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Deposition Processes with Hardcore Behaviour

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Abstract Particles are deposited onto a surface with discrete sites. They are subject to an inhibition by which they cannot pass close by a particle already fixed on the surface. This implies that the surface builds up with vertical gaps between the particles. In this paper it is shown that there is a limiting rate at which the surface grows, and that this is related to the "roughness" of the surface.

Keywords Interacting particle systems

1 Introduction

In a recent paper Fleurke and Külske [1] discuss a model of particle deposition. Suppose that we have a regular graph, G, and that at each site particles arrive as independent Poisson processes, which will be assumed to be at rate 1. The particles can build up at a site, but due to an interaction Fleurke and Külske call "screening", particles cannot pass by neighbouring particles. Thus particles build up at a site with gaps between them. More concretely we have

1. The state-space is $\{0, 1\}^{(G\times\mathbb{N})}$ where $\mathbb{N}$ is the set of non-negative integers. The graph defines for each x a neighbourhood set N_x.
2. The process $\eta_t(x, r) = 1$ if there is a particle at (x, r) at time t, and $= 0$ otherwise for $x \in G, r \in \mathbb{N}$. $\eta_0(x, r) = 0$ for all x, r.
3. The height at a site x at time t is defined to be $h_t(x, \eta) = \max\{r : \eta_t(x, r) = 1\}$.
4. When a particle arrives at site x at time t it deposits at height $h_t(x, \eta) = \max\{h_{t-}(y, \eta), y \in N_x\} + 1$, where neighbourhood set N_x includes x and the sites distance 1 from it. We assume here and throughout that $\mathrm{card}(N_x)$ is the same for all $x \in G$. We denote this quantity by N.

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In this paper we shall chiefly concerned with homogeneous isotropic graphs. In this we differ from the recent paper of Fleurke, Formentin and Külske [8], which considers the process on finite (not necessarily regular) graphs and derives concentration inequalities. It will be shown that these possess a limiting average speed at which the height at a site grows, V, and a limiting density of particles, ρ. We shall show that in these cases

Theorem 1

$$N < \frac{1}{\rho} = V = \lim_{t\to\infty} \frac{h_t(x,\eta)}{t} < k^*,$$

where k^* is such that $(Ne/k^*)^{k^*} = e$. Further we shall show:

Theorem 2

$$\lim_{t\to\infty} \frac{E[h(t)]}{t} = \lim_{t\to\infty} \frac{\int E[d(t)]\,dt}{t} = \lim_{t\to\infty} \frac{\int E[d^+(t)]\,dt}{t} + 1,$$

where $h(t)$ is the height at a single site, d is the mean of the absolute differences in height of a site from its $N-1$ neighbours, and d^+ the maximum of the positive differences.

$d(t)$ can be considered a measure of the roughness of the surface.

2 The Dual Process

We shall drop η where this will not result in ambiguity. Define a path of length n from (x_1, s) to (x,t) to be a set of pairs $(x_1,t_1), (x_2,t_2), \ldots, (x_n,t_n)$, $x = x_n$, $s < t_1 < t_2 < \cdots < t_n < t$, $x_i \in N_{x_{i+1}}$, $i = 1, \ldots, n-1$ and such that for $i \le n$ a particle arrives at site x_i at time t_i.

Lemma 3 *$h_t(x)$ is the length of the longest path to (x,t) from $(z,0)$ maximised over $z \in G$. In particular if there is a path of length n from time 0 for any site z to (x,t) then $h_t(x) \ge n$.*

Proof When a particle arrives at x_{i+1} at time t_{i+1}, the height $h_{t_{i+1}}(x_{i+1})$ must be at least $h_{t_i}(x_i) + 1$. Thus by induction $h_t(x)$ must be at least the maximal length of a path to (x,t). Conversely suppose first that graph G is finite. Initially the relation holds for all x and it can only be violated by the arrival of a particle so, if the claim is false, it must be violated for the first time by the arrival of a particle at a time t at some site y. But at this time the heights of all other sites are unchanged while $h_t(y)$ becomes equal to $h_{t-}(w) + 1$ for a $w \in N_y$. But since the relation is true before time t, there exists a path to (w,s) of length $h_{t-}(w)$ for s close to t. This permits us to construct a path to (y,t) of length $h_t(y)$ and gives a contradiction.

For a general G and a fixed site x we have that for any (x,t) there are only a finite number of z such that there exists path from $(z,0)$ to (x,t) and so we may without loss of generality suppose that the graph G is finite. □

Define $\mathbb{R}(x,t)$ to be the set of paths from $(0,z)$ (for some $z \in G$) to (x,t) and let it have cardinality $R(x,t)$.

Lemma 4 $E[R(x,t)] = e^{(N-1)t}$.

Proof Define $R(y, t-s, x, t)$ to be the number of paths from $(y, t-s)$ to (x, t) and let $R(t-s, x, t) = \sum_y R(y, t-s, x, t)$. Note that at each s the number of nonzero summands is finite. If a particle arrives at y in the interval $(t-s-ds, t-s)$ then, with conditional probability $1 - O(ds)$, no further points for relevant sites arrive in this interval and the number of paths from time $t-s-ds$ to (x, t) is $(N-1)R(y, t-s, x, t)$ greater than the number from time $t-s$ to (x, t). This is because the extra paths are those which use this arrival time and do not start at y. Thus the expected number of paths going backwards in time is increasing at $N-1$ times the number of paths. □

We define a process ξ in which, when a particle arrives at a site, its height increases by 1 and all neighbouring sites are brought up to the same height unless they are already higher. To be more precise: at time 0 there is a single particle at a position we shall call x. We define $h_0(x, \xi) = 0, h_0(y, \xi) = -1, y \neq x$. If a particle arrives at site y at time s then, if $h_{s-}(y, \xi) \geq 0$ and $u \in N_y$, $h_s(u, \xi) = \max\{h_{s-}(u, \xi), h_{s-}(y, \xi) + 1\}$. Define the height of the process $H_t(x, \xi) = \max_{y \in G} h_t(y, \xi)$. Then

Theorem 5 *The distribution of $h_t(x, \eta)$ is the same as the distribution of $H_t(x, \xi)$.*

Proof We have seen that if there is a path $(x_1, t_1), (x_2, t_2), \ldots, (x_n, t_n), x = x_n, s < t_1 < t_2 < \cdots < t_n < t$ then $h_t(x, \eta) \geq n$. To this path there is a corresponding path in the ξ process from $(x, 0)$ to (x_1, t) through pairs $(x_n, t - t_n) \ldots (x_1, t - t_1)$ with particles arriving at site x_i at time $t - t_i$. The height at x_i is then at least 1 more than the height at x_{i+1} at time $t - t_i-$. The height at x_i is then at least n. The maximum height over all such paths starting from x is $H_t(x, \xi)$. □

2.1 Finite and Infinite

We consider relationships between the process η on finite and infinite graphs. We are concerned principally with graphs that are subsets of $\mathbb{Z}^d$ and of two forms, $G^r = [-r, r]^d$ with the usual neighbourhood set, and $G^{r*} = [-r, r]^d$ with added edges making the co-ordinates $-r$ and r neighbours. G^{r*} is spatially homogeneous. Define the expected heights at the point i at time t to be $h^\infty(t), h_i^r(t), h^{r*}(t)$ on the graphs $\mathbb{Z}^d, G^r, G^{r*}$ respectively. We note that there is no dependence on i on the first and third graphs since these graphs are spatially homogeneous. The natural coupling between processes on two graphs is that the Poisson processes of arrivals on sites they have in common are identical. This directly gives

Lemma 6 *If the edges of graph G are contained in the edge set of graph H, then if a site i belongs both to H and G then, under the process η, $h_i^G(t) \leq h_i^H(t)$.*

Simple monotonicity considerations lead to

Corollary 1 *For each $i \in \mathbb{Z}^d$, and each $t \geq 0$, the sequence $h_i^r(t)$ increases to limiting value $h_i(t)$.*

and therefore

Corollary 2 *For each $i \in \mathbb{Z}^d$, and each $t \geq 0$, the sequence $h_i^{r*}(t)$ $(\geq h_i^r(t))$ satisfies* $\liminf_{r \to \infty} h^{r*}(t) \geq h_i(t)$.

In fact we have

Lemma 7

$$h_i(t) = h^{\infty}(t) = \lim_{r\to\infty} h^{r*}(t).$$

Proof Fix t. Consider coupled processes on $\mathbb{Z}^d$ and G^r. The heights at site i will only differ if there is a path back in time from i at t on $\mathbb{Z}^d$ which goes outside G^r. For a path to (i,t), the sequence of sites $\{x_j\}$ must satisfy $|x_j - x_{j-1}| \le 1$. Let us (ignoring time) call a sequence $x_0, x_1, \dots, x_m = i$ a *potential path* to i if this constraint is satisfied. Since for given i, x_m is fixed and given x_j, there are $2d+1$ possibilities for x_{j-1}, we conclude that the number of potential paths of length m to i equals $(2d+1)^m$. For such a fixed path $x_0, x_1, \dots, x_m = i$, we say that it is *realised* (by time t) if there do indeed exist $0 \le t_1 < t_2 < \cdots < t_m \le t$ so that at t_j a particle arrives at site x_{j-1}. The first time a particular potential path (to i) is realised is simply

$$\sum_{j+1}^{m} \tau_j - \tau_{j-1},$$

where $\tau_0 = 0$ and for $j > 0$, τ_j is the first time after τ_{j-1} that there is an arrival at site x_{j-1}. The $\tau_j - \tau_{j-1}$ are i.i.d. exponential random variables of parameter 1 so the probability a particular potential path of length m is realised at time t equals $e^{-t}t^m/m!$. If such a path and no longer path exists, then the height will be m and the maximum difference in heights for η on the two graphs will be less than m. Thus

$$h^{\infty}(t) - h_i^r(t) < \sum_{m>r-|i|} e^{-t}((2d+1)t)^m/m! \to 0 \quad \text{as } r\to\infty.$$

A similar result holds for $h^{r*}(t)$. □

3 The Limiting Speed

In this section we discuss the rate at which the height at a site increases. Most of the results are summarised in Theorem 1.

Lemma 8 *Define k^* such that $(Ne/k^*)^{k^*} = e$ then*

$$\limsup_{t\to\infty} \frac{h_t(x,\eta)}{t} \le k^*.$$

Proof In the η-process we have seen that the height at x at time t is the maximum length of paths proceeding backwards in time from t. We will simply give a crude bound over all potential paths corresponding to a given length. We first note that associated to a sequence $(y_1, y_2, \dots)$ and a positive time t we have a rate 1 Poisson process where the first event $s_1 = \inf\{s > 0 : t - s \text{ is an arrival at } y_1\}$, $s_2 = \inf\{s > s_1 : t - s \text{ is an arrival at } y_2\}$ and so on. (We can assume that the relevant Poisson processes are extended over negative times.) Similarly given a finite sequence $(y_1, y_2, \dots, y_r)$ we have associated a Poisson process run up until the r'th arrival. The probability that (for fixed choice y_i) this process has its r'th arrival before time t is simply the probability that a Poisson random variable of parameter

t has value r or greater. Let us choose $k > k^*$. In particular $k > 1$ and so, as is easily seen, there exists $C = C(k) < \infty$ so that $\forall t$, $\sum_{l=kt}^{\infty} e^{-t}\frac{t^l}{l!} \le C(k)e^{-t}\frac{t^{[kt]}}{[kt]!}$.

Since there are exactly $N^{[kt]-1}$ (reversed) potential paths $(y_1, \ldots, y_{[kt]})$ with $x = y_1$, we have that the probability that for at least one of them, the associated Poisson process has had at least $[kt]$ arrivals is bounded by $N^{[kt]-1}C(k)e^{-t}\frac{t^{[kt]}}{[kt]!} = \frac{C(k)}{N}e^{-t}(Nt)^{[kt]}/[kt]!$.

We have by Stirling's formula that $e^{-t}\frac{(Nt)^{[kt]}}{([kt])!}$ is asymptotically equivalent to

$$\frac{e^{-t}}{\sqrt{2\pi kt}}\left[\left(\frac{Net}{[kt]}\right)\right]^{[kt]},$$

as t becomes large.

This tends to zero geometrically and so summing as t runs over integer values and applying Borel Cantelli gives the result that

$$\limsup_{t\to\infty} \frac{h_t(x,\eta)}{t} \le k.$$

The arbitrariness of k gives the desired result. □

It is interesting to relate this result to a similar result for branching processes. Kingman [2] gives results for the time, B_n, to the first birth in the n'th generation of a branching process where each individual produces offspring over a lifetime. In our model, the number of paths, looking backwards in time, is a branching process in which each individual waits an exponential time and then produces exactly N offspring in the next "generation". In our case the function ϕ introduced by Kingman [2] is given by

$$\phi(\theta) = N\int_0^{\infty} e^{-\theta t}e^{-t}dt = \frac{N}{1+\theta}.$$

He defines

$$\mu(a) = \inf_{\theta} \frac{Ne^{\theta a}}{1+\theta} = aNe^{1-a}.$$

He then shows that

$$\lim_{n\to\infty} B_n/n = \gamma,$$

where γ satisfies $\mu(\gamma) = 1$. The rate of increase of generation number is thus γ. Lemma 8 deals with the number of generations by time t, that is, the reciprocal of γ. Thus the above equation for γ is equivalent to

$$\frac{1}{k^*}Ne^{1-1/k^*} = 1 \quad \Leftrightarrow \quad (Ne/k^*)^{k^*} = e.$$

Thus, in a branching process, the expected number of members in generation n does define the speed. However, this does not appear to be so in the deposition process considered here. The correlation between the paths to two members of the n'th generation in a branching process comes from the last time they were together, usually very early on in the process. However, in the deposition process, the correlations between paths is stronger and k^* is not a very good bound for the speed. In 1-dimension, where $N = 3$, the bound is close to 7.1, but simulations suggest the true value is around 4.

Lemma 9 *For every $x \in G$,*

$$\liminf_{t\to\infty} \frac{h_t(x)}{t} \geq N.$$

Proof Consider the process ξ_t. If the arrival of a particle is such that its position is at the maximum height, then, when it arrives, all of its neighbours must also be at the same height. There are thus always at least N particles at maximum height and the height process dominates a Poisson process rate N. Thus by elementary large deviations and Theorem 5, we have that for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(h_n(x) \leq Nn(1-\epsilon)) = \sum_{n=1}^{\infty} P(H_n(x,\xi) \leq Nn(1-\epsilon)) < \infty.$$

The result now follows from Borel Cantelli and the monotonicity of $h_t(x)$ in time. □

Lemma 10 $\lim_{t\to\infty} \frac{E[h_t(x)]}{t}$ *exists.*

Proof Lemma 3 implies that the height at (x,t) is the length of the maximum path to (x,t). Suppose that one of the maximum paths from time s to $(x,t+s)$ is a path R_2 from (y,s). Call its length n_2. Let the length of the maximum path R_1 from time 0 to (y,s) be n_1. Let the length of the maximum path R from time 0 to $(x,t+s)$ be n. Then $n \geq n_1 + n_2$. Taking expectations we have $E[h_{t+s}(x)] \geq E[h_s(x)] + E[h_t(x)]$. This superadditivity property implies that the limit exists if one restricts to integer times t. Again, the full conclusion follows from the monotonicity of $h_t(x)$ as a function of t. □

Putting Lemmas 8, 9 and 10 together yields

Lemma 11

$$N < V = \lim_{t\to\infty} \frac{E[h_t(x)]}{t} < k^*.$$

In fact the limiting average speed is clearly the same as the limiting average number of particles at maximum height in the ξ process.

The average vertical distance between particles at $x \in G$ in η_t is defined as $v_t(x) = h_t(x,\eta)/\sum_r \eta_t(x,r)$. Since the arrival of particles at a site of G is a Poisson process rate 1, we have $\sum_r \eta_t(x,r)/t \to 1$ as $t \to \infty$. This implies that as t becomes large,

$$v_t(x) \to V.$$

The density of particles in a column tends to $1/V$.

Further, define

$$d_t^+(x) = \max_{y\in N_x} h_t(y,\eta) - h_t(x,\eta).$$

Then the jump made at x when a particle arrives at t equals $d_t^+(x) + 1$. Thus

$$\frac{\int_0^t d_s^+(x)ds}{t} \to V - 1.$$

If the initial distribution of particles is spatially homogeneous, the probability that the height at a position is the maximum of the heights of it and its neighbours is $1/N$. Thus, given that a position is not the maximum in its neighbourhood, the expected height of its maximum neighbour exceeds it by $N(V-1)/(N-1)$.

To prove the next lemma we work on $\mathbb{Z}$ but it will be easy to see that the approach extends to all dimensions. We adopt the method of [7].

Lemma 12 *On $\mathbb{Z}$ the quantity $H_t(0,\xi)$ satisfies*

$$\lim_{t\to\infty}\frac{H_t(0,\xi)}{t}=c\in(0,\infty).$$

Proof Given monotonicity it is sufficient to restrict attention to integer times n. Define X_n to be any $y\in\mathbb{Z}$ so that (with initial point x chosen to be the origin) $h_n(y,\xi)=\max_{u\in\mathbb{Z}} h_n(u,\xi)$ with ties broken by some arbitrary but fixed ordering of the integers. If we define $X_{n,m}$ to be the variable X_m derived from the Poisson system shifted spatially and temporally to have (X_n,n) moved to $(0,0)$, then clearly we have the superadditive relation

$$X_{n+m}\le X_n+X_{n,m}.$$

The conditions of Liggett's subadditive Theorem (Liggett [3]) being satisfied we clearly have that a.s. the non-random limit X_n/n exists. It is easy to see that this limit must be finite. □

We must transfer this law of large numbers to $h_n(0,\eta)$. The problem for us is that while, as noted in Theorem 5, we have for fixed n, $h_n(0,\eta)$ is equal in distribution to $H_n(0,\xi)$, unfortunately for us the processes $\{h_n(0,\eta)\}_{n\ge0}$ and $\{H_n(0,\xi)\}_{n\ge0}$ are not. In particular the subadditivity which gave a simple proof of Lemma 12 appears not to be present. While we know that for any fixed $\epsilon>0$, $P(H_n(0,\xi)<(c-\epsilon)n)$ and $P(H_n(0,\xi)>(c+\epsilon)n)$ tend to zero, we must seek a sufficiently quick rate in order to apply the Borel Cantelli lemma.

We begin with $P(H_n(0,\xi)<(c-\epsilon)n)=P(h_n(0,\eta)<(c-\epsilon)n)$. This is easier than the other bound in that we merely have to show that with high probability there is an appropriate path from $(0,0)$. We will show that for the c of Lemma 12, $\liminf_{n\to\infty}\frac{h_n(0,\eta)}{n}\ge c$. We note that Lemma 12 and Theorem 5 together with the FKG inequality imply that

Lemma 13 *For each $\epsilon>0$, there exists an $M_0<\infty$ so that for all $M\ge M_0$ outside of probability $\epsilon^8/1000$, there exist paths of length at least $(c-\epsilon^4)M$ from $(0,0)$ to (x_1,M) and to (x_{-1},M) with $x_{-1}\le0\le x_1$ and both of absolute value less than $2cM$.*

Proof Consider the events $A_+(M)$, $A_-(M)$ which are respectively that there exists a path from $(0,0)$ to a point in $\mathbb{Z}_+\times\{M\}$, respectively to a point in $\mathbb{Z}_-\times\{M\}$ of length at least $(c-\epsilon^4)M$. By Lemma 12 $\lim_{M\to\infty}P(A_+(M)\cup A_-(M))=1$. In particular for some M_0 and all $M\ge M_0$, $P(A_+(M)\cup A_-(M))\ge1-(\epsilon^8/2000)^2$. Now, both events $A_\pm(M)$ are increasing events with respect to the Poisson (point) processes and so their complements are decreasing events with respect to these processes. Thus by the FKG inequality

$$P(A_+^c(M))P(A_-^c(M))\le P(A_+^c(M)\cap A_-^c(M))\le(\epsilon^8/2000)^2.$$

Therefore, by appeal to symmetry $P(A_+^c(M))$ and $P(A_-^c(M))$ are both less than $\epsilon^8/2000$, which gives the result. □

This begets via simple concatenation

Proposition 14 *There exists M_0 so that for $M \geq M_0$ outside of probability ϵ^2, for each $x \in (-4M, 4M)$, there exits a path from $(0,0)$ to $(x, M/\epsilon^2)$ of length at least $M/\epsilon^2(c-2\epsilon)$ so that the path does not leave $(-6cM, 6cM)$.*

Proof Provisionally let M_0 be as in Lemma 13. We start with $x \in [0, 4M)$, the other case is entirely analogous. Choose $y_1, y_2, \dots, y_{1/\epsilon^2-k}$ for k an integer greater than $5c$ by applying Lemma 13 in succession as follows: we put $y_0 = 0$. If $y_{i-1} \leq 2cM$, then we search for a path from $(y_{i-1}, (i-1)M)$ to (y_i, iM) satisfying the conditions of Lemma 13 with $y_i \geq y_{i-1}$, if $y_{i-1} > 2cM$, then we look for such a path to $y_i \leq y_{i-1}$. By this lemma we have outside of probability $\epsilon^6/1000$ that in all cases we search successfully. By concatenating the paths we obtain a path to $(y_{1/\epsilon^2-k}, M/\epsilon^2 - kM)$ of length at least $(M/\epsilon^2 - kM)(c-\epsilon^2)$ which does not leave $(-6cM, 6cM)$ and with $y_{1/\epsilon^2-k} \in (0, 4cM)$. By the law of large numbers (here we may need to increase N_0) there is such a path from $(y_{1/\epsilon^2-k}, M/\epsilon^2 - kM)$ to $(0, M/\epsilon^2)$ and to $(4M, M/\epsilon^2)$ outside of probability $\epsilon^2/4$. Thus outside of probability $\epsilon^6/1000 + \epsilon^2/4$ we have a suitable path to any $x \in [0, 4M)$. Furthermore this path has length at least $(M/\epsilon^2 - Mk)(c-\epsilon^2) \geq M/\epsilon^2(c-\epsilon)$ if ϵ is sufficiently small. □

We now adapt this to give a lower bound on $h_{nM/\epsilon^2}(0,\eta)$. We construct a 4 dependent site percolation scheme $\Psi(m,n) : (m,n) \in \mathbb{Z} \times \mathbb{Z}_+$ on the basis of Proposition 14 from our Poisson system by specifying that $\Psi(m,n) = 1$ if there are paths from $(2cmM, nM/\epsilon^2)$ to both $(2c(m\pm 1)M, (n+1)M/\epsilon^2)$, both having length at least $M/\epsilon^2(c-\epsilon)$ and not exiting $(2c(m-3)M, 2c(m+3)M)$. We say that (m,n) is *good* if there is a nearest neighbour path of sites (x_i, i) with $(x_n, n) = (m,n)$ and $\forall i\ \Psi(x_i, i) = 1$. This scheme has open site probability at least $1-\epsilon^2$ by Proposition 14. We now use this to get a bound on $h_{nM\epsilon^2}(0,\eta)$. Consider the event $A(\delta, n) \equiv \{\text{there is no } m : |m| < \delta n \text{ with } (x, n(1-\delta)) \text{ good}\}$. By standard percolation theory (see e.g. [4], [5]) we have that (provided ϵ was fixed small) $P(A(\delta,n)) \leq 2e^{-c_1 n}$ for some strictly positive c_1 and all n. If $A(\delta,n)$ does not occur, then there must exist some $|z| \leq 2cMn\delta$ with $h_{M/\epsilon^2 n(1-\delta)}(z,\eta) \geq M/\epsilon^2 n(1-\delta)(c-\epsilon)$. For such a z, the conditional probability that there exists a path from $(z, M/\epsilon^2 n(1-\delta))$ to $(0, nM\epsilon^2)$ is greater than $1-2e^{-c'n}$ for some strictly positive c' if δ is small compared to ϵ. Thus we have shown that

$$P(h_{nM/\epsilon^2}(0,\eta) < M/\epsilon^2 n(1-\delta)(c-\epsilon)) \leq 2(e^{-c'n} + e^{-c_1 n}).$$

Therefore by the Borel Cantelli Lemma we have that for every $\epsilon > 0$ (recall δ is small compared to ϵ),

$$\liminf_{n\to\infty} \frac{h_{nM/\epsilon^2}(0,\eta)}{M/\epsilon^2 n} \geq (1-\epsilon)(c-\epsilon).$$

The result that $\liminf_{t\to\infty} \frac{h_t(0,\eta)}{t} \geq (1-\epsilon)(c-\epsilon)$ is a consequence of monotonicity of $h_t(0,\eta)$ in t. The desired result is now a consequence of the arbitrariness of ϵ.

We now establish the superior bound. We define for our Poisson process $l(x,a,b)$, $x \in \mathbb{Z}$, $0 \leq a < b$ to equal the length of the longest path starting at (x,a) and finishing at (y,b) for some $y \in \mathbb{Z}$. If $a = 0$, then it is simply suppressed from the notation. Thus $l(x,t)$ is simply $H_t(x,\xi)$. For a $A \subset \mathbb{Z}$ (typically an interval), $l(A,a,b) = \max_{x\in A} l(x,a,b)$.

The following is a consequence of similar considerations to those that lead to Lemma 8.

Lemma 15 *There exists finite k_o so that for $k \geq k_o$, $P(l(x,t) \geq kt) \leq e^{-k\log(k)t/2}$.*

Corollary 3 *There exists finite* k_1 *so that* $P(h_t(0,\eta) \geq kt) \leq e^{-kt}$ *for all large* t. *In particular* $\limsup_{t\to\infty} \frac{h_t(0,\eta)}{t} \leq k_1$.

This result means we need only consider paths which have of order t jumps over an interval $(0,t)$.

The following is the crucial building block for our upper bound. It extends the fact that there is a weak law of large numbers for $l(x,n)$ (for x fixed) to there being a weak law (with the same limiting constant) for $l([-kn,kn],n)$. The idea is simply that since paths can be concatenated, then if $l([-kn,kn],n)$ is too large, then this entrains that $l(x,n)$ be too large for some x fixed for a time shift of the system.

Proposition 16 *Fix arbitrary* $\epsilon, \delta > 0$. *For* M *sufficiently large and* k_1 *as above*

$$P(l([-2k_1M, 2k_1M], M) \geq (c+\epsilon)M) < \delta.$$

Proof Given M let $\{x_i\}_{0\leq i\leq 20k_1/\epsilon}$ be points in $[-2k_1M, 2k_1M]$ which are such that no point in $[-2k_1M, 2k_1M]$ is more than $\epsilon M/5$ away from a x_i. By Lemma 12 we have, for M large that

$$P\left(\max_{x_i} l(x_i, M(1+2\epsilon/5)) > Mc(1+3\epsilon/5)\right) < \delta/100.$$

Now suppose that $l([-2k_1M, 2k_1M], \epsilon M, (1+\epsilon)M) \geq (c+\epsilon)M$. Then on (independent) event $A_{M,\epsilon} \equiv \{\forall i\ \exists \text{ paths from } (x_i,0) \text{ to } [x_i - \epsilon M/5, x_i + \epsilon M/5]\}$, we have

$$\max_{x_i} l(x_i, M(1+2\epsilon/5) \geq Mc(1+\epsilon)).$$

Thus we have $P(l([-2k_1M, 2k_1M], M) \geq (c+\epsilon)M) = P(l([-2k_1M, 2k_1M], \epsilon M, (1+\epsilon)M) \geq (c+\epsilon)M) \leq (\delta/100)/P(A_{M,\epsilon}) \leq \delta$ for M large by the law of large numbers for rate one Poisson processes. □

We wish to show that for a given $\epsilon > 0$, $h_t(0,\eta) < (c+\epsilon)t$ for t large. By monotonicity it is sufficient to show that for every ϵ and M, $h_{vM}(0,\eta) < (c+\epsilon)vM$ for integer v large. Thus it will be sufficient to show that $P(h_{vM}(0,\eta) \geq (c+\epsilon)vM)$ converges to zero sufficiently fast. We know already from Corollary 3 that it is sufficient to bound $P(h_{vM}(0,\eta) \in [(c+\epsilon)vM, k_1vM])$. For the event in question to occur, there must be a path of length $L \in [(c+\epsilon)vM, k_1vM]$, $(x_0,0), (x_1,t_1), \ldots, (x_L,t_L)$ for $x_L = 0$ and $t_L \leq vM$. We now discretise this path in two ways; first we define for $j = 0,1,\ldots,v$, $y_j = x_{i_j}$ where $i_j = \max\{k : t_k \leq jM\}$. Secondly we take z_j to equal the integer such that $y_j \in [2k_1Mz_j - k_1M, 2k_1Mz_j + k_1M)$. $z_0, z_1, \ldots, z_v$ is the skeleton of the original path. Note that necessarily $z_v = 0$ since x_L and therefore y_v equal 0 and that there are less than v possibilities for z_0. The following is a simple combinatorial exercise (see e.g. [6], [7]).

Lemma 17 *There are at most* 4^v *skeletons compatible with a path of length less than* k_1vM *whose skeleton has* $z_v = 0$.

Let us denote by $\mathcal{A}(v,M)$ these skeletons. We note that if a path engenders a skeleton $(z_0, z_1, \ldots)$, then the length of the path is bounded by

$$\sum_{k=0}^{v-1} l([2k_1Mz_j - k_1M, 2k_1Mz_j + k_1M], jM, (j+1)M).$$

Thus the event $h_{vM}(0,\eta) \geq (c+\epsilon)vM$ is contained in the event

$$\max_{(z_0,z_1,\ldots)\in\mathcal{A}(v,M)} \sum_{k=0}^{v-1} l([2k_1Mz_j - k_1M, 2k_1Mz_j + k_1M), jM, (j+1)M) \geq (c+\epsilon)vM.$$

We will show that for M fixed (in advance) sufficiently large, this is exponentially small in v. By Lemma 17, it is sufficient to show that for a fixed skeleton the above probability is small compared to 4^{-v}. The argument comes down to showing that with very high probability a very high proportion of the (z_j, j) will be *good* in the sense that $l([2k_1Mz_j - k_1M, 2k_1Mz_j + k_1M), jM, (j+1)M) < (c+\epsilon/4)M$ (see Lemma 18 below), given this one can tolerate a small number of (z_j, j) so that $l([2k_1Mz_j - k_1M, 2k_1Mz_j + k_1M), jM, (j+1)M) \geq (c+\epsilon/4)M$ but reasonable (say bounded by $2k_1M$). It only remains to control the really large deviations. This is done by Lemma 19 below.

Lemma 18 *For any* $\epsilon, \gamma > 0$, *there exists* M *sufficiently large so that for all* v *large and any skeleton* $(z_0, z_1, \ldots)$

$$P\left(\sum_{k=0}^{v-1} I_{(z_k,k)\,\text{good}} \leq v(1-\gamma)\right) < 5^{-v}.$$

Proof For any skeleton the probability in question is the probability that a Binomial random variable (with parameters v and the probability of $(0,0)$ being good) is less than $v(1-\gamma)$. But by standard large deviations for Binomial random variables and Proposition 16 (applied with ϵ replaced by $\epsilon/4$) we can find M large enough to give the bound. □

Next we define random variables

$$L_j = \big(l([2k_1Mz_j - k_1M, 2k_1Mz_j + k_1M), jM, (j+1)M) - 2k_1M\big)_+.$$

The following is true if k_1 was fixed sufficiently large:

Lemma 19 *For any skeleton* $P(\sum_{k=0}^{v-1} L_k \geq vM\epsilon/4) \leq 5^{-v}$.

Proof By bounds on Lemma 15 we have $E[e^{L_j}] \leq 2$ (provided k_1 was fixed large). Thus by usual Chebychev bounds $P(\sum_{k=0}^{v-1} L_k \geq vM\epsilon/4) \leq \frac{2^v}{e^{\epsilon vM/4}} \leq 5^{-v}$, provided M was chosen large. □

Thus we can finally demonstrate our bounds on the probability that $\sum_{j=0}^{v-1} l([2k_1Mz_j - k_1M, 2k_1Mz_j + k_1M), jM, (j+1)M) \geq (c+\epsilon)vM$. The sum is bounded by $vM(c+\epsilon/4) + 2k_1MR(z_0,\ldots,z_v) + \sum_{k=0}^{v-1} L_k$, where $R_(z_0,\ldots,z_v)$ is the number of nongood (z_j, j). By Lemma 19 outside of probability 5^{-v} this is less than $vM(c+\epsilon/2) + 2k_1MR(z_0,\ldots,z_v)$. But, provided we chose $\gamma < \epsilon/8k_1$, by Lemma 18, we have that outside a further probability 5^{-v} this is bounded by $vM(c+3\epsilon/4)$. However by Lemma 17, the number of skeletons is bounded by 4^v and so the probability that among these 4^v objects there exists one that violates any of the above inequalities is bounded by 4^vK5^{-v} for some universal K and we are done.

4 The Relationship Between the Speed and the Roughness

Let G be a finite graph for which site i has n_i neighbours. At each site $i \in G$ there is a height h_i and the process evolves according to η. Define the differences in height between i and its neighbours to be $d_i(1) \geq d_i(2) \geq \cdots \geq d_i(m_i) \geq 0 \geq -d_i(m_i+1) \geq \cdots \geq -d_i(n_i), 0 \leq m_i \leq n_i$. The h's and d's are functions of time. Define

$$H = \sum_i n_i h_i, \qquad D_i = \sum_m d_i(m), \qquad D = \sum_i D_i.$$

Consider the changes to h and D produced by a particle arriving at i. First,

$$h_i \to h_i + 1 + d_i(1)$$

and secondly, for $m \leq m_i$ the difference switches from $d_i(m)$ to $1 + d_i(1) - d_i(m)$ and for $m > m_i$ the difference switches from $d_i(m)$ to $1 + d_i(1) + d_i(m)$, so that

$$\begin{aligned} D_i \to D_i &+ n_i(1 + d_i(1)) - [d_i(1) + \cdots + d_i(m_i)] + d_i(m_i+1) + \cdots + d_i(n_i) \\ &- [d_i(1) + \cdots + d_i(n_i)]. \end{aligned}$$

Thus,

$$n_i h_i - D_i \to n_i h_i - D_i + 2[d_i(1) + \cdots + d_i(m_i)].$$

Summing over all sites, and noting that each $d_i(j) > 0$ appears exactly once, we obtain that the infinitesimal rate of change of $H - D$ equals D. From this follows:

Lemma 20

$$H - D - \int D dt \quad \textit{is a martingale.}$$

We now prove that, in the limit, if the n_i are all the same, the rate of increase of the height at a site is proportional to the Cesaro mean of the difference in heights. This may seem obvious, but some thought will show that it is not.

Lemma 21 *If $f(t) \geq 0$ is continuous and $g(t) = f(t)/t + F(t)/t \to l$ where $F(t) = \int_0^t f(s)ds$ then $f(t)/t \to 0$.*

Proof If it does not converge to 0, then $f(t)/t$ exceeds a value $p > 0$ for arbitrarily large values of t. Define a set of values $t_1 < t_2 < \cdots$ which have the property that $f(t_i-)/t_i- < p$, $f(t_i)/t_i = p$ and $\exists x_{i+1}$ such that $t_i < x_{i+1} < t_{i+1}$, $f(x_{i+1})/x_{i+1} = p/2$ where x_{i+1} is the largest such value. If such an infinite sequence does not exist then there exists a t^* such that $f(t)/t > p/2, t > t^*$ in which case $F(t)/t \to \infty$.

We make some estimates of $g(t_r) - g(x_r)$. The contribution from $f(t)/t$ is $p/2$. For the contribution from $F(t)$ to make $g(t_r) - g(x_r) < p/4$ requires

$$F(t_r)/t_r - F(x_r)/x_r < -p/4 \quad \Rightarrow \quad F(x_r)/x_r > \frac{pt_r}{4(t_r - x_r)}. \tag{1}$$

Suppose $x_r = (1-q)t_r$, then $f(t)/t \geq p/2, (1-q)t_r < t < t_r$ so that

$$F(t_r)/t_r > \max\{pt_r[1-(1-q)^2]/4, p/4q\},$$

where the second inequality comes from (1). The minimum of the above expression occurs when $q[2q-q^2]=1/t_r$ in which case $1/q > \sqrt{t_r}$.

We thus have that either $F(t)/t > p\sqrt{t}/4$ for arbitrarily large t, or we can find arbitrarily large values $t_r > x_r$ for which $g(t_r)-g(x_r) > p/4$. In either case $g(t)$ does not converge. □

Theorem 22 *If the graph G is finite and spatially homogeneous or $G=\mathbb{Z}^k$ we have*

$$\frac{E[h_G(t)]}{t} \to l_G, \qquad \frac{\int E[d_G(t)]\,dt}{t} \to l_G,$$

where $h_G(t)$ is the height at a single site, $d_G(t)$ is the mean difference in height from its neighbours and l_G exists by Theorem 1.

Proof It follows from Lemma 20 that $E[H-D-\int D\,dt]=0$. In the finite case we divide this equation by $n|G|$ where $|G|$ is the number of sites in G. It follows from Theorem 1 that $\mathbb{E}[h_G(t)/t]$ tends to a limit which we shall call l_G. $\mathbb{E}[h_G(t)/t]$ is continuous, and the theorem follows from Lemma 21.

In the case of $\mathbb{Z}^k$ we consider how the process affects $[-r,r]^k$. Define the sum of heights on this set to be H^r and the sum of differences to be D^r. Unlike the sum over all sites above, the negative differences from sites on the boundary will not be counted, so that Lemma 20 becomes

$$H^r - D^r - \int D^r - D_B^{r-}\,dt \quad \text{is a martingale,}$$

where D_B^{r-} is the sum of negative differences from sites on the boundary. Because $\mathbb{Z}^k$ is spatially homogeneous there is an $m<2$ such that $E[D_B^{r-}(t)]=2k(2r+1)^{k-1}mE[d(t)]/2$. Thus we have $H^r-D^r-(1-2km/(n(2r+1)))\int D^r\,dt$ is a martingale, and dividing by $n(2r+1)^k$ and letting $r\to\infty$ the result follows as before. □

5 The Roughness of the Surface

In this section we deal with the process in 1-d so that $G=\mathbb{Z}$. We shall deal with the distributions as $t\to\infty$. In other words we shall assume that certain finite sets of events in the past have occurred with probability 1. We say there is a U at $x+1/2$ if $h(x+1)>h(x)$ and a D there if $h(x+1)<h(x)$. The mechanism by which a particle arrives ensures that the height at neighbouring sites cannot be equal.

On G the probability that a site is a peak, that is, higher than its neighbours, simply equals $1/(n+1)$ where n is the degree of the graph. In 1-d this is $1/3$. We look at the lengths of down sequences from a peak and the distance to the next peak. We shall designate the last arrival time at x, measured backwards in time as t_x. Suppose there is a peak at 0 and that the next trough (lower than all its neighbours) is at r. Then we require $t_0<t_1<\cdots<t_r$, $t_0<t_{-1}$, $t_{r+1}<t_r$. The numbers of ways in which this can happen are as follows: (a) $t_{r+1}<t_0$, in which case there are $r+1$ possible positions for t_{-1}, (b) t_{r+1} is in one the r possible slots greater than t_0 but less than t_r. In this case there are $r+2$ possible places for t_{-1}. Together these total $r+1+r(r+2)=r^2+3r+1$ possibilities. The number of orderings of the arrival times is $(r+3)!$. Define L to be the length of a D sequence from a peak, then

$$P(L=r|0\text{ is a peak}) = \frac{P(L=r\text{ and }0\text{ is a peak})}{P(0\text{ is a peak})} = 3\frac{r^2+3r+1}{(r+3)!}, \qquad r\ge 1.$$

The p.g.f. of L is

$$3e^s\left(\frac{1}{s}-\frac{2}{s^2}+\frac{1}{s^3}\right)-\frac{1}{2}+\frac{3}{2s}+\frac{3}{s^2}-\frac{3}{s^3},$$

giving

$$E[L]=1.5, \qquad \mathrm{Var}(L)=6e-15.75\approx 0.56.$$

We now calculate the probability of a down-slope of length r followed by an up-slope of length s. Put the first peak at 0, the first trough at r and the second peak at $r+s$. There are 4 possibilities:

The first is that t_{-1} is in one of the r positions between t_0 and t_r and t_{r+s+1} is in one of the s positions between t_{r+s} and t_r. We then have the $r+1$ times $t_0, t_{-1}, t_1, \dots, t_{r-1}$ ordered and less than t_r. Similarly we have the $s+1$ times $t_{r+1}, \dots, t_{r+s+1}$ ordered and less than t_r. These $r+s+2$ times can therefore be ordered in $\binom{r+s+2}{r+1}$ ways. The other 3 possibilities depend on the possibilities of t_{-1} being greater than t_r or t_{r+s+1} being less than t_r or both. If L, M respectively are the lengths of a down-slope of length L followed by an up-slope of length M then

$$P(L=r, M=s)=\frac{rs\binom{r+s+2}{r+1}+s\binom{r+s+1}{r}+r\binom{r+s+1}{s}+2\binom{r+s}{s}}{(r+s+3)!}.$$

As would be expected, $E[L+M]=3$. Further, $Corr(L,M)=0.04$, but this is driven by the low values of L, R and is somewhat misleading as, for example, $P(R=8|L=8)=1.9P(R=8)$. However, these values are $O(10^{-6})$.

When a particle arrives at x, the values at $x-1/2$ and $x+1/2$ become UD whatever they were before. $x+1/2$ is thus occupied by a D if the most recent arrival of UD at $x-1/2, x+1/2$ occurred after the most recent arrival of UD at $x+1/2, x+3/2$. Thus the limiting probability of a D equals $1/2$. A UD occupies $x-1/2, x+1/2$ if looking backwards in time $x-1/2, x+1/2$ is the most recent of the 3 events $x-1/2, x+1/2$; $x-3/2, x-1/2$; $x+1/2, x+3/2$. If instead the order had been $x+1/2, x+3/2$; $x-1/2, x+1/2$; $x-3/2, x-1/2$ it would have been DU.

In what follows we look back in time so that "precedes" means "is more recent than". We have the following rules:

- *A U is at* $x+1/2$ *if the last particle to arrive at* $x+1$ *preceded the last at* x.
- *A D is at* $x+1/2$ *if the last particle to arrive at* x *preceded the one at* $x+1$.

Numbering a set of sites $1, 2, 3, \dots$. We have UDD in the first 3 positions if 2 precedes 1, 3 follows 2 and 4 follows 3. The possible orderings of 1, 2, 3, 4 are thus 2134, 2341, 2314. So the probability of UDD is $3/24$.

To calculate UDDU or UDDD we note that the extra letter only concerns the relationship of 5 to 4. We calculate the probability by considering the number of times 4 is in position 1, 2, 3, 4 in the orderings that give UDD.

UDD		UDDU		UDDD	
Position of 4	1 2 3 4	Position of 5	1 2 3 4 5	Position of 5	1 2 3 4 5
Frequency of 4	0 0 1 2	Frequency of 5	3 3 3 2 0	Frequency of 5	0 0 0 1 3

Since, in UDDU 5 precedes 4, it can take positions 1, 2 or 3 for any position of 4, it can only take position 4 when 4 is in position 4 in UDD and 5 cannot take position 5. In UDDD 5 must follow 4.

Theorem 23 *Consider a sequence $x_1x_2\dots x_{m-1}$ where each x_i is U or D. Let $n_{m,r}$ be number of times m appears in position r in the possible orderings of $1,2,\dots,m$. Then for the sequences $x_1x_2\dots x_{m-1}D$ and $x_1x_2\dots x_{m-1}U$, the respective frequencies of $m+1$ are*

$$n^D_{m+1,r} = \sum_{j=1}^{r-1} n_{m,j}, \qquad n^U_{m+1,r} = \sum_{j=r}^{m} n_{m,j}, \quad r = 1,\dots,m+1,$$

where the values are 0 when the summation is not possible. In the above notation

$$P(x_1x_2\dots x_{m-1}) = \sum_{j=1}^{m} n_{m,j}/m!.$$

The particular cases $P(UU\dots U) = P(DD\dots D) = 1/(n+1)!$ where the number of identical symbols is n,

$$P(x_1\dots x_m.y_1\dots y_r) = P(x_1\dots x_m)P(y_1\dots y_r).$$

Proof When the sequence ends in a D, the m must precede the $m+1$, so $m+1$ may appear in position r for all positionings of m up to but not including r. When the sequence ends in a U, the $m+1$ must precede the m, so $m+1$ may appear in position r for all positionings of m from r onwards. It cannot appear in position $m+1$.

To have n U's in a row requires that the $n+1$ arrivals surrounding them came in the order last to first. The last part of the theorem is clear, since the appearance of a U or D at any site is only affected by its immediate neighbours. □

The above formulae provide an algorithm for calculating the probability of any sequence of U's and D's. It can be shown that

$$P(x_1x_2\dots x_{m-1}.U) = P(x_1x_2\dots x_{m-1}UU) + P(x_1x_2\dots x_{m-1}DU) = \frac{1}{2}P(x_1x_2\dots x_{m-1}),$$

a special case of the last proposition in Theorem 23.

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