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Continuous Shearlet Tight Frames

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Abstract Based on the shearlet transform we present a general construction of continuous tight frames for $L^2(\mathbb{R}^2)$ from any sufficiently smooth function with anisotropic moments. This includes for example compactly supported systems, piecewise polynomial systems, or both. From our earlier results in Grohs (Technical report, KAUST, 2009) it follows that these systems enjoy the same desirable approximation properties for directional data as the previous bandlimited and very specific constructions due to Kutyniok and Labate (Trans. Am. Math. Soc. 361:2719–2754, 2009). We also show that the representation formulas we derive are in a sense optimal for the shearlet transform.

Keywords Shearlet · Continuous frames · Representation formulas

1 Introduction

The purpose of this short paper is to give a construction of a system of bivariate functions which has the following desirable properties:

Directionality. The geometry of the set of singularities of a tempered distribution f can be accurately described in terms of the interaction between f and the elements of the system.
Tightness. The system forms a tight frame of $L^2(\mathbb{R}^2)$.
Locality. The representation is *local*. By this we mean that the representation can also be interpreted as a representation with respect to a non tight frame and its dual frame such that both of these frames only consist of compactly supported functions.

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The importance of these criteria is obvious: First, it is widely agreed upon the fact that a large part of the information that a function carries lies in its singularities (just think of the edges in an image). Secondly, any transform should possess a stable analysis and reconstruction operation. This is encoded in the tight frame property. And finally, in many cases it is much easier to work with a local transform than with a non-local one. This is especially true when working with functions over bounded domains like in image processing or numerical PDE theory.

For univariate functions the wavelet transform addresses all these three points perfectly, but as the dimension grows, the geometry of potential singularities becomes too complicated for the wavelet transform to capture.

For this reason a number of new (continuous and discrete) transforms have been introduced in the last years.

The focus of this paper is on the *shearlet transform* which has been introduced in [7, 10] and has become popular in Computational Harmonic Analysis for its ability to sparsely represent bivariate functions. More specifically, we will concentrate on the *continuous shearlet transform*.

An important result is that the magnitude of the coefficients of the continuous shearlet transform of a general tempered distribution characterizes the *Wavefront Set* (see e.g. [8] for the definition) of this distribution [9]. In the same paper it is shown that one can construct a continuous tight frame formula for $L^2(\mathbb{R}^2)$.

While the results in [9] address the directionality and tightness properties, they are based on bandlimited generators and therefore they are not local.

It is the purpose of this paper to find a local substitute for the just mentioned results. Our main findings are a new representation formula for the shearlet transform which is purely local. This is the content of our main Theorem 3.4. Furthermore, we show that in a sense (to be specified below) this new representation formula is optimal, see Theorem 3.6.

Notation We shall use the symbol $|\cdot|$ indiscriminately for the absolute value on $\mathbb{R}, \mathbb{R}^2, \mathbb{C}$ and $\mathbb{C}^2$. We usually denote vectors in $\mathbb{R}^2$ by x, t, ξ, ω and their elements by $x_1, x_2, t_1, t_2, \dots$. In general it should always be clear to which space a variable belongs. The symbol $\|\cdot\|$ is reserved for various function space and operator norms. For two vectors $s, t \in \mathbb{R}^2$ we denote by st their Euclidean inner product. We use the symbol $f \lesssim g$ for two functions f, g if there exists a constant C such that $f(x) \leq Cg(x)$ for large values of x. We will often speak of *frames*. By this we mean continuous frames as defined in [1]. We define $\hat{f}(\omega) = \int f(x) \exp(2\pi i \omega x) dx$ to be the Fourier transform for a function $f \in L^1 \cap L^2$ and continuously extend this notion to tempered distributions.

2 Shearlets

We start by defining what a shearlet is and what the shearlet transform is:

Definition 2.1 A function $\psi \in L^2(\mathbb{R}^2)$ is called *shearlet* if it possesses $M \geq 1$ vanishing moments in x_1-direction, meaning that

$$\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^{2M}} d\omega < \infty.$$

Let $f \in L^2(\mathbb{R}^2)$. The *shearlet transform* of f with respect to a shearlet ψ maps f to

$$\mathcal{SH}_\psi f(a,s,t) := \langle f, \psi_{ast} \rangle,$$

where

$$\psi_{ast}(x) := \psi\left(\frac{x_1 - t_1 - s(x_2 - t_2)}{a}, \frac{x_2 - t_2}{a^{1/2}}\right).$$

The following reproduction formula holds [4]:

Theorem 2.2 *For all $f \in L^2(\mathbb{R}^2)$ and ψ a shearlet*

$$f(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \mathcal{SH}_\psi(a,s,t)\psi_{ast}(x) a^{-3} da ds dt,$$

where equality is understood in a weak sense.

The shearlet transform captures local, scale- and directional information via the parameters t, a, s respectively. A significant drawback of this representation is the fact that the directional parameter runs over the non-compact set $\mathbb{R}$. Also it is easy to see that the distribution of directions becomes infinitely dense as s grows, and therefore there is a strong dependence on the coordinate directions in this representation.

These problems led to the construction of shearlets on the cone [7, 9]. The idea is to restrict the shear parameter s to a compact interval. Since this only allows to caption a certain subset of all possible directions, the function f is split into $f = Pf + P^\nu f$, where P is the frequency projection onto the cone with slope ≤ 1 and only Pf is analyzed with the shearlet ψ while $P^\nu f$ is analyzed with $\psi^\nu(x_1, x_2) := \psi(x_2, x_1)$. The idea behind this splitting is that directional singularities with a direction of slope ≥ 1, (resp. ≤ 1) manifest themselves as slow decay in the frequency cone with slope ≤ 1, (resp. ≥ 1). Therefore the splitting $f = Pf + P^\nu f$ decomposes f in a part with singularities of slope ≤ 1 and a part with singularities of slope ≥ 1. Both of these parts can be analyzed with the parameter s ranging in the compact interval $[-2, 2]$, see below. Also the distribution of the directions becomes almost uniform if restricted to this interval.

For very specific choices of ψ Labate and Kutyniok proved a representation formula

$$\|f\|_2^2 = \int_{\mathbb{R}^2} |\langle f, T_t W \rangle|^2 dt + \int_{\mathbb{R}^2} \int_{-2}^{2} \int_0^1 |\mathcal{SH}_\psi P f(a,s,t)|^2 a^{-3} da ds dt$$

$$+\int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 |\mathcal{SH}_{\psi^\nu} P^\nu f(a,s,t)|^2 a^{-3} da ds dt, \tag{1}$$

T_t denoting the translation operator by t and W being a smooth window function.

The main weakness of this decomposition is its lack of locality: indeed, first of all the need to perform the frequency projection P to f destroys any locality. But also the functions ψ which have been considered in [9] are very specific bandlimited functions which do not have compact support. As a matter of fact no useful local representation via compactly supported functions which is able to capture directional smoothness properties has been found to date. The present paper aims at providing a step towards finding such a representation using two crucial observations: First, in [5] we were able to show that the description of directional smoothness via the decay rate of the shearlet coefficients for $a \to 0$ essentially works for any function which is sufficiently smooth and has sufficiently many vanishing moments in the first direction, hence also for compactly supported functions. The second observation is that actually a full frequency projection P is not necessary to arrive at a useful representation similar to (1). Instead of the operator P we will use a 'localized frequency projection' which is given by Fourier multiplication with a function $\hat{p}_0$ to be defined later.

Our main result Theorem 3.4 will prove a representation formula similar to (1) where ψ is allowed to be compactly supported and the frequency projection is replaced with a local variant. We also show that this local variant of the frequency projection is the best we can do – without it, no useful continuous tight frames (or continuous frames with a structured dual) can be constructed within the scope of the shearlet transform. This is shown in Theorem 3.6.

3 The Construction

The goal of this section is to derive a representation formula

$$\begin{aligned}\|f\|_2^2 = \frac{1}{C_\psi}\Bigg(&\int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 |\langle f, q_0 * \psi_{ast}\rangle|^2 a^{-3} da ds dt \\ &+ \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 |\langle f, q_1 * \psi_{ast}^\nu\rangle|^2 a^{-3} da ds dt \\ &+ \int_{\mathbb{R}^2} |\langle f, T_t\varphi\rangle|^2 dt\Bigg)\end{aligned} \tag{2}$$

for $L^2(\mathbb{R}^2)$-functions f and with some localized frequency projections q_0, q_1 and a window function φ to be defined later. In order to guarantee that the shearlet-part contains the high-frequency part of f and the rest contains low frequencies, it is necessary to ensure that the window function φ in this formula is sufficiently smooth.

3.1 Ingredients

Here we first state the definitions and assumptions that we use in the construction. Then we collect some auxiliary results which we later combine to prove our main results Theorems 3.4 and 3.6. The large part of the results will concern the construction of a useful window function and to ensure its smoothness. We say that a bivariate function f has Fourier decay of order L_i in the i-th variable ($i \in \{1,2\}$) if

$$\hat{f}(\xi) \lesssim |\xi_i|^{-L_i}.$$

We start with a shearlet ψ which has M vanishing moments in x_1-direction and Fourier decay of order L_1 in the first variable. It is clear from the definition of vanishing moments that $\psi = (\frac{\partial}{\partial x_1})^M \theta$ with $\theta \in L^2(\mathbb{R}^2)$. We assume that θ has Fourier decay of order L_2 in the second variable so that the following relation holds:

$$2M - 1/2 > L_2 > M \geq 1. \tag{3}$$

We also set

$$N := 2\min(L_2 - M, L_1), \tag{4}$$

$$C_\psi := \int_{\mathbb{R}^2} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^2} d\omega, \tag{5}$$

and

$$\Delta_\psi(\xi) := \int_{-2}^{2} \int_0^1 |\hat{\psi}(a\xi_1, a^{1/2}(\xi_2 - s\xi_1))|^2 a^{-3/2} da ds. \tag{6}$$

We define functions $\varphi_0,\ \varphi_1$ via

$$|\hat{\varphi}_0(\xi)|^2 = C_\psi - \Delta_\psi(\xi) \quad \text{and} \quad |\hat{\varphi}_1(\xi)|^2 = C_\psi - \Delta_{\psi^\nu}(\xi), \tag{7}$$

where

$$\psi^\nu(x_1, x_2) := \psi(x_2, x_1).$$

We write $\chi_{\mathcal{C}}$ for the indicator function of the cone $\mathcal{C} = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \leq |\xi_1|\}$. Finally, we pick a smooth and compactly supported bump function Φ with $\Phi(0) = 1$ and define functions $p_0,\ p_1$ via

$$\hat{p}_0 = \hat{\Phi} * \chi_{\mathcal{C}} \quad \text{and} \quad \hat{p}_1 = 1 - \hat{p}_0. \tag{8}$$

Clearly, p_0 and p_1 are both compactly supported tempered distributions.

We remark that for any shearlet ψ, the constant C_ψ can also be computed as

$$C_\psi = \int_{-\infty}^{\infty} \int_0^\infty |\hat{\psi}(a\xi_1, a^{1/2}(\xi_2 - s\xi_1))|^2 a^{-3/2} da ds,$$

as a short computation reveals. This fact is related to the inherent group structure of the shearlet transform [4].

The following two lemmas concern the Fourier decay properties of the functions p_i and φ_i respectively ($i = 0, 1$).

Lemma 3.1 *We have*

$$\begin{aligned} |\hat{p}_0(\xi)| &\lesssim |\xi|^{-N} \quad \textit{for } |\xi_1|/|\xi_2| \geq 3/2, \\ |\hat{p}_1(\xi)| &\lesssim |\xi|^{-N} \quad \textit{for } |\xi_2|/|\xi_1| \geq 3/2. \end{aligned} \tag{9}$$

Proof Assume that $\xi = te$ with e a unit vector with $|e_1|/|e_2| \geq 3/2$ and $t > 0$. There exists a uniform $\delta > 0$ such that for all η with $|\eta| < \delta t$ we have $\xi - \eta \in \mathcal{C}^c$, and hence $\chi_{\mathcal{C}}(\xi - \eta) = 0$. It follows that we can write

$$\begin{aligned} |\hat{p}_0(\xi)| &= \left| \int_{\mathbb{R}^2} \chi_{\mathcal{C}}(\xi - \eta)\hat{\Phi}(\eta)d\eta \right| \leq \int_{|\eta|>\delta t} |\hat{\Phi}(\eta)|d\eta \\ &\lesssim t^{-N} = |\xi|^{-N} \end{aligned}$$

if Φ is sufficiently smooth. On the other hand, let $\xi = te$ with e a unit vector with $|e_2|/|e_1| \geq 3/2$ and $t > 0$. There exists a uniform $\delta > 0$ such that for all η with $|\eta| < \delta t$ we have $\xi - \eta \in \mathcal{C}$ and hence $\chi_{\mathcal{C}}(\xi - \eta) = 1$. Now we can estimate

$$\begin{aligned} |\hat{p}_1(\xi)| = |1 - \hat{p}_0(\xi)| &= \left| \int_{\mathbb{R}^2} \hat{\Phi}(\eta)(1 - \chi_{\mathcal{C}}(\xi - \eta))d\eta \right| \\ &= \left| \int_{|\eta|>\delta t} \hat{\Phi}(\eta)(1 - \chi_{\mathcal{C}}(\xi - \eta))d\eta \right| \lesssim t^{-N} = |\xi|^{-N} \end{aligned}$$

again for Φ smooth. Note that in the first equality we have used that $\Phi(0) = 1$. This proves the statement. □

Lemma 3.2 *We have*

$$\begin{aligned} |\hat{\varphi}_0(\xi)|^2 &\lesssim |\xi|^{-N} \quad \textit{for } |\xi_1|/|\xi_2| \leq 3/2, \\ |\hat{\varphi}_1(\xi)|^2 &\lesssim |\xi|^{-N} \quad \textit{for } |\xi_2|/|\xi_1| \leq 3/2. \end{aligned} \tag{10}$$

Proof This follows from [5, Lemma 4.7]. For the convenience of the reader we present a proof here as well. We only prove the assertion for φ_0 since the proof of the corresponding statement for φ_1 is the same. By definition we have

$$\begin{aligned} |\hat{\varphi}_0(\xi)|^2 = \bigg(&\int_{a\in\mathbb{R},\ |s|>2} |\hat{\psi}\big(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1)\big)|^2 a^{-3/2} dads \\ &+ \int_{a>1,\ |s|<2} |\hat{\psi}\big(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1)\big)|^2 a^{-3/2} dads \bigg). \end{aligned}$$

We start by estimating the second integral using the Fourier decay in the first variable:

$$\begin{aligned} \int_{a>1,\ |s|<2} |\hat{\psi}\big(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1)\big)|^2 a^{-3/2} dads &\lesssim 4 \int_{a>1} (a|\xi_1|)^{-2L_1} a^{-3/2} da \\ &\lesssim |\xi_1|^{-2L_1} \lesssim |\xi|^{-2L_1} \end{aligned}$$

for all ξ in the cone with slope $3/2$. To estimate the other term we need the moment condition and the decay in the second variable. We write $\hat{\psi}(\xi) = \xi_1^M \hat{\theta}(\xi)$ and $\xi = (\xi_1, r\xi_1),\ |r| \le 3/2$. We start with the high frequency part:

$$\begin{aligned}
&\int_{a<1,\ |s|>2} |\hat{\psi}\big(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1)\big)|^2 a^{-3/2} da \\
&\quad = \int_{a<1,\ |s|>2} |a\xi_1|^{2M} |\hat{\theta}\big(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1)\big)|^2 a^{-3/2} dads \\
&\quad \lesssim \int_{a<1,\ |s|>2} |a\xi_1|^{2M} |\sqrt{a}(\xi_2 - s\xi_1)|^{-2L_2} a^{-3/2} dads \\
&\quad = \int_{a<1,\ |s|>2} a^{2M-L_2-3/2} |\xi_1|^{2M-2L_2} |r-s|^{-2L_2} dads
\end{aligned}$$

we now use that $|r-s|$ is always strictly away from zero. By assumption $L_2 = 2M - 1/2 - \varepsilon$ for some $\varepsilon > 0$. Hence we can estimate further

$$\cdots = |\xi_1|^{-2(L_2-M)} \int_{a<1,\ |s|>2} a^{-1+\varepsilon} |r-s|^{-2L_2} dads \lesssim |\xi|^{-2(L_2-M)}.$$

The low-frequency part can simply be estimated as follows:

$$\begin{aligned}
&\int_{a>1,\ |s|>2} |\hat{\psi}\big(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1)\big)|^2 a^{-3/2} dads \\
&\quad \lesssim |\xi_1|^{-2L_2} \int_{a>1,\ |s|>2} a^{-3/2-L_2} |r-s|^{-2L_2} dads \\
&\quad \lesssim |\xi|^{-2L_2}.
\end{aligned}$$

Putting these estimates together proves the statement. □

We now show a locality result for the distributions $(|\hat{\varphi}_i|^2)^\vee$, $i = 0, 1$.

Lemma 3.3 *Assume that ψ is compactly supported with support in the ball $\mathcal{B}_A := \{\xi : |\xi| \le A\}$. Then the tempered distributions $(|\hat{\varphi}_i|^2)^\vee$, $i = 0, 1$ are both of compact support with support in the ball $2\sqrt{3+\sqrt{5}}\mathcal{B}_A$.*

Proof Since the inverse Fourier transform of the constant function C_ψ is a Dirac, this follows if we can establish that the functions Δ_ψ and Δ_{ψ^ν} are Fourier transforms of distributions of compact support. We show this only for Δ_ψ, the other case being similar. In what follows we will use the notation $f^-(x) := f(-x)$ for a function f. Since ψ has M anisotropic moments we can write $\psi = \big(\frac{\partial}{\partial x_1}\big)^M \theta$ for some $\theta \in L^2(\mathbb{R}^2)$ with the same support as ψ. Consider the function $\Theta(\xi) := |\hat{\theta}(\xi)|^2$. It is easy to see that this is the Fourier transform of the so-called Autocorrellation function $\theta * \theta^-$ of θ which is compactly supported with support in $\mathcal{B}_{2A}$. Therefore, by the theorem of

Paley-Wiener, the function Θ possesses an analytic extension to $\mathbb{C}^2$ which we will henceforth call F. Furthermore, by the same theorem F is of exponential type:

$$|F(\zeta)| \lesssim (1+|\zeta|)^K \exp(2A|\Im\zeta|), \tag{11}$$

where $\zeta \in \mathbb{C}^2$ and $K \in \mathbb{N}$. We now consider the analytic extension Γ of the function Δ_ψ which is given by

$$\Gamma(\zeta) = \int_{-2}^{2}\int_0^1 (a\zeta_1)^{2M} F(a\zeta_1, a^{1/2}(\zeta_2 - s\zeta_1))a^{-3/2}dads, \quad \zeta \in \mathbb{C}^2.$$

Since $M \geq 1$ by (3) the above integral is locally integrable which implies that Γ is actually an entire function. It is also of exponential type: Writing $M_{as} := \begin{pmatrix} a & 0 \\ -sa^{1/2} & a^{1/2}\end{pmatrix}$ a short computation reveals that

$$\|M_{as}\|_{L^2(\mathbb{C}^2)\to L^2(\mathbb{C}^2)} \leq a^{1/2}\Big(1+\frac{s^2}{2}+(s^2+\frac{s^2}{4})^{1/2}\Big)^{1/2} =: a^{1/2}C(s).$$

Now we estimate

$$\begin{aligned} |\Delta_\psi(\zeta)| &= \left|\int_{-2}^{2}\int_0^1 (a\zeta_1)^{2M} F(M_{as}\zeta)a^{-3/2}dads\right| \\ &\lesssim |\zeta|^{2M}(1+|M_{as}\zeta|)^K \exp(|M_{as}\Im\zeta|) \\ &\lesssim \sup_{s\in[-2,2]} (1+|\zeta|)^{2M+K}\exp(2AC(s)|\Im\zeta|) \\ &\lesssim (1+|\zeta|)^{2M+K}\exp(2AC(2)|\Im\zeta|). \end{aligned}$$

By the Theorem of Paley-Wiener-Schwartz [8] it follows that Δ_ψ is of compact support with support in $\mathcal{B}_{2\sqrt{3+\sqrt{5}}A}$. □

3.2 Main Result

We are now ready to prove our main result, the local representation formula in Theorem 3.4 which is similar to (1) only with local frequency projections (given by convolution with p_0, p_1) and with possibly compactly supported shearlets. In addition, in Theorem 3.6 we show that in a way this is the simplest representation that one can achieve with shearlets – the local frequency projections are necessary in order to wind up with useful systems.

From now on we shall assume that $0 \leq \hat{\Phi}(\xi) \leq 1$ for all $\xi \in \mathbb{R}^2$ and define

$$\hat{q}_i(\xi) := \hat{p}_i(\xi)^{1/2}.$$

Furthermore, we define

$$\hat{\varphi}(\xi) := (\hat{p}_0(\xi)|\varphi_0(\xi)|^2 + \hat{p}_1(\xi)|\varphi_1(\xi)|^2)^{1/2}.$$

Observe that by the positivity assumption above the radicands in the previous definitions are nonnegative and real.

Theorem 3.4 *We have the representation formulas*

$$\|f\|_2^2 = \frac{1}{C_\psi}\Big(\int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 |\langle f, q_0 * \psi_{ast}\rangle|^2 a^{-3} dadsdt + \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 |\langle f, q_1 * \psi_{ast}^\nu\rangle|^2 a^{-3} dadsdt + \int_{\mathbb{R}^2} |\langle f, T_t\varphi\rangle|^2 dt\Big) \tag{12}$$

and

$$f = f^{\mathrm{high}} + f^{\mathrm{low}} := \frac{1}{C_\psi}\Big(\int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 \langle f, \psi_{ast}\rangle p_0 * \psi_{ast} a^{-3} dadsdt + \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 \langle f, \psi_{ast}^\nu\rangle p_1 * \psi_{ast} a^{-3} dadsdt\Big) + \frac{1}{C_\psi}\Big(\int_{\mathbb{R}^2} \langle f, T_t\varphi_0\rangle p_0 * T_t\varphi_0 dt + \int_{\mathbb{R}^2} \langle f, T_t\varphi_1\rangle p_1 * T_t\varphi_1 dt\Big). \tag{13}$$

The function f^{low} satisfies

$$(f^{\mathrm{low}})^\wedge(\xi) \lesssim |\xi|^{-N} \tag{14}$$

for any f. We have the following locality property: *If ψ is compactly supported in $\mathcal{B}_A$ and Φ has support in $\mathcal{B}_B$, then $f^{\mathrm{low}}(t)$ only depends on f restricted to $t + \mathcal{B}_{2\sqrt{3+\sqrt{5}}A+B}$.*

Proof We first prove (13). Taking the Fourier transform of the right hand side yields

$$\frac{1}{C_\psi}\big(\hat{p}_0(\xi)(\Delta_\psi(\xi) + |\hat{\varphi}_0(\xi)|^2) + \hat{p}_1(\xi)(\Delta_{\psi^\nu}(\xi) + |\hat{\varphi}_1(\xi)|^2)\big)\hat{f}(\xi) = \hat{f}(\xi).$$

The proof of (12) is similar. We prove (14): Again taking the Fourier transform of f^{low} gives

$$\frac{1}{C_\psi}\big(\hat{p}_0(\xi)|\hat{\varphi}_0(\xi)|^2 + \hat{p}_1(\xi)|\hat{\varphi}_1(\xi)|^2\big)\hat{f}(\xi).$$

Now, the desired estimate follows from Lemmas 3.1 and 3.2. The last statement follows from the observation that f^{low} can be written as

$$f^{\mathrm{low}} = \frac{1}{C_\psi} f * \big(p_0 * (|\hat{\varphi}_0|^2)^\vee + p_1 * (|\hat{\varphi}_1|^2)^\vee\big)$$

together with Lemma 3.3. □

Remark 3.5 While we could show that the functions $\varphi_i^- * \varphi_i$ and p_i, $i = 1, 2$ are compactly supported, it would be desirable to show that also the functions φ_i, q_i, $i = 1, 2$ have compact support. We currently do not know how to do this. By a result of Boas and Kac [2], in the univariate case there always exists for any compactly supported function g with positive Fourier transform a compactly supported function f with $f * f^- = g$. However, in dimensions ≥ 2 this holds no longer true and things become considerably more difficult.

The previous theorem for the first time gives a completely local representation for square integrable functions which also allows to handle directional phenomena efficiently: by the results of [5] it follows that the decay rate of the coefficients $\langle f, \psi_{ast}\rangle$ for $a \to 0$ accurately describes the microlocal smoothness of f at t in the direction with slope s.

3.3 On the Optimality of the Representation

It is interesting to ask if the frequency projections given by convolution with p_0, p_1 are really necessary, or in other words if it is possible to construct tight frame systems $(T_t\varphi)_{t\in\mathbb{R}^2} \cup (\psi_{ast})_{a\in[0,1],s\in[-2,2],t\in\mathbb{R}^2} \cup (\psi^\nu_{ast})_{a\in[0,1],s\in[-2,2],t\in\mathbb{R}^2}$ for $L^2(\mathbb{R}^2)$. We show that this is actually impossible, meaning that in a sense Theorem 3.4 is the best we can do.

Theorem 3.6 *Assume that $\psi = (\frac{\partial}{\partial x_1})^M\theta$ is a shearlet with $M \geq 1$ vanishing moments in x_1-direction such that either $M > 1$ or $M = 1$ and $\hat{\theta} \in L^\infty(\mathbb{R}^2)$. Furthermore we assume and $L_1 > 0$, $L_2 > M$ with L_1, L_2 defined as in Lemma 3.2. Then there does not exist a window function φ such that*

$$\lim_{\xi\to\infty} \hat{\varphi}(\xi) = 0$$

and such that the system

$$(T_t\varphi)_{t\in\mathbb{R}^2} \cup (\psi_{ast})_{a\in[0,1],s\in[-2,2],t\in\mathbb{R}^2} \cup (\psi^\nu_{ast})_{a\in[0,1],s\in[-2,2],t\in\mathbb{R}^2}$$

constitutes a tight frame for $L^2(\mathbb{R}^2)$, which means that a representation formula

$$\|f\|_2^2 = \frac{1}{C}\left(\int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 |\langle f, \psi_{ast}\rangle|^2 a^{-3} dadsdt \right.$$
$$\left. + \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 |\langle f, \psi^\nu_{ast}\rangle|^2 a^{-3} dadsdt + \int_{\mathbb{R}^2} |\langle f, T_t\varphi\rangle|^2 dt\right) \tag{15}$$

holds for all $f \in L^2(\mathbb{R}^2)$ and some constant C.

Proof In terms of the Fourier transform, (15) translates to

$$\Delta_\psi(\xi) + \Delta_{\psi^\nu}(\xi) + |\hat{\varphi}(\xi)|^2 = C.$$

If we assume that φ has Fourier decay $\lim_{|\xi|\to\infty}\hat{\varphi}(\xi)=0$ this would imply that

$$\lim_{|\xi|\to\infty}\big(\Delta_\psi(\xi)+\Delta_{\psi^\nu}(\xi)\big)=C.$$

Lemma 3.2 implies that $\lim_{|\xi|\to\infty}\Delta_\psi(\xi)=\lim_{|\xi|\to\infty}\Delta_{\psi^\nu}(\xi)=C_\psi$ for $\frac{2}{3}\le\frac{|\xi_1|}{|\xi_2|}\le\frac{3}{2}$. It follows that $C=2C_\psi$. On the other hand, using the moment condition $\hat{\psi}=\xi_1^M\hat{\theta}$ for $M=1$ and $\hat{\theta}\in L^\infty(\mathbb{R}^2)$, we have the following estimate for Δ_ψ and ξ in the strip $\mathcal{S}_\delta:=\{\xi:|\xi_1|\le\delta\}$:

$$|\Delta_\psi(\xi)|\le\delta^2\int_{-2}^{2}\int_0^1 a^2\|\hat{\theta}\|_\infty a^{-3/2}dads\le 8\|\hat{\theta}\|_\infty\delta^2.$$

If we assume that $M>1$, we can write $\hat{\psi}(\xi)=\xi_1^{M-1}\mu(\xi)$ where μ is still a shearlet. We get a similar estimate as above:

$$|\Delta_\psi(\xi)|\le\delta^{2(M-1)}\int_{-2}^{2}\int_0^1|\hat{\mu}(a\xi_1,a^{1/2}(\xi_2-s\xi_1))|^2a^{-3/2}dads\le C_\mu\delta^{2(M-1)}.$$

At any rate, by choosing δ appropriately small, this implies that for $\xi\in\mathcal{S}_\delta$ we have

$$C=\lim_{|\xi|\to\infty}\big(\Delta_\psi(\xi)+\Delta_{\psi^\nu}(\xi)\big)<C,$$

which gives a contradiction. □

It is easy to extend this argument to show that there do not exist shearlet frames such that there exists a dual frame which also has the structure of a shearlet system. By this we mean the existence of functions $\varphi,\tilde{\varphi},\psi,\tilde{\psi}$ such that

$$\begin{aligned}\|f\|_2^2=\frac{1}{C}\bigg(&\int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1\langle f,\psi_{ast}\rangle\langle\tilde{\psi}_{ast},f\rangle a^{-3}dadsdt\\&+\int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1\langle f,\psi_{ast}^\nu\rangle\langle\tilde{\psi}_{ast}^\nu,f\rangle a^{-3}dadsdt\\&+\int_{\mathbb{R}^2}\langle f,T_t\varphi\rangle\langle T_t\tilde{\varphi},f\rangle dt\bigg)\end{aligned}\tag{16}$$

holds for all $f\in L^2(\mathbb{R}^2)$ and some constant C. Indeed, this would lead to the equality

$$\Delta_{\psi,\tilde{\psi}}(\xi)+\Delta_{\psi^\nu,\tilde{\psi}^\nu}(\xi)+\hat{\varphi}(\xi)\overline{\hat{\tilde{\varphi}}(\xi)}=C,$$

where

$$\Delta_{\psi,\tilde{\psi}}(\xi)=\int_{-2}^{2}\int_0^1\hat{\psi}(a\xi_1,a^{1/2}(\xi_2-s\xi_1))\overline{\hat{\tilde{\psi}}(a\xi_1,a^{1/2}(\xi_2-s\xi_1))}a^{-3/2}dads.$$

Now the same arguments as in the proof of Theorem 3.6, with C_ψ replaced by

$$C_{\psi,\tilde{\psi}} := \int_{\mathbb{R}^2} \frac{\hat{\psi}(\omega)\overline{\hat{\tilde{\psi}}(\omega)}}{|\omega_1|^2} d\omega$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty} \hat{\psi}(a\xi_1, a^{1/2}(\xi_2 - s\xi_1))\overline{\hat{\tilde{\psi}}(a\xi_1, a^{1/2}(\xi_2 - s\xi_1))}a^{-3/2} da ds,$$

and Δ_ψ by $\Delta_{\psi,\tilde{\psi}}$, lead to the following result:

Theorem 3.7 *Assume that* $\psi = \left(\frac{\partial}{\partial x_1}\right)^M \theta$, $\tilde{\psi} = \left(\frac{\partial}{\partial x_1}\right)^M \tilde{\theta}$ *are shearlets with* $M \geq 1$ *vanishing moments in* x_1*-direction such that either* $M > 1$ *or* $M = 1$ *and* $\hat{\theta}, \hat{\tilde{\theta}} \in L^\infty(\mathbb{R}^2)$. *Furthermore we assume and* $L_1 > 0$, $L_2 > M$ *with* L_1, L_2 *defined as in Lemma 3.2. Then there do not exist window functions* $\varphi, \tilde{\varphi}$ *such that*

$$\lim_{\xi\to\infty} \hat{\varphi}(\xi) = 0, \quad \lim_{\xi\to\infty} \hat{\tilde{\varphi}}(\xi) = 0$$

and such that (16) holds.

4 Concluding Remarks

In future work we would like to address the problem of constructing continuous tight frames for the so-called 'Hart Smith Transform' [3, 12] where the shear operation is replaced with a rotation. We think that in this case the results might become simpler. One reason for this is that in this case no (smoothed) projection onto a frequency cone is needed. In view of constructing discrete tight frames we think that a simple discretization of a continuous tight frame will not work for non-bandlimited shearlets. The approach that we are currently pursuing in this direction is to construct so-called Shearlet MRA's via specific scaling functions and to try to generalize the 'unitary extension principle' of Ron and Shen [11] to the shearlet setting [6].

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