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Buoyancy Sustained by Viscous Dissipation

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Abstract. The quasi-parallel regime of a Darcy–Boussinesq boundary-layer flow over a permeable vertical flat plate adjacent to a fluid saturated porous medium is considered. 'Quasi-parallel' means here a plane flow with a constant transversal velocity $v = -v_0$ directed perpendicularly towards the vertical surface, where a lateral suction with the same velocity $-v_0$ is applied. The plate is held at a constant temperature T_w which coincides with the ambient temperature T_∞ of the fluid. The heat released by viscous dissipation induces a density gradient in the fluid. Thus, although $T_w = T_\infty$, a thermal convection occurs. The steady regime of this 'self-sustaining buoyant flow' has been examined in detail. Wall jet-like profiles with a continuous but finite spectrum of the momentum flow have been found. These self-sustaining buoyant jets show a universal behavior, that is, there exist certain length, velocity and temperature scales such that the flow characteristics become independent of the (constant) material properties of the fluid and the porous medium as well.

Key words: buoyancy, viscous dissipation, boundary-layer, multiple solutions, buoyant jet, universal behavior.

1. Introduction

A buoyant flow always originates from a density gradient and it is always driven by gravity. In the usual thermal buoyancy problems, the primary density gradients are induced by a hot or a cold surface immersed in, or adjacent to a fluid. In contrast to such situations, the present paper investigates the quasi-parallel regime of a buoyant plane boundary-layer flow over a permeable vertical plate adjacent to a fluid saturated porous medium, assuming that:

(a) the plate is neither 'hot' nor 'cold', but it is held at a constant temperature which coincides with the ambient temperature T_∞ of the fluid,
(b) the primary density gradients are induced in the bulk of the fluid by heat release by viscous dissipation, and
(c) after the transients die out, a steady flow sets on.

Our main interest will be focused on the mechanical and heat transfer characteristics of this self-induced buoyant flow in its quasi-parallel regime. The quasi-parallel flow regime is a universal feature of the suction-controlled steady boundary-layer

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flows over 'long' permeable plane surfaces. It is realized when the convection towards the surface exactly balances the tendency of the boundary-layer to grow in the downstream direction. In viscous flow, the corresponding velocity field is known as the 'asymptotic suction profile'.

The present paper is mainly motivated by the increasing interest paid in the latter time to the effect of viscous dissipation in porous media (Nield, 2000; Tashtoush, 2000; Murthy, 2001; Magyari *et al*. 2002a, b; Magyari and Keller, 2002a, b).

2. Basic Equations

Following Nield and Bejan (1999), we write the mass, momentum and energy conservation equations (of a Darcy–Boussinesq boundary-layer flow) in the form:

$$u_x + v_y = 0 \tag{1}$$

$$u_y = \frac{g\beta K}{\upsilon} T_y \tag{2}$$

$$uT_x + vT_y = \alpha T_{yy} + \frac{\upsilon}{Kc_p} u^2 \tag{3}$$

Here x and y are the Cartesian coordinates along and normal to the plate, respectively (Figure 1), u and v are the velocity components along x and y axes, T is the fluid temperature, K is the permeability of the porous medium, g is the acceleration due to gravity, c_p is the specific heat at constant pressure, α, β and $\upsilon = \mu/\rho$ are the effective thermal diffusivity, thermal expansion coefficient and kinematic viscosity,

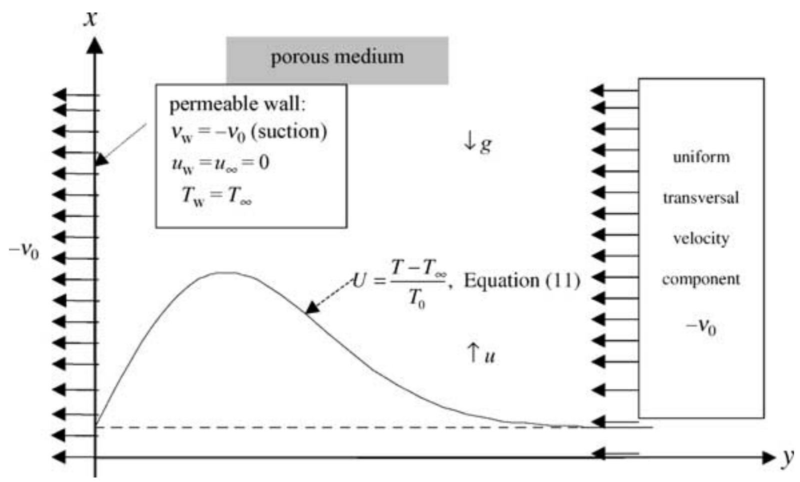

Figure 1. The choice of the coordinate system and the wall jet-like shape of the dimensionless downstream velocity U which, according to Equation (11), coincides with the dimensionless temperature profile $(T - T_\infty)/T_0$. This self-sustaining buoyant flow is quasi-parallel (i.e. $v = \text{const.} = -v_0$) and its thickness is constant (i.e. independent of x).

respectively, and the subscripts x and y indicate partial derivatives. The second term on the right-hand side of Equation (3) is proportional to the volumetric heat generation rate $q''' \equiv \mu u^2/K$ by viscous dissipation.

In the following we consider the quasi-parallel regime of a steady boundary-layer flow over a vertical permeable flat plate adjacent to a fluid saturated porous medium. It is assumed that the temperature of the plate is constant and coincides with the ambient temperature T_∞ of the reservoir fluid. Under 'quasi-parallel regime' we mean a plane flow having a constant transversal velocity component directed toward the plate:

$$v = \text{const.} \equiv -v_0, \quad v_0 > 0 \tag{4}$$

Hence, the boundary conditions accompanying Equations (1)–(3) read:

$$\text{On } y = 0: \quad v = -v_0 \text{ (suction)}, \; T = T_\infty \tag{5a}$$

$$\text{As } y \to \infty: \quad u \to 0, \; T = T_\infty \tag{5b}$$

Equation (2) and the boundary condition (5b) imply

$$T = T_\infty + \frac{\upsilon}{g\beta K} u \tag{6}$$

Thus, in the present case u approaches zero not only for $y \to \infty$ but, due to (5a) also for $y \to 0$, that is, similarly to the free convection of viscous fluids:

$$u|_{y\to 0} = u|_{y\to\infty} = 0 \tag{7}$$

As a consequence of assumption (4), all the physical quantities will depend only on the normal coordinate y. Thus, introducing the dimensionless coordinate Y and the dimensionless velocity field U defined by

$$Y = \frac{y}{L}, \quad L = \frac{\alpha}{v_0} \tag{8}$$

$$U(Y) = \frac{u(y)}{u_0}, \quad u_0 = \frac{c_p v_0^2}{g\beta\alpha} \tag{9}$$

the problem (1)–(5b) reduces to the following nonlinear two-point boundary value problem:

$$U'' + U' + U^2 = 0 \tag{10a}$$

$$U(0) = 0 \tag{10b}$$

$$U(\infty) = 0 \tag{10c}$$

The temperature field (6) in terms of U is given by

$$T(y) = T_\infty + T_0 \cdot U(Y), \quad T_0 = \frac{\upsilon u_0}{g\beta K} \tag{11}$$

In the above equations the primes denote derivatives with respect to Y.

3. Solution

If the effect of viscous dissipation (i.e. the quadratic term of Eq. (10a)) is neglected, the boundary value problem (10a)–(10c) admits only the trivial solution $U \equiv 0$, which in turn implies as expected: $T \equiv T_\infty$ and $u \equiv 0$. If, however, this effect is taken into account, problem (10a)–(10c) admits nontrivial solutions, but unfortunately not in a closed analytical form. They can easily be investigated numerically with the aid of the usual shooting method, by replacing the boundary value problem (10a)–(10c) by the initial value problem

$$U'' + U' + U^2 = 0 \tag{12a}$$

$$U(0) = 0 \tag{12b}$$

$$U'(0) = U'_{\mathrm{w}} \tag{12c}$$

along with the additional condition $U(\infty) = 0$. Now, the quantities of interest are the possible values of the ‘initial velocity’ $U'(0) = U'_{\mathrm{w}}$.

A first valuable information about U'_{w} can be extracted from the integral relationship

$$U'_{\mathrm{w}} = \int_0^\infty U^2(Y)\,\mathrm{d}Y \tag{13}$$

which has been obtained from Equations (12a)–(12c) and $U(\infty) = 0$ by standard operations.

Equation (13) shows that U'_{w} is necessarily positive. This means, in full agreement with physical expectation, that in the present problem heat is transferred everywhere from the fluid to the surface. Indeed, the corresponding wall heat flux and Nusselt number given by

$$q_{\mathrm{w}} = -k\frac{\partial T}{\partial y}\bigg|_{y=0} \equiv \frac{kT_0}{L}\mathrm{Nu}, \tag{14a}$$

$$\mathrm{Nu} = -U'_{\mathrm{w}} \tag{14b}$$

are strictly negative quantities (k denotes the effective thermal conductivity of the porous medium).

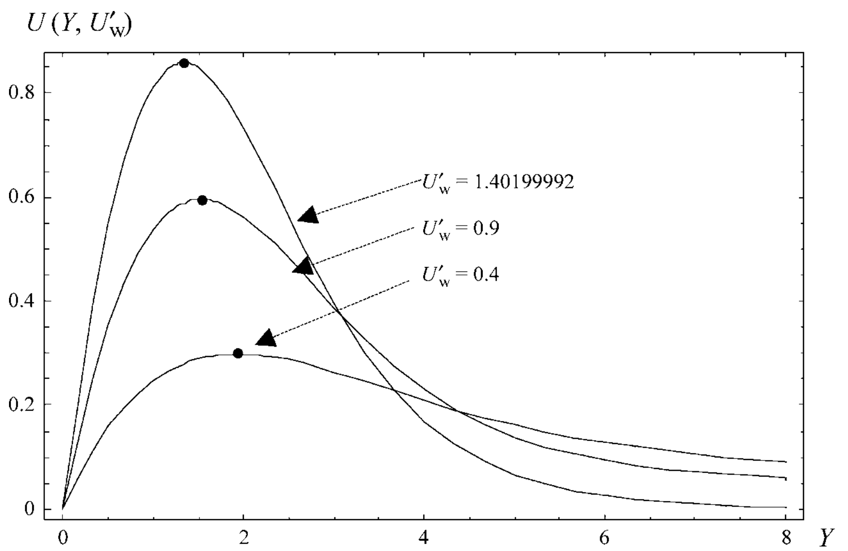

Figure 2. Plot of the multiple solutions $U(Y)$ versus Y for three selected values of U'_{w} belonging to the interval (15).

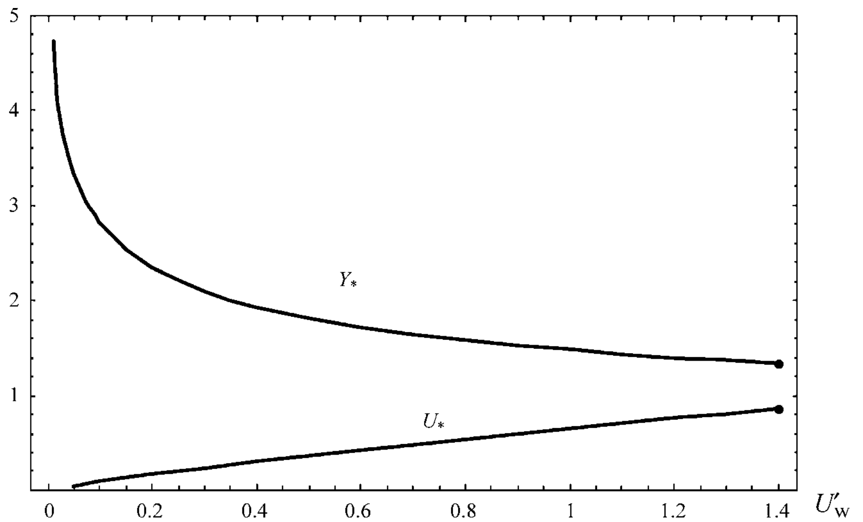

Figure 3. The maximum U_* of the velocity profile $U = U(Y)$ and its distance Y_* from the wall, plotted against U'_{w} in the range (15). The limiting values $U_{*,\max} = 0.858576$ and $Y_{*,\max} = 1.335061$ corresponding to the upper limit of the existence domain (15) are marked by dots.

Now, a detailed (but elementary) numerical approach shows that the initial value problem (12a)–(12c) with the additional condition $U(\infty) = 0$ admits multiple solutions. These multiple solutions correspond to (the nondenumerable set of) values of U'_{w} belonging to the interval

$$0 < U'_{\mathrm{w}} \leqslant U'_{\mathrm{w},\max} = 1.40199992 \cong 1.402 \tag{15}$$

Hence, our boundary value the problem (12a)–(12c) admits infinitely many solutions corresponding to the values of the 'initial velocity U_{w}'' in the interval (15). As an illustration, in Figure 2 some of these multiple solutions are plotted versus Y. One sees that with increasing values of U_{w}', the maximum $U_* = U(Y_*)$ of $U(Y)$ also increases, while its distance from the wall Y_* decreases. For $U_{\mathrm{w}}' = 0.4$ for example, $(U_*, Y_*) = (0.298,\ 1.934)$, while for $U_{\mathrm{w}}' = 0.9$ one has $(U_*, Y_*) = (0.595,\ 1.526)$. As U_{w}' approaches its maximum value, $U_{\mathrm{w,max}}' = 1.40199992$, that is, the upper limit of the existence domain (15), U_* and Y_* approach the limiting values $(U_*, Y_*) = (0.858576,\ 1.335061)$. Conversely, as $U_{\mathrm{w}}' \to 0$, one obtains $U_* \to 0$ and $Y_* \to \infty$, respectively. This behavior of U_* and Y_* is shown in Figure 3 for the whole range (15) of the 'initial velocity' U_{w}'. The above somewhat surprising features of the boundary value problem (15) are 'explained' in Appendix A in terms of a simple point mechanical motion.

4. Discussion

As suggested already by the boundary conditions (7) and shown by the velocity profiles of Figure 2 explicitly, the self-sustaining buoyant flow investigated in this paper possesses the features of a wall jet (moving along a permeable wall with a constant lateral suction applied).

This steady boundary-layer flow shows a universal behavior, that is, watched on the velocity, temperature and length scales u_0, T_0 and L (specified by Eqs. (8), (9) and (11), respectively), its velocity and temperature fields become independent of the material properties both of the fluid and the porous medium:

$$\frac{u(y)}{u_0} = \frac{T(y) - T_\infty}{T_0} = U(Y) \tag{16}$$

The momentum flow $\dot{I}\,(\mathrm{kg\,m/s^2})$ of this self-sustaining buoyant jet across any ($x =$ const.) cross-section of width b is given (in the leading order approximation) by

$$\dot{I} = \rho_\infty b \int_0^\infty u^2(y)\,\mathrm{d}y \tag{17}$$

Thus, its dimensionless counterpart

$$\dot{J} \equiv \frac{\dot{I}}{\rho_\infty u_0^2 b L} = \int_0^\infty U^2(Y)\,\mathrm{d}Y \tag{18}$$

equals according to Equation (13) the wall slope $U_{\mathrm{w}}' = U'(0)$ of the velocity profile, which in turn coincides according to Equation (14b) with the absolute value of the (negative) Nusselt number,

$$\dot{J} = U_{\mathrm{w}}' = |\mathrm{Nu}| \tag{19}$$

Therefore, the existence domain (15) of the solutions of our boundary value problem (10a)–(10c) specifies at the same time the spectrum of the allowed values of the momentum flow and that of the Nusselt number. In physical terms this result means that the quasi-parallel regime of our free convection boundary-layer flow (i.e. the situation in which the tendency of the boundary-layer to an unlimited downstream growth is exactly balanced by the convection towards the wall due to the uniform suction) can be approached on several different ways, which are characterized by different values of the jet momentum of the flow. This latter quantity takes values in a continuous but finite interval which coincides with the existence domain (15) of the boundary-layer flows themselves and at the same time with the negative of the corresponding wall heat flow (Nusselt number). Obviously, the 'option' of our self-sustaining buoyant jet for one or the other of these possibilities, is a 'decision' taken during its transient evolution and cannot be determined from its fully developed steady regime ('memory lost' in processes governed by parabolic equations). The comprehensive discussion of this challenging problem is out of the scope of the present note.

5. Summary and Conclusions

In this paper a new effect of viscous dissipation in fluid saturated porous media has been reported: its ability to give rise along a vertical plat plate to a self-sustaining wall jet-like steady buoyant flow. The plate has neither to be 'hot' nor 'cold' but it is held at a constant temperature which coincides with the ambient temperature of the fluid. However, it must be permeable and a uniform lateral suction must be applied.

In the paper the quasi-parallel regime of this self-sustaining free convection flow has been examined inn detail. This regime is characterized by a constant (i.e. x-independent) width of the boundary-layer which corresponds to an exact balance of its tendency to an unlimited downstream growth, through the convection towards the wall due to the uniform suction. The flow also shows in this regime a universal behavior: its mechanical and heat transfer characteristics (if watched on suitable velocity, temperature and length scales) become independent of the material properties both of the fluid and the porous medium. This universal behavior of the flow is associated with a continuous and finite spectrum of values both of its jet momentum flow and of the Nusselt number as well.

Finally, it is worth mentioning here that, if the (permeable) vertical plane surface is a plate of finite or semi-infinite length, that is, if a leading edge ($x = 0$) exists, the steady flow will approach its quasi-parallel regime $v =$ const. beyond some distance x_0 from the leading edge.

This crossover to quasi-parallel flow regime is a general feature of the two-dimensional boundary-layer flows over 'long' permeable plane surfaces with uniform suction. The corresponding velocity field is known in theory of viscous flow as the 'asymptotic suction profile' (see e.g. Schlichting and Gersten, 1997). The crossover of the Blasius flow (in the presence of a uniform suction) to the quasi-parallel regime takes place at $x_0 = 4\upsilon U_\infty/v_0^2$ as estimated by Iglisch several decades ago (Iglisch, 1944). For water at 20°C with $U_\infty = 25\,\text{cm/s}$ and $v_0 = 1\,\text{cm/s}$, this length amounts to $x_0 \cong 1\,\text{cm}$.

Appendix A

Our aim in this appendix is to discuss the boundary value problem (10a)–(10c) in terms of the analogous point mechanical motion

$$U'' = -U' - \frac{\mathrm{d}W}{\mathrm{d}U} \tag{A.1a}$$

$$U(0) = 0 \tag{A.1b}$$

$$U(\infty) = 0 \tag{A.1c}$$

Equation (A.1a) describes the 1D motion of a particle of coordinate U, mass $M = 1$ and potential energy $W = U^3/3$ in the presence of a viscous friction with friction coefficient 1. The dimensionless normal coordinate Y of our flow problem plays now the role of time variable of the particle motion. Thus, according to (A.1b) and (A.1c) the particle starts at the 'instant' $Y = 0$ in the origin $U = 0$ of the coordinate axis and returns after an infinite time $Y \to \infty$ to this point (Figure 4). It is also required that the terminal velocity of the particle vanishes, $U'(\infty) = 0$. In our flow problem this (tacitly imposed condition) ensures that at infinity the heat flux becomes zero (conservation of energy). Obviously, the particle must be started with a positive initial velocity $U'(0) \equiv U'_\mathrm{w} > 0$, that is, to the right (negative initial velocities lead to unbounded motions toward $U \to -\infty$). The presence of the friction force $-U'$ makes such a motion (which starts and ends at the inflexion point $U = 0$ of the potential energy function W) basically possible.

The particle moves first to the right until it reaches its turning point $U = U_*$ at an instant $Y = Y_*$, where $U'(Y_*) = 0$. In this time interval $[0, Y_*]$ it consumes a part of its initial kinetic energy $U'^2_\mathrm{w}/2$ to overcome the friction while the remaining part is converted in potential energy, that is,

$$\frac{U'^2_\mathrm{w}}{2} = \frac{U_*^3}{3} + \int_0^{Y_*} U'^2(Y)\,\mathrm{d}Y \tag{A.2}$$

Indeed, this energy balance of the particle motion also could be obtained formally by multiplying Equation (A.1a) by U' and then integrating it from $Y = 0$ to $Y = Y_*$. Integrating instead from 0 to ∞, we obtain (for the motion ending in $U = 0$):

$$\frac{U'^2_\mathrm{w}}{2} = \int_0^{\infty} U'^2(Y)\,\mathrm{d}Y \tag{A.3}$$

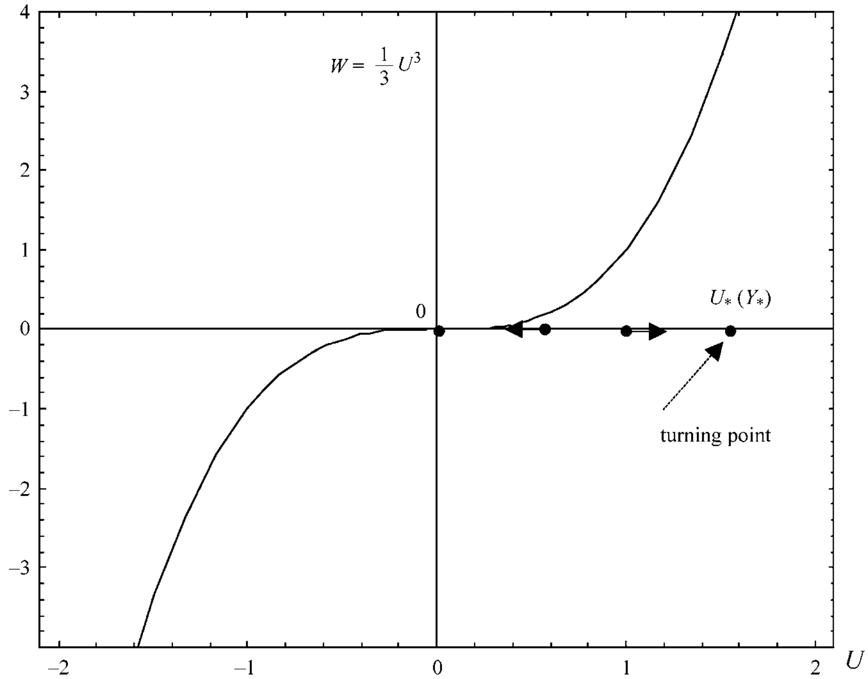

Figure 4. Potential energy of the analogous point mechanical motion. The turning point of the particle motion corresponds to the maximum velocity of the self-sustaining buoyant jet.

This equation shows that the solution of the problem (A.1a)–(A.1c) describes that motion of the point particle in which its whole initial kinetic energy is dissipated to overcome the friction during the motion, $0 \leqslant Y \leqslant \infty$.

Now, it is clear that the maximum velocity U_* of our self-sustaining buoyant jet discussed in Sections 3 and 4 corresponds to the distance of the turning point of the particle motion from the origin, while the distance Y_* from the wall where the maximum jet velocity U_* is reached, corresponds to the instant Y_* at which the turning point is attained. Both U_* and Y_* depend on the initial velocity U'_w of the particle, that is, on the slope of the jet velocity profile at the wall. This dependence of U_* and Y_* on U'_w is shown in Figure 3. As expected, the distance U_* of the turning point from the origin increases with increasing U'_w. However, the smaller the initial velocity U'_w, the longer the time interval $[0, Y_*]$ of this motion from $U = 0$ to $U = U_*$. This latter result (i.e. an increasing Y_* with a decreasing U'_w) is a direct consequence of the velocity dependence of the dissipation force $-U'$.

The physical 'feeling' tells us that this kind of particle motion, which starts and ends on the infinitesimal plateau of the inflexion point $U = 0$ of the potential energy $W = U^3/3$, could be realized not only for a single but for several values of the initial velocity U'_w. Unfortunately, in the present case we could prove this conjecture only numerically, obtaining for the allowed values of the initial velocity the range (15). However, concerning the existence of such multiple solutions, it is instructive to compare the nonlinear problem (A.1a)–(A.1c) to its linear counterpart.

$$U'' + U' + cU = 0 \tag{A.4a}$$

$$U(0) = 0 \tag{A.4b}$$

$$U(\infty) = 0 \tag{A.4c}$$

which describes the damped motion of a linear oscillator of spring constant c. This problem admits solutions (with $U'(\infty) = 0$ and without any overshoot in the negative domain $U < 0$) only for $0 < c \leqslant 1/4$. These solutions are

$$U(Y) = U'_{\mathrm{w}} \cdot Y\,\mathrm{e}^{-Y/2} \quad \text{for } c = \frac{1}{4} \tag{A.5}$$

and

$$U(Y) = \frac{2U'_{\mathrm{w}}}{\sqrt{1-4c}}\,\mathrm{e}^{-Y/2} \sinh\left(\frac{1}{2}\sqrt{1-4c} \cdot Y\right) \quad \text{for } 0 < c < \frac{1}{4} \tag{A.6}$$

and correspond to the critically and overcritically damped regime of the oscillator, respectively. We see therefore that the linear problem (A.4a)–(A.4c), similarly to the nonlinear one (A.1a)–(A.1c) also admits infinitely many solutions: one for every specified value of the initial velocity $U'_{\mathrm{w}} = U'(0) > 0$. However, we are faced here with a ‘dubly infinite’ multiplicity of the solutions since (A.5) and (A.6) exist for any positive U'_{w}. In other words, the solutions are also in the linear case multiple but their domain of existence is not limited by some finite maximum value $U'_{\mathrm{w,max}}$ of the initial velocity, but it extends to $0 < U'_{\mathrm{w}} < \infty$. Another special feature of the linear problem is that U_* increases proportional with U'_{w}, while the corresponding Y_* is independent of U'_{w}.

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