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Calculus of Variations

Optimization problems for weighted Sobolev constants

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Abstract In this paper, we study a variational problem under a constraint on the mass. Using a penalty method we prove the existence of an optimal shape. It will be shown that the minimizers are Hölder continuous and that for a large class they are even Lipschitz continuous. Necessary conditions in form of a variational inequality in the interior of the optimal domain and a condition on the free boundary are derived.

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1 Introduction

Let $D \in \mathbb{R}^N$ be a bounded domain and let $a(x)$ and $b(x)$ be positive, continuous functions in D. Consider for an arbitrary real number $p > 1$ weighted Sobolev constants of the following form

$$S_p(D) = \inf_v \int_D a(x)|\nabla v|^p \, dx, \quad v \in \mathcal{K}(D) \text{ where}$$

$$\mathcal{K}(D) = \left\{ w \in W_0^{1,p}(D) : w \geq 0 \text{ a.e. }, \int_D b(x) w \, dx = 1 \right\}. \tag{1.1}$$

It follows from the Sobolev embedding theorem that there exists a minimizer u which solves the Euler–Lagrange equation

$$\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + S_p(D) b(x) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{1.2}$$

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The first question addressed in this paper is to study the smallest value $s_p(m)$ of $S_p(D)$ when D ranges among all domains contained in a fixed bounded domain $B \subset \mathbb{R}^N$, with prescribed measure $M(D) := \int_D b\,dx = m$. We are mainly interested in the existence of an optimal domain and the regularity of the minimizers.

For this purpose we follow a strategy used in [18] for eigenvalue problems. The idea which goes back to the pioneering papers of Alt and Caffarelli [1] and Alt, Caffarelli and Friedman [2], is to introduce a penalty term depending on m and to consider a variational problem in B without constraints. It has the advantage that it involves only the state function and not the optimal shape which is difficult to grasp. Such a problem appeared for the first time in the literature in connection with the problem of the torsional rigidity of cylindrical beams. In this case D is a simply connected domain in the plane, $p = 2$ and $a(x) = b(x) = 1$ and B is a large circle such that $|B| > m$. It has been conjectured by St.Venant in 1856 and proved by Polyà cf. [14] that the optimal domain is the circle. The same questions have been studied in [6] for the special case $p = 2$ and $a(x) = 1$. A major ingredient there is the isoperimetric inequality which is not available for non constant $a(x)$. Many references and results concerning Sobolev constants with different types of weights can be found in [3,12,15]. For applications to boundary value problems cf. [4,7] and the references cited therein. We shall assume that $a(x)$ and $b(x)$ meet the following assumptions:

(A1) $a(x), b(x) \in C^{0,1}(B)$;
(A2) there exist positive constants a_{min} and a_{max} such that $a_{min} \le a(x) \le a_{max}$;
(A3) there exists a positive constant b_{min} and b_{max} such that $b_{min} \le b(x) \le b_{max}$.

The plan of this paper is as follows. First, we discuss the Sobolev constant $S_p(D)$ in multiply connected domains D. It turns out that it behaves differently from other similar quantities like the smallest eigenvalues. Then, we prove the existence of a minimizer of an auxiliary problem in $W_0^{1,p}(B)$. The next chapter deals with the variational inequality which has to be satisfied by the minimizers, and the characterization of the free boundary between their support and the region where they vanish. In the last chapter we prove regularity results for the minimizers, in particular the Lipschitz continuity. We can then use these results to prove the existence of a minimizer and an optimal domain for $s_p(m)$.

2 Qualitative properties

In this section we list some general properties of $S_p(D)$, where D denotes an open bounded domain in $\mathbb{R}^N$. Instead of (1.1) it will sometimes be more convenient to use the equivalent form

$$S_p(D) = \inf_{W_0^{1,p}(D)} \frac{\int_D a(x)|\nabla v|^p\,dx}{\left(\int_D b(x)|v|\,dx\right)^p}. \tag{2.1}$$

Every minimizer is a multiple of u where u is the unique solution of

$$\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + b(x) = 0 \ \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{2.2}$$

Lemma 1 *$S_p(D)$ is monotone with respect to D in the sense that $S_p(D_1) \ge S_p(D_2)$ for any two open bounded domains D_1 and D_2 in $\mathbb{R}^N$ with $D_1 \subset D_2$.*

Proof The assertion is an immediate consequence of the fact that every admissible function for $S_p(D_1)$, extended as 0 outside of D_1 is an admissible function for $S_p(D_2)$. □

Lemma 2 *Let D_1 and D_2 be two open bounded domains in $\mathbb{R}^N$ such that $D_1 \cap D_2 = \emptyset$. Then*

$$S_p(D_1 \cup D_2)^{-\frac{1}{p-1}} = S_p(D_1)^{-\frac{1}{p-1}} + S_p(D_2)^{-\frac{1}{p-1}}.$$

Proof Let u_{D_1} and u_{D_2} be minimizers for $S_p(D_1)$ or $S_p(D_2)$, resp. which are solutions of (2.2) in D_1 or D_2, resp. Consequently,

$$\int_{D_1} a(x)|\nabla u_{D_1}|^p dx = \int_{D_1} b(x)u_{D_1} dx = S_p^{-\frac{1}{p-1}}(D_1) \text{ and}$$

$$\int_{D_2} a(x)|\nabla u_{D_2}|^p dx = \int_{D_2} b(x)u_{D_2} dx = S_p^{-\frac{1}{p-1}}(D_2)$$

Choosing as a test function in (2.1)

$$v = \begin{cases} u_{D_1} & \text{in } D_1 \\ u_{D_2} & \text{in } D_2 \end{cases}$$

we get

$$S_p(D_1 \cup D_2) \leq \frac{1}{\left(S_p(D_1)^{-\frac{1}{p-1}} + S_p(D_2)^{-\frac{1}{p-1}}\right)^{p-1}}. \tag{2.3}$$

Let u be a minimizer of $S_p(D_1 \cup D_2)$. Then keeping in mind that

$$\int_{D_1} a(x)|\nabla u|^p \, dx \geq S_p(D_1) \left(\int_{D_1} b(x)u \, dx\right)^p$$

$$\int_{D_2} a(x)|\nabla u|^p dx \geq S_p(D_2) \left(\int_{D_2} b(x)u \, dx\right)^p,$$

we find

$$S_p(D_1 \cup D_2) \geq \frac{S_p(D_1)\left(\int_{D_1} b(x)u \, dx\right)^p + S_p(D_2)\left(\int_{D_2} b(x)u \, dx\right)^p}{\left(\int_{D_1} b(x)u \, dx + \int_{D_2} b(x)u \, dx\right)^p}. \tag{2.4}$$

Set $I := \int_{D_1} b(x)u \, dx + \int_{D_2} b(x)u \, dx$, $\int_{D_1} b(x)u \, dx := \lambda I$ and $\int_{D_2} b(x)u \, dx = (1-\lambda)I$. Then

$$S_p(D_1 \cup D_2) \geq S_p(D_1)\lambda^p + S_p(D_2)(1-\lambda)^p =: h(\lambda).$$

This function $h(\lambda)$ achieves its minimum for

$$\lambda = \frac{S_p(D_2)^{1/(p-1)}}{S_p(D_1)^{1/(p-1)} + S_p(D_2)^{1/(p-1)}}.$$

Inserting this expression into $h(\lambda)$ we get

$$S(D_1 \cup D_2) \geq \frac{1}{\left(S_p(D_1)^{-\frac{1}{p-1}} + S_p(D_2)^{-\frac{1}{p-1}}\right)^{p-1}}.$$

This together with (2.3) proves the assertion. □

From this lemma we get immediately the estimate: *If* $S_p(D_1) < S_p(D_2)$ then

$$\frac{S_p(D_1)}{2^{p-1}} \leq S_p(D_1 \cup D_2) \leq \frac{S_p(D_2)}{2^{p-1}}.$$

Remark 1 Notice that the formula for $S_p(D_1 \cup D_2)$ in multiply connected domains differs from the one for the principal eigenvalue

$$\lambda_p(D) = \inf_{W_0^{1,p}(D)} \frac{\int_D a(x)|\nabla v|^p \, dx}{\int_D b(x)|v|^p \, dx}.$$

In this case Lemma 2 has to be replaced by

$$\lambda_p(D_1 \cup D_2) = \lambda_p(D_1), \quad \text{where} \quad \lambda_p(D_1) \leq \lambda_p(D_2).$$

Definition 1 For all positive $M \leq M(B) \int_B b(x)dx$ set

$$s_p(M) := \inf\{S_p(D) : D \subset B \text{ open}, M(D) \leq M\}$$

If for some domain D_0 with measure M we have $s_p(M) = S_p(D_0)$, then D_0 is called optimal domain for $s_p(M)$.

By Lemma 1 the infimum is the same if D' varies in the smaller class of open domains with $M(D') = M$. In the chapter on regularity we shall need the quantity

$$\sigma_p = \inf_{(0,M(B))} M^{p+p/N-1} s_p(M). \tag{2.5}$$

The following lemma will be crucial for our considerations.

Lemma 3 *Assume* (A1), (A3) *and the weaker form of* (A2), *namely*
(A2′) : $0 < a_{min} \leq a(x)$.
Then $\sigma_p > 0$.

Proof We have

$$S_p(D) \geq \frac{a_{min}}{b_{max}^p} \inf_{W_0^{1,p}(D)} \frac{\int_D |\nabla v|^p \, dx}{\left(\int_D |v| \, dx\right)^p}.$$

Let

$$T_p(D) := \inf_{W_0^{1,p}(D)} \frac{\int_D |\nabla v|^p \, dx}{\left(\int_D |v| \, dx\right)^p}.$$

If D^* denotes the ball with the same volume as D then by a symmetrization and a scaling argument we get

$$T_p(D) \geq T_p(D^*) = \left(\frac{|B_1|}{|D|}\right)^{p+p/N-1} T_p(B_1).$$

Hence

$$S_p(D) \geq \frac{a_{min}}{b_{max}^p}|D|^{1-p-p/N}c(N,p) \geq \frac{a_{min}}{b_{max}^p}b_{min}^{p+p/N-1}M^{1-p-p/N}c(N,p),$$
$$\text{where } c(N,p) := |B_1|^{p+p/N-1}T_p(B_1),$$

which implies that

$$\sigma_p \geq \frac{b_{min}^{p+p/N-1}a_{min}}{b_{max}^p}c(N,p) > 0. \tag{2.6}$$

□

More results on σ_p can be found in [5].

3 Existence

Let $B \subset \mathbb{R}^N$ be a bounded fundamental domain, e.g. a large ball, and let $M(B) > t > 0$, $\epsilon > 0$ be arbitrary fixed numbers. We consider the functional $J_{\epsilon,t} : W_0^{1,p}(B) \to \mathbb{R}^+$ given by

$$J_{\epsilon,t}(v) := \frac{\int_B a(x)|\nabla v|^p\, dx}{\left(\int_B b(x)|v|\, dx\right)^p} + f_\epsilon\left(\int_{\{v>0\}} b(x)dx\right),$$

where

$$f_\epsilon(s) = \begin{cases} \frac{1}{\epsilon}(s-t) & : \quad s \geq t \\ 0 & : \quad s \leq t. \end{cases}$$

For $v \equiv 0$ we set $J_{\epsilon,t}(v) = \infty$.

At first we are interested if the following variational problem has a minimizer

$$\mathcal{J}_{\epsilon,t} = \inf_{\mathcal{K}(B)} J_{\epsilon,t}(v). \tag{3.1}$$

Theorem 1 *Under the assumptions* (A1)–(A3) *there exists a function* $u_\epsilon \in \mathcal{K}(B)$, *depending on t such that*

$$J_{\epsilon,t}(u_\epsilon) = \mathcal{J}_{\epsilon,t}.$$

Proof Since the functional is bounded from below there exist minimizing sequences $\{u_k\}_{k\geq 1} \subset \mathcal{K}(B)$. Assume that $J_{\epsilon,t}(u_k) < c_0$ for all k. Without loss of generality we may normalize u_k such that

$$\int_B b(x)u_k\, dx = 1.$$

Therefore $\int_B a|\nabla u_k|^p\, dx < c_0$ and by (A2) also $\|\nabla u_k\|_{L^p(B)}$ is uniformly bounded from above. Hence there exists a function $u \in W_0^{1,p}(B)$ (if no ambiguity occurs we write u instead of u_ϵ) and a subsequence which will again be denoted by $\{u_k\}_{k\geq 1}$, such that

- $\nabla u_k \to \nabla u$ weakly in $L^p(B)$;
- $u_k \to u$ strongly in $L^q(B)$, for $q < Np/(N-p)$ if $p < N$ and for all $q \geq 1$ otherwise;
- $u_k \to u \in \mathcal{K}(B)$ almost everywhere in B.

For the last statement see e.g. [16] Theorem 3.12. This result implies in particular that

$$\int_B b(x)u\,dx = 1. \tag{3.2}$$

Since $\left\{\int_B a(x)|\nabla u|^p\,dx\right\}^{1/p}$ is a norm in $W_0^{1,p}(B)$ and since norms are lower semicontinuous with respect to weak convergence, the inequality

$$\int_B a(x)|\nabla u|^p\,dx \leq \liminf_{k\to\infty}\int_B a(x)|\nabla u_k|^p\,dx \tag{3.3}$$

holds.
For simplicity we shall use in the sequel the following notation: for any $w \in \mathcal{K}(B)$ set

$$D_w := \{x : w(x) > 0 \text{ a.e. }\}, \quad M_w := \int_{D_w} b(x)dx.$$

Next, we want to prove that

$$M_u \leq \liminf_{k\to\infty} M_{u_k}. \tag{3.4}$$

We denote by $|G|$ the Lebesgue measure of a measurable set G. The sequence $\{u_k\}_{k\geq 1}$ satisfies the assumptions for Egoroff's theorem. Hence for any $\delta > 0$ there exists a measurable set E_δ such that $|E_\delta] < \delta$ and such that $\{u_k\}_{k\geq 1}$ converges uniformly on $B \setminus E_\delta$. Set $Q := \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty D_{u_n}$. Since $u_k \to u$ uniformly as $k \to \infty$ on $D_u \setminus E_\delta$ we deduce that $D_u \setminus E_\delta \subseteq Q$. This together with the fact that $\int_Q b\,dx \leq \liminf_k M_{u_k}$ implies

$$M_u = \int_{D_u\setminus E_\delta} b(x)dx + \int_{E_\delta} b(x)dx \leq \liminf_{k\to\infty} M_{u_k} + \delta b_{max}.$$

Moreover, since δ can be chosen arbitrarily small, this establishes (3.4). The assertion now follows from (3.2), (3.3) and (3.4) . □

Observe that u_ϵ does not have to be unique. Next, we study the sequence $\{u_\epsilon\}$ as $\epsilon \to 0$ where u_ϵ is any minimizer of $\mathcal{J}_{\epsilon,t}$.

Lemma 4 *For every positive $t \leq M(B)$ there exists a subsequence $\{u_{\epsilon'}\} \subset \mathcal{K}(\mathcal{B})$ such that*

$$u_{\epsilon'} \to u_0 \text{ weakly in } W_0^{1,p}(B)\ , \quad \int_B b(x)u_0\,dx = 1$$

$$M_{u_0} \leq t \quad \text{and} \quad \int_B a(x)|\nabla u_0|^p\,dx = \mathcal{J}_t,$$

where

$$\mathcal{J}_t = \lim_{\epsilon'\to 0} \mathcal{J}_{\epsilon',t} \leq s_p(t).$$

Proof Let $D' \subset B$ be an open domain in B such that $\int_{D'} b(x)dx = t$. Let $w \in W_0^{1,p}(D')$ be a minimizer of $S_p(D')$ with $w \equiv 0$ in $B \setminus D'$. Then $J_{\epsilon,t}(u_\epsilon) \leq J_{\epsilon,t}(w)$, i.e.

$$\int_B a(x)|\nabla u_\epsilon|^p\, dx + f_\epsilon(M_{u_\epsilon}) \leq S_p(D'). \tag{3.5}$$

Hence $\int_B a(x)|\nabla u_\epsilon|^p\, dx$ and by the assumption $(A3)$ also $\int_B |\nabla u_\epsilon|^p\, dx$ are bounded from above by a constant which is independent of ϵ. Therefore there exists a subsequence $u_{\epsilon'}$ such that

$$u_{\epsilon'} \to u_0, \text{ weakly in } W_0^{1,p}(B), \quad u_{\epsilon'} \to u_0 \text{ strongly in } L^1(B) \text{ as } \epsilon' \to 0.$$

This implies that $u_0 \in \mathcal{K}(B)$. (3.5) also implies

$$(M_{u_\epsilon} - t)_+ \to 0 \quad \text{ as } \epsilon \to 0.$$

Consequently

$$\limsup_{\epsilon \to 0} M_{u_\epsilon} \leq t,$$

and by the same arguments as in Theorem 1, $M_{u_0} \leq t$. It is easy to see that

$$\mathcal{J}_t = \inf_{v \in \mathcal{K}(B)} \int_B a(x)|\nabla v|^p\, dx, \quad \text{ with } M_v \leq t. \tag{3.6}$$

The quantity at the right-hand side of (3.6) could be interpreted as $\mathcal{J}_{0,t}$. Thus by the definition of $s_p(t)$ where the infimum is taken only among functions v such that D_v is open we conclude that $\mathcal{J}_t \leq s_p(t)$. □

Open problem We expect that for ϵ_0 sufficiently small, $\mathcal{J}_{\epsilon,t} = \mathcal{J}_{\epsilon_0,t}$ for all $0 \leq \epsilon \leq \epsilon_0$.

4 Necessary conditions

4.1 First variation

Theorem 2 *Let u_ϵ, $\epsilon \geq 0$, be a minimizer of $\mathcal{J}_{\epsilon,t}$ which is normalized such that $\int_B b(x)u_\epsilon\, dx = 1$. Then for all nonnegative functions $\varphi \in W_0^{1,p}(B)$, the following inequality holds:*

$$\int_B a(x)|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla\varphi\, dx \leq \lambda \int_B b(x)\varphi\, dx,$$

$$\text{where } \lambda := \int_B a(x)|\nabla u_\epsilon|^p\, dx. \tag{4.1}$$

Proof For short we shall write u instead of u_ϵ. Since u is a minimizer we have $\mathcal{J}_{\epsilon,t}(u) \leq \mathcal{J}_{\epsilon,t}((u - \delta\varphi)_+)$ for every $\delta > 0$. Set $v := (u - \delta\varphi)_+$ and note that $D_v \subset D_u$. Hence by the monotonicity of $f_\epsilon(t)$ we have

$$f_\epsilon(M_u) \geq f_\epsilon(M_v)$$

and thus

$$\frac{\int_B a(x)|\nabla u|^p\,dx}{\left(\int_B b(x)u\,dx\right)^p} \le \frac{\int_B a(x)|\nabla v|^p\,dx}{\left(\int_B b(x)v\,dx\right)^p}.$$

Using the normalization we get

$$0 \le \int_B a(x)|\nabla v|^p\,dx - \int_B a(x)|\nabla u|^p\,dx \left(\int_B b(x)v\,dx\right)^p. \tag{4.2}$$

We now discuss the integrals in more detail. Keeping in mind that $\int_B bu dx$ and $\int_B b\varphi dx$ are bounded we find, setting

$$I_0 := \int_{B\cap\{u>\delta\varphi\}} b(x)u\,dx,$$

$$\begin{aligned}\left(\int_B b(x)v\,dx\right)^p &= \left(\int_{B\cap\{u>\delta\varphi\}} b(x)(u-\delta\varphi)dx\right)^p\\ &= I_0^p - p\delta I_0^{p-1}\int_{B\cap\{u>\delta\varphi\}} b(x)\varphi\,dx + O(\delta^2).\end{aligned} \tag{4.3}$$

Next, we compute

$$\begin{aligned}\int_B a(x)|\nabla v|^p\,dx &= \int_{B\cap\{u>\delta\varphi\}} a(x)|\nabla(u-\delta\varphi)|^p\,dx\\ &= \int_{B\cap\{u>\delta\varphi\}} a(x)|\nabla u|^p\,dx - p\delta\int_{B\cap\{u>\delta\varphi\}} a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \eta.\end{aligned} \tag{4.4}$$

The remainder term η contains a finite number of expressions of the form

$$c_{q_1q_2}\delta^{q_1+q_2}\int_B a(x)|\nabla u|^{p-q_1-q_2}|\nabla\varphi|^{q_1}(\nabla u,\nabla\varphi)^{q_2}\,dx$$

with $q_1+q_2 \ge 2$. They can be bounded from above by means of $\int_B a|\nabla u|^p\,dx$ and $\int_B a|\nabla\varphi|^p\,dx$. This implies that

$$\eta = O(\delta^2).$$

Plugging the expressions (4.3) and (4.4) into inequality (4.2) we get

$$\begin{aligned}0 \le &\int_{\{u>\delta\varphi\}} a(x)|\nabla u|^p\,dx - p\delta\int_{\{u>\delta\varphi\}} a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\\ &- \int_B a(x)|\nabla u|^p\,dx\left(I_0^p - p\delta I_0^{p-1}\int_{\{u>\delta\varphi\}} b(x)\varphi\,dx\right) + O(\delta^2).\end{aligned} \tag{4.5}$$

Observe that for small δ,

$$\int_{\{u\le\delta\varphi\}} b(x)u\,dx \le \delta\int_B b(x)\varphi\,dx = O(\delta),$$

and

$$I_0 = 1 - \int_{\{u\le\delta\varphi\}} b(x)u\,dx,$$

$$I_0^p = 1 - p\int_{\{u\le\delta\varphi\}} b(x)u\,dx + O(\delta^2),$$

$$p\delta I_0^{p-1} = p\delta + O(\delta^2).$$

Introducing these expressions into (4.5) and rearranging terms we conclude that

$$p\delta\int_{\{u>\delta\varphi\}} a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx$$

$$\le \int_B a(x)|\nabla u|^p\,dx\left\{p\int_{\{u\le\delta\varphi\}} b(x)u\,dx + p\delta\int_{\{u>\delta\varphi\}} b(x)\varphi\,dx\right\} + O(\delta^2).$$

The expression in the brackets at the right-hand side of this inequality is bounded from above by

$$p\delta\int_B b(x)\varphi\,dx.$$

Hence we obtain, dividing by $p\delta > 0$ and then letting δ tend to 0

$$\int_B a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx \le \int_B a(x)|\nabla u|^p\,dx\int_B b(x)\varphi\,dx.$$

This proves the theorem. □

Corollary 1 *In the interior of D_{u_ϵ}, every normalized minimizer u_ϵ of $\mathcal{J}_{\epsilon,t}$, satisfies the Euler–Lagrange equation*

$$\mathrm{div}(a(x)|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon) + \lambda b(x) = 0, \quad \textit{where } \lambda = \int_B a(x)|\nabla u_\epsilon|^p\,dx,$$

in the weak sense.

Proof Let x_0 be an inner point in D_{u_ϵ} and suppose that the ball $B_\rho(x_0)$ centered at x_0 of radius ρ satisfies $\overline{B_\rho(x_0)} \subset D_{u_\epsilon}$. Let $\varphi \in W_0^{1,p}(B_\rho(x_0))$, extended as zero in $B \setminus B_\rho(x_0)$. In contrast to the previous theorem, φ is allowed to change sign. Choose δ so small that $v := u \pm \delta\varphi > 0$ in $B_\rho(x_0)$. Hence $D_v = D_{u_\epsilon}$. The same arguments as before apply and yield

$$\int_{B_\rho(x_0)} a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx = \lambda\int_{B_\rho(x_0)} b(x)\varphi\,dx.$$

This proves the assertion. □

Remark 2 The proof of the previous Theorem holds also for u_0 which is the minimizer corresponding to $s_p(M)$ [cf. Lemma 4].

4.2 Boundary condition

We derive a necessary condition for the minimizers u_ϵ which has to be satisfied on ∂D_{u_ϵ} where it is smooth. For simplicity we will write u instead of u_ϵ.

Theorem 3 *Let u be a minimizer of $\mathcal{J}_{t,\epsilon}$. Let $A \subset B$ be an open set such that $A \cap \partial D_u$ is smooth and $u \in C^1(A \cap \overline{D_u})$. Then the following identity holds*

$$a(x)|\nabla u|^p = \text{const.}b(x) \quad \text{for } x \in A \cap \partial D_u.$$

Consider the function

$$\tilde{u}(x) := u(x + \delta\eta(x)). \tag{4.6}$$

η denotes a smooth vector field in B with compact support in A satisfying the additional constraint

$$\int_{A\cap\partial D_u} b(x)\eta(x) \cdot \nu \, dS = 0. \tag{4.7}$$

δ denotes a positive constant which is chosen so small, such that $x + \delta\eta(x) \in B$ for all $x \in B$. A consequence of this assumption is

Lemma 5 *Let $\eta \in C_0^\infty(A, \mathbb{R}^N)$ for some open subset $A \subset B$. Then*

$$\int_{D_{\tilde{u}}} b(x)dx = \int_{D_u} b(x)dx + o(\delta). \tag{4.8}$$

Proof The claim follows by direct computation. We set $y = x + \delta\eta(x)$. Then $dx = (1 - \delta\text{div}\eta)dy + o(\delta)$. Hence, we get because of (4.7).

$$\int_{D_{\tilde{u}}} b(x)dx = \int_{D_u} b(y - \delta\eta)(1 - \delta\text{div}\eta)dy + o(\delta) = \int_{D_u} b(y)dy - \delta \int_{A\cap D_u} b(y)\text{div}\eta \, dy$$
$$- \delta \int_{A\cap D_u} \eta \cdot \nabla b(y)dy + o(\delta) = \int_{D_u} b(y)dy + o(\delta).$$

This proves the lemma. □

A consequence of this lemma is, that

$$f_\epsilon(M_{\tilde{u}}) = f_\epsilon(M_u) + o(\delta). \tag{4.9}$$

This will be needed in the following proof.

Proof of the Theorem By our assumption there holds

$$J_{\epsilon,t}(u) \leq J_{\epsilon,t}(\tilde{u}). \tag{4.10}$$

We first make a change of variable $y = x + \delta\eta(x)$ and then expand the terms of the right hand side with respect to δ. We get

$$\int_{D_{\tilde u}} a(x)|\nabla \tilde u|^p\,dx = \int_{D_u} a(y)|\nabla u|^p\,dy - \delta \int_{A\cap D_u} a(y)|\nabla u|^p \operatorname{div}\eta\,dy$$
$$-\delta \int_{A\cap D_u} \eta\cdot\nabla a(y)|\nabla u|^p\,dy$$
$$+\,\delta p \int_{A\cap D_u} a(y)|\nabla u|^{p-2}\nabla u\cdot D\eta\cdot\nabla u\,dy + o(\delta).$$

We integrate by parts, making use the smoothness of ∂D_u locally in A, and we obtain

$$\int_{A\cap D_u} a(y)|\nabla u|^{p-2}\nabla u\cdot D\eta\cdot\nabla u\,dy = -\int_{A\cap D_u} \operatorname{div}(a(y)|\nabla u|^{p-2}\nabla u)\eta\cdot\nabla u\,dy$$
$$-\int_{A\cap D_u} a(y)|\nabla u|^{p-2}\nabla u\cdot D^2u\cdot\eta\,dy$$
$$+\int_{A\cap\partial D_u} a(y)|\nabla u|^{p-2}\nabla u\cdot\nu\eta\cdot\nabla u\,dS$$

Next, we observe that since $u = 0$ on $A\cap\partial D_u$

$$\int_{A\cap\partial D_u} a(y)|\nabla u|^{p-2}\nabla u\cdot\nu\eta\cdot\nabla u\,dS$$
$$= \int_{A\cap\partial D_u} a(y)|\nabla u|^{p-2}(\nabla u\cdot\nu)(\eta\cdot\nu)(\nu\cdot\nabla u)dS$$
$$= \int_{A\cap\partial D_u} a(y)|\nabla u|^{p-2}(\nabla u\cdot\nu)^2(\eta\cdot\nu)dS$$
$$= \int_{A\cap\partial D_u} a(y)|\nabla u|^{p}\eta\cdot\nu\,dS$$

We argue analogously for the other integrals and we obtain the equality:

$$\int_{D_u} a(x)|\nabla \tilde u|^p\,dx = \int_{D_u} a(y)|\nabla u|^p\,dy - \delta p \int_{A\cap D_u} \operatorname{div}(a(y)|\nabla u|^{p-2}\nabla u)\eta\cdot\nabla u\,dy$$
$$+\,\delta(p-1)\int_{A\cap\partial D_u} a(y)|\nabla u|^p\eta\cdot\nu\,dS + o(\delta).$$

Similarly, we have

$$\int_{D_{\tilde u}} b(x)\tilde u\,dx = \int_{D_u} b(y)u(y)dy + \delta p\int_{A\cap D_u} b(y)\eta(y)\cdot\nabla u(y)dy + o(\delta).$$

We insert the above expansions into (4.10) and use (4.9) and Corollary 1. After rearranging terms we get for $\delta \to 0$:

$$\int\limits_{A\cap\partial D_u} a(x)|\nabla u|^p \eta\cdot\nu\, dS = 0 \qquad (4.11)$$

The equality comes from the fact that $\eta\cdot\nu$ can have any sign. Because of (4.7) and the assumption that $u \in C^1(A\cap\overline{D_u})$ this implies the pointwise equality

$$a(x)|\nabla u(x)|^p = \text{const.}\, b(x) \qquad \text{for } x \in A\cap\partial D_u.$$

This proves the theorem. □

5 Regularity

This section is devoted to the regularity of the minimizers of $\mathcal{J}_{\epsilon,t}$. The notation will be the same as in the last section. In particular, we shall need the quantity σ_p defined in (2.5). If $p > N$ it follows immediately from the embedding theorems that the minimizers are Hölder continuous.

Theorem 4 *Every solution u of (4.1) belongs to $L^\infty(B)$ and satisfies*

$$|u|_\infty \leq \left(\frac{\lambda}{\sigma_p}\right)^{\frac{1}{p-1+p/N}} \frac{p+Np-N}{p},$$

provided $\int_B bu\, dx = 1$.

Proof Let t be any positive number. By testing (4.1) with $(u-t)_+$ we obtain, setting $D(t) := \{x \in D : u(x) > t\}$ and $M(t) := M(D(t))$,

$$\int\limits_{D(t)} a(x)|\nabla u|^p dx \leq \lambda \int\limits_{D(t)} b(x)(u-t)dx. \qquad (5.1)$$

Notice that $M(t') = 0$ implies $M(t) = 0$ for all $t > t'$, and in addition $u(x) \leq t'$ a.e. Using the fact that $\sigma_p > 0$ we have, as long as $M(t) \neq 0$

$$\sigma_p \left(\int\limits_{D(t)} (u-t)b(x)dx\right)^p M^{1-\frac{p}{N}-p}(t) \leq \int\limits_{D(t)} a(x)|\nabla u|^p dx$$

This together with (5.1) implies

$$\sigma_p \left(\int\limits_{D(t)} (u-t)b(x)dx\right)^p M^{1-\frac{p}{N}-p}(t) \leq \lambda \int\limits_{D(t)} b(x)(u-t)dx. \qquad (5.2)$$

Integration by parts yields

$$\int\limits_{D(t)} (u-t)b(x)dx = \int\limits_t^\infty M(s)ds =: \hat{M}(t).$$

Inserting this expression into (5.2) we get

$$\left(\frac{\sigma_p}{\lambda}\right)^{\frac{1}{p+p/N-1}} \le -\hat{M}'\hat{M}^{-\frac{p-1}{p+p/N-1}}.$$

Put for short $\gamma = \left(\frac{\sigma_p}{\lambda}\right)^{\frac{1}{p-1+p/N}}$ and $\alpha = \frac{p-1}{p+p/N-1}$. Since $\hat{M}(0) = 1$ we find after integration

$$\gamma(1-\alpha)t \le 1 - \hat{M}(t)^{1-\alpha}.$$

Hence

$$t \le \frac{1}{(1-\alpha)\gamma} = \left(\frac{\lambda}{\sigma_p}\right)^{\frac{1}{p+p/N-1}} \frac{p+Np-N}{p}.$$

This establishes the assertion. □

Next, we we will prove the Hölder continuity of minimizers. For this purpose we need the additional condition on b.
(A4) for all $x \in B$ and all $\mu \ge 1$ there exist $0 < \alpha < N$ such that $b(\frac{x}{\mu}) \le \mu^\alpha b(x)$ holds.

Theorem 5 *Let B be convex and $0 \in B$. Assume* (A1)–(A4) *and* $1 < p < \infty$. *Let $u \in \mathcal{K}(B)$ be any minimizer of $\mathcal{J}_M$. Then $u \in C^{0,\beta}_{loc}(B)$ for all $0 \le \beta < 1$.*

The proof is done in several steps. Let us first collect some useful auxiliary results. Put

$$B_R(x_0) := \{x \in B : |x - x_0| < R\}.$$

In the sequel c denotes a constant which is independent of R. Our arguments rely on a lemma of Morrey (see e.g. [11] Theorem 1.53 and [13]).

Lemma 6 (Morrey's Dirichlet growth theorem) *Let $u \in W^{1,p}(B)$, $1 < p < N$. Suppose that there exist constants $0 < c < \infty$ and $\beta \in (0,1]$ such that for all balls $B_r(x_0) \subset B$*

$$\int_{B\cap B_r(x_0)} |\nabla u|^p dx \le c r^{N-p+\beta p},$$

then $u \in C^{0,\beta}(B)$.

In order to apply the above lemma we shall also need

Lemma 7 *Let $\phi(t)$ be a nonnegative and nondecreasing function. Suppose that*

$$\phi(r) \le \gamma\left[\left(\frac{r}{R}\right)^\alpha + \delta\right]\phi(R) + \kappa R^\beta$$

for all $0 \le r \le R \le R_0$, where γ, κ, α and β are positive constants with $\beta < \alpha$. Then there exist positive constants $\delta_0 = \delta_0(\gamma,\alpha,\beta)$ and $c = C(\gamma,\alpha,\beta)$ such that if $\delta < \delta_0$, then

$$\phi(r) \le c\left(\frac{r}{R}\right)^\beta \left[\phi(R) + \kappa R^\beta\right]$$

for all $0 \le r \le R \le R_0$.

For the proof of this Lemma we refer to [10], Lemma 2.1 in Chapter III. Next, we construct a comparison function for the functional $\mathcal{J}_M$ (cf. (3.6)) which will play an important role in the proof of the Hölder and Lipschitz continuity of the minimizer $u \in \mathcal{K}(B)$. Let $x_0 \in B$ be such that $B_{2R}(x_0) \subset B$ and $B_R(x_0) \cap D_u \neq \emptyset$. Set

$$v(x) = \begin{cases} \hat{v}(x) & \text{if } x \in B_R(x_0) \\ u(x) & \text{if } x \in D_u \setminus B_R(x_0) \end{cases} \tag{5.3}$$

where $\hat{v}$ is the solution of

$$\operatorname{div}(a(x)|\nabla\hat{v}|^{p-2}\nabla\hat{v}) + \lambda b(x) = 0 \text{ in } B_R(x_0), \quad \hat{v} = u \text{ on } \partial B_R(x_0), \tag{5.4}$$
$$\lambda = \int_B a(x)|\nabla u|^p \, dx.$$

Since

$$\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + \lambda b(x) \geq 0 \quad \text{ in } B_R(x_0),$$

the maximum principle gives $\hat{v} \geq u$ in $B_R(x_0)$. Also observe that

$$\int_{B_R(x_0)} a(x)|\nabla\hat{v}|^p \, dx \leq \int_{B_R(x_0)} a(x)|\nabla u|^p \, dx + \lambda \int_{B_R(x_0)} b(x)(\hat{v} - u)dx. \tag{5.5}$$

Since $\hat{v} \geq u$ in $B_R(x_0)$ we have $D_u \subseteq D_v$. Hence in general $v(x)$ cannot be used as a test function in the variational principle for $\mathcal{J}_M$. We therefore define $w(x) := v(\mu x)$ and choose $\mu \geq 1$ such that $M_w = M_u = M$. Since B is convex and contains the origin, it follows that $D_w \subset B$ and $w(x)$ can be used as a test function of the variational characterization of $\mathcal{J}_M$. In the sequel we shall frequently use the notation

$$N_u := B \setminus D_u = \{x \in B : u(x) = 0 \text{ a.e. }\}.$$

The following elementary estimate will be needed later on.

Proposition 1 *Let u be a minimizer and let v and μ be defined as above. Let C be a constant such that*

$$2b_{max} - C(N - \alpha) < 0.$$

Then μ satisfies the estimate

$$1 \leq \mu \leq 1 + C\,\frac{|N_u \cap B_R(x_0)|}{M}. \tag{5.6}$$

Proof To simplify notation we write B_R instead of $B_R(x_0)$. For $\tilde{\mu} \geq 1$ set

$$g(\tilde{\mu}) := \tilde{\mu}^{-N} \int_{D_v} b\left(\frac{x}{\tilde{\mu}}\right) dx.$$

By definition of μ we have

$$g(\mu) = \int_{D_w} b(x)dx = \int_{D_u} b(x)dx = M.$$

On the other hand, by the construction of v we have $g(1) \geq M$. The idea is now to find a $\tilde{\mu}_0 > 1$ such that $g(\tilde{\mu}_0) < M$. Then necessarily the bound $1 \leq \mu \leq \tilde{\mu}_0$ follows.

$$\begin{aligned} g(\tilde{\mu}) &\leq \tilde{\mu}^{\alpha-N} \int_{D_v} b(x)dx \\ &\leq \tilde{\mu}^{\alpha-N} \left(\int_{D_u} b(x)dx + b_{max}|N_u \cap B_R| \right) \\ &= \tilde{\mu}^{\alpha-N} M \left(1 + \frac{b_{max}|N_u \cap B_R|}{M} \right). \end{aligned}$$

If we evaluate the expression above at

$$\tilde{\mu}_0 = 1 + C\frac{|N_u \cap B_R|}{M} =: 1 + C\eta(R),$$

and if we expand $\tilde{\mu}^{\alpha-N}$ w.r.t. $\eta(R)$ we get for sufficiently small $R > 0$

$$\begin{aligned} g(\tilde{\mu}_0) &\leq \left(1 + \frac{1}{2}(\alpha - N)C\eta(R) \right) M \left(1 + b_{max}\eta(R)\right) \\ &\leq \left(1 + \left\{ b_{max} + \frac{1}{2}(\alpha - N)C \right\} \eta(R) \right) M. \end{aligned}$$

Thus for $R > 0$ we find that $g(\tilde{\mu}_0) < M$, if $2b_{max} - C(N - \alpha) < 0$. This proves the assertion. □

Lemma 8 *Let $u \in \mathcal{K}(B)$ be any minimizer of $\mathcal{J}_M$ and let $\hat{v}$ and μ be defined as above. Then for* $1 < p \leq 2$

$$\int_{B_R(x_0)} a(x)|\nabla(u - \hat{v})|^p \, dx \leq cR^{(N-p)(1-\frac{p}{2})}|N_u \cap B_R(x_0)|^{\frac{p}{2}} \tag{5.7}$$

and for $p \geq 2$

$$\int_{B_R(x_0)} a(x)|\nabla(u - \hat{v})|^p \, dx \leq c|N_u \cap B_R(x_0)|. \tag{5.8}$$

Proof By definition we have, with $w(x) = v(\mu x)$ as above,

$$\mathcal{J}_M \leq \frac{\int_B a(x)|\nabla w|^p \, dx}{\left(\int_B b(x)w \, dx\right)^p}. \tag{5.9}$$

Observe that

$$\int_B a(x)|\nabla w|^p \, dx = \mu^{p-N} \int_{D_v} a_\mu(x)|\nabla v|^p \, dx$$

and

$$\int_B b(x)w \, dx = \mu^{-N} \int_{D_v} b_\mu(x)v \, dx,$$

where $a_\mu(x) = a\left(\frac{x}{\mu}\right)$ and $b_\mu(x) = b\left(\frac{x}{\mu}\right)$. From (5.9) and the definition of w it then follows that

$$\mathcal{J}_M \mu^{N-p-Np} \left(\int_{D_v} b_\mu(x) v \, dx \right)^p \leq \int_{D_v} a_\mu(x) |\nabla v|^p \, dx. \tag{5.10}$$

Without loss of generality we can assume that $B_R \not\subseteq D_u$. By the strong maximum principle [17] we have $\hat{v} > 0$ in B_R. We write $D_v = (D_u \setminus B_R) \cup B_R$ and get

$$\begin{aligned}
\int_{D_v} b_\mu(x) v \, dx &= \int_{D_v \setminus B_R} b_\mu(x) v \, dx + \int_{B_R} b_\mu(x) \hat{v} \, dx \\
&= \int_{D_u} b_\mu(x) u \, dx + \int_{B_R} b_\mu(x)(\hat{v} - u) dx \\
&= \int_{D_u} b(x) u \, dx + \int_{D_u} (b_\mu(x) - b(x)) u \, dx \\
&\quad + \int_{B_R} b_\mu(x)(\hat{v} - u) dx \\
&\geq 1 - L_b (\max_B |x|)(\mu - 1) \int_{D_u} u \, dx.
\end{aligned} \tag{5.11}$$

For the last inequality we used the normalization $\int_{D_u} b(x) u \, dx = 1$, the Lipschitz continuity of b with Lipschitz constant L_b and the fact that $\hat{v} \geq u$ in B_R. We estimate

$$\int_{D_u} u \, dx \leq \|u\|_{L^\infty(B)} \frac{M}{b_{min}}$$

where $\|u\|_{L^\infty(B)}$ is estimated in Theorem 4. We now take into account Proposition 1 and choose the constant there as $C = 2 \frac{b_{max} b_{min}}{L_b \max_B |x|}$ we arrive at:

$$\int_{D_v} b_\mu(x) v \, dx \geq 1 - 2 b_{max} \|u\|_{L^\infty(B)} |N_u \cap B_R|. \tag{5.12}$$

In order to estimate the right hand of (5.10) side we use the Lipschitz continuity of a and obtain

$$\begin{aligned}
\int_{D_v} a_\mu(x) |\nabla v|^p \, dx &\leq \int_{D_v} a(x) |\nabla v|^p \, dx + \int_{D_v} |a_\mu(x) - a(x)| |\nabla v|^p \, dx \\
&\leq \int_{D_u} a(x) |\nabla u|^p \, dx + \int_{B_R} a(x) |\nabla \hat{v}|^p \, dx - \int_{B_R} a(x) |\nabla u|^p \, dx + c(\mu - 1).
\end{aligned} \tag{5.13}$$

By Proposition 1 and the definition of $\mathcal{J}_M$ we conclude that

$$\int_{D_v} a_\mu(x) |\nabla v|^p \, dx \leq \mathcal{J}_M + \int_{B_R} a(x)(|\nabla \hat{v}|^p - |\nabla u|^p) dx + c|N_u \cap B_R| \tag{5.14}$$

for R small enough. Thus (5.10) and (5.12) yield

$$\mathcal{J}_M \mu^{N-p-Np} \left(1 - 2b_{max}\|u\|_{L^\infty(B)}|N_u \cap B_R|\right)^p \\ \leq \mathcal{J}_M + \int_{B_R} a(x)(|\nabla\hat{v}|^p - |\nabla u|^p)dx + c|N_u \cap B_R|,$$

and rearranging terms we find for the expression

$$I := \int_{B_R} a(x)(|\nabla u|^p - |\nabla\hat{v}|^p)dx,$$

the estimate

$$I \leq (1 - \mu^{N-p-Np}) \int_{D_u} a(x)|\nabla u|^p \, dx + O(|N_u \cap B_R|). \tag{5.15}$$

Let $u_t(x) := tu(x) + (1-t)\hat{v}(x)$ for $0 \leq t \leq 1$. Then we have

$$I = \int_{B_R} a(x) \int_0^1 \frac{d}{dt}|\nabla u_t|^p \, dt \, dx \\ = p \int_{B_R} a(x) \int_0^1 |\nabla u_t|^{p-2}\nabla u_t \cdot \nabla(u - \hat{v})dt \, dx.$$

Since $\hat{v} \geq u$

$$\int_{B_R} a(x)|\nabla\hat{v}|^{p-2}\nabla\hat{v} \cdot \nabla(u - \hat{v})dx = \lambda \int_{B_R} b(x)(\hat{v} - u)dx \geq 0,$$

and thus

$$I \geq p \int_{B_R} a(x) \int_0^1 \left(|\nabla u_t|^{p-2}\nabla u_t - |\nabla\hat{v}|^{p-2}\nabla\hat{v}\right) \cdot \nabla(u - \hat{v})dt \, dx.$$

Replacing $u - \hat{v}$ by $\frac{1}{t}(u_t - \hat{v})$ we get

$$I \geq p \int_0^1 \frac{1}{t} \int_{B_R} a(x) \left(|\nabla u_t|^{p-2}\nabla u_t - |\nabla\hat{v}|^{p-2}\nabla\hat{v}\right) \cdot \nabla(u_t - \hat{v})dx \, dt. \tag{5.16}$$

Now, we use the following inequalities, which can be found e.g. in [11], Lemma 5.7

$$\left(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'\right) \cdot (\xi - \xi') \geq c(N,p)(|\xi| + |\xi'|)^{p-2}|\xi - \xi'|^2 \quad \text{if } 1 < p \leq 2, \tag{5.17}$$

and

$$\left(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'\right) \cdot (\xi - \xi') \geq c(N,p)|\xi - \xi'|^p \quad \text{if } p \geq 2 \tag{5.18}$$

for all $\xi, \xi' \in \mathbb{R}^N$. Inserting the second inequality into (5.16) we get for $p \geq 2$

$$
\begin{aligned}
I &\geq c(N,p)p \int_0^1 \frac{1}{t} \int_{B_R(x_0)} a(x)|\nabla(u_t - \hat{v})|^p \, dx\, dt \\
&= c(N,p)p \int_0^1 t^{p-1}\, dt \int_{B_R(x_0)} a(x)|\nabla(u - \hat{v})|^p \, dx \\
&= c(N,p) \int_{B_R(x_0)} a(x)|\nabla(u - \hat{v})|^p \, dx.
\end{aligned}
$$

From inequality (5.15) we deduce that

$$
\int_{B_R(x_0)} a(x)|\nabla(u - \hat{v})|^p \, dx \leq O(|N_u \cap B_R(x_0)|). \tag{5.19}
$$

This proves the second assertion (5.8) of the lemma.
For the case $1 < p \leq 2$ we have

$$
\begin{aligned}
I &\geq c(N,p)p \int_0^1 \frac{1}{t} \int_{B_R} a(x)|\nabla(u_t - \hat{v})|^2 \left(|\nabla u_t| + |\nabla \hat{v}|)|\right)^{p-2} \, dx\, dt \\
&\geq c(N,p)\frac{p}{2} \int_0^1 t\, dt \int_{B_R} a(x)|\nabla(u - \hat{v})|^2 \left(|\nabla u| + |\nabla \hat{v}|)|\right)^{p-2} \, dx \\
&= \frac{1}{4} c(N,p) \int_{B_R} a(x)|\nabla(u - \hat{v})|^2 \left(|\nabla u| + |\nabla \hat{v}|)|\right)^{p-2} \, dx.
\end{aligned}
$$

We use Hölder's inequality and get

$$
\begin{aligned}
&\int_{B_R} a(x)|\nabla(u - \hat{v})|^p \, dx \\
&= \int_{B_R} a^{\frac{p}{2}}(x)|\nabla(u - \hat{v})|^p \left(|\nabla u| + |\nabla \hat{v}|)|\right)^{\frac{(p-2)p}{2}} a^{1-\frac{p}{2}}(x) \left(|\nabla u| + |\nabla \hat{v}|)|\right)^{\frac{(2-p)p}{2}} \, dx \\
&\leq \left(\int_{B_R} a(x)|\nabla(u - \hat{v})|^2 \left(|\nabla u| + |\nabla \hat{v}|)|\right)^{p-2} \, dx \right)^{\frac{p}{2}} \left(\int_{B_R} a(x) \left(|\nabla u| + |\nabla \hat{v}|)|\right)^p \, dx \right)^{1-\frac{p}{2}}.
\end{aligned} \tag{5.20}
$$

This together with (5.5) gives

$$\int_{B_R} a(x)|\nabla(u-\hat{v})|^2 \left(|\nabla u| + |\nabla \hat{v}|)|\right)^{p-2}\, dx$$

$$\geq \left(2+p\int_{B_R} b(x)(\hat{v}-u)dx\right)^{1-\frac{2}{p}} \left(\int_{B_R} a(x)|\nabla(u-\hat{v})|^p\, dx\right)^{\frac{2}{p}} \left(\int_{B_R} a(x)|\nabla u|^p\, dx\right)^{1-\frac{2}{p}}.$$

Observe that by the maximum principle, $\hat{v} \leq V$ where

$$\mathrm{div}(a(x)|\nabla V|^{p-2}\nabla V) + \lambda b(x) = 0 \text{ in } B_R, \quad V = |u|_\infty \text{ on } \partial B_R.$$

From the same arguments as for Theorem 4 it follows that $|V|_\infty < \infty$. Thus for $R \leq R'$

$$I \geq c(N,p,R',|u|_\infty) \left(\int_{B_R} a(x)|\nabla(u-\hat{v})|^p\, dx\right)^{\frac{2}{p}} \left(\int_{B_R} a(x)|\nabla u|^p\, dx\right)^{1-\frac{2}{p}}$$

For the case $1 < p \leq 2$ inequality (5.15) then implies

$$\int_{B_R} a(x)|\nabla(u-\hat{v})|^p\, dx \leq c \left(\int_{B_R} a(x)|\nabla u|^p\, dx\right)^{1-\frac{p}{2}} |N_u \cap B_R|^{\frac{p}{2}}$$

The integral $\int_{B_R} a(x)|\nabla u|^p\, dx$ can be estimated by means of a Caccioppoli type inequality, for solutions of the inequality (4.1), as follows. Choose $\varphi = u\eta^p$ for solutions of (4.1) where $\eta \in C_0^\infty(B_{2R})$ such that $\eta \equiv 1$ in B_R and $|\nabla\eta| \leq \frac{c}{R}$ for some positive number $c = c(N)$. Some elementary calculation based on Hölder's and Young's inequalities implies that there exists a constant $c = c(N,p)$ such that for $R \leq 1$

$$\int_{B_R} a(x)|\nabla u|^p\, dx \leq c(n,p) \left(R^{-p}\int_{B_{2R}} u^p\, dx + \lambda \int_{B_{2R}} b(x)u\, dx\right)$$
$$\leq c(n,p)\left(|u|_\infty^p + \lambda b_{max}|\, u|_\infty)R^{N-p}, \tag{5.21}$$

Thus

$$\int_{B_R} a(x)|\nabla(u-\hat{v})|^p\, dx \leq cR^{(N-p)(1-\frac{p}{2})}|N_u \cap B_R|^{\frac{p}{2}} \tag{5.22}$$

for $1 < p \leq 2$. This completes the proof of the lemma. □

The next lemma gives a local estimate for $\hat{v}$.

Lemma 9 *Let $u \in \mathcal{K}(B)$ be any minimizer of $\mathcal{J}_M$ and let $\hat{v}$ be as defined in* (5.4). *Denote by h the unique solution of*

$$\mathrm{div}(|\nabla h|^{p-2}\nabla h) = 0 \quad in\ B_R(x_0), \quad h = u \quad on\ \partial B_R(x_0).$$

Then the following local estimate holds for all $1 < p < \infty$*:*

$$\int_{B_r(x_0)} a(x)|\nabla\hat{v}|^p\,dx \le c\left(\left(\frac{r}{R}\right)^N + R\right)\int_{B_R} |\nabla u|^p\,dx + cR^N, \tag{5.23}$$

where c is some positive constant which is independent of r and R.

Proof We estimate

$$\int_{B_r(x_0)} a(x)|\nabla\hat{v}|^p\,dx \le 2^{p-1}\left\{\int_{B_r(x_0)} a(x)|\nabla(\hat{v}-h)|^p\,dx + \int_{B_r(x_0)} a(x)|\nabla h|^p\,dx\right\}.$$

The first integral is estimated as in the proof of Lemma 5.8 in [11]:

Step 1 Starting with the weak formulation for $\hat{v}$ and h we obtain

$$\begin{aligned}
&\int_{B_R(x_0)} \left(a(x_0)|\nabla\hat{v}|^{p-2}\nabla\hat{v} - a(x_0)|\nabla h|^{p-2}\nabla h\right)\nabla(\hat{v}-h)dx \\
&\le 2L_aR\int_{B_R(x_0)} |\nabla\hat{v}|^{p-1}|\nabla(\hat{v}-h)|\,dx + \lambda\int_{B_R(x_0)} b(x)(\hat{v}-h)dx \\
&\le 2L_aR\,I_1^{\frac{p-1}{p}}\,I_2^{\frac{1}{p}} + \lambda b_{max}c(N,p)R^{N\frac{p-1}{p}+1}\,I_2^{\frac{1}{p}}
\end{aligned}$$

where we set

$$I_1 := \int_{B_R} |\nabla\hat{v}|^p\,dx \quad \text{and} \quad I_2 := \int_{B_R} |\nabla(\hat{v}-h)|^p\,dx,$$

and L_a is the Lipschitz constant of a. For the last inequality we also used Hölder's inequality and the continuity of the embedding of $H^{1,p}(B_R)$ into $L^{p^*}(B_R)$ $\left(p^* = \frac{Np}{N-p}\right)$ (applied to the function $\hat{v}-h$). Hence, we obtain

$$\int_{B_R(x_0)} \left(a(x_0)|\nabla\hat{v}|^{p-2}\nabla\hat{v} - a(x_0)|\nabla h|^{p-2}\nabla h\right)\nabla(\hat{v}-h)dx \le cR\,I_2^{\frac{1}{p}}\left(I_1^{\frac{p-1}{p}} + R^{N\frac{p-1}{p}}\right).$$

c depends on L_a, b_{max}, λ, N and p.

Step 2 We use (5.17), (5.18) and get

$$\begin{aligned}
&\int_{B_R} \left(a(x_0)|\nabla\hat{v}|^{p-2}\nabla\hat{v} - a(x_0)|\nabla h|^{p-2}\nabla h\right)\nabla(\hat{v}-h)dx \\
&\qquad \ge a(x_0)c(N,p)\int_{B_R} \left(|\nabla\hat{v}| + |\nabla h|\right)^{p-2}|\nabla(\hat{v}-h)|^2\,dx
\end{aligned}$$

for $1 < p \leq 2$ and

$$\int\limits_{B_R} \left(a(x_0)|\nabla\hat{v}|^{p-2}\nabla\hat{v} - a(x_0)|\nabla h|^{p-2}\nabla h\right) \nabla(\hat{v} - h)dx$$
$$\geq a(x_0)c(N,p) \int\limits_{B_R} |\nabla(\hat{v} - h)|^p \, dx$$

for $p \geq 2$.

Step 3 We first consider the case $1 < p \leq 2$. We use (5.20), Hölder's inequality and get

$$\int\limits_{B_R} |\nabla(\hat{v} - h)|^p \, dx$$
$$\leq \left(\int\limits_{B_R} \left(|\nabla\hat{v}| + |\nabla h|\right)^{p-2} |\nabla(\hat{v} - h)|^2 \, dx\right)^{\frac{p}{2}} \left(\int\limits_{B_R} \left(|\nabla\hat{v}| + |\nabla h|\right)^p \, dx\right)^{1-\frac{p}{2}}$$

The first integral on the right hand side is estimated with the help of Step 1 and 2. For the second integral we use the fact that $\int_{B_R} |\nabla h|^p \, dx \leq \int_{B_R} |\nabla\hat{v}|^p \, dx$. This gives

$$\int\limits_{B_R} |\nabla(\hat{v} - h)|^p \, dx \leq c\left(R\, I_2^{\frac{1}{p}} \left(I_1^{\frac{p-1}{p}} + R^{N\frac{p-1}{p}}\right)\right)^{\frac{p}{2}} \left(2^p \int\limits_{B_R} |\nabla\hat{v}|^p \, dx\right)^{1-\frac{p}{2}},$$

Thus, we get the inequality

$$I_2 \leq cR^p \left(I_1^{p-1} + R^{N(p-1)}\right) I_1^{2-p} \leq c\left(R\, I_1 + R^{N(p-1)+p} I_1^{2-p}\right) \leq c\left(R\, I_1 + R^{N+2}\right).$$

For the last inequality we used Young's inequality. From (5.5) and the assumption that $R \leq 1$ we derive the inequality

$$\int\limits_{B_R} |\nabla\hat{v}|^p \, dx \leq c\left(R^N + R\int\limits_{B_R} |\nabla u|^p \, dx\right).$$

This together with Step 5 gives (5.23).

Step 4 For $p \geq 2$ we get (using also (5.5))

$$\int\limits_{B_R} |\nabla(\hat{v} - h)|^p \, dx \leq c\left(R^N + R\int\limits_{B_R} |\nabla u|^p \, dx\right),$$

where c depends on the same quantities as the constant in Step 3. Together with Step 5 this gives (5.23)

Step 5 The integral $\int_{B_R} a(x)|\nabla h|^p\, dx$ can be estimated using the following growth result by DiBenedetto [8] Proposition 3.3 (see also [11] Theorem 3.19):

$$\int_{B_r} a(x)|\nabla h|^p\, dx \leq c\|\nabla h\|_{L^\infty(B_r(x_0))} r^N \leq c\left(\frac{r}{R}\right)^N \int_{B_R} |\nabla h|^p\, dx \leq c\left(\frac{r}{R}\right)^N \int_{B_R} |\nabla u|^p\, dx,$$

where $c = c(N, p, a_{max}, a_{min})$. □

After this preparation we are in position to proceed, as in [18], to the proof of the Theorem 5.

Proof of Theorem 5 We use the setting as given by the previous lemmas. In the sequel we assume $x_0 \in \partial D_u$. For $r < R$ and $1 < p < \infty$ we estimate

$$\int_{B_r(x_0)} a(x)|\nabla u|^p\, dx \leq 2^{p-1}\left\{\int_{B_r(x_0)} a(x)|\nabla(u-\hat{v})|^p\, dx + \int_{B_r(x_0)} a(x)|\nabla\hat{v}|^p\, dx\right\}.$$

The first term on the right-hand side is estimated by Lemma 8, while the second is estimated by Lemma 9.
We consider the case $p \geq 2$. Taking into account (5.8) with $|N_u \cap B_R(x_0)| \leq cR^N$ and (5.23) we arrive at

$$\int_{B_r(x_0)} |\nabla u|^p\, dx \leq \gamma\left(\left(\frac{r}{R}\right)^N + R\right)\int_{B_R(x_0)} |\nabla u|^p\, dx + \kappa R^N. \tag{5.24}$$

γ and κ are two constants which do not depend on u. Now, we apply Lemma 7. This gives

$$\int_{B_r(x_0)} |\nabla u|^p\, dx \leq \gamma\left(\frac{r}{R}\right)^\beta \int_{B_R(x_0)} |\nabla u|^p\, dx$$

for all $0 < \beta < N$. From Lemma 6 it follows that $u \in C^{0,\alpha}_{loc}(B)$ for all $0 < \alpha < 1$.
Next, we consider the case $1 < p \leq 2$ and assume that $x_0 \in \partial D_u$. (5.7) with the right hand side replaced by $R^{N-p+\frac{p^2}{2}}$ together with (5.23) gives

$$\int_{B_r(x_0)} |\nabla u|^p\, dx \leq \gamma\left(\left(\frac{r}{R}\right)^N + R\right)\int_{B_R(x_0)} |\nabla u|^p\, dx + \kappa R^{N-p+\alpha_0 p}, \tag{5.25}$$

where $\alpha_0 := \frac{p}{2}$. Lemma 7 now gives

$$\int_{B_r(x_0)} |\nabla u|^p\, dx \leq \gamma\left(\frac{r}{R}\right)^\beta \int_{B_R(x_0)} |\nabla u|^p\, dx$$

for all $0 < \beta < N - p + \alpha_0 p$. Then Lemma 6 gives $u \in C^{0,\alpha}_{loc}(B)$ for all $0 < \alpha < \alpha_0$. This information can now be used to improve estimate (5.21) then (5.22) and consequently (5.7). Let $u \in C^{0,\alpha}_{loc}(B)$ for some $\alpha < \alpha_0$ then the right hand side of (5.21) can be replaced by $cR^{N-p+\alpha p}$, hence the right hand side of (5.22) will be replaced by $cR^{(N-p+\alpha p)(1-\frac{p}{2})}|N_u \cap B_R|^{\frac{p}{2}}$ and consequently the right side of (5.7) is bounded

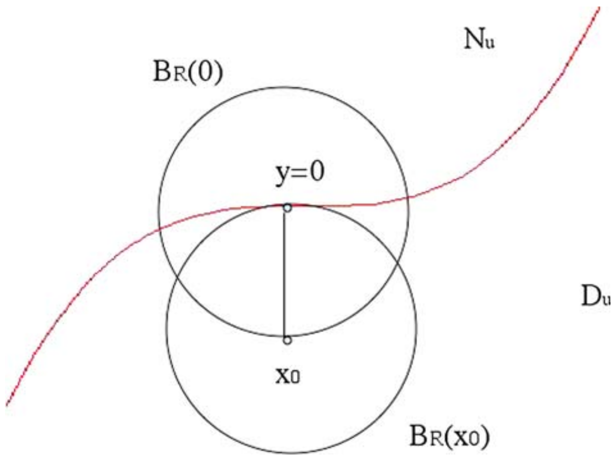

Fig. 1 First illustration in the proof of theorem 7

by $cR^{N-p+p(\frac{p}{2}+\alpha(1-\frac{p}{2}))}$. Set $\alpha_1 := \frac{p}{2} + \alpha(1 - \frac{p}{2})$. This implies that (5.25) holds with α_0 replaced by α_1. Lemma 7 and Lemma 6 then imply $u \in C^{0,\alpha}_{loc}(B)$ for $0 \leq \alpha < \alpha_1$. Thus, we obtain a bootstrap argument which gives in the k -th step: there exists a sequence $(\alpha_k)_{k\geq 1}$ such that inequality (5.24) holds with α_0 replaced by α_k and $\alpha_{k+1} := \frac{p}{2} + \alpha_k(1 - \frac{p}{2})$. Since $\alpha_0 = \frac{p}{2}$ we get $\alpha_k = 1 - (1 - \frac{p}{2})^{k+1}$. Clearly α_k is an increasing sequence with limit 1. This proves $u \in C^{0,\alpha}_{loc}(B)$ for all $0 < \alpha < 1$. □

We are now in position to prove our main theorem.

Theorem 6 *Assume* (A1)–(A4) *and let* $m \leq M(B)$ *be any given positive number. Then there exists an optimal domain* D_0 *with* $M(D_0) \leq m$ *and a minimizer* $u_0 \in W^{1,p}_0(D_0)$ *such that* $S_p(D_0) = s_p(m)$.

Proof By Lemma 4 there exists a minimizer u_0 of $\mathcal{J}_m$. By the preceding theorem D_{u_0} is open. Hence $s_p(m) = \mathcal{J}_m$, see Lemma 4, which establishes the assertion. □

Based on this we now prove the Lipschitz continuity of any minimizer.

Theorem 7 *Assume* (A1)–(A4) *and* $2 \leq p < \infty$. *Let* $u \in \mathcal{K}(B)$ *be a minimizer of* $s_p(M)$. *Then* $u \in C^{0,1}_{loc}(B)$.

Proof The proof follows closely the proof of Theorem 2.3 in [2]. Set $d(x) := dist(x, N_u)$. Since u is continuous the set D_u is open. We will use (5.8):

$$\int_{B_R(x_0)} |\nabla(u - \hat{v})|^p \, dx \leq c|N_u \cap B_R(x_0)|.$$

Let x_0 be any point in B be such that $d(x_0) < \frac{1}{3} dist(x_0, \partial B)$. We will prove, that the estimate $u(x_0) \leq cd(x_0)$ must hold for some positive constant c which does not depend on x_0 (Fig. 1). We set

$$M := \frac{u(x_0)}{d(x_0)} \tag{5.26}$$

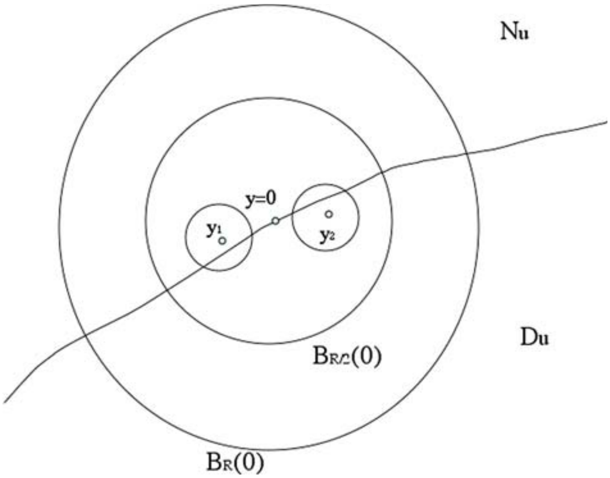

Fig. 2 Second illustration in the proof of theorem 7

and then to derive an upper bound for M. Let $R = d(x_0)$ and consider the ball $B_R(x_0)$. It is contained in D_u. Since

$$\operatorname{div}(a(x)|\nabla u(x)|^{p-2}\nabla u(x)) + \lambda b(x) = 0 \quad \text{in } B_R(x_0) \subset D_u, \tag{5.27}$$

we can apply Harnack's inequality cf. e.g. [9] and we have

$$\inf_{B_{\frac{3}{4}R}(x_0)} u \geq cu(x_0) = cMR. \tag{5.28}$$

by (5.26). c does not depend on x_0. Since $R = d(x_0)$ the boundary $\partial B_R(x_0)$ touches N_u in at least one point. Let $y \in \partial B_R(x_0) \cap N_u$. After translation we may assume that $y = 0$ (see Fig. 1). Next, we consider the ball $B_R(0)$. Let $\hat{v}$ the solution to

$$\begin{aligned} \operatorname{div}(a(x)|\nabla\hat{v}|^{p-2}\nabla\hat{v}) + \lambda b(x) = 0 \quad &\text{in } B_R(0) \\ \hat{v} = u \quad &\text{in } \partial B_R(0) \end{aligned}$$

This is the same function as in (5.3). Thus $\hat{v} \geq u$ in $B_R(0)$ and (5.8) holds. From (5.28) we deduce

$$\hat{v}(x) \geq cMR \quad \text{in } B_{\frac{3}{4}R}(x_0) \cap B_R(0). \tag{5.29}$$

We apply Harnack's inequality once more and get

$$\hat{v}(x) \geq C^* \quad \text{in } B_{\frac{1}{2}R}(0) \tag{5.30}$$

with $C^* = cMR$. We introduce the function

$$w(x) := C^* \left(e^{-\mu x^2} - e^{-\mu R^2}\right)$$

for $\mu > 0$. Direct computation gives

$$\operatorname{div}(a(x)|\nabla w(x)|^{p-2}\nabla w(x)) + \lambda b(x) > 0 \quad \text{in } B_R \setminus B_{\frac{1}{2}R}(0)$$

if μ is sufficiently large but independent of R for $R \leq 1$. Since $w = 0$ in ∂B_R we get

$$w \leq C^* \leq \hat{v} \quad \text{in } \partial B_{\frac{1}{2}R}(0).$$

The maximum principle then implies

$$\hat{v}(x) \geq w(x) \geq C^* \beta \frac{(R - |x|)}{R} \quad \text{in } B_R \setminus B_{\frac{1}{2}R}(0), \tag{5.31}$$

where the last inequality is verified by direct calculations (with $\beta = 2\mu \exp(-\mu)$). From (5.31), (5.30) and the definition of C^* we get

$$\hat{v}(x) \geq cM(R - |x|) \quad \text{in } B_R(0). \tag{5.32}$$

We now use exactly the same construction as in [2] Lemma 2.2. Choose two points y_1 and y_2 in $B_{\frac{1}{2}R}(0)$ such that $B_{\frac{1}{8}R}(y_1) \cap B_{\frac{1}{8}R}(y_1) = \emptyset$ (see Fig. 2). Given a point $R\xi \in \partial B_R(0)$ with $\xi \in \partial B_1(0)$ we consider the segments $L_i(\xi)$ joining $R\xi$ with y_i. Denote by $l_i(\xi) \subset L_i(\xi)$ the largest segment with endpoints $R\xi$ and $\eta_i(\xi)$ such that $\eta_i(\xi) \notin B_{\frac{1}{8}R}(y_i)$ and $u(\eta_i(\xi)) = 0$. We set $\eta_i(\xi) = \xi$ if $u(R\xi) > 0$. Denote by S_i the union of all the segments $l_i(\xi)$ and set $S := S_1 \cup S_2$. We set $x = \eta_i(\xi)$ in(5.32) and compute

$$cM(R - |\eta_i(\xi)|) \leq \hat{v}(\eta_i(\xi)) = \int_{\eta_i(\xi)}^{R} \frac{d}{dr}(u(r\xi) - \hat{v}(r\xi))dr$$

$$\leq \int_{\eta_i(\xi)}^{R} |\nabla(u(r\xi) - \hat{v}(r\xi))|\, dr.$$

Next, we integrate over $\partial B_R(0)$:

$$cM\left|\left(B_R(0) \setminus B_{\frac{1}{8}R}(y_i)\right) \cap N_u\right| \leq c(N) \int_{S_i} |\nabla(u - \hat{v})|\, dx$$

Adding this inequality for $i = 1, 2$ gives the inequality

$$cM|S| \leq \int_S |\nabla(u - \hat{v})|\, dx.$$

This implies

$$cM|S| \leq c(N) \int_S |\nabla(u - \hat{v})|\, dx \leq c(N)|S|^{1-\frac{1}{p}} \left(\int_S |\nabla(u - \hat{v})|^p\, dx\right)^{\frac{1}{p}}.$$

Hence

$$M^p|S| \leq c \int_S |\nabla(u - \hat{v})|^p\, dx.$$

Now, we apply (5.8) in Lemma (8) for $p \geq 2$. Thus

$$M^p|S| \leq c|N_u \cap B_R(0)| \leq c|S|,$$

and this gives an upper bound for M which does not depend on x_0. From this we deduce the Lipschitz continuity as it was done in [2] Theorem 2.3. Let $x \in B' \cap D_u \cap V$, where B' is any subdomain of B with $B' \subset\subset B$. V is a sufficiently small neighbourhood of the free boundary. The smallness of V is such, that by the previous argument we have

$$u(x + d(x)x') \leq cd(x) \quad \text{for all } x' \in B_1(0).$$

This implies that

$$\tilde{u}(x') := \frac{1}{d(x)} u(x + d(x)x') \leq c \quad \text{for all } x' \in B_1(0).$$

The scaled function $\tilde{u}$ solves

$$\operatorname{div}\left(\tilde{a}(x')|\nabla\tilde{u}(x')|^{p-2}\nabla\tilde{u}(x')\right) + \lambda d(x) b\tilde{(}x') = 0 \quad \text{for all } x' \in B_1(0),$$

where $\tilde{a}(x') = a(x + d(x)x')$ and $\tilde{b}(x') = b(x + d(x)x')$. Hence interior regularity gives

$$|\nabla\tilde{u}(0)| \leq c.$$

From this we conclude $|\nabla u(x)| \leq c$. □

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