A Letter of Hermann Amandus Schwarz on Isoperimetric Problems

URS STAMMBACH

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In 1995 (or thereabouts), Eli Maor, then at Loyola University, discovered a little booklet at a used book fair in Chicago containing a reprint of a paper published by Hermann Amandus Schwarz in 1884 [Sch1]. Interestingly, it also contained an original letter by Schwarz written in the old German script. The letter, dated January 28, 1884, began with the words: *Hochgeehrter Herr Director!* Maor could decipher enough of the text to realize that it dealt with isoperimetric problems, which further awakened his interest. With the help of Reny Montandon and Herbert Hunziker, he contacted Günther Frei, who was able to read and transcribe the old German script. Thanks to his detailed knowledge of the history of mathematics of the 19th century, Frei was also able to provide a number of pertinent remarks. Unfortunately, before completing this task, Günther Frei fell gravely ill. The text of the letter together with his remarks was then given to the present author, who agreed to continue the work.

We begin with some background information about Schwarz and the state of research on isoperimetric problems and minimal surfaces. Schwarz became interested in this area of investigations during his student days in Berlin. We next turn to Schwarz's letter itself, given here in an English translation. To make it easier to understand, we provide a number of supplementary explanations, mostly biographical data about the various people Schwarz mentions. Among them, Edvard Rudolf Neovius merits special attention. In the final section we describe the way this booklet, together with Schwarz's letter, found its way from Berlin to Chicago; this in itself is an interesting and surprising story. In an appendix we present a facsimile of Schwarz's letter (courtesy of Eli Maor) along with a German transcript.

To acknowledge Günther Frei's work and to honor him, it is the wish of all involved to dedicate this paper to him. Reny Montandon, who long followed our work on Schwarz's letter with great interest, unexpectedly died in the spring of 2011; it is sad that he did not live to see the publication of this paper.

Schwarz and Isoperimetric Problems

Hermann Amandus Schwarz (1843–1921) took his doctoral degree with Ernst Kummer in 1864 and his habilitation at the University of Berlin in 1866. The next year, he was appointed associate professor in Halle, and in 1869, at just 26 years of age, he became professor at the *Eidgenössische Polytechnikum* (now called *Eidgenössische Technische Hochschule*, ETH) in Zürich. In 1875, he obtained a professorship in Göttingen, and in 1892 succeeded Karl Weierstrass at the University of Berlin, where he remained until 1917. He died in Berlin in 1921. (See [FS], p. 73.)

Although Kummer was Schwarz's thesis adviser/examiner, his work was more closely related to the analytic interests of Weierstrass, and Schwarz always regarded himself chiefly as Weierstrass's pupil. It was during those years

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that Weierstrass began his famous program for building a solid foundation for analysis based on rigorous proofs for the many results he felt had been obtained by dubious reasonings. Closely related to this program were various results on isoperimetric problems and minimal surfaces that his former colleague, Jakob Steiner, had proved by appealing to the methods of synthetic geometry. After Steiner died in 1863, Weierstrass took on the additional task of teaching courses on synthetic geometry, upholding the Steinerian tradition in Berlin. It is easy to understand why Weierstrass took a keen interest in providing analytic and fully rigorous proofs of Steiner's results. Nor is it surprising that his pupil Hermann Amandus Schwarz also began to work in this same direction.

Some of Schwarz's later activity at the *Eidgenössische Polytechnikum* in Zürich sheds further light on this unusual situation. When Schwarz came to Zürich in 1869, he succeeded Elwin Bruno Christoffel, thereby assuming the most prestigious professorship in mathematics at the *Polytechnikum*. His principal task was teaching the large course in differential and integral calculus, the introductory mathematics course required for all beginning students. However, each semester he also taught two or even three more specialized courses on, for example, ordinary differential equations or topics of complex function theory. Some were introductory in nature, but others covered rather advanced topics, such as elliptic functions and their applications. Occasionally he also lectured on fields such as number theory, and in the summer term of 1871 he offered a course on *Analytische Geometrie der Raumkurven und der krummen Flächen*. He was apparently quite successful in attracting students to mathematics, despite the fact that most of his auditors were prospective engineers.

Several eminent names in mathematics have been associated with the *Eidgenössische Polytechnikum*, so Schwarz was hardly alone (see [FS], pp. 71–74). Wilhelm Fiedler was responsible for teaching descriptive geometry. This subject played a central role in the education of engineers, so Fiedler's position, like Schwarz's, carried high prestige. Fiedler was a former pupil of August Ferdinand Moebius; compared with his considerably younger colleagues, he was a rather conservative mathematician. Shortly after Schwarz's arrival, Heinrich Weber was appointed to a chair; like Schwarz, he left Zürich in 1875 to take a position in Germany. The Swiss mathematician Ludwig Stickelberger joined the faculty of the *Polytechnikum* in 1873, even before he took his doctorate in Berlin under Weierstrass. After Schwarz left Zürich, Stickelberger became a close collaborator of his successor, Ferdinand Frobenius, another Berlin product.

For Schwarz, there can be little doubt that his single most important colleague was another Swiss, Carl Friedrich Geiser (1843–1934), who was a grandnephew of Jacob Steiner. Geiser was appointed to a professorship at the *Polytechnikum* in 1869. As a student he spent some time in Berlin with his great-uncle before returning to Switzerland. In Bern he studied under Ludwig Schläfli, a Steiner pupil, and in 1866 Geiser became his first doctoral student. Little wonder that Geiser later went on to compile and edit Steiner's unpublished manuscripts and lecture notes. Steiner was also suitably remembered in 1897 at the First International Congress of Mathematicians in Zürich, an event for which Geiser was a major driving force, both as initiator and organizer (see [FS], pp. 11–13 and p. 74).

During the time Schwarz was in Zürich, Geiser regularly taught courses on synthetic geometry, presumably following the spirit of Steiner's courses rather closely. One can easily imagine that Geiser's close relationship to his great-uncle led to some heated mathematical discussions with Schwarz, who was just as devoted to the rigorous analytic approach championed by Weierstrass. These circumstances help shed light on the remark about Geiser in Schwarz's letter, a remark that is very interesting in many respects. There Schwarz refers to Jacob Steiner's Collected Works [St1], which were edited by Weierstrass in 1882, and in particular to a *Mittheilung* on pages 728 and 729 of Volume 2. On pages 727–729 under the heading "Anmerkungen und Zusätze" (Comments and Additions) we there find some comments by Weierstrass on a paper by Steiner, entitled "Aufgaben und Lehrsätze" (Problems and Theorems). In these comments Weierstrass mainly quotes from a related handwritten table – and apparently also some notes – which were given to him by Geiser and which he explicitly attributes to Steiner. With the remark in his letter Schwarz makes clear that the quote (two paragraphs) in the latter part of Weierstrass's comments is not due to Steiner but to Schwarz himself! And he adds that Geiser has taken responsibility for this mistake.

The classical isoperimetric problem,[1] with which Schwarz's letter is concerned, consists in proving that of all

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URS STAMMBACH received his doctoral degree in 1966 at the ETH in Zürich under the supervision of Beno Eckmann. After two years at Cornell University in Ithaca, New York, he acquired a professorship at the ETH in Zürich in 1969, and since 2005 has enjoyed emeritus professor status there. Urs Stammbach is the author and coauthor of numerous research articles in homological algebra and group theory and of several books, among them *A Course in Homological Algebra* (jointly with Peter Hilton) and *Mathematicians and Mathematics in Zurich, at the University and at the ETH* (jointly with Günther Frei). His interests outside of mathematics include history, literature, and music.

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[1]For details about the isoperimetric problem and its history, see [B] or [CR], Chapter VII, § 8.

Figure 1. H. A. Schwarz (from Hermann Amandus Schwarz zu seinem 50-jährigen Doktorjubiläum, Springer, 1914) and his interlocutors: Jacob Steiner, Karl Weierstrass, Carl Friedrich Geiser, and Edvard Rudolf Neovius (courtesy of the Hamburg Mathematische Gesellschaft, http://www.math.uni-hamburg.de/home/grothkopf/fotos/math-ges/). H. A. Schwarz was drawn to investigate isoperimetric problems as a student in Berlin. There he learned that the famous synthetic proofs of Steiner were faulty and spoke with Weierstrass about analytic methods for proving them. In Zürich, Schwarz discussed these matters with his colleague, Geiser, a relative of Steiner, as well as with Neovius, a student from Finland who was an expert on minimal surfaces.

curves of fixed length, the circle bounds the largest area in the plane. Many eminent mathematicians tried to solve this problem, but none succeeded completely. In 1841 Jacob Steiner presented his proof that the circle has the required isoperimetric property in two long papers published under the title *Sur le maximum et le minimum des figures dans le plan, sur la sphère et dans l'espace en général* (see [St2]). There Steiner dealt not only with the original isoperimetric problem, but also with several generalizations, including analogous questions for curves on spheres as well as the isoperimetric problem in 3-dimensional space: which surface bounds the largest volume? Soon after their publication, several mathematicians criticized the proofs as incomplete.

Steiner's colleague at the University of Berlin, Peter Lejeune Dirichlet, criticized Steiner for presupposing the existence of a solution for the problems in question. Steiner's proof proceeds by showing that a given curve that is not a circle fails to satisfy the required condition, since it is always possible to construct another curve of the same length that bounds a larger area. Clearly, for such an argument to be rigorously valid, one must somehow ensure that a curve actually exists that maximizes the area it bounds. Many other, perhaps less important, points were raised by Weierstrass, Schwarz, and also by Friedrich Edler and Rudolf Sturm.

During the 1870s, Weierstrass apparently presented a series of lectures that aimed to give rigorous proofs of the isoperimetric property of the circle as well as other results of Steiner. For whatever reasons, though, he never published his proofs, so the problem remained open until 1884 when Schwarz unveiled his paper, *Beweis des Satzes, dass die Kugel kleinere Oberfläche besitzt, als jeder andere Körper gleichen Volumens* (see [Sch1]). This is the paper contained in our booklet. As the title suggests, the larger part of this article is devoted to the isoperimetric problem in dimension three.

In the technical portion of the letter that follows, Schwarz critisizes one of Steiner's proofs. His remarks, however, do not point to an actual mathematical error, such as the one Dirichlet noted many years previously. Schwarz only describes a striking simplification that could be made in one of Steiner's arguments; it is an idea so simple that he was surprised Steiner could have overlooked it.

Schwarz's Letter

The original letter consists of three pages written on a single folded sheet of paper in the old German script (see the facsimile of the letter in the Appendix). It should be noted that Schwarz writes an extremely prolix German; his long and complicated sentences make a literal word-by-word translation practically impossible. To make the English text readable, we have therefore broken up some of the long sentences into two or even more parts.

Goettingen, January 28 1884
(Weender Chaussée 17 A.)[2]

Most honorable director![3]
Herewith I return with my best thanks your script of Steiner's course on maxima and minima, which you have lent me so kindly for such a long time. I take the opportunity, to send you some papers of mine as well as the dissertations of two of my students, the academic work of my former auditor Dr. E. Neovius, who is now Professor at the University in Helsingfors, and two copies of my proof that the sphere has the smallest surface area among all bodies of the same volume, which I have presented to our *Gesellschaft der Wissenschaften*. I have also enclosed two copies of the text by Mr. Edler. Please accept the second copy for the library of your institution. The proof that concerns the sphere is no doubt the first for the theorem in question, that is, the first really rigorous proof, since all the other attempts must be regarded as incorrect, since they are based on unproven facts. I may be allowed to reveal to you that the writer of these lines is the author of the statement in volume 2 of the *Gesammelte Abhandlungen* of Steiner (page 728 at the bottom and 729 at the top), where it is mistakenly attributed to Steiner. Professor Geiser has already admitted that he thereby committed an error.

The truly admirable works of Steiner on maxima and minima contain, unfortunately, in addition to the inaccuracies mentioned by Sturm, quite a number of others, which I have discovered while giving public lectures on Steiner's work, which I have repeatedly done at this university.

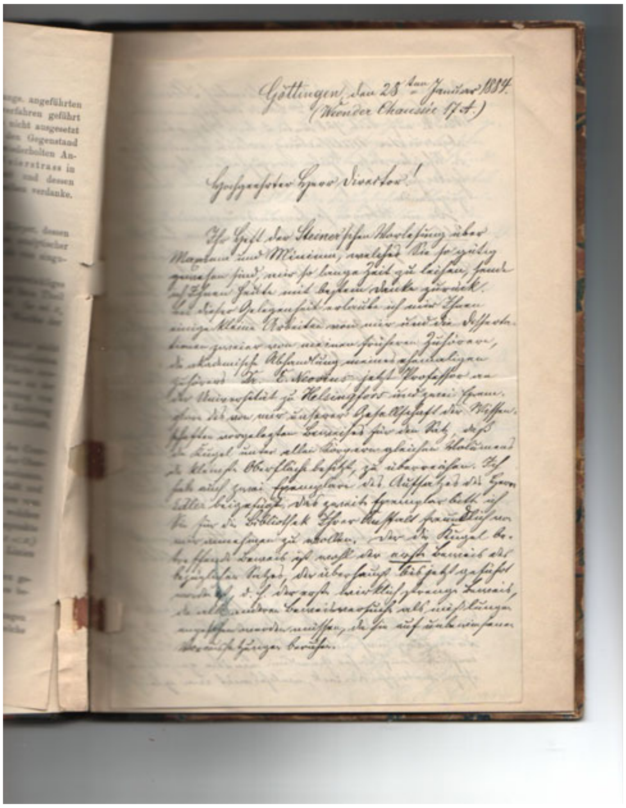
The proof of the theorem that of all triangles the equilateral triangle has the maximal area can be simplified considerably, despite the fact that Steiner has undoubtedly given much attention to his proof. It does not seem acceptable to use two intermediate steps, as Steiner does, when one intermediate step is sufficient.

The very simple thing that Steiner does not seem to have noticed – for otherwise he would have mentioned it – is the following:

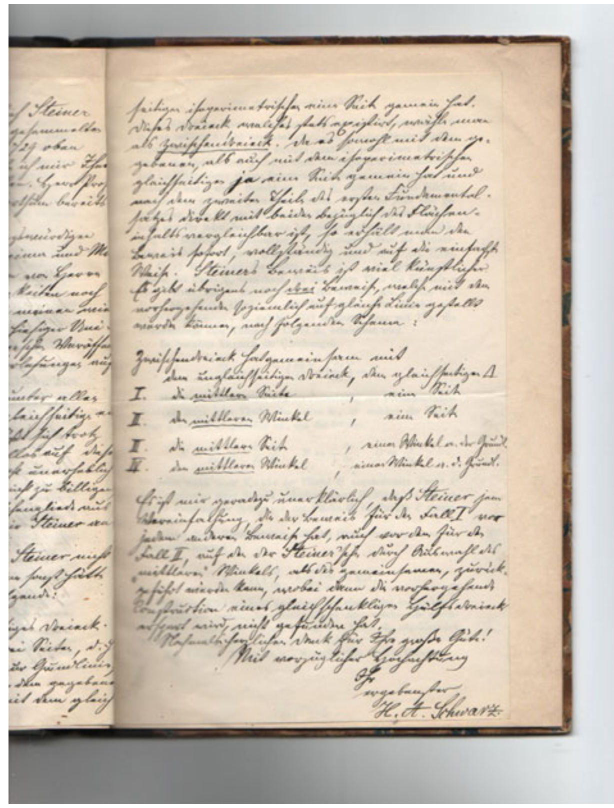
Let a triangle be given which is *not* equilateral. Choose as baseline the intermediate side, that is, the side of intermediate length; on this baseline construct a triangle which is isoperimetric to the given one and which also has a common side with the isoperimetric equilateral triangle. This triangle, which always exists, is chosen as the triangle for the intermediate step in the proof; since it has a common side with the original triangle as well as with the isoperimetric equilateral triangle, the second part of the first fundamental theorem can be applied to directly compare the areas of these triangles. This immediately provides the proof in a complete and simple way. Steiner's proof is much more complicated and it uses additional constructions. By the way, there exist three more proofs that are in the same spirit, following the scheme:

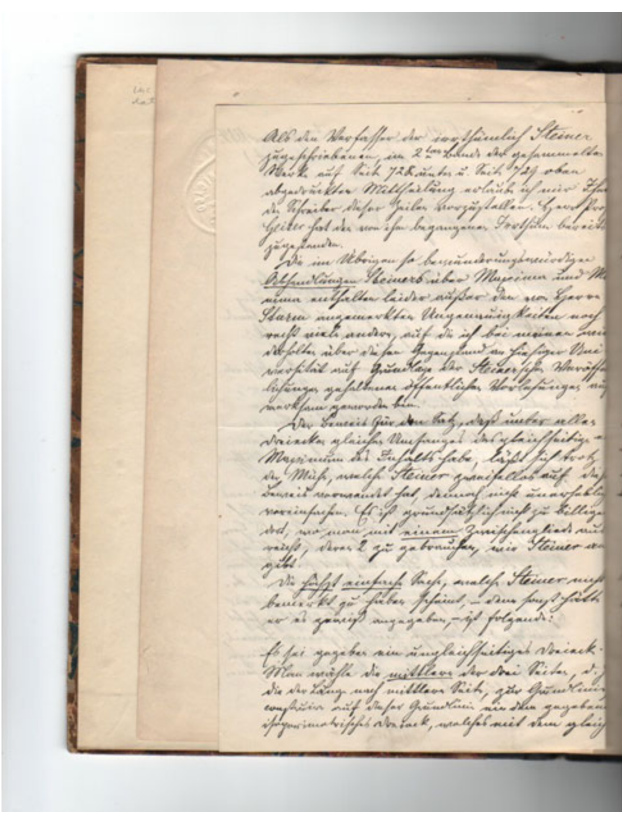
[2]Schwarz gives Weender Chaussée 17 A as his home address. This was the house next to Weender Chaussée No. 17, where Ida Riemann, the only surviving sister of Bernhard Riemann, lived together with Riemann's widow and her daughter, following Riemann's death in 1866.

[3]We suspect that the addressee was the director of a *Gymnasium*, but we have not been able to verify this, nor have we been able to identify the person. From the first sentences of the letter, one can assume that the addressee had attended some of the courses on maxima and minima by Steiner.


Göttingen, den 28ten Januar 1884.
(Weender Chaussee 17A.)

Hochgeehrter Herr Doctor!


H. A. Schwarz



Intermediate triangle has in common with

	the nonequilateral triangle,	the equilateral triangle
I.	the intermediate side	one side
II.	the intermediate angle	one side
III.	the intermediate side	one angle at the baseline
IV.	the intermediate angle	one angle at the baseline

I cannot really understand why Steiner has not noticed this simplification which *case I* provides with respect to any other proof, even with respect to *case II*, to which Steiner's proof can be reduced by choosing the intermediate angle as the common one; in this way [in Steiner's proof] the intermediate step of constructing an isosceles triangle can be avoided. Again many thanks for your kindness!

With highest esteem.
Your devoted
H. A. Schwarz

It is interesting to note that in his letter Schwarz does not simply try to promote his own work – although he includes a reprint of his paper [Sch1] – but also calls attention to the work of his students and associates. In particular he mentions the work of two of his doctoral students (whom we were unable to identify) and underscores very prominently the *academic work* of Edvard Rudolf Neovius.

Edvard Rudolf Neovius (see [Elf], [L][4]) lived from 1851 until 1917. His father, Edvard Engelbert Neovius, was a teacher of mathematics at the military school in Hamina (Swedish: Fredrikshamn). Like his older brother Lars, Edvard Rudolf

[4]We are grateful to Professor Olli Lehto for making us aware of these two sources of information, and to Dr. Yvonne Voegeli from the ETH-library for helping us to uncover the traces of Neovius's stay in Zürich.

joined this school and then served in the Russian army, but he was soon dissatisfied with military life and decided to leave service for university studies. Together with his brother Lars he went to Zürich, and during the following three years, from 1871 until 1874, he studied mechanical engineering at the *Eidgenössische Polytechnikum*. After that, he spent a year at the Technical University in Dresden before returning to Zürich for a year between 1875 and 1876 (see [Elf]). Finally, he was awarded a doctoral degree in 1880 at the Kejserliga Alexander Universitetet (Imperial Alexanders-University) in Helsingfors (Helsinki) with his dissertation "Kurvor af tredje och fjerde graden betraktade sø asom alster af tvänne projektiviska involutioner" (Curves of degree three and four considered as a product of two projective involutions). His dissertation is written in Swedish, the official language at the time. From 1883 until 1900 Neovius was Professor for mathematics at the University in Helsingfors.

During his first stay in Zürich, Neovius apparently came into close contact with Schwarz. In a letter from October 2, 1882, to Angelo Genocchi in Turin, Schwarz refers to Neovius as *un de mes anciens élèves de Zurich*. Schwarz and Neovius remained in steady contact. In our letter Schwarz refers to the paper "Bestimmung zweier speciellen (sic!) periodischen (sic!) Minimalflächen, auf welchen unendlich viele gerade Linien und unendlich viele ebene geodätische Linien liegen" (Determination of two special periodic minimal surfaces on which there lie infinitely many straight lines and infinitely many plane geodetic lines) (see [N]). In this paper, which is his "Habilitationsschrift" at the University in Helsingfors, Neovius constructed a minimal surface that today bears his name. (For an image of the surface see [WW], a reproduction of the original image in [N].)

Neovius visited Schwarz several times when the latter taught in Göttingen, and Schwarz sometimes took summer vacations in Helsinki. During a visit to Göttingen in 1885, Neovius noticed that Schwarz was completely overworked and longed for some quiet time to recover. Schwarz thus joined him on the trip back to Helsinki, where he stayed in his home. It was during this stay with Neovius that he completed his famous paper *Über ein die Flächen kleinsten Flächeninhaltes betreffendes Problem der Variationsrechnung. Festschrift zum Jubelgeburtstag des Herrn Karl Weierstrass* (On a problem of the calculus of variation concerning surfaces of minimal area) (see [Sch2]), a Festschrift marking the occasion of Weierstrass's 70th birthday. Many experts regard this paper as the most important of all of Schwarz's work.

In 1897 Neovius accepted an appointment as president of an important bank, and soon afterward became a member of its board. In 1900 he resigned from the university; he became a member of the Finnish cabinet as head of the country's finance department. The first years of the new century proved to be a politically difficult time for Finland. Following a pan-Slavic course, Tsar Nicholas II of Russia sought to gain more influence over internal Finnish matters, hoping to absorb the autonomous Grand Duchy of Finland into the Russian Empire. Finland was faced with the delicate question of how to react to the unilateral measures of its powerful Russian neighbour. Some Finns favoured a cautious policy, others preferred passive or even active resistance. The clash between these two attitudes grew more and more bitter, eventually culminating with the murder of the Russian governor-general Bobrikov in 1904, followed by a general strike in the autumn of 1905. Only a few months before this, Neovius left his cabinet post in bitterness, harshly criticized by those who favoured a stronger policy toward Russia. Unable to regain his chair at the university – it was now occupied by Ernst Lindelöf – he decided to leave Finland and spend the rest of his life in Denmark, his wife's native country. There he continued to follow his mathematical interests, pursuing research on minimal surfaces, and occasionally giving courses at the university of Copenhagen.

Edvard Rudolf was born into a mathematical family: his father and his uncle were both mathematicians, and two of his brothers became mathematicians as well. Moreover, he was the nephew of Leonard Lorenz Lindelöf, who was professor of mathematics at the University in Helsingfors. The latter's son, Ernst Lindelöf, became more famous than his father as a professor at this university. It is interesting to note that the mathematical genes in the wider family of Edvard Rudolf Neovius were passed to the next generation. Around 1906 some members of the family changed their name to Nevanlinna. A brother of Edvard Rudolf, Otto Wilhelm Neovius-Nevanlinna – one of the two brothers who became mathematicians – had two sons, Rolf and Frithiof Nevanlinna. Both went on to become world-famous mathematicians.

In his letter, Schwarz also mentions a text by Friedrich Edler, probably referring to the paper [E2]. Edler gives a simplified and rigorous proof of the result that for every polygon in the plane there exists a circle with smaller circumference but larger area. The paper is a sequel to his earlier [E1], in which he gave a rigorous proof of the isoperimetric property of the circle. One of the numerous lemmas leading up to the final result states that among the polygons with $2n$ edges and given circumference, the regular polygon has the largest area. In [E2] Edler presents a simplified proof.

Friedrich Edler, born in 1855 in Mühlhausen (Thüringen), began studies of mathematics in 1877 at the University in Halle, where he received his doctorate in 1882.[5] We were unable to find any information about his later life.

Schwarz's reference to *inaccuracies* in Steiner's work on maxima and minima mentions the criticism of Friedrich Otto Rudolf Sturm. Sturm received his doctorate 1863 in Breslau, taught in Darmstadt and Münster for the next twenty years, and then returned to the University of Breslau. He published a number of papers on synthetic geometry in the spirit of Steiner and wrote several textbooks on descriptive and synthetic geometry. In his paper [Stu1] he explicitly lists several gaps in Steiner's proofs. Later he published a little booklet, *Maxima und Minima in der elementaren Geometrie* (see [Stu2]), in which he gives a complete and correct presentation of Steiner's results.

Finally, at the close of his letter, Schwarz explains his own criticism of Steiner's proof of his "first fundamental theorem" for the equilateral triangle in [St2], which reads as follows:

[5] We thank Ms Karin Keller from the *Archiv der Universität Halle* for this kind information.

De deux de ces triangles [i.e., triangles with the same baseline], *celui qui aura l'angle le plus petit ou le plus grand à la base, ou bien dont l'un des côtés sera le plus petit ou le plus grand, sera le plus petit lui-même, et réciproquement*. Steiner's proof takes about half a printed page and requires a rather complicated figure. By contrast, Schwarz sketches a direct and extremely simple proof, so simple that Schwarz found it strange that Steiner could have overlooked this argument. As noted previously, Schwarz's critique does not concern a real mistake, it merely points to sloppiness on Steiner's part.

The *Berlin Collection*[6]

A document like this long-lost letter obviously sheds new light on the history of early work on isoperimetric problems. But how did the booklet containing Schwarz's paper and letter reach the shores of Lake Michigan? Here, again, we have managed to trace a line back to the year 1891 when the University of Chicago was first founded. In that same year its president, William Rainey Harper, made a trip to Germany. In Berlin, he stopped to visit the bookshop *S. Cavaltry* at Unter den Linden 17, where he discovered a large collection of books for sale. This bookshop was owned by a certain G. Heinrich Simon, a man of advanced age who wanted to retire. According to Simon, the collection consisted of about 300,000 books and 150,000 smaller booklets, including dissertations and the like. Among them were rare and distinguished works going back to the 15th and 16th centuries, but also more recent scholarly studies in philosophy, classical philology, Greek and Roman archaeology, and the sciences.

Harper realized that this collection would make an ideal acquisition for the new library at the University of Chicago. He decided to purchase the entire collection, even though this meant finding sponsors to pay for it when he returned to the United States. The contract he signed was for 180,000 marks, or about U.S. $ 45,000, a considerable sum of money. Yet it seemed to Harper a good buy. He had contacted several specialists who confirmed that the market value of the collection was significantly higher. After his return to Chicago, he was gratified that members of the Board of Trustees of the University of Chicago privately pledged large sums in order to make the purchase possible.

Later, once the transport of the books had begun, certain difficulties arose. It seems that the contract had not specified with sufficient clarity what was meant by a "volume". In fact, the number had to be revised downward: instead of 300,000 there were actually only 120,000 books and about 80,000 booklets. It is not clear how many books were actually delivered in the end; Harper's Presidential Reports of the year 1897 to 1898 mention 175,000 volumes, but this number could not be verified later. Of course, the fact that a smaller number of books was delivered resulted in a reduction of the price: only part of the pledged 180,000 marks had finally to be paid.

The purchase of this large and precious collection of books was widely publicized all over the United States, and it received considerable attention. It was seen as a very important step in the building of a university system. The books of the so called "Berlin Collection" thus became part of the University Library in Chicago; they were marked as such with a special label. It does not seem possible to determine when the booklet containing Schwarz's publication and enclosed letter was removed from the library stacks. Yet, somehow it surfaced at a used book sale in Chicago more than 100 years after President Harper purchased it in Berlin. The handwritten letter, written in old-fashioned German script, aroused the interest of an open-eyed mathematician, a remarkable and fortunate accident for historians of mathematics.

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[6]The information about the history of the *Berlin Collection* stems from the detailed account by Robert Rosenthal: *The Berlin Collection. A History*, see [RR].

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Appendix: German transcription of the letter

Göttingen, den 28ten Januar 1884
(Weender Chaussée 17 A.)

Hochgeehrter Herr Director!
Ihr Heft der Steiner'schen Vorlesung über Maxima und Minima, welches Sie so gütig gewesen sind, mir so lange Zeit zu leihen, sende ich Ihnen heute mit bestem Danke zurück. Bei dieser Gelegenheit erlaube ich mir, Ihnen einige kleine Arbeiten von mir und die Dissertationen zweier von meinen früheren Zuhörern, die akademische Abhandlung meines ehemaligen Zuhörers Dr. E. Neovius, jetzt Professor an der Universität zu Helsingfors, und zwei Exemplare des von mir unserer Gesellschaft der Wissenschaften vorgelegten Beweises für den Satz, dass die Kugel unter allen Körpern gleichen Volumens die kleinste Oberfläche besitzt, zu überreichen. Ich habe auch zwei Exemplare des Aufsatzes des Herrn Edler beigefügt. Das zweite Exemplar bitte ich Sie für die Bibliothek Ihrer Anstalt freundlich von mir annehmen zu wollen. Der die Kugel betreffende Beweis ist wohl der erste Beweis des bezüglichen Satzes, der überhaupt bis jetzt geführt worden ist, d.h. der erste wirklich strenge Beweis, da alle anderen Beweisversuche als mißlungen angesehen sehen werden müssen, da sie auf unbewiesenen Voraussetzungen beruhen. Als den Verfasser der irrthümlich Steiner zugeschriebenen, im 2-ten Bande der gesammelten Werke auf Seite 728 unten u[nd] Seite 729 oben abgedruckten Mittheilung erlaube ich mir, Ihnen den Schreiber dieser Zeilen vorzustellen. Herr Prof. Geiser hat den von ihm begangenen Irrthum bereits zugestanden. Die im Übrigen so bewunderungswürdigen Abhandlungen Steiners über Maxima, und Minima, enthalten leider, außer den von Herrn Sturm angemerkten Ungenauigkeiten noch recht viele andere, auf die ich bei meinen wiederholten über diesen Gegenstand an hiesiger Universität auf Grundlage der Steinerschen Veröffentlichungen gehaltenen öffentlichen Vorlesungen aufmerksam geworden bin.

Der Beweis für den Satz, daß unter allen Dreiecken gleichen Umfanges das gleichseitige ein Maximum des Inhalts habe, läßt sich trotz der Mühe, welche Steiner zweifellos auf diesen Beweis verwendet hat, dennoch nicht unerheblich vereinfachen. Es ist grundsätzlich nicht zu billigen dort, wo man mit einem Zwischenglied ausreicht, davon 2 zu gebrauchen, wie Steiner angibt.

Die höchst einfache Sache, welche Steiner nicht bemerkt zu haben scheint, – denn sonst hätte er es gewiß angegeben, – ist folgende:

Es sei gegeben ein ungleichseitiges Dreieck. Man wähle die mittlere der drei Seiten, d.h. die der Länge nach mittlere Seite, zur Grundlinie, construire auf dieser Grundlinie ein dem gegebenen isoperimetrisches Dreieck, welches mit dem gleichseitigen isoperimetrischen eine Seite gemein hat. Dieses Dreieck, welches stets existiert, wähle man als Zwischendreieck, dass es sowohl mit dem gegebenen, als auch mit dem isoperimetrischen gleichseitigen je eine Seite gemein hat und nach dem zweiten Theile des ersten Fundamentalsatzes direkt mit beiden bezüglich des Flächeninhalts vergleichbar ist, so erhält man den Beweis sofort, vollständig und auf die einfachste Weise. Steiners Beweis ist viel künstlicher. Es gibt übrigens noch drei Beweise, welche mit dem vorhergehenden so ziemlich auf gleiche Linie gestellt werden können, nach folgendem Schema: Zwischendreieck hat gemeinsam mit

	dem ungleichseitigen Dreieck,	dem gleichseitigen Δ [Dreieck]
I.	die mittlere Seite	eine Seite
II.	den mittleren Winkel	eine Seite
III.	die mittlere Seite	einen Winkel a[n] der Grundl[inie]
IV.	den mittleren Winkel	einen Winkel a[n] d[er] Grundl[inie]

Es ist mir geradezu unerklärlich, daß Steiner jene Vereinfachung, die der Beweis für den Fall I vor jedem anderen Beweis hat, auch vor dem für den Fall II, auf den der Steinersche durch Auswahl des "mittleren" Winkels, als des gemeinsamen, zurückgeführt werden kann, wobei dann die vorhergehende Construction eines gleichschenkligen Hülfsdreiecks erspart wird, nicht gefunden hat. Nochmals herzlichen Dank für Ihre große Güte!
Mit vorzüglicher Hochachtung

Ihr
ergebenster
H. A. Schwarz