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Differential operators on toric varieties and Fourier transform

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Abstract. We show that Fourier transforms on the Weyl algebras have a geometric counterpart in the framework of toric varieties, namely they induce isomorphisms between twisted rings of differential operators on regular toric varieties, whose fans are related to each other by reflections of one-dimensional cones. The simplest class of examples is provided by the toric varieties related by such reflections to projective spaces. It includes the blow-up at a point of the affine space and resolution of singularities of varieties appearing in the study of the minimal orbit of $\mathfrak{sl}_{n+1}$.

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1. Introduction

Let $\pi\colon \widetilde{\mathbb{A}^n} \to \mathbb{A}^n = \mathbb{C}^n$ ($n \geq 2$) be the blow-up at the origin of the n-dimensional affine space over the complex numbers (or an algebraically closed field of characteristic 0) and let $E = \pi^{-1}(0)$ be the exceptional divisor. Consider the ring $\mathcal{D}_{\mathcal{O}(mE)}(\widetilde{\mathbb{A}^n})$ of global regular differential operators twisted by (i.e., acting on sections of) $\mathcal{O}(mE)$, the line bundle associated with the divisor mE, $m \in \mathbb{Z}$. Our first observation is that this ring is isomorphic to the ring $\mathcal{D}_{\mathcal{O}(m-n)}(\mathbb{P}^n)$ of differential operators on the n-dimensional projective space twisted by the line bundle $\mathcal{O}(m-n)$. To describe the isomorphism we realise both rings as subrings of the ring

$$\mathcal{D}(\mathbb{A}^n) = \mathbb{C}[z_1, \ldots, z_n; \partial_1, \ldots, \partial_n]$$

of differential operators in n variables with polynomial coefficients. Namely, any section σ_{mE} with divisor mE can be used to trivialize the bundle over the complement $\widetilde{\mathbb{A}^n} \smallsetminus E \simeq \mathbb{A}^n \smallsetminus \{0\}$ of E and thus we have an injective restriction map $\mathcal{D}_{\mathcal{O}(mE)}(\widetilde{\mathbb{A}^n}) \hookrightarrow \mathcal{D}(\mathbb{A}^n \smallsetminus \{0\})$. Since $\{0\}$ is of codimension at least 2 in $\mathbb{A}^n$ (recall that $n \geq 2$), we have an isomorphism $\mathcal{D}(\mathbb{A}^n \smallsetminus \{0\}) = \mathcal{D}(\mathbb{A}^n)$. Thus we get a ring monomorphism

$$i_m \colon \mathcal{D}_{\mathcal{O}(mE)}(\widetilde{\mathbb{A}^n}) \hookrightarrow \mathcal{D}(\mathbb{A}^n).$$

Similarly, the bundle $\mathcal{O}(\ell)$ is associated with, say, a multiple of the hyperplane in $\mathbb{P}^n$ defined by the vanishing of some homogeneous coordinate, whose complement is $\mathbb{A}^n$. So again we have an injective restriction map

$$j_\ell \colon \mathcal{D}_{\mathcal{O}(\ell)}(\mathbb{P}^n) \hookrightarrow \mathcal{D}(\mathbb{A}^n).$$

Let finally $F \colon \mathcal{D}(\mathbb{A}^n) \to \mathcal{D}(\mathbb{A}^n)$, the *Fourier transform*, be the ring automorphism such that $F(z_i) = \partial_i, F(\partial_i) = -z_i$.

Theorem 1.1. *Let $m \in \mathbb{Z}$ and $n \geq 2$. Then the Fourier transform restricts to a ring isomorphism*

$$\phi_m \colon \mathcal{D}_{\mathcal{O}(m-n)}(\mathbb{P}^n) \to \mathcal{D}_{\mathcal{O}(mE)}(\widetilde{\mathbb{A}^n}).$$

That is, we have $F \circ j_{m-n} = i_m \circ \phi_m$.

Recall that $\mathbb{P}^n = SL_{n+1}/P$ is a homogeneous space with parabolic isotropy subgroup P and that $\mathcal{O}(\ell)$ is the line bundle associated with a character of P. Thus the results of Borho and Brylinski [3, Theorem 3.8 and Remark 3.9] apply: there is a surjective algebra homomorphism

$$U(\mathfrak{sl}_{n+1}) \to \mathcal{D}_{\mathcal{O}(\ell)}(\mathbb{P}^n),$$

given by the infinitesimal action of the group SL_{n+1} and the kernel is the annihilator J_ℓ of a generalized Verma module.

Corollary 1.2. *There is a surjective algebra homomorphism*

$$U(\mathfrak{sl}_{n+1}) \to \mathcal{D}_{\mathcal{O}(mE)}(\widetilde{\mathbb{A}^n})$$

with kernel J_{m-n}.

The $\mathbb{C}$-algebra $\mathcal{D}_{\mathcal{O}(\ell)}(\mathbb{P}^n)$, viewed as a subalgebra of $\mathcal{D}(\mathbb{A}^n)$, is generated by 1 and the first order operators $\partial_i, z_i\partial_j + \delta_{ij}(e-\ell), z_i(e-\ell)$ given by the action of the fundamental vector fields. Here $e = \sum_i z_i\partial_i$ denotes the Euler vector field.

Corollary 1.3. *The $\mathbb{C}$-algebra $\mathcal{D}_{\mathcal{O}(mE)}(\widetilde{\mathbb{A}^n})$, viewed as a subalgebra of $\mathcal{D}(\mathbb{A}^n)$, is generated by* 1 *and the second order operators $z_i, z_i\partial_j, \partial_i(e+m)$, $i,j = 1,\dots,n$.*

The algebra of differential operators twisted by a line bundle acts on the space of global section of the line bundle. By Corollary 1.2 the space of global sections $\Gamma(\widetilde{\mathbb{A}^n}, \mathcal{O}(mE))$ is a module over $U(\mathfrak{sl}_{n+1})$. Let $\mathfrak{sl}_{n+1} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the Cartan decomposition into lower triangular, traceless diagonal and upper triangular matrices. Denote by $M(\lambda)$ the Verma module with highest weight $\lambda \in \mathfrak{h}^*$ and by $L(\lambda)$ the irreducible quotient of $M(\lambda)$ (see [7]). The fundamental weights are $\varpi_i \colon x \mapsto x_1 + \cdots + x_i$, $x = \mathrm{diag}(x_1,\dots,x_{n+1}) \in \mathfrak{h}$.

Theorem 1.4. *The $U(\mathfrak{sl}_{n+1})$-module $\Gamma(\widetilde{\mathbb{A}^n}, \mathcal{O}(mE))$ is isomorphic to the module $L(-(m+1)\varpi_n)$ if $m \geq 0$ and to $L((m-1)\varpi_n - m\varpi_{n-1})$ if $m < 0$.*

We deduce Theorem 1.1 from a more general theorem (Theorem 3.2) on toric varieties, stating that suitably twisted algebras of differential operators on nonsingular toric varieties whose fans are related by reflections of one-dimensional cones (see Fig. 1 and Section 3) are isomorphic via a Fourier transform. The proof of this theorem uses Musson's description [9] of the algebra of differential operators on a toric variety and partial Fourier transforms on Weyl algebras as in the works of Musson and Rueda [11] and Rueda [12].

In this way we obtain several families of toric varieties with line bundles whose twisted algebras of differential operators are isomorphic. Even in the simplest case of projective spaces, which we treat in detail in Section 4, the situation is rather rich. Additionally to the projective n-space and the blow-up at the origin of the affine n-space there are many other varieties related by such reflection. They include resolutions of singularities of the varieties considered by Levasseur, Smith and Stafford [8] in the case of $\mathfrak{sl}_{n+1}$ and Musson [10]. Our results extend and unify their results. For these varieties we also compute the action of $\mathfrak{sl}_{n+1}$ on the cohomology with coefficients in any line bundle and find that cohomology groups form irreducible modules. They are described in Theorems 4.2 and 4.8. Theorem 1.4 is a special case of this more general result.

Let us note that the problem of studying isomorphisms of algebras of differential operators is also actively studied in different contexts, mostly for singular affine varieties; see [2] for a recent review.

2. Differential operators on toric varieties and Fourier transform

2.1. Line bundles on toric varieties

We refer to the book [5] for an introduction to toric varieties and line bundles on them, and use mostly the same notations. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n and $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice. Recall that a *fan* in N is a finite collection of strongly convex rational polyhedral cones $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ such that each face of a cone in Δ is a cone in Δ and the intersection of cones in Δ is a face of each. To each fan Δ there corresponds a toric variety $X = X(\Delta)$, a normal algebraic variety with an action of the torus $T_N = N \times_{\mathbb{Z}} \mathbb{C}^\times \simeq (\mathbb{C}^\times)^n$ with a dense orbit U_0. The functions on U_0 are spanned by Laurent monomials χ_μ with exponent $\mu \in M$. To each $\sigma \in \Delta$ there corresponds an invariant affine open set U_σ and an orbit closure $V(\sigma)$ of codimension equal to the dimension of σ. The orbit closures $D_1, \ldots, D_d$ associated with the one-dimensional cones are T_N-invariant Weil divisors and there is an exact sequence of groups ([5, Proposition on p. 63])

$$M \to \bigoplus_{i=1}^{d} \mathbb{Z} D_i \to A_{n-1}(X) \to 0,$$

where $A_{n-1}(X)$ is the group of Weil divisors modulo divisors of rational functions. Let $k_1, \ldots, k_d$ be the nonzero lattice vectors closest to the origin on the one-dimensional cones. The left map sends $\mu \in M$ to $\sum \langle \mu, k_i \rangle D_i$, which is the divisor of the function χ_μ. The sequence is also exact on the left if the vectors in Δ span $N_{\mathbb{R}}$.

We will make the following simplifying assumptions about Δ.

Definition 2.1. A fan Δ is *regular* if

(i) Each cone in Δ is generated by part of a basis of N.
(ii) The one-dimensional cones in Δ are generated by vectors $k_1, \dots, k_d$ spanning the lattice N over $\mathbb{Z}$.

Assumption (i) means that we consider nonsingular toric varieties so that the map $\mathrm{Pic}(X) \to A_{n-1}(X)$ sending a line bundle to the class of the divisor of a nonzero rational section is an isomorphism. Assumption (ii) implies in particular that the k_i span N and thus there is an exact sequence of (free Abelian) groups

$$0 \to M \to \sum_{i=1}^{d} \mathbb{Z} D_i \to A_{n-1}(X) \to 0, \qquad A_{n-1}(X) \simeq \mathrm{Pic}(X). \tag{1}$$

Under these assumptions, each cone of Δ is generated by a subset of the lattice vectors $k_1, \dots, k_d$, called the *generating vectors* of the fan.

Example 2.2. Let $\epsilon_1, \dots, \epsilon_n$ be a basis of N and let $\epsilon = \epsilon_1 + \cdots + \epsilon_n$. Then the fan in N whose cones are generated by all proper subsets of $\{\epsilon_1, \dots, \epsilon_n, -\epsilon\}$ gives the toric variety $\mathbb{P}^n$ and the fan whose cones are generated by all proper subsets of $\{\epsilon_1, \dots, \epsilon_n, \epsilon\}$ except $\{\epsilon_1, \dots, \epsilon_n\}$ gives the toric variety $\widetilde{\mathbb{A}^n}$. In Fig. 1 the case $n = 2$ is depicted.

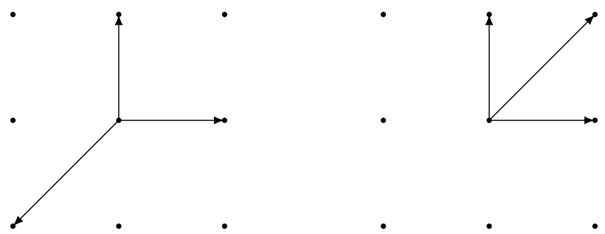

Fig. 1. The generating vectors of the fans of $\mathbb{P}^2$ (left) and $\widetilde{\mathbb{A}^2}$ (right).

2.2. Musson's construction

Musson [9] gave a description of the ring of twisted differential operators on an arbitrary toric variety in terms of generators and relations. We recall here his construction in the special case of varieties with regular fan, for which some simplifications occur. Let as above $k_1, \dots, k_d \in N$ be the generating vectors of a regular fan Δ. Let $p\colon \mathbb{Z}^d \to N$ be the group homomorphism sending the i-th standard basis vector ϵ_i to k_i, $i = 1, \dots, d$. For each $\sigma \in \Delta$ let $\hat\sigma \in \mathbb{R}^d$ be the cone spanned by the vectors ϵ_i such that $k_i \in \sigma$. The cones $\hat{\sigma}$ form a fan in $\mathbb{R}^d$ and define a toric variety Y with the action of $(\mathbb{C}^\times)^d$. We have an exact sequence

$$0 \to K \to \mathbb{Z}^d \to N \to 0. \tag{2}$$

The torus $(\mathbb{C}^\times)^d$ acting on Y has thus a subgroup $G = T_K$ and it is proved in [9] (and also elsewhere, see [1], [4] and references therein) that $X = Y//G$ is the algebro-geometric quotient of Y by G. The variety Y is the union of G-invariant open sets V_σ in $\mathbb{C}^m$ associated with the cones $\hat{\sigma}$, $\sigma \in \Delta$:

$$V_\sigma = \{Q \in \mathbb{C}^d \mid Q_i \neq 0, \text{whenever } k_i \notin \sigma\}.$$

Thus Y is the complement in $\mathbb{A}^d$ of the union of subspaces of codimension ≥ 2. Moreover, to each character $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ there corresponds a sheaf $\mathcal{L}_\chi$ on X: the space of sections over $U_\sigma = V_\sigma/G$ is the $\mathcal{O}_X(U_\sigma) = \mathcal{O}_Y(V_\sigma)^G$-module

$$\mathcal{L}_\chi(U_\sigma) = \{f \in O_Y(V_\sigma) \mid f(gx) = \chi(g)f(x),\ g \in G\}. \tag{3}$$

Under our assumptions, $\mathcal{L}_\chi$ is the sheaf of sections of a line bundle and each line bundle can be obtained in this way. To see this notice that the exact sequence (2) is dual to the exact sequence (1) so

$$\mathrm{Hom}(G, \mathbb{C}^\times) \simeq \mathrm{Hom}(K, \mathbb{Z}) \simeq A_{n-1}(X).$$

Explicitly, the character $\chi(z) = z^a = z_1^{a_1} \cdots z_d^{a_d}$, $z \in T_K \subset (\mathbb{C}^\times)^d$ corresponds to the class in $A_{n-1}(X)$ of the divisor $\sum a_i D_i$. The fundamental vector fields of the infinitesimal action of G on Y form a Lie algebra $\mathfrak{g} \simeq K \otimes_{\mathbb{Z}} \mathbb{C}$ and χ defines a Lie algebra homomorphism χ_*, from $\mathfrak{g}$ to $\mathbb{C}$. The group G acts on $\mathcal{O}_Y$ and on $\mathcal{D}(Y)$. For each $\sigma \in \Delta$, the restriction of a G-invariant differential operator on Y to V_σ maps $\mathcal{L}_\chi(U_\sigma)$ to itself, compatibly with inclusions of open sets. Thus there is a map $\mathcal{D}(Y)^G \to \mathcal{D}_{\mathcal{L}_\chi}(X)$.

Theorem 2.3 (Musson, [9, Theorem 5]). *The natural map $\mathcal{D}(Y)^G \to \mathcal{D}_{\mathcal{L}_\chi}(X)$ is surjective with kernel generated by $\{x - \chi_*(x)1 \mid x \in \mathfrak{g}\}$.*

Using this theorem, Musson gives a description of $\mathcal{D}_{\mathcal{L}_\chi}(X)$ in terms of generators and relations. Let us describe it under our assumptions. The complement of Y in $\mathbb{A}^d$ has codimension at least 2 so

$$\mathcal{D}(Y)^G \simeq \mathcal{D}(\mathbb{A}^d)^G.$$

The Weyl algebra $\mathcal{D}(\mathbb{A}^d)$ is generated by the coordinate functions Q_i and vector fields $P_i = \partial/\partial Q_i$. A basis is formed by the monomials $Q^\lambda P^\mu = Q_1^{\lambda_1} \cdots P_d^{\mu_d}$, $\lambda, \mu \in \mathbb{N}^d$, and $g \in G \subset (\mathbb{C}^\times)^d$ acts on these monomials by multiplying them by $g^{\lambda-\mu}$. Thus $\mathcal{D}_{\mathcal{L}_\chi}(X)$ is isomorphic to the quotient of the subalgebra of $\mathcal{D}(\mathbb{A}^d)$ generated by $Q^\lambda P^\mu$, $\lambda - \mu \in K^\perp$, by the ideal generated by $\sum_i m_i Q_i P_i - \chi_*(m)1$, $m \in K$. For the line bundle $\mathcal{O}(D)$ associated with the T_N-invariant divisor $D = \sum_i a_i D_i$, we have $\chi_*(m) = \sum_i a_i m_i$.

Example 2.4. Let $X = \mathbb{P}^n$ with fan as in Example 2.2. Then one has a projection $p : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$, defined via $p(\epsilon_i) = \epsilon_i$, $i \le n$, $p(\epsilon_{n+1}) = -\epsilon$. The kernel K is spanned by $(1, \ldots, 1)$, $Y = \mathbb{C}^{n+1} \smallsetminus \{0\}$ with coordinates $Q_1, \ldots, Q_{n+1}$, $G = \mathbb{C}^\times$ acts by $Q_i \mapsto \lambda Q_i$, $\lambda \in \mathbb{C}^\times$ and $\mathfrak{g}$ is spanned by the vector field $\sum_{i=1}^{n+1} Q_i P_i$. Then $\mathcal{D}_{\mathcal{O}(\ell)}(\mathbb{P}^n)$ is the quotient of the algebra of operators $Q^\lambda P^\mu$, $\sum_i \lambda_i = \sum_i \mu_i$, $\lambda_i, \mu_i \ge 0$, by the ideal generated by $\sum Q_i P_i - \ell$.

3. Fourier transforms

Let $d \ge n \ge 2$ and $I \subset \{1, \ldots, d\}$. Let us say that two regular fans Δ, Δ' with the same underlying lattice $N \simeq \mathbb{Z}^n$ and the same number d of rays (= one-dimensional cones) are *related by an I-reflection* if there is a numbering $k_1, \ldots, k_d$, $k'_1, \ldots, k'_d$ of the generating vectors of Δ, Δ' such that

$$k'_i = \begin{cases} -k_i & \text{if } i \in I, \\ k_i & \text{otherwise.} \end{cases}$$

Let $X = X(\Delta)$, $X' = X(\Delta')$ be the corresponding toric varieties. The groups $A_{n-1}(X)$, $A_{n-1}(X')$ of classes of Weil divisors are generated by the classes of the T_N-invariant divisors D_i, D'_i associated with the rays $\mathbb{R}_{\ge 0} k_i$, $\mathbb{R}_{\ge 0} k'_i$. Let us introduce an *affine* isomorphism $\phi_I : A_{n-1}(X) \to A_{n-1}(X')$ defined on representatives by

$$\phi_I : \sum_{i=1}^d a_i D_i \mapsto \sum_{i \notin I} a_i D'_i - \sum_{i \in I} (a_i + 1) D'_i. \tag{4}$$

Lemma 3.1. *The map ϕ_I is well defined, i.e., independent of the choice of representatives.*

Proof. The group $A_{n-1}(X)$ is the quotient of $\sum_{i=1}^d \mathbb{Z} D_i$ by the image of M, embedded via $\mu \mapsto \sum_{i=1}^d \langle \mu, k_i \rangle D_i$, and similarly for $A_{n-1}(X')$. Changing representative thus means replacing a_i by $a_i + \langle \mu, k_i \rangle$. The image under ϕ_I changes then by $\sum_{i \notin I} \langle \mu, k_i \rangle D'_i - \sum_{i \in I} \langle \mu, k_i \rangle D'_i = \sum_i \langle \mu, k'_i \rangle D'_i$, which belongs to the image of M in $\sum \mathbb{Z} D'_i$. $\square$

Let $F_I \in \mathrm{Aut}(\mathcal{D}(\mathbb{A}^d))$ be the automorphism acting on generators as

$$F_I(Q_i) = \begin{cases} P_i & \text{if } i \in I, \\ Q_i & \text{otherwise,} \end{cases} \qquad F_I(P_i) = \begin{cases} -Q_i & \text{if } i \in I, \\ P_i & \text{otherwise.} \end{cases}$$

These automorphisms were first considered in this setting by Musson and Rueda in [11]. The next theorem is an extension of a result of these authors ([11, Lemma 5.2]).

Theorem 3.2. *Suppose that Δ, Δ' are regular fans related by an I-reflection for some $I \subset \{1, \ldots, d\}$. Let $[D] \in A_{n-1}(X)$ be a Weil divisor and let $D = \sum a_i D_i$ be a T_N-invariant representative. Then the Fourier transform F_I restricts to an* *isomorphism* $\mathcal{D}(Y)^G \to \mathcal{D}(Y')^{G'}$ *of the corresponding Musson algebras, which in turn descends to an algebra isomorphism*

$$\mathcal{D}_{\mathcal{O}(D)}(X) \to \mathcal{D}_{\mathcal{O}(\phi_I(D))}(X')$$

of the algebras of differential operators.

Proof. The first part of the proof is a matter of going through Musson's construction and is taken from [11, Lemma 5.2]. The involution $\sigma_I : \mathbb{Z}^d \to \mathbb{Z}^d$, changing sign to the coordinates labeled by I, maps K to the kernel K' of the map $p' : \mathbb{Z}^d \to N$ and thus induces an automorphism σ_I of $T_d = (\mathbb{C}^\times)^d$ mapping the subtorus G to G'. This automorphism maps z to z' with $z'_i = z_i^{-1}$ if $i \in I$ and $z'_i = z_i$ otherwise. The torus T_d acts on generators Q_i, $P_i = \partial/\partial Q_i$ of $\mathcal{D}(\mathbb{A}^d)$ via $z \cdot Q_i = z_i Q_i$, $z \cdot P_i = z_i^{-1} P_i$. From this and the formula for F_I it follows that $F_I(z \cdot x) = \sigma_I(z) \cdot F_I(x)$, $z \in T_d$, $x \in \mathcal{D}(\mathbb{A}^d)$. Therefore F_I induces an isomorphism of the algebras of invariants.

It remains to show that the kernel of the map $\mathcal{D}(Y)^G \to \mathcal{D}_{\mathcal{O}(D)}(X)$ is mapped to the kernel of the corresponding map for X'. By Musson's theorem this kernel is generated by the operators $\xi_m = \sum_i m_i Q_i P_i - \chi_*(m)1$, where m runs over $K \subset \mathbb{Z}^d$ and if $D = \sum_i a_i D_i$, then $\chi_*(m) = \sum_i a_i m_i$. Since $m \in K$ if and only if $m' = \sigma_I(m) \in K'$ we have:

$$\begin{aligned}
F_I(\xi_m) &= \sum_{i=1}^{d} m_i (F_I(Q_i P_i) - a_i) \\
&= \sum_{i \notin I} m_i Q_i P_i - \sum_{i \in I} m_i P_i Q_i - \sum_{i=1}^{d} a_i m_i \\
&= \sum_{i=1}^{d} m'_i Q_i P_i - \sum_{i \notin I} a_i m_i - \sum_{i \in I} (a_i + 1) m_i \\
&= \sum_{i=1}^{d} m'_i Q_i P_i - \sum_{i \notin I} a_i m'_i + \sum_{i \in I} (a_i + 1) m'_i .
\end{aligned}$$

Thus F_I maps the generators ξ_m of the kernel to the generators $\xi_{m'}$ of the kernel of $\mathcal{D}(Y')^{G'} \to \mathcal{D}_{\mathcal{O}(\phi_I(D))}(X')$. □

Remark 3.3. More generally, this theorem has a version valid for singular toric varieties. In this case, however, not all Weil divisors are Cartier divisors and thus they do not all correspond to line bundles. It can then happen that ϕ_I maps a Cartier divisor to a divisor which is not Cartier. Suppose first that only assumption (ii) holds, and that both D and $\phi_I(D)$ are Cartier divisors. Then the statement of Theorem 3.2 holds. If assumption (ii) is not satisfied, then there are factors of $\mathbb{C}^\times$ in Y and the algebra $\mathcal{D}(Y)$ has generators $Q_1, \ldots, Q_k, Q_{k+1}^{\pm 1}, \ldots, Q_d^{\pm 1}$ and $P_i = \partial/\partial Q_i$. The Fourier transform F_I is defined for $I \subset \{1, \ldots, k\}$. Theorem 3.2 holds for these subsets I and for Cartier divisors D, $\phi_I(D)$.

4. The case of the projective space

4.1. Varieties related by reflection to the projective space

The projective n-dimensional space $\mathbb{P}^n = (\mathbb{A}^{n+1} \smallsetminus \{0\})/\mathbb{C}^\times$ is a toric variety with generating vectors $k_1, \dots, k_{n+1} \in N \simeq \mathbb{Z}^n$ such that $\sum_{i=1}^{n+1} k_i = 0$. Varieties related to $\mathbb{P}^n$ by an I-reflection thus have generating vectors generating N and obeying

$$\sum_{i \in I} k_i = \sum_{i \notin I} k_i.$$

The set S_I of regular fans with these generating vectors is a finite set partially ordered by inclusion. Here are some interesting examples.

(a) *Minimal fans.* For each I, S_I has a minimal element: the fan consisting of just the one-dimensional cones spanned by the vectors k_i. The variety corresponding to this minimal fan is contained as an open set, with complement of codimension at least 2, in all varieties $X(\Delta)$, $\Delta \in S_I$.

(b) *Projective spaces.* The projective space $\mathbb{P}^n$ has a maximal fan in S_I, with $I = \emptyset$ or $I = \{1, \dots, n+1\}$.

(c) *Blow-ups.* The blow-up at the origin of $\mathbb{A}^n$ is obtained from a maximal fan in S_I, for I a set with $r = 1$ or $r = n$ elements.

(d) *Matrices of rank one and of rank ≤ 1.* Let $I = \{1, \dots, r\}$, $2 \leq r \leq n-1$. A variety with regular fan in S_I is $Z_r = (\mathbb{C}^r \smallsetminus \{0\}) \times (\mathbb{C}^{n+1-r} \smallsetminus \{0\})/\mathbb{C}^\times$, where $z \in \mathbb{C}^\times$ acts by $z \cdot (x, y) = (zx, z^{-1}y)$. It may be identified via $(x, y) \mapsto x^T y$ with the variety of r by $n+1-r$ matrices of rank 1, arising in the study of the orbit $O_{\min}$ of the highest weight vector in the adjoint representation of $\mathfrak{sl}_{n+1}$, studied in [8]. The corresponding fan consists of the cones generated by $\{k_i \mid i \in J\}$ for a subset $J \subset \{1, \dots, n+1\}$ whose complement contains at least one element in I and at least one element not in I. The closure $\bar{Z}_r = Z_r \cup \{0\}$ in the space of r by $n+1-r$ matrices is a singular affine toric variety. Its fan (it is not in S_I) is obtained from the fan of Z_r simply by adding the cone generated by $k_1, \dots, k_{n+1}$. These varieties are the irreducible components of $\bar{O}_{\min} \cap \mathfrak{n}_+$ (see [8]).

(e) *Resolution of singularities of the above.* Let again $I = \{1, \dots, r\}$, $2 \leq r \leq n-1$. Let C_i be the n-dimensional cone spanned by the basis $(k_j)_{j \neq i}$. Let Δ_I^+, resp. Δ_I^-, be the fan consisting of the cones C_i with $i \in I$, resp. $i \notin I$, and their corresponding faces. One can check that both are regular maximal fans. Both are subdivisions of the fan of Z_r and thus, by the results of Section 2.6 of [5], the corresponding toric varieties $\tilde{Z}_r^+$ and $\tilde{Z}_r^-$ give resolutions of singularity $\tilde{Z}_r^\pm \to \bar{Z}_r$.

For any $X = X(\Delta)$, $\Delta \in S_I$, the group $A_{n-1}(X) \simeq \operatorname{Pic}(X)$ of Weil divisors modulo linear equivalence is the quotient of the group $\sum \mathbb{Z} D_i$ of T_N-invariant Weil divisors by the relations $D_i = D_j$ if either $i, j \in I$ or $i, j \notin I$, and $D_i = -D_j$ otherwise. Thus $A_{n-1}(X) \simeq \mathbb{Z}$ generated by any D_i.

Theorem 4.1. *Let $X = X(\Delta)$, $\Delta \in S_I$, be an n-dimensional toric variety whose fan is regular with $n+1$ generating vectors obeying $\sum_{i\in I} k_i = \sum_{i\neq I} k_i$ for some subset $I \subset \{1,\dots,n+1\}$ with $|I|$ elements. Let D_0 be a divisor linearly equivalent to D_i, $i \notin I$, or equivalently to $-D_i$, $i \in I$. Then for all $\ell \in \mathbb{Z}$, F_I induces an isomorphism*

$$\mathcal{D}_{\mathcal{O}(\ell)}(\mathbb{P}^n) \to \mathcal{D}_{\mathcal{O}((\ell+|I|)D_0)}(X).$$

Proof. This is the special case of Theorem 3.2 in which one of the varieties is $\mathbb{P}^n$. The line bundle $\mathcal{O}(\ell)$ is isomorphic to $\mathcal{O}(\ell D_i)$ for any i. Take $i \in I$ (if I is empty there is nothing to prove). The map ϕ_I of Theorem 3.2 sends ℓD_i to $-\ell D_i - \sum_{j\in I} D_j$ (see (4)), which is equivalent to $-(\ell+|I|)D_i$. □

If $I = \{1,\dots,n\}$, X is the blow-up $\widetilde{\mathbb{A}^n}$, the exceptional divisor E is D_{n+1} and we obtain Theorem 1.1. The formula for F_I in affine coordinates is then the isomorphism F of the Introduction in this case.

4.2. The module of global sections

Let X, Δ, I and D_0 be as in Theorem 4.1. The algebra $\mathcal{D}_{\mathcal{O}(mD_0)}(X)$, $\Delta \in S_I$, acts on the space $\Gamma(X, \mathcal{O}(mD_0))$ of global sections. Let $\ell = m - |I|$. The homomorphism $U(\mathfrak{sl}_{n+1}) \to \mathcal{D}_{\mathcal{O}(\ell D_0)}(\mathbb{P}^n)$ composed with the isomorphism of Theorem 4.1 gives $\Gamma(X, \mathcal{O}(mD_0))$ the structure of an $\mathfrak{sl}_{n+1}$-module. Let ρ be the half-sum of the positive roots of $\mathfrak{sl}_{n+1}$, and let $\varpi_i\colon x \mapsto x_1 + \cdots + x_i$, $x = \mathrm{diag}(x_1,\dots,x_{n+1}) \in \mathfrak{h}$, be the fundamental weights of $\mathfrak{sl}_{n+1} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with respect to the standard Cartan decomposition. Let $s_i \in \mathrm{End}(\mathfrak{h}^*)$, $i = 1,\dots,n$, denote the simple reflections in the Weyl group.

Theorem 4.2. *Let $I = \{1,\dots,r\}$ and X, D_0 be as in Theorem 4.1. The $\mathfrak{sl}_{n+1}$-module $\Gamma(X, \mathcal{O}((\ell+r)D_0))$ is an irreducible highest weight module with highest weight*

$$\lambda_r(\ell) = \begin{cases} s_r \cdots s_2 s_1(\ell\varpi_1 + \rho) - \rho & \text{if } \ell + r \geq 0 \text{ and } r \leq n,\\ s_{r-1} \cdots s_2 s_1(\ell\varpi_1 + \rho) - \rho & \text{if } \ell + r < 0 \text{ and } r \geq 1,\\ 0 & \text{if } \ell + r = 0 \text{ and } r = n+1. \end{cases}$$

In the remaining cases (i.e. $\ell + r \geq 0, r = n+1$ and $\ell + r < 0, r = 0$), we have $\Gamma(X, \mathcal{O}(\ell+r)D_0) = 0$.

Proof. We use the description (3) of the sheaf of sections of a line bundle with divisor $D = \sum a_i D_i$. The exact sequence (2) is

$$0 \to K \to \mathbb{Z}^{n+1} \xrightarrow{p} \sum_{i=1}^{n} \mathbb{Z}k_i \to 0,$$

The kernel K of the map $p\colon \epsilon_i \mapsto k_i$ is spanned by $\sum_{i\notin I} \epsilon_i - \sum_{i\in I} \epsilon_i$. The character χ is (the restriction to T_K of) the character $z \mapsto z^a = z_1^{a_1} \cdots z_{n+1}^{a_{n+1}}$ of $(\mathbb{C}^\times)^{n+1}$. The sections on the open set U_0 (the open orbit of T_N) are then spanned by the monomials $Q^{\mu+a}$ where Q_i are the coordinates on $Y \subset \mathbb{C}^{n+1}$ and μ runs over

$K^\perp = \{\mu \in \mathbb{Z}^{n+1} \mid \sum_{i\notin I}\mu_i = \sum_{i\in I}\mu_i\}$. The sections $Q^{\mu+a}$ extend to global regular sections if and only if $\mu_i + a_i \geq 0$ for all i. Let $\sigma_I\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ be the involution which changes the sign of the coordinates labeled by I. Then we get the following (well-known) description of $\Gamma(X, \mathcal{O}(\sum a_i D_i))$: a basis is

$$\Big\{Q^\nu \,\Big|\, \sum_{i\notin I}\nu_i - \sum_{i\in I}\nu_i = m,\, \nu_i \geq 0,\, i = 1,\dots,n+1\Big\}, \tag{5}$$

where $m = \sum_{i\notin I} a_i - \sum_{i\in I} a_i$. This basis consists of weight vectors. Let us compute their weight in the weight lattice $P = \mathbb{Z}^{n+1}/\mathbb{Z}\cdot(1,\dots,1)$: $x = \mathrm{diag}(x_1,\dots,x_{n+1}) \in \mathfrak{h}$ acts as $F_I(\sum x_i Q_i P_i) = \sum_{i\notin I} x_i Q_i P_i - \sum_{i\in I} x_i P_i Q_i$, thus

$$x\cdot Q^\nu = \Big(\sum_{i\notin I}\nu_i x_i - \sum_{i\in I}(\nu_i+1)x_i\Big)Q^\nu.$$

Thus Q^ν has weight $\lambda = \sigma_I(\nu) - \sum_{i\in I}\epsilon_i$ modulo $\mathbb{Z}\cdot(1,\dots,1)$. Now suppose that the divisor $D = \phi_I(D^0)$ is obtained from a divisor $D^0 = \sum a_i^0 D_i$ on $\mathbb{P}^n$, so that $a = \sigma_I(a^0) - \sum_{i\in I}\epsilon_i$. Let $\ell = \sum a_i^0$ so that $m = \ell + r$. Then the weights appearing in $\Gamma(X, \mathcal{O}(D))$ are the classes in P of the integer vectors λ such that

$$\sum_{i=1}^{n+1}\lambda_i = \ell, \qquad \lambda_i \geq 0 \quad \text{if } i \notin I, \quad \lambda_i \leq -1 \text{ if } i \in I,$$

where $\ell = \sum_i a_i^0 = m - |I|$. The corresponding weight spaces are one-dimensional, spanned by Q^ν with $\lambda = \sigma_I(\nu) - \sum_{i\in I}\epsilon_i$.

Let now $I = \{1,\dots,r\}$. We need to identify the primitive weight vectors in $\Gamma(X, \mathcal{O}(D))$, that is, the monomials Q^ν annihilated by the action of $\mathfrak{n}_+$.

Lemma 4.3. *The only primitive monomial in* $\Gamma(X,\mathcal{O}(D))$ *is* 1 *if* $m = 0$, Q_r^{-m} *if* $m < 0$, $r \geq 1$, *and* Q_{r+1}^m *if* $m > 0$, $r \leq n$. *If* $m < 0$ *and* $r = 0$, *or if* $m > 0$ *and* $r = n+1$, *then* $\Gamma(X, \mathcal{O}(D)) = 0$.

Proof. The monomial Q^ν is primitive if and only if $e_i \cdot Q^\nu = 0$ for the Chevalley generators e_i, $i = 1,\dots,n$, associated with simple roots. These generators act as

$$e_i = F_I(Q_i P_{i+1}) = \begin{cases} -P_i Q_{i+1}, & i < r, \\ P_i P_{i+1}, & i = r, \\ Q_i P_{i+1}, & i > r, \end{cases}$$

The only solutions of $e_i \cdot Q^\nu = 0$ are powers of Q_r and of Q_{r+1}. The condition on the sum of exponents in (5) implies that the exponents are $-m$ and m, respectively. The last assertion follows from the fact that there is no homogeneous polynomial of negative degree. □

If $m = \ell + r \geq 0$ then the weight of Q_{r+1}^m is

$$\lambda_r(\ell) = \sigma_I((\ell+r)\epsilon_{r+1}) - \sum_{i\in I}\epsilon_i = (\ell+r)\epsilon_{r+1} - \sum_{i=1}^r \epsilon_i.$$

If $m < 0$ then the weight of Q_r^m is

$$\lambda_r(\ell) = \sigma_I(-(\ell+r)\epsilon_r) - \sum_{i\in I}\epsilon_i = (\ell+r-1)\epsilon_r - \sum_{i=1}^{r-1}\epsilon_i.$$

To identify these weights with the vector in the ρ-shifted Weyl orbit of $\ell\varpi_1$, recall that $\varpi_1 = \epsilon_1$, that s_i is transposition of ϵ_i and ϵ_{i+1}, and that $s_i\rho - \rho = -\alpha_i = \epsilon_{i+1} - \epsilon_i$. It follows that there is a nonzero module homomorphism $M(\lambda_r(\ell)) \to \Gamma(X, \mathcal{O}(D))$ from the Verma module with highest weight $\lambda_r(\ell)$. The image is irreducible by Lemma 4.3, since any proper submodule would contain a primitive monomial distinct from the highest weight vectors. It remains to show that this homomorphism is surjective. This follows from the next lemma.

Lemma 4.4. *The $\mathcal{D}_{\mathcal{O}(D)}(X)$-module $\Gamma(X, \mathcal{O}(D))$ (when $\neq 0$) is generated by the primitive monomial of Lemma 4.3.*

Proof. By Musson's theorem, $\mathcal{D}_{\mathcal{O}(D)}(X)$ is spanned by the images of the monomial differential operators $Q^\lambda P^\mu$ such that $\tau = \lambda - \mu$ obeys $\sum_{i\notin I}\tau_i = \sum_{i\in I}\tau_i$. If $m \geq 0$, a general monomial Q^ν of the basis (5) can be obtained by the action of the differential operator $Q^\nu P_{r+1}^m/m!$ on the highest weight vector Q_{r+1}^m. If $m < 0$, take $Q^\nu P_r^{-m}/(-m)!$ and apply it to the highest weight vector Q_r^{-m}. □

This concludes the proof of Theorem 4.2. □

Remark 4.5. The symmetric group S_{n+1} of $\mathfrak{sl}_{n+1}$ acts on $U(\mathfrak{sl}_{n+1})$ by automorphisms (it is the Weyl group) and on $\mathbb{P}^n$ and $\mathcal{O}(\ell)$ by permutations of homogeneous coordinates. The map $U(\mathfrak{sl}_{n+1}) \to \mathcal{D}_{\mathcal{O}(\ell)}(\mathbb{P}^n)$ is S_{n+1}-equivariant. The modules $\Gamma(X, \mathcal{O}((\ell+r)D_0))$ corresponding to the other subsets I are related by the Weyl automorphism of $U(\mathfrak{sl}_{n+1})$ associated with any permutation sending I to $\{1,\dots,r\}$ to the modules of the theorem. They are thus highest weight modules for other Cartan decompositions.

Remark 4.6. In the case of the variety of rank one matrices Z_r, $2 \leq r \leq n-1$, only the trivial vector bundle extends to the closure $\bar{Z}_r$ in the space of all matrices, considered in [8]. In this case we recover the result of [8] that $\Gamma(Z_r, \mathcal{O}) \simeq L(-\omega_r)$.

4.3. Higher cohomology groups

Finally, we want to discuss the $\mathfrak{sl}_{n+1}$-module structure of higher cohomology groups for X a regular toric variety of the kind considered in (c) and (e) in Subsection 4.1, related to $\mathbb{P}^n$ by an I-reflection, with $I = \{1,\dots,r\}$ and $2 \leq r \leq n$: in particular, we consider $X = \tilde{Z}_r^+$, for $2 \leq r \leq n-1$, or $X = \widetilde{\mathbb{A}^n}$, for $r = n$. Let as above D_0 be a generator of the Picard group of X; for the subsequent computations, assume $D_0 = D_{n+1}$. Finally, let m be an integer number.

Theorem 4.7. *For I, X, D_0 and m as above we have*

$$H^*(X, \mathcal{O}(mD_0)) \cong \bigoplus_{\lambda\in\mathbb{N}^{n-r+1}} H^*(\mathbb{P}^{r-1}, \mathcal{O}(-m+|\lambda|)).$$

Proof. We compute the cohomology of X with values in $\mathcal{O}(mD_0)$ by means of Čech cohomology. For this purpose, recall that X is a torus quotient and that it has an open covering by sets $U_i = \{Q_i \neq 0\}$, $i = 1, \dots, r$, with Q_i denoting affine coordinates on $\mathbb{A}^{n+1}$. Further, the Weil divisor mD_0 has a Cartier representative of the form $f_{i,m}(Q) = Q_i^m Q_{n+1}^m$, $i = 1, \dots, r$: the transition functions of the corresponding line bundle $\mathcal{O}(mD_0)$ have the form $(Q_i/Q_j)^m$ on $U_i \cap U_j$, $1 \leq i \neq j \leq r$. Finally, the variety X admits local affine coordinates over U_i, namely

$$[(Q_1, \dots, Q_r, Q_{r+1}, \dots, Q_{n+1})] \mapsto \left(\frac{Q_1}{Q_i}, \dots, \frac{Q_r}{Q_i}, Q_i Q_{r+1}, \dots, Q_i Q_{n+1}\right),$$

where we omit the i-th entry. We denote by $(z_1, \dots, z_{r-1}, w_1, \dots, w_{n-r+1})$ the corresponding local coordinates on $\mathbb{A}^n$.

With respect to the open covering $\{U_i\}$, a Čech p-cochain on X with values in $\mathcal{O}(mD_0)$ is represented by a family $\sigma_{(i_0,\dots,i_p)}$ of regular sections of $\mathcal{O}(mD_0)$ on the intersection $U_{i_0} \cap \dots \cap U_{i_p}$. E.g. in the affine coordinates over U_{i_0}, the component $\sigma_{(i_0,\dots,i_p)}$ is represented by a regular function $P_{(i_0,\dots,i_p)}(z,w)$ on $\bigcap_{i=1}^p \{z_i \neq 0\}$, admitting an expansion of the form

$$P_{(i_0,\dots,i_p)}(z,w) = \sum_{\lambda \in \mathbb{N}^{n-r+1}} P^\lambda_{(i_0,\dots,i_p)}(z) w^\lambda,$$

where $P^\lambda_{(i_0,\dots,i_p)}$ are now regular functions on $\bigcap_{i=1}^p \{z_i \neq 0\}$.

The cocycle and coboundary conditions can now be rewritten in terms of the previous expansion with respect to w, using the previous transition functions for $\mathcal{O}(mD_0)$: denoting the degree of $\lambda \in \mathbb{N}^{n-r+1}$ by $|\lambda|$, since neither a change of coordinates on X nor a change of trivialization on $\mathcal{O}(mD_0)$ affects the degree $|\lambda|$ of w, the cocycle and coboundary conditions for Čech cohomology on X can be rewritten as cocycle and coboundary conditions for $P^\lambda_{(i_0,\dots,i_p)}$, viewed as sections of a Serre bundle $\mathcal{O}(-m+|\lambda|)$ over $\mathbb{P}^{r-1}$, for any multiindex λ as above. Notice that the transition functions of $\mathcal{O}(mD_0)$ depend only on the homogeneous coordinates Q_i for $i = 1, \dots, r$, and when $Q_i = 0$, $i = r+1, \dots, n+1$, they correspond to the transition functions of the Serre bundle $\mathcal{O}(-m)$. Finally, notice that the affine coordinates w_i are all multiplied by some coordinate z_j, when performing a corresponding coordinate change on $\mathbb{P}^{r-1}$, whence the shift by the degree $|\lambda|$ in the Serre bundles.

Summarizing, any Čech cocycle on X of degree p with values in $\mathcal{O}(mD_0)$ is equivalent to a family of Čech cocycles of the same degree on $\mathbb{P}^{r-1}$ with values in certain Serre bundles related to $\mathcal{O}(-m)$ by shifts of the parameter; the same holds true for Čech coboundaries, and these two facts yield the above isomorphism. □

In particular, since the cohomology of the projective space $\mathbb{P}^{r-1}$ is concentrated in degree 0 and $r-1$, the cohomology of X with values in $\mathcal{O}(mD_0)$ is also concentrated in degree 0 and $r-1$.

We can describe explicitly the isomorphism of Theorem 4.7 using homogeneous coordinates Q_i on X, knowing the description of the cohomology of projective spaces. Namely, if $k \geq 0$, the 0-th cohomology of $\mathbb{P}^{r-1}$ with values in $\mathcal{O}(k)$ is generated by the monomials $Q_1^{\mu_1} \cdots Q_r^{\mu_r}$, $\mu_i \geq 0$ and $\sum_{i=1}^r \mu_i = k$, while, if $k \leq -r$, the $r-1$-th cohomology with values in $\mathcal{O}(k)$ is generated (modulo coboundaries) by the monomials of the same kind, with $\mu_i < 0$ and $\sum_{i=1}^r \mu_i = k$ (in all other cases, the cohomologies are trivial). Then the isomorphism of Theorem 4.7 can be written as

$$H^*(\mathbb{P}^{r-1}, \mathcal{O}(-m+|\lambda|)) \ni Q_1^{\mu_1} \cdots Q_r^{\mu_r} \mapsto Q_1^{\mu_1} \cdots Q_r^{\mu_r} Q_{r+1}^{\lambda_{r+1}} \cdots Q_{n+1}^{\lambda_{n+1}} \in H^*(X, \mathcal{O}(mD_0))$$

for any $\lambda \in \mathbb{N}^{n-r+1}$. In particular, a basis of the cohomology of X with values in $\mathcal{O}(mD_0)$ is given (i) (in degree 0) by the monomials Q^ν with $\sum_{i=1}^r \nu_i + m = \sum_{i=r+1}^{n+1} \nu_i$ and $\nu_i \geq 0$, $i = 1, \ldots, n+1$, (see also Subsection 4.2) and (ii) (in degree $r-1$) by the monomials Q^ν with $\sum_{i=1}^r \nu_i + m = \sum_{i=r+1}^{n+1} \nu_i$ and $\nu_i < 0$ for $i = 1, \ldots, r$, and $\nu_i \geq 0$ for $i = r+1, \ldots, n+1$. It follows immediately that the 0-th cohomology of X with values in $\mathcal{O}(mD_0)$ is always infinite-dimensional and nontrivial, while the $r-1$-th cohomology is always finite-dimensional and is nontrivial exactly when $m \geq r$.

The $\mathfrak{sl}_{n+1}$-module structure on the 0-th cohomology was discussed in Subsection 4.2; we now consider the cohomology of degree $r-1$. We set $m = \ell + r$ for an integer ℓ; then, by Theorem 4.7 and the discussion thereafter, we need only discuss the case $\ell \geq 0$.

Theorem 4.8. *For X, I, D_0 as in Theorem 4.7, and $\ell \geq 0$, there is an isomorphism*

$$H^{r-1}(X, \mathcal{O}((\ell+r)D_0)) \cong L(\ell\varpi_1)$$

of $\mathfrak{sl}_{n+1}$-modules.

Proof. We first compute all possible primitive vectors in $H^{r-1}(X, \mathcal{O}((\ell+r)D_0))$; for this purpose, recall the expressions for the Chevalley generators e_i in Lemma 4.3. A basis element of $H^{r-1}(X, \mathcal{O}((\ell+r)D_0))$, represented uniquely by a monomial Q^ν, as described above, is annihilated by e_i, $i = r, \ldots, n$, if and only if $\nu_i = 0$ for $i = r+1, \ldots, n+1$: namely, we can use the explicit expressions computed in the proof of Lemma 4.3 for the Chevalley generators e_i, whence we get

$$e_i \cdot Q^\nu = \begin{cases} \nu_r \nu_{r+1} Q^{\nu - \varepsilon_r - \varepsilon_{r+1}}, & i = r, \\ \nu_{i+1} Q^{\nu + \varepsilon_i - \varepsilon_{i+1}}, & i > r. \end{cases} \tag{6}$$

Using the isomorphism of Theorem 4.7, the cohomology class $e_i \cdot Q^\nu$, by (6), is nontrivial, unless (since $\nu_r < 0$) ν_i vanish for $i = r+1, \ldots, n+1$. Thus, Q^ν has the form $Q_1^{\nu_1} \cdots Q_r^{\nu_r}$ with $\nu_i < 0$ and $\sum_{i=1}^r \nu_i = -\ell - r$. We consider the remaining Chevalley generators e_i, $i = 1, \ldots, r-1$; their action on Q^ν as above can also be easily computed:

$$e_i \cdot Q^\nu = -\nu_i Q^{\nu - \varepsilon_i + \varepsilon_{i+1}}, \quad i = 1, \ldots, r-1. \tag{7}$$

Since $\nu_i < 0$, $i = 1, \dots, r-1$, using again the isomorphism of Theorem 4.7, the cohomology class on the right hand-side of (7) vanishes if and only if $\nu_{i+1} \geq -1$, $i = 1, \dots, r-1$, using the well-known expressions for the $r-1$-th cohomology of $\mathbb{P}^{r-1}$. This implies immediately that $\nu_i = -1$, $i = 2, \dots, r$; the remaining exponent is automatically $\nu_1 = -\ell - 1$. Thus, for any $\ell \geq 0$, the only primitive vector in $H^{r-1}(X, \mathcal{O}((\ell + r)D_0))$ is the monomial $Q_1^{-\ell-1} Q_2^{-1} \cdots Q_r^{-1}$.

The primitive vector $Q_1^{-\ell-1} Q_2^{-1} \cdots Q_r^{-1}$ is also a weight vector: its weight is readily computed, since

$$x \cdot (Q_1^{-\ell-1} Q_2^{-1} \cdots Q_r^{-1}) = \ell x_1 (Q_1^{-\ell-1} Q_2^{-1} \cdots Q_r^{-1})$$

for any $x = \mathrm{diag}(x_1, \dots, x_{n+1}) \in \mathfrak{sl}_{n+1}$, which acts in this situation as the differential operator $-\sum_{i=1}^r x_i Q_i P_i + \sum_{i=r+1}^{n+1} x_i Q_i P_i - \sum_{i=1}^r x_i$ (see also the proof of Theorem 4.2). Therefore, $H^{r-1}(X, \mathcal{O}((\ell + r)D_0))$ has a unique primitive vector of weight $\ell \varpi_1$.

Moreover, the module $H^{r-1}(X, \mathcal{O}((\ell + r)D_0))$ is generated by the primitive monomial $Q_1^{-\ell-1} Q_2^{-1} \cdots Q_r^{-1}$. To see this, note that a general differential operator on $\mathcal{D}_{\mathcal{O}((\ell+r)D_0)}(X)$ is of the form $Q^\lambda P^\mu$, with λ, μ positive multiindices such that $\tau = \lambda - \mu$ satisfies $\sum_{i=1}^r \tau_i = \sum_{i=r+1}^{n+1} \tau_i$. On the other hand, Q^ν represents a unique generator of $H^{r-1}(X, \mathcal{O}((\ell + r)D_0))$ if $\nu_i < 0$, $i = 1, \dots, r$, $\nu_i \geq 0$, $i = r+1, \dots, n+1$, and $\sum_{i=1}^r \nu_i + \ell + r = \sum_{i=r+1}^{n+1} \nu_i$. For such a weight ν, the differential operator

$$D_\nu = \frac{(-1)^{\sum_{i=1}^r \nu_i + r}}{\prod_{j=0}^{-\nu_1-2} (\ell + 1 + j) \prod_{i=2}^r (-(\nu_i + 1))!} Q_1^\ell Q_{r+1}^{\nu_{r+1}} \cdots Q_{n+1}^{\nu_{n+1}} P_1^{-(\nu_1+1)} \cdots P_r^{-(\nu_r+1)}$$

belongs, by a direct computation, to $\mathcal{D}_{\mathcal{O}((\ell+r)D_0)}(X)$, and it is easy to verify that

$$Q^\nu = D_\nu (Q_1^{-\ell-1} Q_2^{-1} \cdots Q_r^{-1}).$$

All these computations show, by the very definition of Verma modules, that there is a surjective, nontrivial module homomorphism $M(l\varpi_1) \to H^{r-1}(X, \mathcal{O}((\ell+r)D_0))$, whose image is irreducible (since there is only one primitive vector up to multiplication by $\mathbb{C}^\times$); this yields the above isomorphism. $\square$

Notice that the highest weight $\ell\varpi_1$ is dominant, which reproves the fact that $H^{r-1}(X, \mathcal{O}((\ell + r)D_0))$, when nontrivial, is always finite-dimensional.

Remark 4.9. Theorem 4.8 refers only to $X = \tilde{Z}_r^+$ with $2 \leq r \leq n$ or $X = \widetilde{\mathbb{A}^n}$: the reason is that (i) the resolution of singularities $\tilde{Z}_r^-$, corresponding to $2 \leq r \leq n-1$, are related to $\tilde{Z}_{n-r}^+$ by an isomorphism of the corresponding fans, which lifts to the Chevalley involution of $\mathfrak{sl}_{n+1}$ on the corresponding (twisted) rings of differential operators, and (ii) if $r = 1$, then $X = \widetilde{\mathbb{A}^n}$, and the two blow-ups $r = 1$ and $r = n$ are related to each other by an automorphism of the corresponding fan, which also lifts to the Chevalley involution at the level of differential operators. Hence, by means of the Chevalley involution, we can deduce the $\mathfrak{sl}_{n+1}$-module structure of the twisted cohomologies of $\tilde{Z}_r^-$, $2 \le r \le n-1$, and of $\widetilde{\mathbb{A}^n}$, for $r = 1$, from Theorem 4.7 and 4.8.

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