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ELEMENTARY PARTICLES AND FIELDS
Theory

Entanglement Production in Quantum Decision Making*

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Abstract—The quantum decision theory introduced recently is formulated as a quantum theory of measurement. It describes prospect states represented by complex vectors of a Hilbert space over a prospect lattice. The prospect operators, acting in this space, form an involutive bijective algebra. A measure is defined for quantifying the entanglement produced by the action of prospect operators. This measure characterizes the level of complexity of prospects involved in decision making. An explicit expression is found for the maximal entanglement produced by the operators of multimode prospects.

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1. INTRODUCTION

Entanglement is a quantum property that is very important for quantum information processing and quantum computing [1, 2]. It is one of the key features for creating artificial intelligence based on quantum rules. Our recent formulation of Quantum Decision Theory (QDT) [3, 4] is based on the recognition that entanglement is also a characteristic property of human decision making. Indeed, any real decision making procedure deals with composite prospects, composed of many intended actions, which produce naturally entanglement due to correlations between particular actions. These correlations need only to be subjectively felt in the brain of the decision maker to affect his/her choices, strongly colored by the emotional effects and aversion to risk and uncertainty. This has led us to use the mathematics of quantum theory, in order to develop a decision theory of nonquantum objects, such as human decision makers [3, 4]. This approach can be also applied to physical devices of quantum information processing [5].

With the understanding that almost any decision procedure involves entanglement, it then becomes necessary to quantify in some way the level of produced entanglement. While we have modeled the phenomenon of entanglement to explain and quantify many classical paradoxes arising in standard utility theory, a systematic measurement of entanglement has not been developed in the previous publications on QDT [3–5]. It is the aim of the present paper to analyze the problem of entanglement production that can be generated in the process of decision making.

2. ALGEBRA OF PROSPECT OPERATORS

In order to construct the mathematics of QDT, we employ the techniques of quantum theory of measurement [6, 7], with the specifications appropriate for describing the process of decision making. The primary objects of a decision procedure are the intended actions, whose totality forms the *action ring*

$$\mathcal{A} = \{A_n : n = 1, 2, \ldots, N\}. \tag{1}$$

Each action, generally, is composed of several representations, called *action modes*,

$$A_n = \bigcup_{\mu=1}^{M_n} A_{n\mu} \qquad (A_{n\mu}A_{n\nu} = \delta_{\mu\nu}). \tag{2}$$

A prospect is a conjunction of several actions,

$$\pi_j = \bigcap_{n=1}^{N} A_{j_n} \qquad (A_{j_n} \in \mathcal{A}). \tag{3}$$

It can be simple, if each action is represented by a single mode, or composite, when there is at least one composite action in the conjunction. The family of all prospects forms a lattice

$$\mathcal{L} = \{\pi_j : j = 1, 2, \ldots, N_L\}, \tag{4}$$

endowed with the binary operations $<$ ("less preferred than"), $>$ ("more preferred than"), and $=$ ("equivalent to" or "indifferent with"), so that each two prospects from $\mathcal{L}$ are connected either as $\pi_i \leq \pi_j$ or as $\pi_i \geq \pi_j$, or as $\pi_i = \pi_j$. Elementary prospects are defined as simple disjoint prospects

$$e_\alpha = \bigcap_{n=1}^{N} A_{i_n\mu_n} \qquad (e_\alpha e_\beta = \delta_{\alpha\beta}), \tag{5}$$

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containing only single modes and labelled with a multi-index $\alpha = \{i_n, \mu_n : n = 1, 2, ..., N_L\}$. The cardinality of the set $\{\alpha\}$ is card$\{\alpha\} = \prod_{n=1}^{N} M_n$.

Each mode $A_{n\mu}$ corresponds to a mode state $|A_{n\mu}\rangle$, which is a complex-valued function with an orthonormalized scalar product $\langle A_{n\mu}|A_{n\nu}\rangle = \delta_{\mu\nu}$. The closed linear envelope, spanning all mode states, is the *mode space*

$$\mathcal{M}_n = \text{Span}\{|A_{n\mu}\rangle : \mu = 1, 2, \ldots, M_n\}, \quad (6)$$

with the dimensionality $\dim \mathcal{M}_n = M_n$.

An elementary prospect e_α corresponds to a *basic state* $|e_\alpha\rangle$, which is a complex function

$$|e_\alpha\rangle = |A_{i_1\mu_1} A_{i_2\mu_2} \ldots A_{i_N\mu_N}\rangle = \bigotimes_{n=1}^{N} |A_{i_n\mu_n}\rangle. \quad (7)$$

The basic states are orthonormalized, such that $\langle e_\alpha|e_\beta\rangle = \delta_{\alpha\beta}$. The closed linear envelope, spanning all basic states, is the *mind space*

$$\mathcal{M} = \text{Span}\{|e_\alpha\rangle : \alpha \in \{\alpha\}\} = \bigotimes_{n=1}^{N} \mathcal{M}_n, \quad (8)$$

whose dimensionality is $\dim \mathcal{M} = \text{card}\{\alpha\} = \prod_{n=1}^{N} M_n$.

A prospect π_j is represented by a prospect state $|\pi_j\rangle$, which is a member of the mind space $\mathcal{M}$. That is, it can be expanded over the basic states,

$$|\pi_j\rangle = \sum_\alpha b_j(e_\alpha)|e_\alpha\rangle, \quad (9)$$
$$b_j(e_\alpha) = \langle e_\alpha|\pi_j\rangle.$$

The prospect operator is defined as

$$\hat{P}(\pi_j) = |\pi_j\rangle\langle\pi_j|, \quad (10)$$

with the condition that the sum

$$\sum_{j=1}^{N_L} \hat{P}(\pi_j) = \hat{1}_{\mathcal{M}},$$

over the prospect lattice, is a unity operator in the weak sense, with respect to the matrix element over a fixed strategic state characterizing the considered decision maker. The involution is given by the Hermitian conjugation. By their definition, the prospect operators (10) are self-adjoint. Hence, the family of the prospect operators forms an involutive bijective algebra.

3. PROSPECT PRODUCED ENTANGLEMENT

We now introduce a measure of the amount of entanglement produced by a prospect operator. To understand how the prospect operators entangle the mind states, we need first to classify the latter into entangled or disentangled states. A mind state is disentangled if it can be represented as a product state or *factor state*

$$|f\rangle \equiv \bigotimes_{n=1}^{N} |f_n\rangle \qquad (|f_n\rangle \in \mathcal{M}_n). \quad (11)$$

The ensemble of all such factor states forms the *disentangled set*

$$\mathcal{D} \equiv \left\{ |f\rangle = \bigotimes_{n=1}^{N} |f_n\rangle \in \mathcal{M} \right\}, \quad (12)$$

hence, $\mathcal{D} \subset \mathcal{M}$. The complement $\mathcal{M} \setminus \mathcal{D}$ composes the entangled set.

The entangling properties of the prospect operator can be understood by comparing its action with that of its nonentangling counterpart composed as a product of the partially traced prospect operators

$$\hat{P}_n(\pi_j) \equiv \text{Tr}_{\{\mathcal{M}_m : m \neq n\}} \hat{P}(\pi_j), \quad (13)$$

where the trace is over all $\mathcal{M}_m$ except $m = n$. The *nonentangling prospect operator*

$$\hat{P}^{\otimes}(\pi_j) \equiv \frac{\text{Tr}_{\mathcal{M}} \hat{P}(\pi_j)}{\text{Tr}_{\mathcal{M}} \bigotimes_{n=1}^{N} \hat{P}_n(\pi_j)} \bigotimes_{n=1}^{N} \hat{P}_n(\pi_j) \quad (14)$$

is the product of the partially traced operators (13), defined so that to preserve the normalization condition

$$\text{Tr}_{\mathcal{M}} \hat{P}(\pi_j) = \text{Tr}_{\mathcal{M}} \hat{P}^{\otimes}(\pi_j). \quad (15)$$

The following equalities hold for the traces:

$$\text{Tr}_{\mathcal{M}} \hat{P}(\pi_j) = \text{Tr}_{\mathcal{M}_n} \hat{P}_n(\pi_j) = \sum_\alpha |b_j(e_\alpha)|^2,$$

$$\text{Tr}_{\mathcal{M}} \bigotimes_{n=1}^{N} \hat{P}_n(\pi_j) = \prod_{n=1}^{N} \text{Tr}_{\mathcal{M}_n} \hat{P}_n(\pi_j) \quad (16)$$
$$= \left(\sum_\alpha |b_j(e_\alpha)|^2 \right)^N.$$

As a result, the nonentangling operator (14) takes the form

$$\hat{P}^{\otimes}(\pi_j) = \left(\sum_\alpha |b_j(e_\alpha)|^2 \right)^{1-N} \bigotimes_{n=1}^{N} \hat{P}_n(\pi_j). \quad (17)$$

The entangling properties of the prospect operator (10) are the most clearly pronounced in the action of the prospect operator on the disentangled set (12). On this set, we may define the restricted norm

$$||\hat{P}(\pi_j)||_{\mathcal{D}} \equiv \sup_{|f\rangle\in\mathcal{D}} \frac{|\langle f|\hat{P}(\pi_j)|f\rangle|}{|\langle f|f\rangle|}, \qquad (18)$$

which is a kind of a subnorm [8, 9]. In particular, we have

$$||\hat{P}(\pi_j)||_{\mathcal{D}} = \sup_{\alpha} |b_j(e_\alpha)|^2,$$

$$||\hat{P}^{\otimes}(\pi_j)||_{\mathcal{D}} = \frac{||\bigotimes_{n=1}^{N}\hat{P}_n(\pi_j)||_{\mathcal{D}}}{\left(\sum_\alpha |b_j(e_\alpha)|^2\right)^{N-1}},$$

$$\left|\left|\bigotimes_{n=1}^{N}\hat{P}_n(\pi_j)\right|\right|_{\mathcal{D}} = \prod_{n=1}^{N} ||\hat{P}_n(\pi_j)||_{\mathcal{M}_n}. \qquad (19)$$

The measure of entanglement production [10–12], generated by the prospect operator (10), is defined as

$$\varepsilon(\pi_j) \equiv \log \frac{||\hat{P}(\pi_j)||_{\mathcal{D}}}{||\hat{P}^{\otimes}(\pi_j)||_{\mathcal{D}}}, \qquad (20)$$

where the logarithm can be defined with respect to any base, say, to the base two. Taking into account Eqs. (16) and (19) yields

$$\varepsilon(\pi_j) \qquad (21)$$
$$= \log \frac{\sup_\alpha |b_j(e_\alpha)|^2 \left(\sum_\alpha |b_j(e_\alpha)|^2\right)^{N-1}}{\prod_{n=1}^{N} ||\hat{P}_n(\pi_j)||_{\mathcal{M}_n}}.$$

In order to evaluate the maximal entanglement that could be generated by the prospect operators, we should consider the maximally entangled prospect states $|\pi_j\rangle$. For the simplest case of two actions with two modes each, the maximally entangled state is of the Bell type

$$|\pi_B\rangle = b_1|A_{11}A_{21}\rangle + b_2|A_{12}A_{22}\rangle.$$

When there are N two-mode actions, the prospect state is a multicat state

$$|\pi_C\rangle = b_1|A_{11}A_{21}\ldots A_{N1}\rangle + b_2|A_{12}A_{22}\ldots A_{N2}\rangle.$$

The general case of a maximally entangled state is a multimode state

$$|\pi_M\rangle = \sum_{\mu=1}^{M} b_\mu |A_{1\mu}A_{2\mu}\ldots A_{N\mu}\rangle, \qquad (22)$$

corresponding to N actions with M modes. The related prospect operator, characterizing the multimode prospect, is

$$\hat{P}(\pi_M) = |\pi_M\rangle\langle\pi_M|. \qquad (23)$$

For this operator, we have

$$||\hat{P}(\pi_M)||_{\mathcal{D}} = ||\hat{P}_n(\pi_M)||_{\mathcal{M}_n} = \sup_\mu |b_\mu|^2.$$

Therefore, measure (21) transforms into

$$\varepsilon(\pi_M) = (N-1)\log \frac{\sum_{\mu=1}^{M}|b_\mu|^2}{\sup_\mu |b_\mu|^2}. \qquad (24)$$

Expression (24) acquires its maximal value when all modes are equally probable, such that $|b_\mu| = |b| =$ const. Then the measure of entanglement production (24) becomes

$$\varepsilon(\pi_M) = (N-1)\log M. \qquad (25)$$

In this way, the maximal measure of entanglement that can be generated by a prospect, consisting of N actions, corresponds to the case when all actions possess the same number of equiprobable modes M. Then the measure of entanglement production is proportional to the number of actions and logarithmically depends on the number of modes.

4. CONCLUSION

We have formulated quantum decision theory as the theory of quantum measurements. We have suggested a method for evaluating the entanglement generated by the prospect operators in QDT. The measure of entanglement production depends on the structure of the prospects involved. This measure can be employed for quantifying the complexity of prospects in decision theory. It can also be used for characterizing the complexity of operations in quantum information processing.

The most effective information processing requires a high level of produced entanglement [1, 2, 13]. We have shown that the maximal entanglement production is characterized by formula (25). The developed theory can be applied to nonquantum decision makers [3, 4] as well as to quantum objects of different physical nature [5]. Except spin systems, multimode coherent states of trapped atoms [14, 15] seem to be good candidates for realizing quantum-information processing devices.

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