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ORIGINAL ARTICLE

The Geneva stellar evolution code

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Abstract This paper presents the Geneva stellar evolution code with special emphasis on the modeling of solar-type stars. The basic input physics used in the Geneva code as well as the modeling of atomic diffusion is first discussed. The physical description of rotation is then presented. Finally, the modeling of magnetic instabilities and transport of angular momentum by internal gravity waves is briefly summarized.

Keywords Stars: evolution

1 Introduction

The Geneva evolution code was mainly developed and used for the computation of models of massive stars. These efforts resulted in the publication of grids of stellar models computed for a wide range of masses and metallicities which have been extensively used by the astronomical community (Maeder and Meynet 1987; Schaller et al. 1992; Schaerer et al. 1993; Charbonnel et al. 1996; Meynet et al. 1994; Mowlavi et al. 1998). Recently, rotational effects have been included in the Geneva code (see Meynet and Maeder 1997, and other papers of the series). These rotating models are found to successfully reproduce many observational features of massive stars (see for instance Maeder and Meynet 2004a).

The Geneva evolution code was also used for the computation of stars at very different evolutionary stages, from pre-sequence evolutionary models including the effects of accretion (Behrend and Maeder 2001) to pre-supernova evolution of rotating massive stars (Hirschi et al. 2004). We also mention the study of rotating stars at very low metallicity (see e.g. Meynet et al. 2006; Ekström et al. 2006). Concerning low mass stars, specific grids of models have been computed with the Geneva code (see Charbonnel et al. 1996, 1999) as well as detailed models of solar-type stars for which p-mode frequencies have been observed (Eggenberger et al. 2004, 2005a, Carrier et al. 2005; Eggenberger and Carrier 2006).

In this paper, the input physics introduced in the Geneva evolution code for the modeling of solar-type stars is discussed. Section 2 is dedicated to the description of the basic input physics. The modeling of atomic diffusion is discussed in Sect. 3. The physical description of rotation is presented in Sect. 4. The modeling of the Tayler–Spruit dynamo and the transport of angular momentum by internal gravity waves is summarized in Sect. 5, while the conclusion is given in Sect. 6.

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2 Basic input physics

2.1 Equations of stellar evolution

Four equations describe the evolution of the structure of the star. Since rotational effects are included in the Geneva code, spherical symmetry is no longer valid and the effective gravity (sum of the centrifugal force and gravity) can no longer be derived from a potential. By assuming that there is a strong horizontal (along isobars) turbulence, the angular velocity is then constant on isobars (Zahn 1992). The case is referred to as shellular rotation and is described in (Meynet and Maeder 1997) (see also Sect. 4). In this scheme, the four structure equations are the following:

- Hydrostatic equilibrium:

$$\frac{\partial P}{\partial M_P} = -\frac{GM_P}{4\pi r_P^4} f_P. \tag{1}$$

- Continuity equation:

$$\frac{\partial r_P}{\partial M_P} = \frac{1}{4\pi r_P^2 \bar{\rho}}. \tag{2}$$

- Energy conservation:

$$\begin{aligned}\frac{\partial L_{\mathrm{P}}}{\partial M_{\mathrm{P}}} &= \epsilon_{\mathrm{nucl}} - \epsilon_\nu + \epsilon_{\mathrm{grav}} \\ &= \epsilon_{\mathrm{nucl}} - \epsilon_\nu - c_{\mathrm{P}} \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}.\end{aligned} \tag{3}$$

- Energy transport equation:

$$\frac{\partial \ln \overline{T}}{\partial M_P} = -\frac{GM_P}{4\pi r_P^4} f_P \min\left[\nabla_{\mathrm{ad}}, \nabla_{\mathrm{rad}} \frac{f_T}{f_P}\right] \tag{4}$$

where

$$\nabla = \nabla_{\mathrm{ad}} = \frac{P\delta}{\overline{T}\bar{\rho} c_{\mathrm{P}}} \quad \text{in convective zones,} \quad \text{and}$$

$$\nabla = \nabla_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{\kappa l P}{m\overline{T}^4} \quad \text{in radiative zones,}$$

$$f_P = \frac{4\pi r_P^4}{GM_P S_P}\frac{1}{\langle g^{-1}\rangle},$$

$$f_T = \left(\frac{4\pi r_P^2}{S_P}\right)^2 \frac{1}{\langle g\rangle\langle g^{-1}\rangle}.$$

$\langle x\rangle$ is x average on an isobaric surface, $\overline{x}$ is x average in the volume separating two successive isobars and the index P refers to the isobar with a pressure equal to P, while other variables have their usual meaning (see Meynet and Maeder 1997).

To solve these equations, the following physical ingredients are required:

- the nuclear reaction rates in order to evaluate ϵ_{nucl} and ϵ_ν (see Sect. 2.2);
- the equation of state to determine ρ and the needed thermodynamic quantities (see Sect. 2.4);
- the opacities to calculate ∇_{rad} (see Sect. 2.3);
- a treatment of convection to compute the convective flux (see Sect. 2.5).

On top of that, the equations of the evolution of chemical elements abundances are to be followed. In the Geneva evolution code, these equations are calculated separately from the structure equations according to time splitting method. Moreover, when atomic diffusion is included in the computation of a stellar model, equations describing the variation of the chemical composition due to diffusion are needed (see Sect. 3).

2.2 Nuclear reactions

2.2.1 Nuclear networks

For hydrogen burning the *pp* chains and the CNO tri-cycle are calculated in detail and the evolution of the main nuclear species is followed explicitly. For helium burning, we take into account the following reactions:

- the 3α reaction,
- $^{12}\mathrm{C}(\alpha,\gamma)\,^{16}\mathrm{O}(\alpha,\gamma)\,^{20}\mathrm{Ne}(\alpha,\gamma)\,^{24}\mathrm{Mg}$,
- $^{13}\mathrm{C}(\alpha,n)\,^{16}\mathrm{O}$,
- $^{14}\mathrm{N}(\alpha,\gamma)\,^{18}\mathrm{F}(\beta,\nu)\,^{18}\mathrm{O}(\alpha,\gamma)\,^{22}\mathrm{Ne}(\alpha,n)\,^{25}\mathrm{Mg}$,
- $^{17}\mathrm{O}(\alpha,n)\,^{20}\mathrm{Ne}$,
- $^{22}\mathrm{Ne}(\alpha,\gamma)\,^{26}\mathrm{Mg}$.

The system of nuclear reactions and the abundances variations are then determined for 15 isotopes: H, ^{3}He, ^{4}He, ^{12}C, ^{13}C, ^{14}N, ^{15}N, ^{16}O, ^{17}O, ^{18}O, ^{20}Ne, ^{22}Ne, ^{24}Mg, ^{25}Mg, and ^{26}Mg. The values of the following isotopic ratios ^{3}He/He, C/^{13}C, ^{14}N/^{15}N, O/^{18}O, ^{18}O/^{17}O, ^{21}Ne/^{20}Ne, ^{22}Ne/^{20}Ne, ^{25}Mg/^{24}Mg, and ^{26}Mg/^{24}Mg used in the Geneva code are listed in Maeder (1983) who chose, when available, the ratios given by radioastronomical observations of the interstellar material. The code can also account for the Ne–Na and Mg–Al chains in H-burning regions (see e.g. Meynet et al. 1997) and for the neutron capture reactions during He-burning (see e.g. Meynet and Arnould 2000). Note that for the computation of massive PopIII stars, a proper treatment of H-burning has been set up (see Ekström et al. 2006, for more details).

The Geneva code is also used to compute stellar models during the advanced stages of evolution (see Hirschi et al. 2004, for more details). The list of elements followed during C-, Ne-, O- and Si-burnings is then: α, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ^{28}Si, ^{32}S, ^{36}Ar, ^{40}Ca, ^{44}Ti, ^{48}Cr, ^{52}Fe and ^{56}Ni.

2.2.2 Reaction rates

The main inputs of nuclear reaction networks are the thermonuclear reaction rates. In the Geneva code, the NACRE nuclear reaction rates (Angulo et al. 1999) are used. Note that the numerical tables of the NACRE compilation are used and not the analytic fits. Screening factors are included according to the prescription by Graboske et al. (1973).

2.2.3 Numerical scheme

The chemical changes due to secular evolution are computed implicitly: we start with a model at time t^n for which the correct structure S^n (pressure profile, temperature profile, etc.) and chemical composition C^n is known. In order to compute the same quantities at time $t^{n+1} = t^n + \delta t$, an initial estimate of the chemical composition C_0^{n+1} is made. The approximate solution for the structure is then improved during the first iteration (the Henyey method is used) leading to S_1^{n+1}. These refined profiles of pressure and density are then used to compute a new chemical composition C_1^{n+1}. The same scheme is iterated as long as necessary to obtain the desired accuracy. Since the nuclear reaction network is resolved at each iteration, this procedure guarantees a good consistency between the structure and chemical composition at each iteration. Details regarding the computation of the nuclear network and the associated abundance variations are given in (Maeder 1983).

2.3 Opacities

The opacities are needed to calculate the energy transport by the radiative transfer and to determine the radiative gradient $\nabla_{\rm rad}$. Two groups, the OPAL group (Iglesias and Rogers 1996) and the Opacity Project (OP) group (Seaton et al. 1994), obtain very similar results using a different approach for computing opacities. In the Geneva evolution code, we use the opacity tables from the OPAL group complemented at low temperatures with the molecular opacities of Alexander and Ferguson (1994). Note that opacity tables are available for the standard solar abundances of (Grevesse and Noels 1993) and (Grevesse and Sauval 1998) as well as for the new solar abundances of (Asplund et al. 2004).

2.4 Equation of state

The equation of state describes the relation between the three physical parameters (p, T, ρ). For a given chemical composition, it enables the determination of the third physical parameter from the two others and also the determination of the thermodynamic quantities needed for the computation of a stellar model ($\nabla_{\rm ad}$, δ, $c_{\rm p}$, Γ_1, ...). In the Geneva evolution code, the equation of state usually used is a general equation of state (see Schaller et al. 1992). Although this general equation of state is perfectly suited for models of massive stars, the computation of reliable models of solar-type stars required a more specific and realistic equation of state. Indeed, in the case of low mass stars, non ideal effects, such as Coulomb interactions become important. Two different equations of state are included in the Geneva evolution code for solar-type stars models: the MHD equation of state (Hummer and Mihalas 1988; Mihalas et al. 1988; Daeppen et al. 1988) and the OPAL equation of state (Rogers et al. 1996; Rogers and Nayfonov 2002).

2.5 Convection and overshooting

A prescription for the energy transport by convection is required to calculate the temperature gradient in a convective zone. In the Geneva code, the stability of a given layer is determined by using the Schwarzschild criterion. In the outer convective zone, the standard mixing-length formalism for convection is used (Böhm-Vitense 1958). The basic idea of the mixing-length theory (MLT) is to express the parameters of the non local phenomena of convection in terms of local quantities. The value of the mixing length l, usually expressed as a dimensionless parameter $\alpha \equiv l/H_{\rm p}$ (with $H_{\rm p}$ the pressure scale height), is a free parameter of this formalism. Concerning the stellar core, an overshoot of the convective core into the radiative zone on a distance $d_{\rm ov} \equiv \alpha_{\rm ov} \min[H_{\rm p}, r_{\rm core}]$ can be included in the computation.

We also note that for red supergiants with luminosities brighter than $M_{\rm bol} > -8.5$ ($M_{\rm ini} \gtrsim 25\ {\rm M}_\odot$), the structure of the outer convective envelope is complex. The acoustic flux seems to be the dominant mode of energy transport. In that case, the treatment of convection in the Geneva code includes turbulent pressure, acoustic flux and a density scale height (see Maeder 1987).

In the advanced stages of evolution, we also mention that convective diffusion replaces instantaneous convection from oxygen burning onwards because the mixing timescale becomes longer than the evolution timescale at that point. The numerical method used for this purpose is the method used for rotational diffusive mixing (Meynet et al. 2004). The mixing length theory is then used to derive the corresponding diffusion coefficient.

2.6 Mass loss

For low mass stars ($M \lesssim 7\,{\rm M}_\odot$) the main sequence evolution is calculated at constant mass. On the red giant branch and on the asymptotic giant branch, the mass loss becomes however non negligible and is taken into account by using the prescription by Reimers (1975): $\dot{M} = 4 \times 10^{-13} \eta L R/M$ (in ${\rm M}_\odot\ {\rm yr}^{-1}$) with $\eta = 0.5$ (see Maeder and Meynet 1989).

Mass loss plays a key role in the physics and evolution of massive stars (see e.g. Maeder et al. 2005). For these stars, the prescription of Vink et al. (2001) is adopted. Note that the expressions by de Jager et al. (1988) are used when the prescription by Vink et al. does not apply. Mass loss rates follow a scaling relation with the metallicity Z of the type

$$\dot{M}(Z) = \left(\frac{Z}{Z_\odot}\right)^{\alpha} \dot{M}(Z_\odot) \tag{5}$$

where $\dot{M}(Z)$ is the mass loss rate when the metallicity is equal to Z and $\dot{M}(Z_\odot)$ is the mass loss rate for the solar metallicity. In the metallicity range from 1/30 to 3.0 times solar, the value of α is between 0.5 and 0.8 according to stellar wind models (see Kudritzki et al. 1987; Leitherer et al. 1992; Vink et al. 2001).

2.6.1 Anisotropic stellar winds

The Geneva evolution code also includes anisotropies of the mass loss by stellar winds (Maeder 2002). It is indeed interesting to recall that a rotating star has a non uniform surface brightness, and the polar regions are those which have the most powerful radiative flux. Thus one expects that the star will lose mass preferentially along the rotational axis. This is correct for hot stars, for which the dominant source of opacity is electron scattering. In that case the opacity only depends on the mass fraction of hydrogen and does not depends on other physical quantities such as temperature. Thus rotation induces anisotropies of the winds (Maeder and Desjacques 2001; Dwarkadas and Owocki 2002). The quantity of mass lost through radiatively driven stellar winds is enhanced by rotation. The ratio of the mass loss rate of a star with a surface angular velocity Ω to that of a non-rotating star, of the same initial mass, metallicity and lying at the same position in the HR diagram is given by (see Maeder and Meynet 2000):

$$\frac{\dot{M}(\Omega)}{\dot{M}(0)} \simeq \frac{(1-\Gamma)^{1/\alpha-1}}{[1-\frac{4}{9}(v/v_{\mathrm{crit},1})^2-\Gamma]^{1/\alpha-1}} \tag{6}$$

where Γ is the electron scattering opacity for a non-rotating star with the same mass and luminosity and α is a force multiplier (Lamers et al. 1995).

3 Atomic diffusion

Atomic diffusion on H, He, C, N, O, Ne and Mg is included in the Geneva evolution code by using the routines developed for the Toulouse–Geneva version of the code (see for example Richard et al. 1996).

The chemical changes due to diffusion are computed separately from the changes due to nuclear reactions. For the 15 isotopes included in the nuclear network (see Sect. 2.2), the abundance variations due to the nuclear reactions are first computed. The diffusion equation is then solved separately for all the isotopes except hydrogen.

Diffusion due to concentration and thermal gradients is included, but the radiative acceleration is neglected as it is negligible for the structure of the low-mass stellar models with extended convective envelopes (Turcotte et al. 1998). The computation of atomic diffusion is based on the Boltzmann equation for dilute collision-dominated plasmas with the use of the Chapman–Enskog method (Chapman and Cowling 1970) to solve this equation.

3.1 The diffusion equations

The diffusion equation describes the evolution of the number concentration c of a given element in the stellar interiors. Its general form in spherical coordinates can be written as:

$$\rho\frac{\partial c}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho D\frac{\partial c}{\partial r} - r^2\rho c V\right) - \lambda\rho c \tag{7}$$

where D is the diffusion coefficient, V the atomic diffusion velocity and λ the nuclear reaction rate. As mentioned above, the chemical changes due to nuclear reactions are calculated separately from the changes due to diffusion. Thus, λ is only included in the diffusion equations for lithium and beryllium, which are treated separately from the nuclear network.

In the formalism of the Chapman–Enskog method used in the Geneva code, the diffusion equations for the different isotopes are written in Lagrangian coordinates as:

$$\frac{\partial c_i}{\partial t} = D'_{1i}\frac{\partial^2 c_i}{\partial m_r^2} + \left(\frac{\partial D'_{1i}}{\partial m_r} - V'_{1i}\right)\frac{\partial c_i}{\partial m_r} - \left(\frac{\partial V'_{1i}}{\partial m_r} + \lambda_i\right)c_i \tag{8}$$

where, as before, c_i is the number concentration of element i and λ_i is the nuclear reaction rate. The diffusion coefficient D'_{1i} is given by

$$D'_{1i} = (4\pi\rho r^2)^2\,(D_{\mathrm{turb}} + D_{1i}) \tag{9}$$

where D_{turb} corresponds to the effective macroscopic diffusion coefficient (see Sect. 4), while D_{1i} represents the atomic diffusion coefficient of element i relative to hydrogen (element 1). V'_{1i} is then given by

$$V'_{1i} = (4\pi\rho r^2)V_{1i}, \tag{10}$$

with

$$V_{1i} = -D_{1i}\left(A_i - \frac{Z_i}{2} - \frac{1}{2}\right)\left(\frac{m_H}{kT}\frac{Gm_r}{r^2}\right) + D_{1i}\alpha_{1i}\nabla\ln T, \tag{11}$$

where α_{1i} is the thermal diffusion coefficient, A_i the atomic mass number of element i, Z_i the charge of element i, and m_{H} is the atomic hydrogen mass.

3.2 Computation of the diffusion coefficients

The atomic diffusion coefficient D_{1i} and the thermal diffusion coefficient α_{1i} are computed with the formalism of Paquette et al. (1986). Using the Chapman–Enskog method, the atomic diffusion coefficient for collisions involving particles s and t can be expressed as:

$$D_{st} = \frac{3E}{2nm(1-\Delta)} \tag{12}$$

where n is the total particle number density and m is the sum of the masses of the particles. The terms E and Δ contain the values of the collision integrals $\Omega_{st}^{(i,j)}$ according to (9–17) of Paquette et al. (1986). In the same way, the thermal diffusion coefficient α_{st} is given by

$$\alpha_{st} = \frac{5C(c_s S_s - c_t S_t)}{c_s^2 Q_s + c_t^2 Q_t + c_s c_t Q_s t} \tag{13}$$

where c_i is the number concentration of particles of species i ($i = s, t$), while the terms C, S_s, S_t, Q_s, Q_t, and Q_{st} contain the values of the collision integrals $\Omega_{st}^{(i,j)}$ and are defined in (9–17) of Paquette et al. (1986). In order to compute the diffusion coefficients D_{st} and α_{st}, one needs to determine the values of the collision integrals.

The collision integrals $\Omega_{st}^{(i,j)}$ are given by (Paquette et al. 1986):

$$\Omega_{st}^{(i,j)} = \left(\frac{kT}{2\pi m M_s M_t}\right)^{(1/2)} \int_0^\infty e^{-g^2} g^{2j+3} \phi_{st}^{(i)} \,\mathrm{d}g, \tag{14}$$

with

$$\phi_{st}^{(i)} = 2\pi \int_0^\infty (1 - \cos^i \chi_{st}) b \,\mathrm{d}b \quad \text{and} \tag{15}$$

$$\chi_{st} = \pi - 2 \int_{r_{st}^{\min}}^\infty b \,\mathrm{d}r \left\{ r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{st}(r)}{g^2 kT} \right]^{1/2} \right\}^{-1}, \tag{16}$$

with $V_{st}(r)$ the interaction potential and $r_{st}^{\min}$ the distance of closest approach given by the solution of the equation:

$$1 - \frac{b^2}{(r_{st}^{\min})^2} - \frac{V_{st}(r_{st}^{\min})}{g^2 kT} = 0. \tag{17}$$

In the formalism of Paquette et al., the collision integrals are computed for a static screened potential

$$V_{st}(r) = Z_s Z_t e^2 \frac{e^{-r/\lambda}}{r} \tag{18}$$

where the screening length λ is taken as the larger of the Debye length

$$\lambda_{\mathrm{D}} = \left(\frac{kT}{4\pi e^2 \sum_i n_i Z_i^2} \right)^{1/2}$$

or the average interionic distance $\lambda_i = (3/4\pi n_i)^{1/3}$.

The value of the collision integrals are given by Paquette et al. in the form of high-accuracy analytic fits. These fits are provided for the dimensionless collision integrals $F_{st}^{(ij)}$ defined as

$$F_{st}^{(ij)} = \frac{\Omega_{st}^{(ij)}}{\epsilon_{st}}, \tag{19}$$

with

$$\epsilon_{st} = \pi \left(\frac{Z_s Z_t e^2}{2kT} \right)^2 \left(\frac{kT}{2\pi m M_s M_t} \right)^{1/2}. \tag{20}$$

The value of these integrals depend uniquely upon Z_s, Z_t, λ, and T. The independent variable for the fits to the dimensionless collision integrals is

$$\psi_{st} = \ln\bigl[\ln(1 + \gamma_{st}^2)\bigr], \tag{21}$$

where the dimensionless parameter γ_{st} is given by

$$\gamma_{st} = \frac{4kT\lambda}{Z_s Z_t e^2}. \tag{22}$$

Tables 1–8 of Paquette et al. contain the values of the coefficients of the analytic fits and are used to determine the values of the collision integrals. These values are then introduced in the expressions for the diffusion coefficients (see (12, 13)) as well as in (11) for the diffusion velocity. The diffusion equation is then solved by using the numerical method described below.

3.3 Numerical method

Two different numerical methods are included in the Geneva code to solve the diffusion equations: the Crank–Nicholson finite differences method and the implicit finite elements method. Models of solar-type stars are usually computed by using the implicit finite elements method (see Meynet et al. 2004, for a comparison between these different numerical methods).

3.3.1 Implicit finite elements method

The detailed description of this method is presented in (Schatzman et al. 1981) (see the appendix by Glowinsky and Angrand). The basic idea of this method is to decompose the unknown function as a linear combination of well chosen independent functions.

The radiative zone of the star is divided in K shells, with the Lagrangian mass coordinate of the ith mesh point being m_i (m_i is the mass inside the sphere of radius r_i). We introduce K functions $v_i(m_r)$ defined by

$$v_i(m_r) = \begin{cases} \dfrac{m_r - m_{i-1}}{m_i - m_{i-1}} & \text{if } m_r \in [m_{i-1}, m_i], \\ \dfrac{m_r - m_{i+1}}{m_i - m_{i+1}} & \text{if } m_r \in [m_i, m_{i+1}], \\ 0 & \text{if } m_r \notin [m_{i-1}, m_{i+1}]. \end{cases}$$

The function v_i is equal to one at $m_r = m_i$, is equal to zero at m_{i+1} and m_{i-1} and varies linearly as a function of m_r inbetween. By multiplying the diffusion equation (8) for a given chemical element by each of the functions $v_i(m_r)$, one obtains K equations. Each of these K equations is then integrated over the volume Ω corresponding to the radiative zone of the star. By using the following general relations:

$$\mathrm{div}(a\mathbf{v}) = a\,\mathrm{div}(\mathbf{v}) + \mathbf{grad}(a)\cdot\mathbf{v}, \tag{23}$$

$$\int_V \mathrm{div}(\mathbf{v})\mathrm{d}V = \int_\Gamma \mathbf{v}\cdot\mathrm{d}\mathbf{S} \tag{24}$$

where a is a scalar, $\mathbf{v}$ a vector and Γ the surface corresponding to the volume Ω, one obtains (see Appendix B of Talon 1997 for more details):

$$\int_{M_1}^{M_2} \frac{\partial c}{\partial t} v_i \,\mathrm{d}m + \int_{M_1}^{M_2} D' \frac{\partial c}{\partial m}\frac{\partial v_i}{\partial m}\,\mathrm{d}m - \int_{M_1}^{M_2} V' c \frac{\partial v_i}{\partial m}\,\mathrm{d}m + \int_{M_1}^{M_2} \lambda c v_i \,\mathrm{d}m + \frac{\partial}{\partial t}(cM_{\mathrm{zc}_1}) - \frac{\partial}{\partial t}(cM_{\mathrm{zc}_2}) = 0 \tag{25}$$

where the boundary conditions at the edge of the convective zones M_1 and M_2 have been used (M_{zc} is the mass of the convective zone). D' and V' are the diffusion coefficient and velocity as defined in (9 and 10).

Finally, the unknown quantity c is expressed as a linear combination of the functions v_i: $c = \sum_j C_j v_j$. Integrals in (25) can then be written as:

$$\int_{M_1}^{M_2} c v_i \,\mathrm{d}m \cong \int_{M_1}^{M_2} \left(\sum_j C_j v_j\right) v_i \,\mathrm{d}m = \sum_j C_j \int_{M_1}^{M_2} v_j v_i \,\mathrm{d}m = \sum_j M_{ji} C_j, \tag{26}$$

$$\int_{M_1}^{M_2} D' \frac{\partial c}{\partial m}\frac{\partial v_i}{\partial m}\,\mathrm{d}m \cong \sum_j C_j \int_{M_1}^{M_2} D' \frac{\partial v_j}{\partial m}\frac{\partial v_i}{\partial m}\,\mathrm{d}m = \sum_j N_{ji} C_j, \tag{27}$$

$$\int_{M_1}^{M_2} V' c \frac{\partial v_i}{\partial m}\,\mathrm{d}m \cong \sum_j C_j \int_{M_1}^{M_2} V' v_j \frac{\partial v_i}{\partial m}\,\mathrm{d}m = \sum_j P_{ji} C_j. \tag{28}$$

The values of these integrals can then be calculated for every i and j with $1 \le i \le K$ and $1 \le j \le K$ (see (B7) of Talon 1997). Using (26–28), the diffusion equation (25) is finally expressed as:

$$\frac{\partial}{\partial t}(M_{j,i} C_j) + N_{j,i} C_j - P_{j,i} C_j + \delta_{1j}\frac{\partial}{\partial t}(C_1 M_{\mathrm{zc}_1}) - \delta_{Kj}\frac{\partial}{\partial t}(C_K M_{\mathrm{zc}_2}) = 0. \tag{29}$$

This linear system of equations is then solved by using the LAPACK routines. Note that the diffusion equation has to be solved simultaneously for all the considered elements. The order of magnitude of the time scales generally implies the computation of many iterations of the diffusion process for a single evolutionary time step.

3.4 Application to the solar case

Using the input physics described above, a solar calibration was performed with the Geneva stellar evolution code. In order to reproduce the solar luminosity, radius and surface chemical composition of Grevesse and Noels (1993) at the age of the Sun (4.57 Gyr), we obtain an initial helium mass fraction $Y_\mathrm{i} = 0.2735$, an initial ratio between the mass fraction of heavy elements and hydrogen $(Z/X)_\mathrm{i} = 0.0274$ and a mixing-length parameter $\alpha = 1.7998$. The surface helium mass fraction of this model is $Y_\mathrm{s} = 0.2426$, while the bottom of the convective zone is located at $r_{\mathrm{bzc}} = 0.712\ \mathrm{R}_\odot$. The comparison between the sound speed profile of this model and helioseismological measurements is shown in Fig. 1.

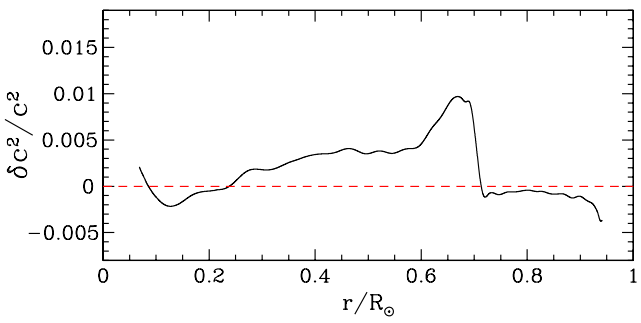

Fig. 1 Relative sound speed differences between helioseismological results and a standard solar model computed with the Geneva stellar evolution code

4 Modeling of rotation

In this section, we briefly summarize the basic physical ingredients of the numerical models of rotating stars.

4.1 Shellular rotation

In the radiative interiors of rotating stars, meridional circulation is generated as a result of the thermal imbalance induced by the breaking of the spherical symmetry (Eddington 1925; Vogt 1926). This large scale circulation transports matter and angular momentum. As a result, differential rotation takes place in the radiative zones making the stellar interior highly turbulent. The turbulence is very anisotropic, with a much stronger geostrophic-like transport in the horizontal than in the vertical direction (Zahn 1992). The horizontal turbulent coupling favours an essentially constant angular velocity Ω on the isobars. In the context of this hypothesis of shellular rotation, every quantity depends solely on pressure and can be split into a mean value and its latitudinal perturbation

$$f(P,\theta) = \overline{f}(P) + \tilde{f}(P)P_2(\cos\theta) \tag{30}$$

where $P_2(\cos\theta)$ is the second Legendre polynomial.

4.2 Transport of angular momentum

For shellular rotation, the transport of angular momentum obeys an advection–diffusion equation written in Lagrangian coordinates (Zahn 1992; Maeder and Zahn 1998):

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}(r^2\Omega)_{M_r} = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\rho r^4\Omega U(r)\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho D r^4\frac{\partial\Omega}{\partial r}\right), \tag{31}$$

r being the radius, ρ the density and $\Omega(r)$ the mean angular velocity at level r. It is worthwhile to recall here that meridional circulation is treated as a truly advective process in the Geneva evolution code. The vertical component $u(r,\theta)$ of the velocity of the meridional circulation at a distance r to the center and at a colatitude θ can be written

$$u(r,\theta) = U(r)P_2(\cos\theta). \tag{32}$$

Only the radial term $U(r)$ appears in (31); its expression is given below in (35). The quantity D is the total diffusion coefficient representing the various instabilities which transport the angular momentum: convection, semiconvection and shear turbulence. In convective regions, a very large diffusion coefficient implies a rotation law which is not far from solid body rotation. In radiative zones, we take $D = D_{\mathrm{shear}}$, since we consider shear mixing and meridional circulation as extra-convective mixing. The expression of the coefficient D_{shear} is given below in (36).

The full solution of 31 taking into account $U(r)$ and D gives the non-stationary solution of the problem. The expression of $U(r)$ (35) involves derivatives up to the third order; (31) is thus of the fourth order and implies four boundary conditions. The first boundary conditions impose momentum conservation at convective boundaries (Talon et al. 1997)

$$\frac{\partial}{\partial t}\left[\Omega\int_{r_t}^{R} r^4\rho\,\mathrm{d}r\right] = -\frac{1}{5}r^4\rho\Omega U + \mathcal{F}_\Omega \quad \text{for } r = r_t,$$

$$\frac{\partial}{\partial t}\left[\Omega\int_{0}^{r_b} r^4\rho\,\mathrm{d}r\right] = \frac{1}{5}r^4\rho\Omega U \quad \text{for } r = r_b.$$

The other conditions are determined by requiring the absence of differential rotation at convective boundaries

$$\frac{\partial\Omega}{\partial r} = 0 \quad \text{for } r = r_t, r_b, \tag{33}$$

r_t and r_b correspond respectively to the top (surface) and bottom (center) of the radiative zone. When the star has no convective core, momentum conservation is then simply equivalent to $U = 0$.

$\mathcal{F}_\Omega$ represents the torque applied at the surface of the star. For solar-type stars, this torque corresponds to the magnetic coupling at the stellar surface. Indeed, these stars are assumed to undergo magnetic braking while arriving on main sequence. In the Geneva code, we adopt the braking law of Kawaler (1988) corresponding to a field geometry intermediate between a dipolar and a radial field

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \begin{cases} -K\Omega^3\left(\dfrac{R}{R_\odot}\right)^{1/2}\left(\dfrac{M}{M_\odot}\right)^{-1/2} & (\Omega \le \Omega_{\mathrm{sat}}), \\ -K\Omega\Omega_{\mathrm{sat}}^2\left(\dfrac{R}{R_\odot}\right)^{1/2}\left(\dfrac{M}{M_\odot}\right)^{-1/2} & (\Omega > \Omega_{\mathrm{sat}}). \end{cases}$$

The constant K is related to the magnitude of the magnetic field strength; it is usually calibrated on the Sun and taken to be a constant in all stars (see Bouvier et al. 1997, for instance). Ω_{sat} expresses the fact that magnetic field generation saturates at some critical value (Saar 1996, and references therein). This saturation is required in order to retain a sufficient amount of fast rotators in young clusters, as originally suggested by Stauffer and Hartmann (1987). Ω_{sat} is usually fixed to $14\,\Omega_\odot$ (see Bouvier et al. 1997). A scaling in $\tau_{\mathrm{conv}}^{-1}$ of Ω_{sat} (τ_{conv} is the global convective time-scale as defined by Kim and Demarque 1996) is also suggested for low mass stars (see Palacios et al. 2003, Sect. 3.3 and references therein). Such a scaling of the value of Ω_{sat} can also be included in the Geneva code. Note that by neglecting the evolution of stellar structure and assuming solid body rotation, this braking law leads to the classical Skumanich law in $t^{-1/2}$ for the surface velocity.

4.3 Transport of chemical elements

The vertical transport of chemicals through the combined action of vertical advection and strong horizontal diffusion can be described as a pure diffusive process (Chaboyer and Zahn 1992). The advective transport is then replaced by a diffusive term, with an effective diffusion coefficient

$$D_{\rm eff} = \frac{|rU(r)|^2}{30D_{\rm h}} \tag{34}$$

where $D_{\rm h}$ is the diffusion coefficient associated to horizontal turbulence (see Sect. 4.6). The vertical transport of chemical elements then obeys a diffusion equation which, in addition to this macroscopic transport, also accounts for (vertical) turbulent transport, nuclear reactions, and atomic diffusion (see Sect. 3).

4.4 Meridional circulation

The velocity of the meridional circulation in the case of shellular rotation was initially derived by Zahn (1992). The effects of the vertical μ-gradient ∇_μ and of the horizontal turbulence on meridional circulation were taken into account by Maeder and Zahn (1998). They found

$$U(r) = \frac{P}{\rho g C_P T[\nabla_{\rm ad} - \nabla + (\varphi/\delta)\nabla_\mu]} \left\{ \frac{L}{M}(E_\Omega + E_\mu) \right\}. \tag{35}$$

P is the pressure, C_P the specific heat, E_Ω and E_μ are terms depending on the Ω- and μ-distributions respectively, up to the third order derivatives and on various thermodynamic quantities (see Maeder and Zahn 1998, for more details).

4.5 Shear turbulence

The diffusion by shear instabilities is expressed by a coefficient $D_{\rm shear}$, namely

$$D_{\rm shear} = \frac{4(K + D_{\rm h})}{[\frac{\varphi}{\delta}\nabla_\mu(1 + \frac{K}{D_{\rm h}}) + (\nabla_{\rm ad} - \nabla_{\rm rad})]} \times \frac{H_{\rm p}}{g\delta}\left[\frac{\alpha}{4}\left(f\Omega\frac{{\rm d}\ln\Omega}{{\rm d}\ln r}\right)^2 - (\nabla' - \nabla)\right] \tag{36}$$

where f is a numerical factor equal to 0.8836, K is the thermal diffusivity and $(\nabla' - \nabla)$ expresses the difference between the internal nonadiabatic gradient and the local gradient (Maeder and Meynet 2001).

4.6 Horizontal turbulence

The usual expression for the coefficient $D_{\rm h}$ is, according to Zahn (1992),

$$D_{\rm h} = \frac{1}{c_{\rm h}} r\left|2V(r) - \alpha U(r)\right| \tag{37}$$

where $U(r)$ is the vertical component of the meridional circulation velocity, $V(r)$ the horizontal component, $c_{\rm h}$ a constant of order unity and $\alpha = \frac{1}{2}\frac{{\rm d}\ln r^2\bar{\Omega}}{{\rm d}\ln r}$.

By expressing the balance between the energy dissipated by the horizontal turbulence and the excess of energy present in the differential rotation on an equipotential that can be dissipated in a dynamical time, Maeder (2003) recently derived a new expression for the diffusion coefficient $D_{\rm h}$:

$$D_{\rm h} = Ar\left(r\bar{\Omega}(r)V[2V - \alpha U]\right)^{\frac{1}{3}}, \tag{38}$$

with

$$A = \left(\frac{3}{400n\pi}\right)^{\frac{1}{3}}. \tag{39}$$

Mathis et al. (2004) borrowed another prescription for the horizontal turbulence from torque measurements in the classical Couette–Taylor experiment. They find

$$D_{\rm h} = \left(\frac{\beta}{10}\right)^{\frac{1}{2}}\left(r^2\bar{\Omega}(r)\right)^{\frac{1}{2}}\left[r|2V - \alpha U|\right]^{\frac{1}{2}}, \tag{40}$$

with $\beta \cong 1.5\,10^{-5}$ (Richard and Zahn 1999).

These three expressions for the horizontal turbulence are included in the Geneva evolution code. Stellar models are usually computed by using the recent prescription by Maeder.

5 Magnetic fields and internal gravity waves

Meridional and rotational turbulent diffusion are not able to account for all observed properties of solar-type stars and in particular for the rotation profile of the radiative interior of the Sun as deduced from helioseismic measurements. Indeed, helioseismological results indicate that the angular velocity $\Omega(r)$ is constant as a function of the radius r between about 20% and 70% of the total solar radius (Brown et al. 1989; Kosovichev et al. 1997; Couvidat et al. 2003), while meridional and rotational turbulent diffusion produce an insufficient internal coupling to ensure solid body rotation (Pinsonneault et al. 1989; Chaboyer et al. 1995). This suggests that other effects intervene. In the Geneva evolution code, the effects of the Tayler–Spruit dynamo (Spruit 2002; Maeder and Meynet 2004b) as well as the transport of angular momentum by internal gravity waves (Zahn et al. 1997; Talon et al. 2002; Talon and Charbonnel 2005; Charbonnel and Talon 2005) have been included.

5.1 Tayler–Spruit dynamo

In this section we briefly summarize the consistent system of equations for the Tayler–Spruit dynamo (Spruit 2002; Maeder and Meynet 2004b).

The energy density u_B of a magnetic field of intensity B per unit volume is

$$u_B = \frac{B^2}{8\pi} = \frac{1}{2}\rho r^2 \omega_A^2 \quad \text{with } \omega_A = \frac{B}{(4\pi\rho)^{\frac{1}{2}} r} \tag{41}$$

where ω_A is the Alfvén frequency in a spherical geometry. If due to magnetic field or rotation, some unstable displacements of vertical amplitude $l/2$ occur around an average stable position, the restoring buoyancy force produces vertical oscillations around the equilibrium position with a frequency equal to the Brunt–Väisälä frequency N.

The restoring oscillations will have an average density of kinetic energy

$$u_N \simeq f_N \rho l^2 N^2, \tag{42}$$

where f_N is a geometrical factor of the order of unity. If the magnetic field produces some instability with a vertical component, one must have $u_B > u_N$. Otherwise, the restoring force of gravity which acts at the dynamical timescale would immediately counteract the magnetic instability. From this inequality, one obtains $l^2 < \frac{1}{2f_N} r^2 \frac{\omega_A^2}{N^2}$. If $f_N = \frac{1}{2}$, we have the condition for the vertical amplitude of the instability (Spruit 2002, (6)),

$$l < r\frac{\omega_A}{N} \tag{43}$$

where r is the radius. This means that there is a maximum size of the vertical length l of a magnetic instability. In order to not be quickly damped by magnetic diffusivity, the vertical length scale of the instability must satisfy

$$l^2 > \frac{\eta}{\sigma_B} = \frac{\eta\Omega}{\omega_A^2} \tag{44}$$

where Ω is the angular velocity and σ_B the characteristic growth-rate of the magnetic field. In a rotating star, this growth-rate is $\sigma_B = (\omega_A^2/\Omega)$ due to the Coriolis force (Spruit 2002; Pitts and Tayler 1985). The combination of the limits given by (43) and (44) gives for the case of marginal stability,

$$\left(\frac{\omega_A}{\Omega}\right)^4 = \frac{N^2}{\Omega^2}\frac{\eta}{r^2\Omega}. \tag{45}$$

The equality of the amplification time of Tayler instability $\tau_a = N/(\omega_A \Omega q)$ with the characteristic frequency σ_B of the magnetic field leads to the equation (Spruit 2002)

$$\frac{\omega_A}{\Omega} = q\frac{\Omega}{N} \quad \text{with } q = -\frac{\partial \ln \Omega}{\partial \ln r}. \tag{46}$$

By eliminating the expression of N^2 between (45) and (46), we obtain an expression for the magnetic diffusivity,

$$\eta = \frac{r^2\Omega}{q^2}\left(\frac{\omega_A}{\Omega}\right)^6. \tag{47}$$

Equations (45) and (46) form a coupled system relating the two unknown quantities η and ω_A. The fact that the ratio η/K is very small allows us to bring these coupled equations to a system of degree 4 (Maeder and Meynet 2004b),

$$\frac{r^2\Omega}{q^2 K}\left(N_T^2 + N_\mu^2\right)x^4 - \frac{r^2\Omega^3}{K}x^3 + 2N_\mu^2 x - 2\Omega^2 q^2 = 0 \tag{48}$$

where $x = (\omega_A/\Omega)^2$. The solution of this equation, which is easily obtained numerically, provides the Alfvén frequency and by (47) the thermal diffusivity.

The azimuthal component of the magnetic field is much stronger that the radial one in the Tayler–Spruit dynamo. We have for these components (Spruit 2002)

$$B_\varphi = (4\pi\rho)^{\frac{1}{2}} r\omega_A \quad \text{and} \quad B_r = B_\varphi(l_r/r) \tag{49}$$

where ω_A is the solution of the general equation (48) and l_r is given by (43).

Turning towards the transport of angular momentum by magnetic field, we first write the azimuthal stress by volume unity due to the magnetic field

$$S = \frac{1}{4\pi}B_r B_\varphi = \frac{1}{4\pi}\left(\frac{l_r}{r}\right)B_\varphi^2 = \rho r^2\left(\frac{\omega_A^3}{N}\right). \tag{50}$$

Then, the viscosity ν for the vertical transport of angular momentum can be expressed in terms of S (Spruit 2002),

$$\nu = \frac{S}{\rho q \Omega} = \frac{\Omega r^2}{q}\left(\frac{\omega_A}{\Omega}\right)^3\left(\frac{\Omega}{N}\right). \tag{51}$$

This is the general expression of ν with ω_A given by the solution of (48). We have the full set of expressions necessary to obtain the Alfvén frequency ω_A and the magnetic diffusivity η. Let us recall that η also expresses the vertical transport of the chemical elements, while the viscosity ν determines the vertical transport of angular momentum by the magnetic field.

Stellar models computed with the Tayler–Spruit dynamo show that the evolution of a rotating star with magnetic fields leads to an equilibrium value of the differential rotation. This equilibrium is determined by the magnetic coupling, which favours a constant rotation profile, and meridional circulation which tends to build differential rotation (Maeder and Meynet 2003, 2004b, 2005; Eggenberger et al. 2005b). For a 1 $M_\odot$ star, the global equilibrium stage is close to solid body rotation between about 0.7 and 0.2 $R_\odot$, in good agreement with helioseismic measurements (Eggenberger et al. 2005b).

5.2 Internal gravity waves

In this section, the modeling of the effects of internal gravity waves is very briefly summarized. For more details, the reader is referred to (Zahn et al. 1997; Talon et al. 2002; Talon and Charbonnel 2005; Charbonnel and Talon 2005).

Gravity waves are excited at the base of the convective zone by Reynolds stresses and/or convective plumes. These waves lead to two different features:

- They produce a shear layer, that generates turbulence close to the bottom of the convection zone;
- They deposit negative (positive) momentum throughout the radiative interior when the convection zone rotates slower (faster) than the radiative zone.

When meridional circulation, turbulence and waves are taken into account, the evolution of angular momentum is then described by the following equation (Talon et al. 2002)

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}[r^2\Omega] = \frac{1}{5r^2}\frac{\partial}{\partial r}[\rho r^4\Omega U] \tag{52}$$

$$+\frac{1}{r^2}\frac{\partial}{\partial r}\left[\rho(\nu_t+\nu_{\mathrm{waves}})r^4\frac{\partial\Omega}{\partial r}\right] - \frac{3}{8\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\mathcal{L}_J(r) \tag{53}$$

where ν_{waves} is the diffusion coefficient associated with wave-induced turbulence (ν_t is the turbulent viscosity due to differential rotation away from the shear layer) and $\mathcal{L}_J(r)$ is the net momentum deposition of angular momentum associated to the gravity waves. To obtain this net momentum deposition, one must follow the local momentum luminosity

$$\mathcal{L}_J(r) = \sum_{\sigma,\ell,m}\mathcal{L}_{J\,\ell,m}(r_{\text{shear layer}})\exp\bigl[-\tau(r,\sigma,\ell)\bigr] \tag{54}$$

where the local damping rate takes into account the mean molecular weight stratification

$$\tau = \bigl[\ell(\ell+1)\bigr]^{\frac{3}{2}}\int_r^{r_c}(K+\nu_t)\frac{NN_T^2}{\sigma^4}\left(\frac{N^2}{N^2-\sigma^2}\right)^{\frac{1}{2}}\frac{\mathrm{d}r}{r^3} \tag{55}$$

where $N^2 = N_T^2 + N_\mu^2$ is the total Brunt–Väisälä frequency, N_T^2 is its thermal part and N_μ^2 is due to the mean molecular weight stratification (Zahn et al. 1997). σ is the local, Doppler shifted frequency

$$\sigma(r) = \omega - m\bigl(\Omega(r)-\Omega_{\mathrm{cz}}\bigr) \tag{56}$$

and ω is the wave frequency in the reference frame of the convection zone. The spectrum of internal gravity waves strongly depends on the structure of the convection zone. In order to determine the net mean luminosity below the shear layer $\mathcal{L}_{J\,\ell,m}(r_{\text{shear layer}})$, we have to evaluate the global momentum luminosity in waves, but also the dynamics of the shear layer and the resulting differential filtering. In the Geneva code, this is done by interpolating in tables which give the filtered luminosities for fixed values of the differential rotation just below the shear layer. These tables are obtained by performing numerical simulations of the dynamics of the shear layer (see Talon and Charbonnel 2005, for more details).

It is very interesting to note that stellar models including internal gravity waves are found to successfully reproduce both the rotation profile and the surface abundance of lithium in solar-type stars of various ages (see Talon et al. 2002; Talon and Charbonnel 2005; Charbonnel and Talon 2005, for more details).

6 Conclusion

We conclude that the input physics used in the Geneva stellar evolution code enables the computation of models including a detailed treatment of many physical processes at work in stellar interiors. Note that these models can be computed for stars situated at various evolutionary stages and for a wide range of masses and metallicities. In the specific case of solar-type stars, standard stellar models can be produced with the Geneva code, as well as more sophisticated models including meridional circulation, turbulence, transport of angular momentum by internal gravity waves and magnetic instabilities.

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