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Embedding of two-colored right-angled Coxeter groups into products of two binary trees

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Abstract. We prove that every finitely generated 2-colored right-angled Coxeter group Γ can be quasi-isometrically embedded into the product of two binary trees. Moreover we show that the natural extension of this embedding to n-colored groups is not for every group quasi-isometric.

1. Introduction

A group Γ with a finite generator set S is a right-angled Coxeter group iff all elements s of S have order two and all relations in Γ are composed of relations of the form $st = ts$ with $s, t \in S$.

We call a group Γ *n-colored* iff the chromatic number of Γ is n, i.e. we need at least n colors to color the elements of S so that if $s, t \in S$ commute, they have different colors.

We want to discuss embedding results of the Cayley graph $C(\Gamma, S)$ into products of binary trees. On graphs and trees we consider the simplicial metric, i.e. every edge has length 1. On a product of trees we consider the l_1-metric, so the distance in the product is the sum of the distances in the factors.

In [1] it was proven that the Cayley graph of a finitely generated n-colored right-angled Coxeter group admits a bilipschitz embedding into a product of n locally finite trees.

In [2] this result was improved for the hyperbolic case. So if a Coxeter group is additionally hyperbolic, then there exists a quasi-isometric embedding of it into a product of n binary trees, which are trees of finite valence.

Now we modify this embedding and achieve that the hyperbolicity is not necessary in the two-colored case:

Theorem 1.1. *Let Γ be a finitely generated two-colored right-angled Coxeter group and T be a binary tree. Then there exists a quasi-isometric embedding $\psi : C(\Gamma, S) \to T \times T$ of it into the product of two binary trees.*

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Unfortunately the embedding method we use in the proof does not work for higher chromatic numbers:

Theorem 1.2. *The natural extension of the introduced map ψ to the n-colored case with $n > 2$ is not for every group quasi-isometric.*

It is an open question whether there exists a quasi-isometric embedding of a finitely generated right-angled Coxeter group with chromatic number $n>2$ into a product of trees of finite valance or if the embedding into a product of locally compact trees is already optimal.

This paper will form a part of my Ph.D. thesis. I would like to express my deepest gratitude to my advisor, Prof. Dr. Viktor Schroeder, for his support and guidance throughout the research.

2. Preliminaries

In this section we will give an outline of basic facts about Coxeter groups, Cayley graphs and rooted trees. Because we need nearly the same basics, we repeat the necessary definitions and properties from [2] without proofs and additional remarks. Moreover, in Sect. 2.3 a square-free sequence will be described, which we need for modifying of the embedding.

2.1. Right-angled Coxeter groups

A *Coxeter matrix* $(m_{s,t})_{s,t\in S}$ is a symmetric $S \times S$ matrix with 1 s on the diagonal and nonnegative integers different from 1 elsewhere. A Coxeter matrix defines a *Coxeter group* Γ generated by the set S with relations $(st)^{m_{s,t}} = 1$ for all $s, t \in S$. Here $m_{s,t} = 0$ means: there is no relation between s and t. A Coxeter group Γ is *finitely generated* if S is finite. The group Γ is called *right-angled* if all entries of the corresponding Coxeter matrix are 0, 1, 2.

A right-angled Coxeter group Γ with generating set S is completely described by a graph with vertex set S, where we connect two vertices s and t iff $m_{s,t} = 2$.

The *chromatic number* of the group Γ is defined as the chromatic number of the corresponding graph, which is the minimal number of colors needed to color the vertices in such a way that the adjacent vertices have different colors.

Assume that the chromatic number of the group Γ equals n (we say also Γ is n-colored). If we denote the colors with letters $a, b, c, \ldots$, then we can write S as a disjoint union $S = S_a \bigsqcup S_b \bigsqcup S_c, \ldots$, where $S_a \subset S$ are the elements with color a and so on. Note that two elements of the same color do not commute, because two elements s and t commute iff $m_{s,t} = 2$.

Remark 2.1. We can easily see that there exists one-to-one correspondence between right-angled Coxeter groups and undirected graphs. The set of two-colored right-angled Coxeter groups is, further, bijective to the bipartite undirected graphs, i.e. all the circles in the graph have an even length.

Example 2.2. 1. The corresponding graph to the two-colored right-angled Coxeter group generated by reflections on sides of a square in $\mathbb{R}^2$ is the circle with four vertices.

2. The graph of the two-colored right-angled (hyperbolic) Coxeter group generated by reflection on sides of right-angled hexagon in $\mathbb{H}^2$ is the circle with six vertices.
3. The free product of these two groups with amalgamation on arbitrary two commuting generators, which is a two-colored right-angled non hyperbolic Coxeter group, has as the corresponding graph these two circles glued together at one edge. Note this group does not have an obvious embedding into the product of two binary trees.

In a right-angled Coxeter group Γ two reduced words representing the same element can be transformed in each other only by replacing subwords of the form st by ts if $m_{s,t} = 2$. These words have therefore the same length and are formed from the same set of letters [2]. Therefore the following notations are well defined. The length of a reduced representation W of an element $\gamma \in \Gamma$ is denoted by $\ell(\gamma)$ and is called the *norm* of γ. If a is a color and $\gamma \in \Gamma$, then we denote by $\ell_a(\gamma)$ the number of letters with color a in a reduced representation of γ and we have $\ell_a(\gamma) + \ell_b(\gamma) + \ell_c(\gamma) + \cdots = \ell(\gamma)$.

The Cayley graph $C(\Gamma, S)$ is a geodesic metric space with the *word metric* $d(\gamma, \overline{\gamma}) = \ell(\gamma^{-1}\overline{\gamma})$ and *geodesics* between two points $\alpha, \beta \in \Gamma$ given by a sequence $\alpha = \gamma_0, \ldots, \gamma_k = \beta$ with $d(\gamma_i, \gamma_j) = |i - j|$ for all $0 \leq i, j \leq k$.

The following lemma (Lemma 2.3 in [2]) discusses the property of $C(\Gamma, S)$ to span a tripod between any three points. This property is essential for the proof of the main theorem.

Lemma 2.3. *Let Γ be a right-angled Coxeter group and $\alpha, \beta, \gamma \in \Gamma$, then there exists $\delta \in \Gamma$ such that the following holds:*

$$d(\alpha, \delta) + d(\delta, \beta) = d(\alpha, \beta),$$
$$d(\beta, \delta) + d(\delta, \gamma) = d(\beta, \gamma),$$
$$d(\gamma, \delta) + d(\delta, \alpha) = d(\gamma, \alpha).$$

2.2. *Rooted trees*

Let Ω be a finite set. We associate to Ω a rooted simplicial tree T_Ω in the following way: the set of vertices is the set of finite sequences $(\omega_1, \ldots, \omega_k)$ with $\omega_i \in \Omega$. The empty sequence defines the root vertex. Two vertices are connected by an edge in T_Ω if they have the form $(\omega_1, \ldots, \omega_k)$ and $(\omega_1, \ldots, \omega_k, \omega_{k+1})$ with $k \in \mathbb{N}$ and $\omega_1, \ldots, \omega_{k+1} \in \Omega$.

On trees we consider the simplicial metric, i.e. every edge has length 1. We denote by $|v|$ the distance from the root to the vertex v and by $|vw|$ the distance from vertex v to vertex w. On products of trees we consider the l_1-metric, i.e. the distance in the product is equal to the sum of the distances in the factors. The notation for the l_1-metric is analogous to the simplicial metric with $|\cdot|_1$ and $|\cdot\cdot|_1$, respectively.

Because the elements of Ω can be represented by binary sequences of length $\leq \log_2(|\Omega|) + 1$, one can easily prove:

Remark 2.4. The tree $T_{\{0,1\}}$ is also called the *binary tree*. If Ω is finite and $|\Omega| \geq 2$, then T_Ω is quasi-isometric to the binary tree.

2.3. Square-free sequence

In the following we will need a square-free sequence, i.e. a sequence without two consecutive identical segments. Therefore the Morse–Thue sequence will be introduced and modified. This was introduced and studied first by Thue [5] and later independently by Morse [4].

The *Morse–Thue sequence* $(t_i)_{i\in\mathbb{N}_0} = 0110100110010110\ldots$ is defined by $t_i = 0$ if the sum of the binary digits of i is even and $t_i = 1$ otherwise.

The Morse–Thue sequence has a remarkable overlap-free property, see e.g. [3]:

Proposition 2.5. *There is no substring of the form ababa in the Morse–Thue sequence where a and b are words in* 0 *and* 1 *and a is not empty.*

The following sequence is also discovered by Thue and has the desired square-free property. We define $(v_i)_{i\in\mathbb{N}_0} = 21020121012\ldots$ as following: v_i is the number of 1s between the ith and $(i+1)$th 0 in $(t_i)_{i\in\mathbb{N}_0}$. Note that the elements v_i are in $\{0, 1, 2\}$ for all $i \in \mathbb{N}_0$.

Corollary 2.6. *The sequence* $(v_i)_{i\in\mathbb{N}_0}$ *is square-free.*

Proof. Assume there exists a substring ww in $(v_i)_{i\in\mathbb{N}_0}$ with $w = w_1 \cdots w_n$, $n > 0$. Let $a = 0$ and $b = 1^{w_1}01^{w_2}0\cdots01^{w_n}$. Then the string *ababa* is a substring of the Morse–Thue sequence in contradiction to the overlap-free property. □

3. Construction of the embedding ψ

In this section, we consider a finitely generated two-colored right-angled Coxeter group Γ with a generator set S which can be decomposed as $S = S_a \bigsqcup S_b$ such that elements within S_a have the color a and do not commute with each other and the same holds for elements of S_b and color b.

The idea of the map ψ was introduced in [2]. We take the same map ψ'^{dia} with another decoration. For the convenience of the reader, we recall the construction and properties of the map ψ'^{dia} without repeating of the proofs and we then introduce the differing decoration with the modified Morse–Thue sequence $(v_i)_{i\in\mathbb{N}_0}$.

3.1. Canonical representations

Fix a color, say a. The *reduced left a-representation* of $\gamma \in \Gamma$ is a reduced word

$$W = W_1 a_1 W_2 a_2 \cdots W_r a_r W_{r+1}$$

representing γ in which the $W_i = w^i_{r_i} \cdots w^i_1$ are the words with letters in S_b and the last entry of W_i does not commute with a_i, i.e. the letters of color a are moved as left as possible. The reduced left a-representation of γ is unique and thus we call it also the *canonical a-representation* of γ.

We now study in general the situation where we have two canonical a-representations U and V such that the composition UV is a reduced word. We choose the notation such that

$$\begin{aligned} U &= U_1 a_1 \cdots U_p a_p U_{p+1}, \\ V &= V_{p+1} a_{p+1} \cdots V_{p+m} a_{p+m} V_{p+m+1}. \end{aligned}$$

Then let

$$W = W_1 a_1 W_2 a_2 \cdots W_{p+m} a_{p+m} W_{p+m+1}$$

be the canonical a-representation of the word UV, then $W_{p+1} a_{p+1} U_R$ is the canonical a-representation of the word $U_{p+1} V_{p+1} a_{p+1}$, with the word U_R consisting of the letters of $U_{p+1} V_{p+1}$ which can be moved right of a_{p+1}. In [2] the following is proven (Proposition 3.1).

Proposition 3.1. 1. *If $\ell(V_{p+1}) = 0$, then $U_{p+1} = W_{p+1} U_R$.*
2. *If $\ell(V_{p+1}) > 0$, then $U_R = \emptyset$ and $W_{p+1} = U_{p+1} V_{p+1}$ and consequently UV is in the canonical a-representation.*

Consider two elements $\gamma, \overline{\gamma} \in \Gamma$. We investigate the situation where the canonical a-representations of these elements differ. By Lemma 2.3 1, γ and $\overline{\gamma}$ span a tripod. Thus there are words U, V, $\overline{V}$ given in canonical a-representation, such that UV is a reduced representation of γ, $U\overline{V}$ is a reduced representation of $\overline{\gamma}$ and $V^{-1}\overline{V}$ is a reduced representation of $\gamma^{-1}\overline{\gamma}$.

We then write

$$\begin{aligned} U &= U_1 a_1 \cdots U_p a_p U_{p+1}, \\ V &= V_{p+1} a_{p+1} \cdots V_{p+m} a_{p+m} V_{p+m+1}, \\ \overline{V} &= \overline{V}_{p+1} \overline{a}_{p+1} \cdots \overline{V}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{V}_{p+\overline{m}+1}. \end{aligned}$$

Let

$$\begin{aligned} W &= W_1 a_1 \cdots W_{p+m} a_{p+m} W_{p+m+1}, \\ \overline{W} &= \overline{W}_1 \overline{a}_1 \cdots \overline{W}_{p+\overline{m}} \overline{a}_{p+\overline{m}} \overline{W}_{p+\overline{m}+1} \end{aligned}$$

be the canonical a-representations of $\gamma, \overline{\gamma}$. Clearly $W_i a_i = \overline{W}_i \overline{a}_i = U_i a_i$ for $1 \le i \le p$ and furthermore, the following is shown in [2], Lemma 3.2.

Lemma 3.2. $W_{p+1} a_{p+1} \neq \overline{W}_{p+1} \overline{a}_{p+1}$.

3.2. Diary map

Let $\kappa \in \mathbb{N}$. Define $E = S_a \cup S_b \cup \{\varnothing\}$. Let $T_a^{\prime\text{dia}} = T_{E^\kappa \times S_a}$ be the rooted tree with label set $E^\kappa \times S_a$. The tree $T_a^{\prime\text{dia}}$ is called *"augmented" diary tree*. In the following, we define a map $\psi_a^{\prime\text{dia}} : C(\Gamma, S) \to T_a^{\prime\text{dia}}$. For trees and maps we keep the notations from [2].

Let $\gamma \in \Gamma$ be given by the canonical a-representation

$$W = W_1 a_1 \cdots W_r a_r W_{r+1}.$$

By $W|_i$ the i*-cut* is denoted

$$W|_i = W_1 a_1 \cdots a_{i-1} W_i.$$

Recall that the vertices of the tree $T_a^{\prime\text{dia}}$ are finite sequences of elements of $E^\kappa \times S_a$. We define $\psi_a^{\prime\text{dia}}(\gamma) = ((\alpha_1, a_1), \ldots, (\alpha_r, a_r))$ as a sequence of length r by induction on i.

Let α_1 be the string of the last κ symbols in the chain

$$\varnothing \cdots \varnothing W_1,$$

considered as a word in the alphabet E. We assume here that we have enough symbols $\varnothing$ in front. We define α_i as the string of the last κ symbols of the word in the alphabet E that is obtained from the word

$$\varnothing \cdots \varnothing W|_i$$

by removing the letters from $\bigcup_{j<i} \alpha_j$.

We collect the maps $\psi_a^{\prime\text{dia}}$ and the corresponding $\psi_b^{\prime\text{dia}} : C(\Gamma, S) \to T_b^{\prime\text{dia}} = T_{E^\kappa \times S_b}$ to a common map

$$\psi^{\prime\text{dia}} : C(\Gamma, S) \to T_a^{\prime\text{dia}} \times T_b^{\prime\text{dia}}.$$

Now a general reconstruction result can be proven. This is a slight modification of the Proposition 3.5 in [2] and the proof is almost the same.

Proposition 3.3. *Let* $W = W_1 a_1 \cdots W_{j+r} a_{j+r} W_{j+r+1}$ *be a canonical a-representation and assume that*

$$k := \kappa(r+1) - \ell(W_{j+1} a_{j+1} \cdots W_{j+r}) \geq 1.$$

Then one of the following holds:

1. We can reconstruct the subword $W_j a_j$ from the diary entries $((\alpha_j, a_j), \ldots, (\alpha_{j+r}, a_{j+r}))$.
2. The word $W_j a_j$ has $\geq k$ letters and we can reconstruct from $((\alpha_j, a_j), \ldots, (\alpha_{j+r}, a_{j+r}))$ the last k letters of $W_j a_j$.

However, the modification of the diary map is not complete. It does not fulfill the requirements yet [2].

Remark 3.4. The map $\psi^{\prime\text{dia}}$ is 1-Lipschitz but it is not necessarily quasi-isometric.

This is shown in [2], in particular in Remark 3.8 it is given a counter-example and it is observed, that the essential problem is the periodicity within certain words. To exclude this, a decoration with the Morse–Thue sequence is used in the original construction. Because this unfortunately does not work in our case, we will use a different decoration with the square-free sequence $(v_i)_{i\in\mathbb{N}_0}$.

3.3. *Square-free decoration of the diary map*

Let Ω be a set, we define the decorated set $\widetilde{\Omega} = \cup_{\omega\in\Omega}\{\omega^0, \omega^1, \omega^2\}$, note that the elements of the square-free sequence $(v_i)_{i\in\mathbb{N}_0}$ are in $\{0, 1, 2\}$. Let $E_d = \widetilde{S} \cup \{\varnothing\}$ and let T_a be the rooted tree with label set $E_d^\kappa \times \widetilde{S}_a$.

We define the map ψ_a from $C(\Gamma, S)$ into the tree T_a in the following way. As above we take the canonical a-representation W of an element $\gamma \in \Gamma$ and decorate the occurrences of every letter by $(v_i)_{i\in\mathbb{N}_0}$, i.e. for every letter $s \in S$, the ith occurrence of the letter s gets the decoration v_i, so we get s^{v_i}. This can be better seen in an example.

Example 3.5. We decorate the word $a_1a_2a_1b_1a_2b_1b_2b_1a_1b_1b_2b_1a_2b_1$ in two colors a and b. To simplify matters, we first decorate the occurrence of one generator, e.g. b_1:

$$a_1a_2a_1b_1^2a_2b_1^1b_2b_1^0a_1b_1^2b_2b_1^0a_2b_1^1.$$

Then we decorate the rest and receive:

$$a_1^2a_2^2a_1^1b_1^2a_2^1b_1^1b_2^2b_1^0a_1^0b_1^2b_2^1b_1^0a_2^0b_1^1.$$

After that we apply the diary map $\psi_a'^{\mathrm{dia}}$ to the decorated word.

Analogously we define $T_b = T_{E_d^\kappa\times\widetilde{S}_b}$ and the map $\psi_b : C(\Gamma, S) \to T_b$. Let

$$\psi = (\psi_a, \psi_b) : C(\Gamma, S) \to T_a \times T_b.$$

This map fulfills our requirements, which will be proven in the next part.

4. Proof of Theorem 1.1

Now we prove that ψ is quasi-isometric. Since the trees T_a and T_b are quasi-isometric to binary trees, see Remark 2.4, Theorem 1.1 results from the following:

Theorem 4.1. *Let* Γ *be a finitely generated two-colored right-angled Coxeter group and let* $\kappa \geq 16$. *Then the map* ψ *is quasi-isometric.*

Proof. To prove this we need to show two inequalities. One of them follows from the 1-Lipschitz property of ψ. The proof of this is the same as the proof of this property for ψ'^{dia} in [2] (Lemma 3.6).

It is left to prove the lower estimate:

$$\exists\lambda_1, \lambda_2 \geq 0\ \forall\gamma, \overline{\gamma} \in \Gamma : \quad \frac{1}{\lambda_1}d(\gamma, \overline{\gamma}) - \lambda_2 \leq |\psi(\gamma)\psi(\overline{\gamma})|_1.$$

Let be $\gamma \neq \overline{\gamma} \in \Gamma$ and $c := d(\gamma, \overline{\gamma})$. For corresponding words we use the notations as in Lemma 3.2, only with the difference, that we now consider square-free decorated words $U, V, \overline{V}, W, \overline{W}$. We will show that if $c \geq 4$ then $|\psi(\gamma)\psi(\overline{\gamma})|_1 \geq \frac{c}{12}$. Clearly the claim implies the desired estimate.

We can assume without loss of generality that in a reduced representation of $\gamma^{-1}\overline{\gamma}$ the number of letters with color a is larger or equal than the number of letters with color b. This means that $m + \overline{m} \geq \frac{c}{2}$.

We first show that the claim is true in the case $m < \frac{c}{6}$ or $\overline{m} < \frac{c}{6}$. We may assume that $m < \frac{c}{6}$. Since $m + \overline{m} \geq \frac{c}{2}$ we have $\overline{m} - m \geq \frac{c}{6}$.

Since $\psi_a(\gamma)$ has by definition $\ell_a(\gamma)$ terms, we have

$$|\psi_a(\overline{\gamma})\psi_a(\gamma)| \geq |\psi_a(\overline{\gamma})| - |\psi_a(\gamma)| = \ell_a(\overline{\gamma}) - \ell_a(\gamma) = (p + \overline{m}) - (p + m)$$
$$= \overline{m} - m \geq \frac{c}{6}$$

and hence $|\psi_a(\gamma)\psi_a(\overline{\gamma})| \geq \frac{c}{12}$. Thus we can assume for the rest of the proof that $m \geq \frac{c}{6}$ and $\overline{m} \geq \frac{c}{6}$.

We prove the claim by contradiction, so assume $|\psi_a(\gamma)\psi_a(\overline{\gamma})| < \frac{c}{12}$. This assumption together with $m \geq \frac{c}{6}$ and $\overline{m} \geq \frac{c}{6}$ implies

$$\psi_a(W|_{p+r}a_{p+r}) = \psi_a(\overline{W}|_{p+r}\overline{a}_{p+r}) \text{ for some } r \geq \frac{c}{8}. \tag{1}$$

In particular for the diary entries this means

$$\psi_a^{\text{dia}}(W|_{p+r}a_{p+r}) = (\alpha_1, \ldots, \alpha_{p+r}) = (\overline{\alpha}_1, \ldots, \overline{\alpha}_{p+r}) = \psi_a^{\text{dia}}(\overline{W}|_{p+r}a_{p+r}).$$

In the case $V_{p+1} = \emptyset = \overline{V}_{p+1}$ we can directly see a contradiction to the Eq. (1). Because $V^{-1}\overline{V}$ is reduced we know that $a_{p+1} \neq \overline{a}_{p+1}$ and hence $\psi_a(W|_{p+1}a_{p+1}) \neq \psi_a(\overline{W}|_{p+1}\overline{a}_{p+1})$, by definition of the map $\psi_a'^{\text{dia}}$.

So we study the case $V_{p+1} \neq \emptyset$ or $\overline{V}_{p+1} \neq \emptyset$. Without loss of generality $\ell(\overline{V}_{p+1}) \geq \ell(V_{p+1})$, so $\overline{V}_{p+1} \neq \emptyset$. In this case $\overline{W}_j = \overline{V}_j$ for all $j \geq p+2$ by Proposition 3.1(2). Therefore

$$\ell(\overline{W}_{p+2}\overline{a}_{p+2} \cdots \overline{W}_{p+r}) \leq \ell(\overline{V}) \leq \ell(V^{-1}\overline{V}) = c$$

and hence (since $\kappa \geq 16$ and $r \geq \frac{c}{8}$)

$$k = \kappa r - \ell(\overline{W}_{p+2}\overline{a}_{p+2} \cdots \overline{W}_{p+r}) \geq c.$$

By Proposition 3.3 we can reconstruct either the whole word $\overline{W}_{p+1}\overline{a}_{p+1}$ or at least the last k letters of this word from $((\overline{\alpha}_{p+1}, \overline{a}_{p+1}), \ldots, (\overline{\alpha}_{p+r}, \overline{a}_{p+r}))$. We prove then here the second holds. Assume we can reconstruct the whole word. Then since

$$((\alpha_{p+1}, a_{p+1}), \ldots, (\alpha_{p+r}, a_{p+r})) = ((\overline{\alpha}_{p+1}, \overline{a}_{p+1}), \ldots, (\overline{\alpha}_{p+r}, \overline{a}_{p+r})),$$

we have $W_{p+1}a_{p+1} = M\overline{W}_{p+1}\overline{a}_{p+1}$, where M is a word with only b-colored letters. If M is empty, then we have $W_{p+1}a_{p+1} = \overline{W}_{p+1}\overline{a}_{p+1}$, in contradiction to Lemma 3.2. Otherwise it follows

$$\ell(W_{p+1}) > \ell(\overline{W}_{p+1}). \tag{2}$$

On the other hand $\overline{W}_{p+1} = U_{p+1}\overline{V}_{p+1}$ by Proposition 3.1 and $W_{p+1} = U_{p+1}V_{p+1}$ or $W_{p+1} = U'_{p+1}$ for $V_{p+1} = \emptyset$, U'_{p+1} denotes U_{p+1} without U_R. Then, $\ell(U'_{p+1}) \leq \ell(U_{p+1}V_{p+1}) \leq \ell(U_{p+1}\overline{V}_{p+1})$, since $\ell(\overline{V}_{p+1}) \geq \ell(V_{p+1})$. So $\ell(W_{p+1}) \leq \ell(\overline{W}_{p+1})$. This is a contradiction to (2).

It follows that $W_{p+1}a_{p+1}$ and $\overline{W}_{p+1}\overline{a}_{p+1}$ have at least $k \geq c$ letters and the last k letters of the words coincide. Since

$$\ell(\overline{V}_{p+1}) < \ell(\overline{V}_{p+1}\overline{a}_{p+1}) \leq \ell(\overline{V}) = c \leq k,$$

the last $\ell(\overline{V}_{p+1})$ letters of W_{p+1} and $\overline{W}_{p+1}$ are $\overline{V}_{p+1}$, more precisely $\overline{W}_{p+1} = U_{p+1}\overline{V}_{p+1}$ and $W_{p+1} = M\overline{V}_{p+1}$with a word M of color b.

If $V_{p+1} = \emptyset$, we have

$$U_{p+1} = W_{p+1}U_R = M\overline{V}_{p+1}U_R.$$

Hence

$$\overline{W}_{p+1} = U_{p+1}\overline{V}_{p+1} = M\overline{V}_{p+1}U_R\overline{V}_{p+1}.$$

Let $\overline{V}_{p+1} = b_1 \cdots b_n$ with $b_i \in S_b$, $1 \leq i \leq n$. Now we look at the square-free decoration of the letter b_n in $\overline{W}_{p+1}$. In U_R there are no letters b_n because b_n does not commute with $\overline{a}_{p+1}$, since $\overline{V}$ is in the canonical a-representation, and on the other hand $a_{p+1} = \overline{a}_{p+1}$ commutes with all letters in U_R, by the definition of U_R. Let the word d be the v_i-decoration of appearances of the generator b_n in $\overline{V}_{p+1}$, then the word dd is a substring of $(v_i)_{i\in\mathbb{N}_0}$ in contradiction to the square-free property.

If $V_{p+1} \neq \emptyset$, we have by Proposition 3.1 that $\overline{W}_{p+1} = U_{p+1}\overline{V}_{p+1}$ and $W_{p+1} = U_{p+1}V_{p+1}$. Since the last $k-1 \geq \ell(\overline{V}_{p+1}) \geq \ell(V_{p+1})$ letters of $\overline{W}_{p+1}$ and W_{p+1} coincide, there exists a word H from color b with $\overline{V}_{p+1} = HV_{p+1}$. Since

$$U_{p+1}V_{p+1} = W_{p+1} = M\overline{V}_{p+1} = MHV_{p+1},$$

we have

$$U_{p+1} = MH.$$

Thus

$$\overline{W}_{p+1} = U_{p+1}\overline{V}_{p+1} = MHHV_{p+1}.$$

In the word HH we have v_i-decoration $d'd'$ of an arbitrary letter b in H, note H is not empty, otherwise $\overline{V}_{p+1} = V_{p+1}$. Hence, $d'd'$ is a substring of $(v_i)_{i\in\mathbb{N}_0}$, in contradiction to the square-free property. □

5. Counter-example for more colors

Now it is reasonable to try to extend the embedding to more than two colors. For this we only need to extend the definition of the canonical representation to n colors. The remaining procedure of embedding does not depend on the number of colors.

For the canonical a-representation of a word we move all the letters of color a as left as possible. If we have more than two colors, this representation is not unique, we have to determine the order of letters in the subwords between the a-letters. One option to do this is to move all b-letters in the subwords as left as possible, then the c-letters in the subsubwords and so on. Another, more elaborate, possibility to define the order in the subwords was introduced in [2] for the proof of the hyperbolic case with $n \geq 3$.

However, the following example does not depend on the order of letters in the subwords, so we can apply it to the extension of ψ with an arbitrary canonical representation.

Now we show that this extension is not necessarily quasi-isometric.

Proposition 5.1. *There exists a finitely generated three-colored right-angled Coxeter group* Γ *with generator set* S *such that* $\psi : C(\Gamma, S) \to T_a \times T_b \times T_c$ *is not quasi-isometric.*

Proof. We first explain the idea of the counter-example and afterwards we look at a concrete group with the required properties.

Thus consider a finitely generated 3-colored right-angled Coxeter group with the generator set S. We denote the colors of S by a, b and c. Consider two words $\gamma, \overline{\gamma} \in \Gamma$ with $\gamma = BCa_1A$ and $\overline{\gamma} = B\overline{B}Ca_1A$ where $a_1, A, B, \overline{B}, C$ are chosen by the following:

(a) The word A is formed by a-letters, a_1 is an a-letter.
(b) The words B and $\overline{B}$ have the color b and the last κ letters of B and $B\overline{B}$ coincide even in the square-free decoration. The last letter of B (and $\overline{B}$) does not commute with a_1.
(c) The word C has the color c and it commutes with $a_1, B, \overline{B}$. The last letter of C does not commute with the first letter of A.

Consider additionally that $\ell(\overline{B}) \ll \ell(A)$ and $\ell(A) \ll \ell(C)$. We will show that

$$|\psi(\gamma)\psi(\overline{\gamma})|_1 \ll d(\gamma^{-1}\overline{\gamma}), \tag{3}$$

which is a contradiction to the quasi-isometry property.

We first investigate the diary maps by the colors. The canonical a-representations of the words are the following:

$$\gamma = Ba_1CA, \quad \overline{\gamma} = B\overline{B}a_1CA.$$

The diary entries of a_1 in γ and $\overline{\gamma}$ are the last κ decorated letters of B and $\overline{B}$, respectively, so they are the same by choice of $B, \overline{B}$. The diary entries of A are also the same, because they only reconstruct parts of C and A but not of B or $\overline{B}$ by reason of the length of C. Hence $|\psi_a(\gamma)\psi_a(\overline{\gamma})| = 0$.

The canonical b-representations of the words are the following:

$$\gamma = BD, \quad \overline{\gamma} = B\overline{B}D, \quad \text{where } D \text{ is the word } Ca_1A \text{ in a certain order.}$$

The first $\ell(B)$ entries are the same. Now $\psi_b(\overline{\gamma})$ has $\ell(\overline{B})$ additional entries, so $|\psi_b(\gamma)\psi_b(\overline{\gamma})| = \ell(\overline{B})$.

The canonical c-representations of the words are the following:

$$\gamma = CBa_1A, \quad \overline{\gamma} = CB\overline{B}a_1A.$$

The diary entries of C are the same and therefore $|\psi_c(\gamma)\psi_c(\overline{\gamma})| = 0$.

Altogether we have

$$|\psi(\gamma)\psi(\overline{\gamma})|_1 = |\psi_a(\gamma)\psi_a(\overline{\gamma})| + |\psi_b(\gamma)\psi_b(\overline{\gamma})| + |\psi_c(\gamma)\psi_c(\overline{\gamma})| = \ell(\overline{B}).$$

Now consider the distance between γ and $\overline{\gamma}$:

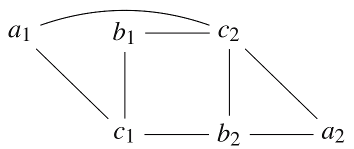
$$d(\gamma, \overline{\gamma}) = \ell(A^{-1}a_1C^{-1}B^{-1}BC\overline{B}a_1A) = \ell(A^{-1}a_1\overline{B}a_1A) = 2\ell(A) + 2 + \ell(\overline{B}).$$

Consequently we can show (3):

$$|\psi(\gamma)\psi(\overline{\gamma})|_1 = \ell(\overline{B}) \ll \ell(A) \leq 2\ell(A) + 2 + \ell(\overline{B}) = d(\gamma^{-1}\overline{\gamma}).$$

More concrete, we can look at a minimal example of such group. Consider the group $\Gamma = \langle a_1, a_2, b_1, b_2, c_1, c_2 \mid a_1^2 = a_2^2 = b_1^2 = b_2^2 = c_1^2 = c_2^2 = 1, [a_1c_1] = [a_1c_2] = [a_2b_2] = [a_2c_2] = [b_1c_1] = [b_1c_2] = [b_2c_1] = [b_2c_2] = 1\rangle$.

We use the following diagram to visualize the relations between the generators (connected elements commute):



And the words we need are:

$$\begin{aligned} A &= a_2a_1a_2 \cdots a_2a_1, \\ B &= b_2b_1b_2 \cdots b_2b_1, \\ \overline{B} &= b_2b_1b_2 \cdots b_2b_1, \\ C &= c_2c_1c_2 \cdots c_2c_1. \end{aligned}$$

Choose the length of $B, \overline{B}$ such that the decoration of the last κ letters of B is equal to the decoration of the last κ letters of $\overline{B}$ (this is possible for combinatorial reasons). The words A and C can be chosen arbitrary long. □

Remark 5.2. By adding further generators and relations we can extend this counter-example also to $n > 3$ colors.

Theorem 1.2 follows from Proposition 5.1 because of this Remark and Remark 2.4.

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