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Weak Necessity on Weak Kleene Matrices

FABRICE CORREIA

ABSTRACT. A possible world semantics for standard modal languages is presented, where the valuation functions are allowed to be partial, the truth–functional connectives are interpreted according to weak Kleene matrices, and the necessity operator is given a "weak" interpretation. Completeness and incompleteness results for some (axiomatic) systems are then established. Extensions of these modal logics in which figure "statability" operators are also examined.

Introduction

Consider a standard modal language L with $\neg$, $\wedge$ and $\Box$ as primitive operators.[1] Suppose now that L is interpreted in *partial* Kripke models, i.e. in triples $< \mathcal{W}, R, V >$ where $\mathcal{W}$ (worlds) is a non–empty set, R (accessibility) is a binary relation on $\mathcal{W}$, and V (valuation) is a (maybe not total) function from $atoms(\mathsf{L}) \times \mathcal{W}$ to $\{0, 1\}$. Then in order to have a full interpretation of L, it suffices to specify truth and falsity clauses for its three primitive operators. Or alternatively, you can specify truth and undefinedness clauses for them.

The most radical view one can imagine about when a complex formula is undefined – and this is the view I shall endorse – is the *Contamination Principle* (CP):

> Truth–value gaps are unconditionally transmitted from simples to complexes – i.e. a formula is undefined provided that

[1]Essentially the same paper was presented at the ESSLLI 2000 students session held in august 2000 at the University of Birmingham (UK). The ESSLLI version did not incorporate the contents of sections 5, 6 and 7 below.

at least one of its constituent formulae is.

The CP may seem to be difficult to justify. For e.g. if proposition p is neither true nor false and q is false, then one is tempted to say that the conjunction $p \wedge q$ should be false. However, the intuitive correctness of the Principle depends on what one takes to be the source of truth–value gaps. If the source of gaps is, say, lack of meaning, then the Principle seems to be correct; for instance, if p is meaningless, then arguably so is any conjunction $p \wedge q$.

Assuming the CP forces one to interpret negation and conjunction according to the following three–valued matrices, known as weak Kleene or Bochvar's:

$\neg$	
0	1
1	0
2	2

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

where 2 is for 'undefined'. Thus, the modal logics I shall focus on depart from the well investigated modal logics based on the strong Kleene matrices.[2]

As to the truth–clause for necessity, there are two natural possible choices:

- $\Box A$ is true at w iff A is defined at w and A is true at every world accessible from w;
- $\Box A$ is true at w iff A is defined at w and A is true at every world accessible from w at which A is defined.

The first clause defines what one may call *strong necessity*, and the second *weak necessity*. In this paper, I focus on weak necessity, and the main question I will address is simply:

> What logical systems does one get when language L is interpreted as previously indicated?

Strong necessity will indirectly enter the scene in section 6, when the basic modal language will be enriched with "statability" operators: as we shall see, strong necessity is definable in terms of weak necessity and any one of these operators.[3]

[2]See (Thijsse 1990), (Thijsse 1992) and (Jaspars and Thijsse 1996).

[3]If we decide not to follow the CP, there appears to be two extra natural truth–clauses for necessity:

- $w \models \Box A$ iff for every v such that wRv and A is defined at v, $v \models \Box A$;
- $w \models \Box A$ iff for every v such that wRv, $v \models \Box A$.

The first defines another form of weak necessity, the second another form of strong necessity. The existing studies on modal logics based on the strong Kleene matrices all

The history of modal logics based on weak Kleene matrices dates back from (Prior 1957), where Prior gives a semantic presentation of his celebrated system Q.[4] As far as I know, the only completeness studies about these logics are to be found in (Bull 1964), (Segerberg 1967) and (Correia *To appear*). Bull axiomatizes Q. Segerberg establishes completeness for a number of modal logics based on weak Kleene matrices. And finally I do the same for some Q–like systems. Prior's semantics for system Q, as well as the semantics presented in (Correia *To appear*) and in the present paper, respects the CP. Segerberg's semantics does not.[5]

Recently, systems for the logic of essentialist claims have been proved sound and complete with respect to partial possible world semantics respecting the CP.[6] My interest for that kind of semantics actually stems from my own work on this topic. In these logics, the modal operators are indexed by expressions (proper names, predicates, class–terms, ...) intended to isolate one or more objects. The intended meaning of e.g. '$\Box_a A$' – where 'a' is a proper name – is that it is true in virtue of the nature of a that A; and the use of predicates or class–terms as indices allows one to express claims to the effect that some given proposition is true in virtue of the natures of several objects taken together, with no restriction on how many these objects might be. It is not the place here to go into details about the logics of essence. Suffices it to say that, as I hope to show elsewhere, the indexed essentialist operators are definable in terms of weak necessity and conditional statability (see section 6 for that notion).

The structure of the paper is the following. Section 1 presents the grammar and the semantics for the basic modal language. Section 2 introduces the family of (axiomatic) systems I am primarily dealing with in this paper. Section 3 presents two canonical constructions. Completeness results are given in section 4, and incompleteness results in section 5. In section 6, the basic modal language is enriched with statability operators and some completeness facts are established. Finally, section 7 is devoted to consequence relations.

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use exclusively this last clause. For that reason, a comparison between these studies and the present investigations on (the first form of) weak necessity appears to be pointless.

[4]Pages 85–86.

[5]In fact, Segerberg's language contains a "truth–operator", T, whose semantics violates the CP: TA is false whenever A is false or has no truth–value. Comparing the present study with Segerberg's paper, the reader will realize that the absence of Segerberg's truth–operator induces very significant differences, in the systems (e.g. Segerberg's systems are closed under a *restricted* version of the rule Modus Ponens, not under full Modus Ponens), as well as in the completeness proofs.

[6]See (Fine 1995), (Correia 2000) and (Fine 2000).

1 Grammar and Semantics

Thus, our language is a standard modal language, with $\neg$ (negation), $\wedge$ (conjunction) and $\Box$ (necessity) as primitive operators. Operators $\vee$ (disjunction), $\to$ (material implication), and $\Diamond$ (possibility) are standardly defined. I shall use A, B, ... (resp. p, q, ...) with or without indices for arbitrary formulae (resp. atoms) of the language. I adopt the following useful definitions:

- at is the set of all atoms;
- $at(A)$ is the set of all atoms in A;
- Where Γ is a set of formulae, $at(\Gamma)$ is the set of all atoms of all formulae in Γ;
- Where X, Y are formulae or sets of formulae, '$X \leq Y$' is used for '$at(X) \subseteq at(Y)$'.

A *frame* is as usual defined as a pair $< \mathcal{W}, R >$, where $\mathcal{W}$ (worlds) is a non–empty set and R (accessibility) is a binary relation on $\mathcal{W}$.

A *valuation* on a set S will be a total function from $at \times S$ to $\{0,1,2\}$.

A *model* is any triple $< \mathcal{W}, R, V >$, with $< \mathcal{W}, R >$ a frame and V a valuation on $\mathcal{W}$. Model $< \mathcal{W}, R, V >$ is said to be *based on* frame $< \mathcal{W}, R >$.

Where $\mathcal{M} = < \mathcal{W}, R, V >$ is a model and w is in $\mathcal{W}$, A is said to be *defined* at w in $\mathcal{M}$ – in short: $d(A, w, \mathcal{M})$ – iff for every atom p in A, $V(p, w) \neq 2$. When no confusion threatens, mention of the model will be omitted. Sometimes '$d(A, B, w)$' will be used for '$d(A, w)$ and $d(B, w)$'.

The truth–predicate $\models$ is defined on every model $< \mathcal{W}, R, V >$ by the following recursion:

- $w \models p$ iff $V(p, w) = 1$;
- $w \models \neg A$ iff $d(A, w)$ and $w \not\models A$;
- $w \models (A \wedge B)$ iff $w \models A$ and $w \models B$;
- $w \models \Box A$ iff $d(A, w)$ and $\forall v \in \mathcal{W}$, if wRv and $d(A, v)$, then $v \models A$.

We have then:

- $d(A, w)$ iff $w \models A$ or $w \models \neg A$;
- $w \models (A \vee B)$ iff $d(A, B, w)$ and ($w \models A$ or $w \models B$);
- $w \models (A \to B)$ iff $d(A, B, w)$ and ($w \not\models A$ or $w \models B$);
- $w \models \Diamond A$ iff $d(A, w)$ and $\exists v \in \mathcal{W}$ wRv and $v \models A$.

Finally, *validity on a model* is defined as non–falsehood at every world of that model; *validity on a frame* is validity on every model based on that frame; and *validity on a class* of models or frames is validity on every model or frame which belong to that class. A formula schema is said to be valid on a model, or frame, or class of models, or class of frames, when all its instances are.

A word on the definition of validity on a model is required. If validity on a model had been defined as truth at every world of that model, as it is in standard modal logic, then whatever the frame, no formula – and in particular no propositional tautology – would be valid on it; for one can assign 2 to each atom at every world of that frame. According to our definition, every propositional tautology is valid on every model – and hence on every frame. Moreover, given any frame, the non–modal formulae valid on that frame are exactly the propositional tautologies.

2 The Systems

Normal modal system K may be defined as the closure of the set of all propositional tautologies and all instances of schema

K $\quad \Box(A \to B) \to (\Box A \to \Box B)$

under the rules Modus Ponens (A, $A \to B$ / B) and Necessitation (A / $\Box A$). It is not difficult to see that this system is not sound with respect to the class of all frames. In fact, schema **K** is not valid on every frame. Let p, q be distinct atoms, and let $\mathcal{M}_0 = < \{v, w\}, R, V >$ be a model such that R is universal and $V(p, v) = V(q, v) = 1$, $V(p, w) = 2$, $V(q, w) = 0$. Then $\mathcal{M}_0, v \models \Box(p \to q) \land \Box p \land \neg\Box q$.

However, consider system k, obtained from K by replacing axiom schema **K** by

k $\quad \Box(A \to B) \to (\Box A \to \Box B)$, *provided* $A \leq B$.

Proposition 2.1 *k is sound with respect to the class of all frames.*

Proof. It is easy to verify that every tautology is valid on every frame, as well as **k**, and trivially the rule Necessitation preserves validity on a model (and so, validity on a frame as well). The case of the rule Modus Ponens is not so simple. In fact, Modus Ponens does not preserve validity on a model (p and $p \to q$ are valid on model $\mathcal{M}_0$ mentioned above, but q is not). But it does preserve validity on a frame, as the following argument shows. Let $< \mathcal{W}, R >$ be a frame, and let A and $A \to B$ be valid on it. Let $\mathcal{M} = < \mathcal{W}, R, V >$ be a model. If for every world w of $\mathcal{W}$, B is not defined at w, then B is trivially valid on $\mathcal{M}$. Now suppose that B is defined at some world w. We want to prove that B is true at w in $\mathcal{M}$. Suppose first that every atom in A is in B. Then both A and $A \to B$ are defined at w. So, since both A and $A \to B$ are valid on $< \mathcal{W}, R >$, they both are true at w in $\mathcal{M}$. And by the properties of truth–at–a–world, so is B. Suppose now that some atoms in A are not in B, and let p_1, ..., p_n be these atoms. Consider the model $\mathcal{N} = <\mathcal{W}, R, V'>$ whose valuation V' is defined by:

- $V'(p_i, w) = 1$ for every $i \in \{1, ..., n\}$;
- $V'(p, w') = V(p, w')$ if $w' \neq w$ or p is not one of p_1, ..., p_n.

One can then show (by induction on the length of C) that for every formula C whose atoms are all in B: $\forall w' \in \mathcal{W}, \mathcal{N}, w' \vDash C$ iff $\mathcal{M}, w' \vDash C$. As a consequence, $\mathcal{N}, w \vDash B$ iff $\mathcal{M}, w \vDash B$. Now by construction, both A and $A \rightarrow B$ are true at w in $\mathcal{N}$ and thus by the properties of truth–at–a–world, so is B. By the previous result, then, B is true at w in $\mathcal{M}$. $\dashv$

Every system distinct from k to be considered in this paper will be obtained by adding new axioms to k. Of course, since every instance of **k** is an instance of **K**, system k is included in system K; and more generally, each extension of k is included in the corresponding extension of K.

3 Two Canonical Models

In this section, two canonical models for systems extending k are defined, which differ only by their accessibility relation. Each of them will provide us with completeness results we cannot get with the other. In the sequel, S is any system containing k.

Proposition 3.1 *(i)* $\vdash_S \Box A \rightarrow (\Box B \rightarrow \Box(A \wedge B))$; *(ii) If* $\vdash_S A \rightarrow B$ *and* $A \leq B$, *then* $\vdash_S \Box A \rightarrow \Box B$.

Proof. (i) Use the fact that $A \rightarrow (B \rightarrow (A \wedge B))$ is a tautology, and **k**. (ii) Use Necessitation and **k**. $\dashv$

The canonical constructions in standard modal logics are based on the notion of maximal consistent sets of formulae. The property of maximality for sets in a canonical model corresponds in the semantics to the fact that every formula is either true or false at every world of every model. In the present context, maximality is too strong. We need a finer notion, that of *r–maximality*. A set Γ of formulae will be said to be *r–maximal* when for every A such that $A \leq \Gamma$, $A \in \Gamma$ or $\neg A \in \Gamma$. And where Γ is a set of formulae and Σ a set of atoms, Γ will be said to be Σ*–maximal* when it is r–maximal and $at(\Gamma) = \Sigma$.

Propositions 3.2, 3.3 and 3.4 below actually hold when S is any system containing classical propositional logic. Their proofs are easy and omitted.

Proposition 3.2 *Let* Γ *be a consistent set of formulae, and let* Σ *be a set of atoms such that* $\Gamma \leq \Sigma$. *Then* Γ *has an extension which is* *both S–consistent and Σ–maximal. (Γ is S–consistent iff it is not S–inconsistent. Γ is S–inconsistent iff there are $A_1, \ldots, A_n$ in Γ such that $\vdash_S \neg(A_1 \wedge \ldots \wedge A_n)$.)*

I call $\mathcal{W}^S$ the set of all r–maximal S–consistent sets of formulae. In this section, I shall use w, w', and so on for arbitrary elements of $\mathcal{W}^S$.

Proposition 3.3

1. *If $\vdash_S A$ and $A \leq w$, then $A \in w$;*
2. *If $A \in w$ and $(A \to B) \in w$, then $B \in w$;*
3. *If $\vdash_S (A \to B)$ and $A \in w$ and $B \leq w$, then $B \in w$.*

Proposition 3.4

1. *$\neg A \in w$ iff $A \leq w$ and $A \notin w$;*
2. *$(A \wedge B) \in w$ iff A, $B \in w$.*

I adopt the following definitions:

- $\mu_X(w) = \{B : \Box B \in w$ and $B \leq X\}$, for X a formula or a set of formulae;
- $wQ^S w'$ iff for every formula A, if $\Box A \in w$, then $\neg A \notin w'$;
- $wR^S w'$ iff for every formula A, if $A \in w'$, then $\Diamond A \in w$.

Proposition 3.5

1. *$wQ^S w'$ iff $\mu_{w'}(w) \subseteq w'$;*
2. *$wR^S w'$ iff $wQ^S w'$ and $w' \leq w$.*

Proof. 1. (a) From left to right, by the definitions of Q^S and μ, and the fact that w' is r–maximal. (b) For the other direction, suppose $\mu_{w'}(w) \subseteq w'$ and let $\Box A$ be in w. Then either $A \not\leq w'$ or $A \leq w'$. We want to prove that in both cases, $\neg A \notin w'$. If $A \not\leq w'$, then since w' is r–maximal, $\neg A \notin w'$. If on the other hand $A \leq w'$, then $A \in \mu_{w'}(w)$ and so $A \in w'$. Since w' is S–consistent, it follows that $\neg A \notin w'$.
2. (a) Suppose $wR^S w'$. Let $\Box A \in w$. Then by S–consistency, $\neg\Box A \notin w$, and so, $\neg A \notin w'$. Thus, $wQ^S w'$. On the other hand, let p be any atom such that $p \leq w'$. Then by Proposition 3.3 (1), $(p \vee \neg p) \in w'$. So, $\Diamond(p \vee \neg p) \in w$, and consequently $p \leq w$. Thus, $w' \leq w$. (b) Suppose now $wQ^S w'$ and $w' \leq w$. Let A be in w'. Then by Proposition 3.3 (3), $\neg\neg A \in w'$, and given that $wQ^S w'$, $\Box\neg A \notin w$. Now since $w' \leq w$ and $A \leq w'$, $A \leq w$. It follows by r–maximality that $\neg\Box\neg A \in w$, i.e. $\Diamond A \in w$. Thus, $wR^S w'$. $\dashv$

Proposition 3.6 *If $\neg\Box A \in w$, then $\{\neg A\} \cup \mu_A(w)$ is S–consistent.*

Proof. Suppose $\neg\Box A \in w$. Then

$$(\mathbf{a}) \qquad \Box A \leq w.$$

Suppose now that $\{\neg A\} \cup \mu_A(w)$ is S–inconsistent. Then one can find $B_1, \ldots, B_n$ in $\mu_A(w)$ such that $\vdash_S (B_1 \wedge \ldots \wedge B_n) \rightarrow A$. Now since $B_1, \ldots, B_n$ are in $\mu_A(w)$, $(B_1 \wedge \ldots \wedge B_n) \leq A$, and so by Proposition 3.1 (ii)

$$(\mathbf{b}) \qquad \vdash_S \Box(B_1 \wedge \ldots \wedge B_n) \rightarrow \Box A.$$

Since $B_1, \ldots, B_n$ are in $\mu_A(w)$, $\Box B_1, \ldots, \Box B_n$ are in w. It follows by Propositions 3.1 (i) and 3.3 (3) that

$$(\mathbf{c}) \qquad \Box(B_1 \wedge \ldots \wedge B_n) \in w.$$

From (**a**), (**b**), (**c**) and Proposition 3.3 (3), it follows that $\Box A \in w$, and so $\neg\Box A \notin w$. Contradiction. So, $\{\neg A\} \cup \mu_A(w)$ is S–consistent. $\dashv$

Proposition 3.7

1. *$\Box A \in w$ iff $A \leq w$ and for every w' such that $wQ^S w'$, $\neg A \notin w'$;*
2. *$\Box A \in w$ iff $A \leq w$ and for every w' such that $wR^S w'$, $\neg A \notin w'$.*

Proof. From left to right, easy. For the other direction, suppose that $\Box A \notin w$ and $A \leq w$. Then by r–maximality, $\neg\Box A \in w$. So by Proposition 3.6, $\Gamma = \{\neg A\} \cup \mu_A(w)$ is S–consistent. Now, $at(\Gamma) = at(A)$. So by Proposition 3.2, Γ has an $at\{A\}$–maximal S–consistent extension w'. Since $\mu_{w'}(w) = \mu_A(w) \subseteq \Gamma \subseteq w'$, $wQ^S w'$ (Proposition 3.5 (1)). Moreover, since $at(w') = at(A)$ and $A \leq w$, $w' \leq w$. So, $wR^S w'$ (Proposition 3.5 (2)). Finally, $\neg A \in w'$. $\dashv$

Let V^S be the valuation on $\mathcal{W}^S$ defined by $V^S(p, w) = 0$ iff $\neg p \in w$, $V^S(p, w) = 1$ iff $p \in w$, and $V^S(p, w) = 2$ iff $p, \neg p \notin w$. Our two canonical models are $\mathcal{M}^S =< \mathcal{W}^S, Q^S, V^S >$ and $\mathcal{N}^S =< \mathcal{W}^S, R^S, V^S >$.

Proposition 3.8 $A \in w$ *iff* $\mathcal{M}^S, w \vDash A$
iff $\mathcal{N}^S, w \vDash A$.

Proof. By induction on the length of A, using Propositions 3.4, 3.7 and the fact that for every formula B, $d(B, w, \mathcal{M}^S)$ iff $d(B, w, \mathcal{N}^S)$ iff $B \leq w$. $\dashv$

Theorem 3.9 *A is a theorem of S iff A is valid on $\mathcal{M}^S$*
iff A is valid on $\mathcal{N}^S$.

Proof. From left to right, by Proposition 3.3 (1). For the other direction, suppose $\nvdash_S A$. Then by Proposition 3.2, $\{\neg A\}$ has an *at*–maximal S–consistent extension. The result follows from Proposition 3.8. $\dashv$

4 Soundness + Completeness

Following a standard terminology, a system is said to be *determined* by a class of models or frames when that system is both sound and complete with respect to it. Proposition 2.1 and Theorem 3.9 establish that:

Theorem 4.1 *k is determined by the class of all frames.*

The aim is now to obtain determination results for systems containing k. I shall use the usual terminology to refer to these systems: kX is k enriched by axiom schema **X**, kXY is k enriched by axiom schemata **X** and **Y**, etc. In this section, I shall focus on systems obtained by enriching k with one or more items from the following classical list:

T $\Box A \to A$
B $A \to \Box\Diamond A$
4 $\Box A \to \Box\Box A$
E $\Diamond A \to \Box\Diamond A$

If a formula is valid (in our semantics) on a frame, then it is valid *in the sense of normal modal logic* on that frame. Indeed, if A is valid on frame $\mathcal{F}$, then A is valid on every model based on $\mathcal{F}$ whose valuation nowhere gets the value 2, and from this fact one can easily show that A is valid in the normal sense on every Kripke model based on $\mathcal{F}$. As a consequence, given any correspondence of type 'A is valid on a frame iff that frame satisfies condition φ' holding in normal modal logic, half of that condition holds in the present context, namely 'if A is valid on a frame, then that frame satisfies φ'. For each of the four axiom schemata I am dealing with in this section, the other half of correspondence holds as well:

Proposition 4.2

1. **T** *is valid on a frame iff that frame is reflexive;*
2. **B** *is valid on a frame iff that frame is symmetrical;*
3. **4** *is valid on a frame iff that frame is transitive;*
4. **E** *is valid on a frame iff that frame is euclidean.*

Proof. I give the proof for **E** as an illustration. Suppose $< \mathcal{W}, R >$ is euclidean, i.e. such that for all $x, y, z \in \mathcal{W}$ if (xRy and xRz), then yRz. Then let $\mathcal{M} = < \mathcal{W}, R, V >$ be a model, and assume for a *reductio* that for some $w \in \mathcal{W}$, both (i) $\mathcal{M}, w \models \Diamond A$ and (ii) $\mathcal{M}, w \not\models \Box\Diamond A$. By (i), $d(\Box\Diamond A, w, \mathcal{M})$, and so by (ii) we have: (iii) $\mathcal{M}, w \models \Diamond\Box\neg A$. From (i), we know that there is some $w' \in \mathcal{W}$ such that wRw' and $\mathcal{M}, w' \models A$. On the other hand, from (iii), we know that there is some $w'' \in \mathcal{W}$ such that wRw'' and $\mathcal{M}, w'' \models \Box\neg A$. But since $< \mathcal{W}, R >$ is euclidean, $w''Rw'$, and so, $\mathcal{M}, w' \not\models \neg\neg A$, i.e. $\mathcal{M}, w' \not\models A$. Contradiction. So, if (i), then (ii). As a consequence, **E** is valid on $< \mathcal{W}, R >$. $\dashv$

Things become interesting when one looks at the canonical frames defined in the previous section. In normal modal logic, the (usual) canonical frame of a system S containing any one of **T**, **B**, **4** or **E** satisfies the corresponding condition. As a result, any system obtained from the basic K by adding one or more axiom schemata from the list is determined by the class of all frames satisfying the corresponding conditions. In the present context, things are not so simple. Let us start with a positive fact:

Proposition 4.3

1. *If S contains* **T**, *then both Q^S and R^S are reflexive;*
2. *If S contains* **B**, *then Q^S is symmetrical;*
3. *If S contains* **4**, *then R^S is transitive.*

Proof. The proof for **B** will serve as an illustration. Assume S contains **B**, and let $w, w' \in \mathcal{W}^S$ be such that wQ^Sw'. Suppose now $\neg A \in w$. Then by **B**, $\Box\Diamond\neg A \in w$. So, $\neg\Diamond\neg A \notin w'$, i.e. $\Box A \notin w'$. Thus by contraposition, for every formula A, if $\Box A \in w'$, then $\neg A \notin w$. So $w'Q^Sw$. $\dashv$

By Propositions 4.2 and 4.3, we have the following

Theorem 4.4 *Each of the systems kT, kB, $k4$, kTB and $kT4$ is determined by the same class of frames as its normal counterpart.*

Unfortunately, one can prove neither that if S contains **B** then R^S is symmetrical, nor that if S contains **4** then Q^S is transitive. Worse, by Proposition 4.6 below, both are *false*. Proposition 4.6 settles the case of axiom schema **E** as well.

Proposition 4.5 *Suppose S is sound with respect to some frame $\mathcal{F}$. Then for every atom p, $\{p\}$, $\{\neg p\}$ and $\{\Box p\}$ are S–consistent. If moreover some worlds of $\mathcal{F}$ are not dead ends (i.e. worlds which are related to no world), then for all atoms p and q, $\{\neg\Box p, \neg\Box q\}$ is S–consistent.*

Proof. Let $< \mathcal{W}, R >$ be a frame with respect to which S is sound, and let p, q be atoms. (i) Let V be a valuation on $\mathcal{W}$ such that $V(p, w) = 1$ for some $w \in \mathcal{W}$. Then $\neg p$ is not valid on $< \mathcal{W}, R, V >$. By soundness, $\{p\}$ is S–consistent. (ii) Let V be a valuation on $\mathcal{W}$ such that $V(p, w) = 0$ for some $w \in \mathcal{W}$. Then p is not valid on $< \mathcal{W}, R, V >$. By soundness, $\{\neg p\}$ is S–consistent. (iii) Let V be a valuation on $\mathcal{W}$ such that $V(p, w) = 1$ for every $w \in \mathcal{W}$. Then $\neg\Box p$ is not valid on $< \mathcal{W}, R, V >$. By soundness, $\{\Box p\}$ is S–consistent. (iv) Suppose that world w in $\mathcal{W}$ is not a dead end, and let V be a valuation on $\mathcal{W}$ such that $V(p, w') = V(q, w') = 0$ for every $w' \in \mathcal{W}$. Then $\neg\Box p$ and $\neg\Box q$ are both true at w. So $\Box p \vee \Box q$ is not valid on $< \mathcal{W}, R, V >$. By soundness, $\{\neg\Box p, \neg\Box q\}$ is S–consistent. ⊣

Proposition 4.6 *Suppose S is sound with respect to some frame $\mathcal{F}$. Then Q^S is neither transitive nor euclidean. If moreover some worlds of $\mathcal{F}$ are not dead ends, then R^S is neither symmetrical nor euclidean.*

Proof. (1) Let p, q be two distinct atoms, and assume that S is sound with respect to some frame. By Proposition 4.5, $\{\neg p\}$, $\{q\}$ and $\{\Box p\}$ are S–consistent. Let w be a $\{p\}$–maximal S–consistent extension of $\{\Box p\}$, let w' be a $\{q\}$–maximal S–consistent extension of $\{q\}$, let w'' be a $\{p\}$–maximal S–consistent extension of $\{\neg p\}$. Then trivially, wQ^Sw', $w'Q^Sw$ and $w'Q^Sw''$. But since $\Box p \in w$ and $\neg p \in w''$, it is not the case that wQ^Sw''. So Q^S is neither transitive nor euclidean. (2) Let p, q be two distinct atoms, and assume that S is sound with respect to some frame some of whose worlds are not dead ends. Then by Proposition 4.5, $\{\neg\Box p, \neg\Box q\}$ is S–consistent. Let w be an at–maximal S–consistent extension of it. One can then find w' and w'' in $\mathcal{W}^S$ such that wR^Sw', wR^Sw'', $at(w') = \{p\}$ and $at(w'') = \{q\}$ (use Proposition 3.6). Now since w is at–maximal, it is not the case that $w'R^Sw$. So R^S is not symmetrical. Moreover, it is trivially not the case that $w'R^Sw''$. So R^S is not euclidean. ⊣

So, by Theorem 4.4, we have a determination result for every system obtained from k by adding one or more of **T**, **B** and **4**, except $kB4$ and $kTB4$. By Proposition 4.6, completeness for these two systems, cannot be established by means of our canonical constructions. In fact, by Proposition 4.2, both $kB4$ and $kTB4$ are sound with respect to equivalence (i.e. reflexive, symmetrical and transitive) frames, which by reflexivity do not contain dead ends. The situation for systems containing **E** is even more desperate. It should be noted here however that in (Correia *To appear*), system kTE (which is identical to $kTB4$) is proved to be determined by the class of all universal models (i.e. the models defined without accessibility relations, or equivalently, the models whose accessibility relation is universal).

5 Incompleteness

In the previous section, we saw that the usual correspondences for axiom schemata **T**, **B**, **4** and **E** hold in the present context. The problem was just that our two canonical constructions cannot help us to establish determination for some of the systems one can build from these axiom schemata. In this section, we shall see that the usual correspondences do not always hold in our three–valued context; and we shall also see that there are systems extending k which are *incomplete* (i.e. determined by no class of frames), while their normal counterparts are not.

Let us start with a general fact:

Proposition 5.1 *Let A_1, ..., A_n be any formulae, and let B be any non–modal formula, satisfiable in the sense of propositional calculus. Then $\neg(\Box A_1 \wedge ... \wedge \Box A_n \wedge B)$ is valid on a frame only if that frame is reflexive.*

Proof. Let $< \mathcal{W}, R >$ be a non–reflexive frame, and let $w \in \mathcal{W}$ be such that it is not the case that wRw. Let V be any valuation such that:

- V makes B true,
- For every atom p in every A_i, $V(p, w) \neq 2$, and
- For every atom p of the language and every world $w' \neq w$, $V(p, w') = 2$.

Then it is easy to check that $\Box A_1 \wedge ... \wedge \Box A_n \wedge B$ is true at w according to V. ⊣

Now, let us focus on axiom schema

D $\Box A \rightarrow \Diamond A$

In normal modal logic, **D** is valid on a frame $< \mathcal{W}, R >$ iff that frame is serial (i.e. such that for every $x \in \mathcal{W}$, there is a $y \in \mathcal{W}$ such that xRy). But in the present context, this is not the case. Indeed, if **D** is valid on a frame, then so is any formula of type $\neg(\Box A \wedge \Box\neg A \wedge p)$ where p is an atom,[7] and so by the previous Proposition, the frame must be reflexive. But not all serial frames are reflexive. So, here we have a case where the standard correspondence does not hold.

But is there any correspondence fact for **D** at all? Yes. We have the following correspondence on *models*:

(co) **D** is valid on a model iff that model is serial*,

[7] For, of course, in general if a formula A is valid on a frame, then so is $A \vee B$ for every formula B.

where a model $< \mathcal{W}, R, V >$ is serial* iff for all $x \in \mathcal{W}$ and formula A such that $d(A, x, \mathcal{M})$, there is a $y \in \mathcal{W}$ such that $d(A, y, \mathcal{M})$ and xRy. And we also have a correspondence on *frames*:

D is valid on a frame iff that frame is reflexive.

This correspondence can be proved from the fact that a frame is reflexive iff all the models based on it are serial*.

Now, let S be any system extending k containing as a theorem some formula of the type described in Proposition 5.1. From that Proposition and the fact that **T** is valid on every reflexive frame, it follows that if S is determined by some class of frames, then **T** must be a theorem of S. That is, we have the following:

Proposition 5.2 *System kXY... is incomplete if the following two conditions are satisfied:*

1. **T** *is not a theorem of kXY...;*
2. *Some formula of the type described in Proposition 5.1 is a theorem of kXY...*

Given that for every atom p, $\mathbf{D} \to \neg(\Box A \wedge \Box\neg A \wedge p)$ is a theorem of k, we have:

Theorem 5.3 *Every extension of k containing* **D** *but not* **T** *is incomplete.*

Thus, for instance, system kD is incomplete. Indeed, if **T** were a theorem of kD, then it would be a theorem of KD as well (since $kD \subseteq KD$), and we know that this is not the case.

Of course, incompleteness here is *frame*–incompleteness. The frame–incompleteness of a system is no bar to that system's being determined by some class of *models*. Actually, since kD has a canonical model, it is determined by the class consisting of this model only. But we have a more interesting characterization of kD in terms of models. Indeed, since kD has a canonical model, we have by the correspondence (co) that kD is determined by the class of all serial* models.

6 Adding Statability Operators

In this section I examine extensions of the previous logics obtained by adding "statability" operators to the basic modal language. I shall focus on two such operators, Prior's unary necessary statability operator $\mathcal{S}$ and the binary conditional statability operator $\rhd$. Their truth–clauses are the following:

- $w \models \mathcal{S}A$ iff $d(A,w)$ and $\forall v \in \mathcal{W}$ such that wRv, $d(A,v)$;
- $w \models A \rhd B$ iff $d(A,B,w)$ and $\forall v \in \mathcal{W}$ such that wRv and $d(A,v)$, $d(B,v)$.

The first operator is interesting in particular because strong necessity (see the introduction) can be defined as weak necessity plus necessary statability: it is strongly necessary that $A = \mathcal{S}A \wedge \Box A$. The second operator gives us even more expressive power: $\mathcal{S}A$ can be defined as $t \rhd A$, where t is a truth–constant (defined to be true at every world of every model) one may decide to introduce in the language.

I define two systems $k^{\mathcal{S}}$ and $k^{\rhd}$. System $k^{\mathcal{S}}$ is k plus

k$^{\mathcal{S}}$ $(\mathcal{S}p_1 \wedge ... \wedge \mathcal{S}p_n) \wedge \Box(A \to B) \to (\Box A \to \Box B)$, where p_1, ..., p_n are all the atoms of A not in B,

plus the following axioms for $\mathcal{S}$:

NS1 $\mathcal{S}A \to \mathcal{S}p$, where p is any atom of A;

NS2 $(\mathcal{S}p_1 \wedge ... \wedge \mathcal{S}p_n) \to \mathcal{S}A$, where p_1, ..., p_n are all the atoms of A.

System $k^{\rhd}$ is k with axiom **k** replaced by

k$^{\rhd}$ $B \rhd A \wedge \Box(A \to B) \to (\Box A \to \Box B)$,

plus the following axioms for $\rhd$:

CS1 $A \rhd B$, if all atoms in B are in A;

CS2 $(A \rhd B \wedge B \rhd C) \to A \rhd C$;

CS3 $(A \rhd B \wedge A \rhd C) \to A \rhd (B \wedge C)$.

Obviously, $k \subseteq k^{\mathcal{S}}$ and $k \subseteq k^{\rhd}$. I shall use the same notations for extensions of these two systems as for the extensions of **k**.

Both $k^{\mathcal{S}}$ and $k^{\rhd}$ are sound with respect to the class of all frames. Completeness can easily be proved by slight modifications of the canonical constructions of section 3, *but for $k^{\mathcal{S}}T$ and $k^{\rhd}T$*: the proof of proposition 6.2 below makes use of axiom **T**. I go through the proofs for $k^{\rhd}T$, leaving the case of $k^{\mathcal{S}}T$ to the reader.

Propositions 3.1–3.4 still hold for S any extension of $k^{\rhd}$, since these systems contain k. We shall also need the following

Proposition 6.1 *Let S be any extension of $k^{\rhd}$. Then*

1. *If $\vdash_S A \to B$, then $\vdash_S B \rhd A \to (\Box A \to \Box B)$;*
2. *If $C_1 \leq A$, ..., $C_n \leq A$, then $\vdash_S (C_1 \rhd B_1 \wedge ... \wedge C_n \rhd B_n) \to A \rhd (B_1 \wedge ... \wedge B_n)$.*

Proof. 1. Use Necessitation and **k**$^{\rhd}$.

2. Use **CS1**, **CS2** and **CS3**. $\dashv$

The definitions of the accessibility relations and μ change:

- $\mu_X(w) = \{B : \Box B \in w \text{ and } C \rhd B \in w \text{ for some } C \leq X\}$;
- $wQ^S w'$ iff for every formula A,
  - if $\Box A \in w$, then $\neg A \notin w'$;
  - if $A \rhd B \in w$ and $A \leq w'$, then $B \leq w'$;
- $wR^S w'$ iff for every formula A,
  - if $A \in w'$, then $\Diamond A \in w$;
  - if $A \rhd B \in w$ and $A \leq w'$, then $B \leq w'$.

Propositions 3.5–3.7 still hold with the new definitions. In the proof for Proposition 3.5 (1) (right–to–left direction), one has to use the fact that $A \rhd A$ is a theorem. The proof for Proposition 3.6 makes use of Proposition 6.1. In the proof for Proposition 3.7, $at(\Gamma) = \{p : A \rhd p \in w\}$. Proposition 3.8 and Theorem 3.9 can then be proved for S any extension of $k^{\rhd}T$ with the help of the following

Proposition 6.2 *Let S be any extension of $k^{\rhd}T$. Then*

1. *$A \rhd B \in w$ iff $A, B \leq w$ and for every w' such that $wQ^S w'$ and $A \leq w'$, $B \leq w'$;*
2. *$A \rhd B \in w$ iff $A, B \leq w$ and for every w' such that $wR^S w'$ and $A \leq w'$, $B \leq w'$.*

Proof. 1. From left to right, by the definition of Q^S. For the other direction, suppose that $A \rhd B \notin w$ and $A, B \leq w$. Then by r–maximality, $\neg(A \rhd B) \in w$. Now, $\mu_A(w)$ is consistent. For suppose $\mu_A(w)$ is inconsistent. Then one can find B_1, ..., B_n such that (i) $\Box B_1$, ..., $\Box B_n \in w$ and (ii) $\vdash_S \neg(B_1 \wedge ... \wedge B_n)$. It follows that (i) $\Box(B_1 \wedge ... \wedge B_n) \in w$ and (ii) $\Box\neg(B_1 \wedge ... \wedge B_n) \in w$. But by axiom **T**, it follows that w is inconsistent. Thus, $\mu_A(w)$ is consistent. Moreover, $at(\mu_A(w)) = \{p : A \rhd p \in w\}$, and so $at(B) \nsubseteq at(\mu_A(w))$. Let w' be an $at(\mu_A(w))$–maximal consistent extension of $\mu_A(w)$. Then $wQ^S w'$ and $A \leq w'$ but $B \nleq w'$.

2. From left to right, by the definition of R^S. For the other direction, it suffices to note that in the proof for the previous point, $at(w') = at(\mu_A(w)) \subseteq at(w)$. $\dashv$

It is easy to see that for S any extension of $k^{\rhd}T$, both Q^S and R^S are reflexive. Thus, $k^{\rhd}T$ is sound and complete with respect to the class of all reflexive frames. The same is true of $k^S T$.

If an extension S of $k^{\rhd}$ contains both modal axiom **4** and

CS4 $\quad A \rhd B \to \Box(A \rhd B)$.

then R^S is transitive. On the other hand, both axioms are valid on every transitive frames. So, $k^{\rhd}T4$ plus **CS4** is sound and complete with respect to the class of all reflexive and transitive frames. The same is true of $k^{\mathcal{S}}T4$ plus the following axiom:

NS3 $\quad \mathcal{S}A \to \Box\mathcal{S}A.$

Completeness results for other extensions of $k^{\triangleright}$ and $k^{\mathcal{S}}$ are hard to establish with our constructions. For instance it is not clear that there is an axiom for $\triangleright$ which, together with axiom **B**, makes the Q–accessibility relation symmetrical. It is worth noting here that in (Correia *To appear*) both $k^{\triangleright}TB4$ plus **CS4** and $k^{\mathcal{S}}TB4$ plus **NS3** are proved sound and complete with respect to the class of all universal frames.

7 Consequence Relations

As we saw, the standard notion of validity on a class of frames is not appropriate in the present context. In contradistinction, there is room here for the standard notion of *semantic consequence* on a class of frames. Let us use the symbol $\Vdash$ for standard semantic consequence. Thus, where $\mathcal{C}$ is a class of frames, Γ a set of formulae, and A a formula, I put:

> $\Gamma \Vdash_{\mathcal{C}} A$ iff for every world w of every model based on a frame in $\mathcal{C}$, if $w \vDash \Gamma$ then $w \vDash A$.

The standard notion of consequence is based on the idea that a proposition is a consequence of some propositions when the former *must be true* when the latter are all true. Yet, there is another intuitive notion of consequence: a proposition is a consequence of some propositions when the former *cannot be false* when the latter are all true. Let us use the symbol $\Vvdash$ for this notion of consequence:

> $\Gamma \Vvdash_{\mathcal{C}} A$ iff for every world w of every model based on a frame in $\mathcal{C}$, if $w \vDash \Gamma$ then $w \nvDash \neg A$.

It turns out that a formula A is valid on a class $\mathcal{C}$ of frames iff $\varnothing \Vvdash_{\mathcal{C}} A$.

As one can easily see, the deduction theorem holds for the second consequence relation:

$$\Gamma, A \Vvdash_{\mathcal{C}} B \text{ iff } \Gamma \Vvdash_{\mathcal{C}} A \to B.$$

The following Proposition shows that $\Vdash$ is definable in terms of $\Vvdash$:

Proposition 7.1 *Let $\mathcal{C}$ be a class of frames, Γ a set of formulae, and A a formula. Then:*

1. *If Γ is not satisfiable in $\mathcal{C}$, then both $\Gamma \Vdash_{\mathcal{C}} A$ and $\Gamma \Vvdash_{\mathcal{C}} A$;*
2. *If Γ is satisfiable in $\mathcal{C}$, then $\Gamma \Vdash_{\mathcal{C}} A$ iff $\Gamma \Vvdash_{\mathcal{C}} A$ and $A \leq \Gamma$.*

Proof. The first point is trivial. For the second point, the right–to–left direction is almost immediate. For the other direction, suppose that

$\Gamma \Vdash_{\mathcal{C}} A$. Then trivially, $\Gamma \Vvdash_{\mathcal{C}} A$. Remains to prove that $A \leq \Gamma$. Suppose for a *reductio* that $A \not\leq \Gamma$, and let p be an atom in A which is foreign to any formula in Γ. Then let $\mathcal{M} = < \mathcal{W}, R, V >$ be a model based on a frame in $\mathcal{C}$ and let w be a world of that model such that $\mathcal{M}, w \vDash \Gamma$ (such a model–world pair exists since Γ is satisfiable in $\mathcal{C}$). We define a new model $\mathcal{N}$ based on the same frame, and whose valuation V' is identical to V unless $V(p, w) \neq 2$ – in this case, we put $V'(p, w) = 2$. Since p is foreign to Γ, the truth–values of the formulae in Γ do not change, and so we have: $\mathcal{N}, w \vDash \Gamma$. And of course, we also have: $\mathcal{N}, w \nvDash A$. Thus, $\Gamma \nVdash_{\mathcal{C}} A$. But this contradicts the hypothesis. So, $A \leq \Gamma$. $\dashv$

Further Investigations

They should go in two directions. First, it would be nice to get a proper understanding about how much correspondence and determination facts for propositional extensions of k depart from those we have in normal modal logic, and why. Second, quantificational extensions of k should also be studied, if only because among the logics of essence mentioned in the introduction to this paper, the most natural ones are quantified.

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