X_1, Y_1, Z_1) = (0, 0, 0)$ corresponding to the thermal conduction regime, $\det(M) = 0$ yields

$$(\lambda + 4\gamma)\,[(\lambda + 1)\,(\lambda + \alpha) - \alpha R] = 0 \tag{13}$$

We see that, as predicted above, the eigenvalue equation (13) does not contain the feedback parameter K at all. It coincides with Eq. 26 of MH2009 which in turn coincides with Eq. 22 of Vadasz and Olek (1998).

The origin of several erroneous results of MH2009 resides in the fact that in case of the stationary points $(X_{2,3}, Y_{2,3}, Z_{2,3})$ corresponding to homogeneous convection states, instead of the correct expression given by Eq. 12, in MH2009, for $K \neq 0$ the flawed expressions $\left(X_{2,3}, Y_{2,3}, Z_{2,3}\right) = (\pm 1, \pm 1, 1)$, has been processed (see the 6th line below of Eq. 25 of MH2009). However, setting in (11) the correct expressions of $\left(X_{2,3}, Y_{2,3}, Z_{2,3}\right)$ given by Eq. 12, one arrives to the correct eigenvalue equation (8) which is independent of K, too, in full agreement with our qualitative reasoning presented above. Thus, it is proven explicitly that Eq. 27 of MH2009 is false when $K \neq 0$.

As an immediate consequence of the erroneous eigenvalue, equation 27 of MH2009, also the plots shown in their Figs. 1 and 2 are faulty for $K \neq 0$. A further consequence of the erroneous eigenvalue equation (27) of MH2009 is that their Eq. 28 (i.e., the above Eq. 7) is also in error. The critical Rayleigh number R_{c2}, which marks the transition to the chaotic evolution of the system, corresponds to the situation in which the two complex roots of the eigenvalue equation arising for $R > 1$ become purely imaginary. In this case, as pointed out already by Lorenz (1963), the product of the coefficients of λ^2 and λ equals the free term of the eigenvalue equation. This equality leads in turn to the exact analytical expression of R_{c2}. Performing this operation with reference to the erroneous Eq. 6 of MH2009, one arrives to their erroneous result (7). Using, however, the correct Eq. 8, one obtains the correct result (9), which has already been reported by Vadasz and Olek (1998).

Therefore, we may summarize that the feedback control considered by MH2009 has no influence on the eigenvalues and the critical Rayleigh numbers which specify the stability limits of the attractors (point attractors, limit cycles, and the strange attractors) of the Vadasz–Olek model. Thus, the change in the chaotic behavior interpreted by MH2009 as an effect of the feedback control on the transition threshold to chaos is in fact a consequence of the high sensitivity of the chaotic dynamics with respect to the initial conditions, i.e., a manifestation of the *Butterfly Effect.* The effect of the feedback control on the Vadasz–Olek model is only the rescaling of the pertinent initial conditions and the convection fixed points according to Eqs. (5) and (12), respectively. Not more, and not less. The erroneous equations and interpretations of MH2009 can be traced back to their expressions for the convection fixed points $\left(X_{2,3}, Y_{2,3}, Z_{2,3}\right) = (\pm 1, \pm 1, 1)$, which are false when $K \neq 0$. There seems that these expressions were simply pasted from the article of Vadasz and Olek (1998) (where they are correct) into the manuscript of MH2009. Similarly, astonishingly, large parts of the article of Vadasz and Olek (1998) were transferred by MH2009 in their paper literally, without any further ado.

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