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Finite element approximation of multi-scale elliptic problems using patches of elements

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Abstract In this paper we present a method for the numerical solution of elliptic problems with multi-scale data using multiple levels of not necessarily nested grids. The method consists in calculating successive corrections to the solution in patches whose discretizations are not necessarily conforming. This paper provides proofs of the results published earlier (see C. R. Acad. Sci. Paris, Ser. I 337 (2003) 679–684), gives a generalization of the latter to more than two domains and contains extensive numerical illustrations. New results including the spectral analysis of the iteration operator and a numerical method to evaluate the constant of the strengthened Cauchy-Buniakowski-Schwarz inequality are presented.

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1 Introduction

The numerical approximation of the solution of elliptic partial differential equations in domains such that in certain regions a "better" precision on the solution is needed leads to many interesting issues. Efficient approaches include adaptive mesh refinement techniques, domain decomposition methods and multigrid methods. The objective of this paper is to present a method to solve numerically elliptic problems with multi-scale data using multiple levels of not necessarily nested grids.

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A motivation for developing such a method can be found, for example, in air quality management. Pollution emission sources, and in particular point source plumes, give rise to models needing careful examination of the space-scale. Getting an accurate simulation on large scales is linked to a simulation in subregions around the pollution sources using finer grids. Such a method can be applied straightforwardly to boundary layer problems through the use of patches in critical regions, or in the coupling of problems with nonconforming grids for example.

We solve the problem on a domain Ω and consider therein patches $\Lambda_1, \Lambda_2, \ldots$ wherein we would like to obtain more accuracy (see Fig. 1). Thus we calculate successively corrections to the solution in the patches. The discretizations of the latter are not necessarily conforming. The method is a domain decomposition method with complete overlapping. It resembles the Fast Adaptive Composite grid (FAC) method (see, e.g., [31]) or possibly a hierarchical method (see [16] for example). However it is of much more flexible use in comparison to the latter.

In Section 2 we first introduce the correction algorithm (Algorithm 1) in the case of two scales, i.e. with the domain Ω and one only patch Λ. We give an *a priori* error estimate for the h-convergence in Proposition 1. The convergence properties of the two-scale algorithm are stated in Proposition 2 through the iteration operator (10).

In Section 3 we prove some convergence results in an abstract setting. In the first paragraph we analyze some properties of vector spaces. Next we introduce an operator (22) that is to be identified with the iteration operator (10) of Algorithm 1. In Proposition 5 we recall the upper bound of its norm presented in [22]. A spectral analysis of the operator yields an exact formula for its spectral radius and norm given in Proposition 6.

In Section 4 we discuss the constant γ of the strengthened Cauchy-Buniakowski-Schwarz (C.B.S.) inequality which appears in our convergence analysis. In the case of a patch Λ included in only one triangle of the coarse triangulation of Ω we give a bound for γ in the case of a scalar product corresponding to the bilinear form of an elliptic operator. We also develop some upper bounds in particular cases (see Fig. 2) and give a new method to estimate it numerically.

In Section 5 we generalize Algorithm 1 and some results obtained in Section 3 for two spaces to multiple spaces. We establish a generalization of Proposition 5 in Proposition 9 which is used to prove the convergence of the multi-scale algorithm in Proposition 8.

Section 6 gives some numerical results. In §6.1 we present numerical estimates of γ for some cases treated in Section 4 (Table 2) and new grid constellations

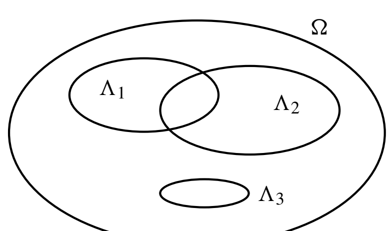


Fig. 1 Domain Ω with patches

(Table 3) that we use in §6.2 to illustrate the convergence behavior of the algorithm. In §6.3 we illustrate the *a priori* estimate for the convergence in the mesh size (Fig. 6).

2 Two-scale algorithm

Let $\Omega \subset \mathbb{R}^2$ be an open polygonal domain and consider a bilinear, symmetric, continuous and coercive form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}.$$

The usual $H^1(\Omega)$-norm is equivalent to the a-norm defined by $||v|| = a(v,v)^{\frac{1}{2}}$, $\forall v \in H_0^1(\Omega)$. If $f \in H^{-1}(\Omega)$, due to Riesz' representation Theorem there exists a unique $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f|v\rangle, \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where $\langle\cdot|\cdot\rangle$ denotes the duality $H^{-1}(\Omega) - H_0^1(\Omega)$. Let us point out that (1) is the weak formulation of a problem of type $\mathcal{L}(u) = f$ in Ω, $u = 0$ on the boundary $\partial\Omega$ of Ω, where $\mathcal{L}(\cdot)$ is a second order, linear, symmetric, strongly elliptic operator. An approximation of u by the finite element method of order r consists in introducing a regular triangulation $\mathcal{T}_H$ of $\overline{\Omega}$ (see [20], Sect. 17), defining

$$\begin{aligned} V_H = \{g : \overline{\Omega} \to \mathbb{R} \text{ continuous such that } & g|_K \in \mathbb{P}_r(K), \forall K \in \mathcal{T}_H \\ & \text{and } g = 0 \text{ on } \partial\Omega\}, \end{aligned} \tag{2}$$

where $\mathbb{P}_r(K)$ is the space of polynomials of degree $\leq r$ on triangle $K \in \mathcal{T}_H$, and calculating $u_H \in V_H$ satisfying

$$a(u_H, v) = \langle f|v\rangle, \quad \forall v \in V_H. \tag{3}$$

Consider now $\Lambda \subset \Omega$ another open polygonal domain wherein we would like to obtain a better precision on the solution u than the one given by u_H. Take note that $\overline{\Lambda}$ is not necessarily the union of several triangles K of $\mathcal{T}_H$. Besides Λ can be determined in practice by an a priori knowledge or an a posteriori error estimator, for example. Let $\mathcal{T}_h$ be a regular triangulation of $\overline{\Lambda}$ and consider

$$\begin{aligned} V_h = \{g : \overline{\Omega} \to \mathbb{R} \text{ continuous such that } & g|_K \in \mathbb{P}_s(K), \forall K \in \mathcal{T}_h \\ & \text{and } g = 0 \text{ on } \overline{\Omega} \setminus \Lambda\}. \end{aligned}$$

We call $H = \max_{K \in \mathcal{T}_H} \operatorname{diam}(K)$ and $h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K)$. Setting $V_{Hh} = V_H + V_h$ we search as approximation for u the function $u_{Hh} \in V_{Hh}$ satisfying

$$a(u_{Hh}, v) = \langle f|v\rangle, \quad \forall v \in V_{Hh}. \tag{4}$$

Let us observe that in practice, it is not possible to determine a finite element basis of V_{Hh} since, in principle, $V_H \cap V_h$ is not necessarily reduced to zero. Before to show how to compute u_{Hh}, we establish the following *a priori* estimate:

Proposition 1 *Let* $q = \max(r, s) + 1$ *and suppose that the solution* u *of (1) is in* $H^q(\Omega)$. *Then the approximation* u_{Hh} *to* u *satisfies the* a priori *error estimate*

$$||u - u_{Hh}|| \leq C\left(H^r ||u||_{H^q(\Omega\setminus\overline{\Lambda})} + h^s ||u||_{H^q(\Lambda)}\right),$$

where C *is a constant independent of* H *and* h.

Proof The boundary $\partial\Lambda$ being locally Lipschitz, due to the Stein Extension Theorem (see Adams and Fournier [3], Thm. 5.24), there exists a bounded extension operator $E : H^q(\Omega \setminus \overline{\Lambda}) \to H^q(\Omega)$, i.e. $Ev|_{\Omega\setminus\overline{\Lambda}} = v|_{\Omega\setminus\overline{\Lambda}}, \forall v \in H^q(\Omega \setminus \overline{\Lambda})$. Let u be the solution of (1). We define $\tilde{u}$ the extension of $u|_{\Omega\setminus\overline{\Lambda}}$ to Ω such that $\tilde{u} = Eu$ if $||Eu||_{H^q(\Lambda)} \leq ||u||_{H^q(\Lambda)}$ and $\tilde{u} = u$ otherwise. We have that $\tilde{u} = u$ in $\Omega \setminus \overline{\Lambda}$,

$$||\tilde{u}||_{H^q(\Omega)} \leq C||u||_{H^q(\Omega\setminus\overline{\Lambda})}, \tag{5}$$

where here, like in the sequel, C denotes a generic constant, and

$$||\tilde{u}||_{H^q(\Lambda)} \leq ||u||_{H^q(\Lambda)}. \tag{6}$$

Note that $u - \tilde{u} \in H_0^q(\Lambda)$. Let r_H and r_h be the standard interpolants to the spaces V_H and V_h respectively. We introduce $\tilde{u}_H = r_H\tilde{u}$ and $\tilde{u}_h = r_h(u - \tilde{u})$. Define $\tilde{u}_{Hh} = \tilde{u}_H + \tilde{u}_h$ and $v_{Hh} = u_{Hh} - \tilde{u}_{Hh}$. By the definitions of u and u_{Hh} we have $a(u, v_{Hh}) = a(u_{Hh}, v_{Hh})$. This and the previous definitions lead to the equality $a(v_{Hh}, v_{Hh}) = a(u - \tilde{u}_{Hh}, v_{Hh})$, from which we derive using the Cauchy-Schwarz inequality that $||v_{Hh}||^2 \leq ||u - \tilde{u}_{Hh}|| ||v_{Hh}||$. Thus

$$||u_{Hh} - \tilde{u}_{Hh}|| \leq ||u - \tilde{u}_{Hh}||. \tag{7}$$

With $u - u_{Hh} = (u - \tilde{u}_{Hh}) + (\tilde{u}_{Hh} - u_{Hh})$ and (7), we have

$$||u - u_{Hh}|| \leq ||u - \tilde{u}_{Hh}|| + ||u_{Hh} - \tilde{u}_{Hh}|| \leq 2||u - \tilde{u}_{Hh}||. \tag{8}$$

Writing $u - \tilde{u}_{Hh} = (\tilde{u} - \tilde{u}_H) + [(u - \tilde{u}) - \tilde{u}_h]$, we get by standard interpolation results

$$\begin{aligned} ||u - \tilde{u}_{Hh}|| &\leq ||\tilde{u} - \tilde{u}_H|| + ||(u - \tilde{u}) - \tilde{u}_h|| \\ &\leq C\left(H^r ||\tilde{u}||_{H^q(\Omega)} + h^s ||u - \tilde{u}||_{H^q(\Lambda)}\right), \end{aligned}$$

and furthermore, with $||u - \tilde{u}||_{H^q(\Lambda)} \leq ||u||_{H^q(\Lambda)} + ||\tilde{u}||_{H^q(\Lambda)}$ and using the relations (5) and (6), we obtain

$$||u - \tilde{u}_{Hh}|| \leq C\left(H^r ||u||_{H^q(\Omega\setminus\overline{\Lambda})} + h^s ||u||_{H^q(\Lambda)}\right). \tag{9}$$

Hence, combining the results (8) and (9) completes the proof. □

As we have mentioned above, a priori $V_H \cap V_h$ does not necessarily reduce to the element zero and it is impossible, practically speaking, to exhibit a finite element basis of the space V_{Hh} and consequently to compute directly u_{Hh}. It is the reason for which we suggest the following algorithm for computing u_{Hh}.

Algorithm 1 1. Set $u^0 = u_H \in V_H$ and choose $\omega \in (0; 2)$.
2. For $n = 1, 2, 3, \ldots$ find
 (i) $w_h \in V_h$ such that $a(w_h, v) = \langle f|v\rangle - a(u^{n-1}, v), \quad \forall v \in V_h$;
 $u^{n-\frac{1}{2}} = u^{n-1} + \omega w_h$;
 (ii) $w_H \in V_H$ such that $a(w_H, v) = \langle f|v\rangle - a(u^{n-\frac{1}{2}}, v), \quad \forall v \in V_H$;
 $u^n = u^{n-\frac{1}{2}} + \omega w_H$.

When implementing the algorithm, the coarse and the fine parts of u^n and $u^{n-\frac{1}{2}}$ are stored separately. In practice this is efficient for calculating the scalar product $a(\cdot,\cdot)$ in the right hand side of (i) and (ii).

It is readily seen that this algorithm is a Schwarz type domain decomposition method [33] with complete overlapping but without any conformity between the meshes $\mathcal{T}_H$ and $\mathcal{T}_h$ (see the work by Chan et al. [19]). This multiplicative Schwarz method is similar to the Gauss-Seidel method and is called by Xu successive subspace correction algorithm (see, e.g., [41]). The spaces V_H and V_h defined on the arbitrary triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$ are not necessary orthognal nor share the only element zero as intersection. Note in particular that the sum which defines V_{Hh} is, a priori, not a direct sum. This property makes the above algorithm different from most known iterative schemes (see for example the scheme by Laydi [26]). The algorithm resembles the FAC method, see for example the works from McCormick et al. [29–31], or possibly a hierarchical method (see for example the papers from Yserentant [44,45], Bank et al. [10] and Bank and Smith [11]) with a mortar method (see [2]). It is also similar to the Chimera or overset grid method [17,35]. The new aspect we introduce here is to link the speed of convergence of this algorithm to the parameter $\tilde{\gamma}$, introduced here below, corresponding to an abstract angle between the spaces V_h and V_H.

We shall now analyze the convergence of the two-scale algorithm.

If $P_h : V_{Hh} \to V_h$ and $P_H : V_{Hh} \to V_H$ are orthogonal projectors from V_{Hh} onto V_h and V_H respectively with regard to the scalar product $a(\cdot,\cdot)$, we have

$$u_{Hh} - u^n = (I - \omega P_H)(I - \omega P_h)(u_{Hh} - u^{n-1}),$$

where I denotes the identity operator in V_{Hh}. Setting

$$B = (I - \omega P_H)(I - \omega P_h), \tag{10}$$

we obtain that $u_{Hh} - u^n = B^n(u_{Hh} - u^0)$.

We set $V_{Hh0} = V_H \cap V_h$ and $V_{Hh0}^{\perp}$ the orthogonal complement of V_{Hh0} in V_{Hh}.

Proposition 2 *If $\omega \in (0; 2)$, then the algorithm (i), (ii) converges, i.e.* $\lim_{n\to\infty} ||u^n - u_{Hh}|| = 0$. *The convergence factor in the norm induced by the scalar product $a(\cdot,\cdot)$ is bounded by*

$$||B|| = \frac{1}{2}\omega\,(2-\omega)\,\tilde{\gamma} + \sqrt{\frac{1}{4}\omega^2\,(2-\omega)^2\,\tilde{\gamma}^2 + (\omega-1)^2} < 1,$$

where $\tilde{\gamma} \in [0; 1]$ is defined by

$$\tilde{\gamma} = \begin{cases} \sup\limits_{\substack{v_h \in V_h \cap V_{Hh0}^{\perp}, v_h \neq 0 \\ v_H \in V_H \cap V_{Hh0}^{\perp}, v_H \neq 0}} \frac{(v_h, v_H)}{||v_h||\,||v_H||}, & \text{if } V_h \neq V_{Hh0} \text{ and } V_H \neq V_{Hh0}, \\ 0, & \text{otherwise.} \end{cases}$$

We prove Proposition 2 at the end of Section 3 after studying some properties of vector spaces and doing an abstract analysis of the iteration operator B.

3 Abstract analysis of the iteration operator B

3.1 Theoretical preliminaries: some properties of vector spaces

Let V be a Hilbert space with scalar product $(\cdot, \cdot)$ and denote by $||\cdot||$ the induced norm. Consider V_1, V_2 two closed subspaces of V.

We introduce the number

$$\gamma = \begin{cases} \sup_{\substack{v_1 \in V_1, v_1 \neq 0 \\ v_2 \in V_2, v_2 \neq 0}} \frac{(v_1, v_2)}{||v_1|| ||v_2||}, & \text{if } V_1 \neq \{0\} \text{ and } V_2 \neq \{0\}, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

which is the constant from the corresponding strengthened C.B.S. inequality. The constant γ is the cosine of the abstract angle between the two subspaces V_1 and V_2. We have the obvious properties for γ:

1. Constant γ is necessarily included in the interval [0; 1].
2. If $V_1 \cap V_2 \neq \{0\}$, then we have $\gamma = 1$.
3. Constant $\gamma = 0$ if and only if V_1 is orthogonal to V_2.

We set $V_0 = V_1 \cap V_2$ and $V_0^{\perp}$ the orthogonal complement of V_0 in V. The second property suggests to introduce the number

$$\tilde{\gamma} = \begin{cases} \sup_{\substack{v_1 \in V_1 \cap V_0^{\perp}, v_1 \neq 0 \\ v_2 \in V_2 \cap V_0^{\perp}, v_2 \neq 0}} \frac{(v_1, v_2)}{||v_1|| ||v_2||}, & \text{if } V_1 \neq V_0 \text{ and } V_2 \neq V_0, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

In the sequel we assume that the following hypothesis is satisfied:

Hypothesis (H) There exists a constant C_0 such that for all $v \in V$ there exist $v_1 \in V_1$, $v_2 \in V_2$ satisfying $v = v_1 + v_2$ and

$$||v_1||^2 + ||v_2||^2 \leq C_0^2 ||v||^2. \tag{13}$$

Let us observe that:

1. If (H) is satisfied, we have necessarily $V = V_1 + V_2$.
2. If $V_1 \neq V_2$, we have necessarily $C_0 \geq 1$.
3. In the case $V_1 = V_2 = V$ the optimal constant C_0 in (13) is equal to $1/\sqrt{2}$ (it suffices to take $v_1 = v_2 = \frac{1}{2} v$, $v \in V$).
4. If V_1 is orthogonal to V_2, we can take $C_0 = 1$ from Pythagore's Theorem.

Proposition 3 *If $V = V_1 + V_2$ then Hypothesis (H) is satisfied and $\tilde{\gamma} < 1$. If, moreover, $V_1 \neq V_2$ then*

$$C_0^{\text{opt}} = \sqrt{\frac{1}{1 - \tilde{\gamma}}}, \tag{14}$$

is the optimal constant in (13).

Proof Let us denote $\tilde{V}_j = V_j \cap V_0^{\perp}$, $j = 1, 2$, then $V_0^{\perp} = \tilde{V}_1 \oplus \tilde{V}_2$ and $V = V_0 \oplus \tilde{V}_1 \oplus \tilde{V}_2$. The Corollary of the Open Mapping Theorem (see Yosida [43], §II.5) for the one-to-one mapping $(\tilde{v}_1, \tilde{v}_2) \in \tilde{V}_1 \times \tilde{V}_2 \to \tilde{v}_1 + \tilde{v}_2 \in V_0^{\perp}$ yields the existence of $\tilde{C}_0 < +\infty$ such that $\forall \tilde{v}_j \in \tilde{V}_j$, $j = 1, 2$, we have $||\tilde{v}_1||^2 + ||\tilde{v}_2||^2 \leq \tilde{C}_0^2 ||\tilde{v}_1 + \tilde{v}_2||^2$. We can take $\tilde{C}_0 \geq 1$.

For all $v \in V$ we have a unique decomposition

$$v = v_0 + \tilde{v}_1 + \tilde{v}_2 \text{ with } v_0 \in V_0,\ \tilde{v}_j \in \tilde{V}_j,\ j = 1, 2. \tag{15}$$

Hence, we can put

$$v_1 = v_0 + \tilde{v}_1 \in V_1 \text{ and } v_2 = \tilde{v}_2 \in V_2, \tag{16}$$

so that $v = v_1 + v_2$ and

$$\begin{aligned} ||v_1||^2 + ||v_2||^2 &= ||v_0||^2 + ||\tilde{v}_1||^2 + ||\tilde{v}_2||^2 \\ &\leq \tilde{C}_0^2 (||v_0||^2 + ||\tilde{v}_1 + \tilde{v}_2||^2) = \tilde{C}_0^2 ||v||^2, \end{aligned}$$

i.e. Hypothesis (H) is satisfied with $C_0 = \tilde{C}_0$.

Let us now consider the case $V_1 \neq V_0$ and $V_2 \neq V_0$. Using Definition (12), there exists a sequence $v^m = \tilde{v}_1^m + \tilde{v}_2^m$ with $\tilde{v}_1^m \in \tilde{V}_1$, $\tilde{v}_2^m \in \tilde{V}_2$ and $||\tilde{v}_1^m|| = ||\tilde{v}_2^m|| = 1$ such that

$$(\tilde{v}_1^m, \tilde{v}_2^m) \to -\tilde{\gamma}. \tag{17}$$

Suppose *ad absurdum* that $\tilde{\gamma} = 1$. Thus

$$\frac{||\tilde{v}_1^m||^2 + ||\tilde{v}_2^m||^2}{||v^m||^2} = \frac{1}{1 + (\tilde{v}_1^m, \tilde{v}_2^m)} \to +\infty,$$

which contradicts Hypothesis (H). Hence $\tilde{\gamma} < 1$.

Using again the decomposition (15) for any $v \in V$ and setting $v_1 \in V_1$, $v_2 \in V_2$ as in (16), we have $||v_1||^2 + ||v_2||^2 \leq ||v_0||^2 + ||\tilde{v}_1 + \tilde{v}_2||^2 + 2|(\tilde{v}_1, \tilde{v}_2)| \leq ||v||^2 + 2\tilde{\gamma}||\tilde{v}_1||||\tilde{v}_2||$. Since $2||\tilde{v}_1||||\tilde{v}_2|| \leq ||\tilde{v}_1||^2 + ||\tilde{v}_2||^2 \leq ||v_1||^2 + ||v_2||^2$, we get

$$||v_1||^2 + ||v_2||^2 \leq \frac{1}{1 - \tilde{\gamma}} ||v||^2.$$

Thus we can choose $C_0 = \sqrt{\frac{1}{1-\tilde{\gamma}}}$ in (13). It suffices to use (17) in (12) to show that $\sqrt{\frac{1}{1-\tilde{\gamma}}}$ is the best constant we can choose.

In the case $V_1 = V_0$ or $V_2 = V_0$ we have that $\tilde{\gamma} = 0$ and if, moreover, $V_1 \neq V_2$, then $C_0^{\text{opt}} = 1$, i.e. (14) is also valid. □

We introduce $P_j : V \to V_j \subset V$ the orthogonal projectors from V upon V_j, $j = 1, 2$, and call $V_j^{\perp}$ the orthogonal complement of V_j in V.

Proposition 4 *Let V be of finite dimension and $V = V_1 + V_2$. There exist $2p$ ($p \geq 0$) vectors $v_1^{(m)} \in V_1$ and $v_2^{(m)} \in V_2$, $m = 1, \dots, p$, such that*

$$||v_1^{(m)}|| = ||v_2^{(m)}|| = 1, \quad (v_1^{(m)}, v_2^{(m)}) = \gamma_m, \quad m = 1, \dots, p,$$

with

$$1 > \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p > 0, \tag{18}$$

and V can be decomposed into the direct sum

$$V = (V_1 \cap V_2) \oplus (V_1^{\perp} \cap V_2) \oplus (V_1 \cap V_2^{\perp}) \oplus L_1 \oplus \cdots \oplus L_p, \tag{19}$$

where $L_m = \text{span}\{v_1^{(m)}, v_2^{(m)}\}$, $m = 1, \dots, p$, and all the summands in (19) are mutually orthogonal subspaces of V, which are invariant with respect to both operators P_1 and P_2, i.e. $P_j L_m \subset L_m$, $j = 1, 2$.

Proof Let us prove that for any integer k, $0 \leq k \leq p$ with p to be identified later in the proof, the space V can be decomposed into a direct sum with mutually orthogonal summands

$$V = V_0 \oplus W_k \oplus L_1 \oplus \cdots \oplus L_k \tag{20}$$

where $V_0 = V_1 \cap V_2$, the spaces L_m are the two-dimensional subspaces of V appearing in (19) and all the subspaces V_0 and $L_1, \dots, L_k$, $W_k \subset V_0^{\perp}$ are invariant with respect to both operators P_1 and P_2. The decomposition (20) will be constructed by induction on k.

We start with $k = 0$ and set $W_0 = V_0^{\perp}$. Note that V_0 and W_0 are invariant subspaces of operators P_1 and P_2. On the k-th step of our construction ($k \geq 1$) we suppose that (20) is established for $k-1$. Let $V_1^{(k)} = V_1 \cap W_{k-1}$, $V_2^{(k)} = V_2 \cap W_{k-1}$ and define

$$\gamma_k = \begin{cases} \max_{\substack{v_1 \in V_1^{(k)}, v_2 \in V_2^{(k)} \\ ||v_1||=||v_2||=1}} (v_1, v_2), & \text{if } V_1^{(k)} \neq \{0\} \text{ and } V_2^{(k)} \neq \{0\}, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

If $\gamma_k = 0$ we stop the induction and set $p = k - 1$. Indeed, it is easy to see that in this case, any vector from $V_1^{(k)}$ is orthogonal to V_2 and any vector from $V_2^{(k)}$ is orthogonal to V_1, i.e.

$$W_{k-1} \subseteq (V_1^{\perp} \cap V_2) \oplus (V_1 \cap V_2^{\perp}),$$

which gives in combination with (20) the desired decomposition (19).

Assume now $\gamma_k \neq 0$ and let us construct L_k and W_k. Note that $0 < \gamma_k < 1$. Indeed, if $\gamma_k = 1$ there would exist a non-zero vector $v \in V_1^{(k)} \cap V_2^{(k)} = V_1 \cap V_2 \cap W_{k-1} \subseteq V_0 \cap V_0^{\perp}$, which is impossible. Let $v_1^{(k)} \in V_1^{(k)}$ and $v_2^{(k)} \in V_2^{(k)}$, $||v_1^{(k)}|| = ||v_2^{(k)}|| = 1$, be the vectors that give the maximum in (21) and $L_k = \text{span}\{v_1^{(k)}, v_2^{(k)}\}$. The vector $P_1 v_2^{(k)}$ belongs to $V_1^{(k)}$ since $v_2^{(k)} \in W_{k-1}$ and W_{k-1} is the invariant subspace of P_1 by induction hypothesis. Suppose that $P_1 v_2^{(k)}$ is not parallel to $v_1^{(k)}$. We have then the inequality

$$(v_1^{(k)}, v_2^{(k)}) = (v_1^{(k)}, P_1 v_2^{(k)}) < ||P_1 v_2^{(k)}|| = \left(\frac{P_1 v_2^{(k)}}{||P_1 v_2^{(k)}||}, v_2^{(k)} \right),$$

which contradicts the definition of $v_1^{(k)}$ and $v_2^{(k)}$. This means that $P_1 v_2^{(k)}$ is parallel to $v_1^{(k)}$, hence $P_1 L_k \subset L_k$. One can prove in the same manner that $P_2 v_1^{(k)}$ is parallel to $v_2^{(k)}$, hence $P_2 L_k \subset L_k$. Let $W_k = (V_0 \oplus W_{k-1} \oplus L_1 \oplus \cdots \oplus L_k)^{\perp}$. The subspace $V_0 \oplus W_{k-1} \oplus L_1 \oplus \cdots \oplus L_k$ is invariant with respect to P_1 and P_2 and so is the subspace W_k since operators P_1 and P_2 are symmetric.

Note at last that $W_{k-1} = W_k \oplus L_k$ hence for $k > 1$, $V_1^{(k)} \subset V_1^{(k-1)}$ and $V_2^{(k)} \subset V_2^{(k-1)}$, i.e. $\gamma_k \leq \gamma_{k-1}$ according to (21). Thus we have result (18). □

3.2 Norm and spectral radius of an operator involving the orthogonal projectors P_1 and P_2

If $\mathcal{L}(V)$ is the space of linear and continuous operators from V into V, we denote by $||B|| = \sup_{v \in V, ||v||=1} ||Bv||$ the norm of $B \in \mathcal{L}(V)$. If I denotes the identity operator in V and ω is a real parameter, we define the operator $B \in \mathcal{L}(V)$ by

$$B = (I - \omega P_2)(I - \omega P_1). \tag{22}$$

In this paragraph we formulate first a result for the norm of the operator B in order to get an estimate as presented in [22]. The idea of Proposition 5 and its proof come originally from Bramble et al. [16]. In their work, an abstract analysis of product iterative methods is presented and similar convergence estimates are given. Comparable results proved using the technique from [16] can be found, for example, in early papers from Xu [39,40] and Yserentant [46] appended by the work of Griebel and Oswald [23], in the article of Cai and Widlund [18] or Wang [36], and in an abstract theory presented by Widlund in [37]. More recent reports include the framework of the successive subspace correction algorithm by Xu and Zikatanov [42] and Xu [41]. Some estimates in the framework of an abstract convergence analysis of Schwarz methods are presented in textbooks, e.g., by Quarteroni and Valli [32] (§4.6), Smith et al. [34] (§5.2) and Wohlmuth [38] (§2.1).

Proposition 5 *If Hypothesis (H) is satisfied and if* $0 < \omega < 2$, *then the norm of the operator* B *given by (22) verifies*

$$||B|| \leq \left(1 - \frac{(2-\omega)\omega}{C_0^2(1+\omega\gamma)^2}\right)^{\frac{1}{2}} < 1. \tag{23}$$

Proof The proof is adapted from [16] to the present setting and we establish it for the convenience of the reader. Introduce $R_1 = I - \omega P_1$ and $R_2 = (I - \omega P_2)(I - \omega P_1) = B$. We begin by proving

$$(2-\omega)\omega\left(||P_1 v||^2 + ||P_2 R_1 v||^2\right) = ||v||^2 - ||Bv||^2, \qquad \forall v \in V. \tag{24}$$

As $v = R_1 v + \omega P_1 v$, $||v||^2 = ||R_1 v||^2 + \omega^2 ||P_1 v||^2 + 2\omega(R_1 v, P_1 v)$, and by definition $(R_1 v, P_1 v) = ((I - \omega P_1)v, P_1 v) = (1-\omega)||P_1 v||^2$. Hence

$$||v||^2 - ||R_1 v||^2 = \left[\omega^2 + 2\omega(1-\omega)\right] ||P_1 v||^2 = (2-\omega)\omega||P_1 v||^2. \quad (25)$$

Furthermore, $R_1 v = R_2 v + \omega P_2 R_1 v$ implies $||R_1 v||^2 = ||R_2 v||^2 + \omega^2 ||P_2 R_1 v||^2 + 2\omega(R_2 v, P_2 R_1 v)$ and by definition $(R_2 v, P_2 R_1 v) = ((I - \omega P_2)R_1 v, P_2 R_1 v) = (1-\omega)||P_2 R_1 v||^2$. Hence

$$||R_1 v||^2 - ||R_2 v||^2 = (2-\omega)\omega||P_2 R_1 v||^2. \quad (26)$$

Summing (25) and (26), we get (24).

We next prove

$$||P_1 v||^2 + ||P_2 v||^2 \leq (1+\gamma\omega)^2 \left(||P_1 v||^2 + ||P_2 R_1 v||^2\right), \qquad \forall v \in V. \quad (27)$$

Starting from $I - R_1 = \omega P_1$, we get

$$(P_2 v, v) - (P_2 v, R_1 v) = \omega(P_2 v, P_1 v),$$

which implies that $||P_2 v||^2 = (P_2 v, R_1 v) + \omega(P_2 v, P_1 v)$. Hence

$$\begin{aligned} ||P_1 v||^2 + ||P_2 v||^2 &= (P_1 v, P_1 v) + (P_2 v, P_2 R_1 v) + \omega(P_2 v, P_1 v) \\ &\leq \left(||P_1 v||^2 + ||P_2 v||^2\right)^{\frac{1}{2}} \left(||P_1 v||^2 + ||P_2 R_1 v||^2\right)^{\frac{1}{2}} \\ &\quad + \omega(P_1 v, P_2 v). \end{aligned}$$

From the Definition (11) of γ we get

$$\begin{aligned} |(P_1 v, P_2 v)| &\leq \gamma ||P_1 v|| ||P_2 v|| \leq \gamma \left(||P_2 v|| ||P_1 v|| + ||P_1 v|| ||P_2 R_1 v||\right) \\ &\leq \gamma \left(||P_1 v||^2 + ||P_2 v||^2\right)^{\frac{1}{2}} \left(||P_1 v||^2 + ||P_2 R_1 v||^2\right)^{\frac{1}{2}}. \end{aligned}$$

Thus we have

$$\begin{aligned} &||P_1 v||^2 + ||P_2 v||^2 \\ &\quad \leq (1+\omega\gamma) \left(||P_1 v||^2 + ||P_2 v||^2\right)^{\frac{1}{2}} \left(||P_1 v||^2 + ||P_2 R_1 v||^2\right)^{\frac{1}{2}}, \end{aligned}$$

which leads to (27).

Finally, we show that Hypothesis (H) implies

$$||v||^2 \leq C_0^2 \left(||P_1 v||^2 + ||P_2 v||^2\right), \qquad \forall v \in V. \quad (28)$$

When $v \in V$, there exist $v_1 \in V_1$, $v_2 \in V_2$ such that $v = v_1 + v_2$ and $||v_1||^2 + ||v_2||^2 \leq C_0^2 ||v||^2$ (see Hypothesis (H)). Hence $||v||^2 = (v_1, v) + (v_2, v) = (v_1, P_1 v) + (v_2, P_2 v)$. Result (28) thus follows from:

$$\begin{aligned} ||v||^2 &\leq ||v_1|| ||P_1 v|| + ||v_2|| ||P_2 v|| \\ &\leq \left(||v_1||^2 + ||v_2||^2\right)^{\frac{1}{2}} \left(||P_1 v||^2 + ||P_2 v||^2\right)^{\frac{1}{2}} \\ &\leq C_0 ||v|| \left(||P_1 v||^2 + ||P_2 v||^2\right)^{\frac{1}{2}}. \end{aligned}$$

The proof of Proposition 5 is now straightforward. Combining (24) and (27), we get for all $v \in V$,

$$\frac{(2-\omega)\omega}{(1+\gamma\omega)^2}\left(||P_1 v||^2 + ||P_2 v||^2\right) \leq ||v||^2 - ||Bv||^2,$$

and finally, (28) yields

$$\frac{(2-\omega)\omega}{C_0^2(1+\gamma\omega)^2}||v||^2 \leq ||v||^2 - ||Bv||^2.$$

Thus $||Bv||^2 \leq \left(1 - \frac{(2-\omega)\omega}{C_0^2(1+\gamma\omega)^2}\right)||v||^2$, i.e. $||B|| \leq \left(1 - \frac{(2-\omega)\omega}{C_0^2(1+\omega\gamma)^2}\right)^{\frac{1}{2}}$ which is strictly bounded by one if $0 < \omega < 2$. □

It is readily seen that the estimate of Proposition 5 is not optimal even in the case where $V = V_1 \oplus V_2$. In particular, if the space V is two-dimensional and V_1 and V_2 are one-dimensional subspaces of V, then $||B|| = \gamma$ for $\omega = 1$. Indeed, $\forall v \in V$ we have in this case $||Bv||^2 = |(Bv, (I - P_1)v)| = \gamma||Bv||||(I - P_1)v||$ since $(I - P_1)v \in V_1^{\perp}$, $Bv \in V_2^{\perp}$ and the angle between $V_1^{\perp}$ and $V_2^{\perp}$ is equal to the angle between V_1 and V_2. However, estimate (23) with the best choice of C_0 (14) gives only $||B|| \leq \sqrt{\gamma(\gamma+3)}/(1+\gamma)$, which is optimal only if $\gamma = 0$. The non-optimality of (23) is also discussed, for example, by Griebel and Oswald in the concluding remarks of [23].

In the case where V_1 and V_2 are of finite dimension, an analysis of the spectral properties of B leads to exact formulas for its spectral radius and its norm. Hereafter we present these new results.

For $\tilde{\gamma}$ and $\omega \in (0; 2)$ we define the functions

$$\rho(\tilde{\gamma}, \omega) = \begin{cases} \frac{\omega^2\tilde{\gamma}^2}{2} - \omega + 1 + \frac{\omega\tilde{\gamma}}{2}\sqrt{\omega^2\tilde{\gamma}^2 - 4\omega + 4}, & \text{if } \omega \leq \omega_0(\tilde{\gamma}), \\ \omega - 1, & \text{otherwise}, \end{cases} \tag{29}$$

where

$$\omega_0(\tilde{\gamma}) = \begin{cases} \frac{2-2\sqrt{1-\tilde{\gamma}^2}}{\tilde{\gamma}^2}, & \text{for } \tilde{\gamma} \in (0; 1], \\ 1, & \text{for } \tilde{\gamma} = 0, \end{cases}$$

and

$$N(\tilde{\gamma}, \omega) = \frac{1}{2}\omega\,(2-\omega)\,\tilde{\gamma} + \sqrt{\frac{1}{4}\omega^2\,(2-\omega)^2\,\tilde{\gamma}^2 + (\omega-1)^2}. \tag{30}$$

Proposition 6 *Let V be of finite dimension, $V = V_1 + V_2$ and $\tilde{\gamma}$ be defined by (12). The spectral radius of operator B given by (22) is a function of $\tilde{\gamma}$ and $\omega \in (0; 2)$ given by $\rho(B) = \rho(\tilde{\gamma}, \omega)$. The norm of B is a function of $\tilde{\gamma}$ and $\omega \in (0; 2)$ given by $\|B\| = N(\tilde{\gamma}, \omega)$.*

Proof The idea of the proof is to establish first all the results in the two-dimensional case and to use then decomposition (19) to extend the results to the general case. Therefore, we assume first that the space V is two-dimensional and V_1 and V_2 are one-dimensional subspaces of V spanned by the vectors v_1 and v_2, respectively. Without loss of generality, we can assume that $\|v_1\| = \|v_2\| = 1$ and $(v_1, v_2) = \tilde{\gamma}$. We can verify that the linear operator B is represented in the basis $\{v_1, v_2\}$ by the matrix

$$\mathbf{B} = \begin{pmatrix} 1-\omega & -\omega\tilde{\gamma} \\ \omega\,(\omega-1)\,\tilde{\gamma} & \omega^2\tilde{\gamma}^2 + 1 - \omega \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \lambda^2 - \left(\omega^2\tilde{\gamma}^2 - 2\,\omega + 2\right)\lambda + (\omega-1)^2.$$

If $\tilde{\gamma} > 0$ and $\omega \in (\omega_0(\tilde{\gamma}); 2)$, $p(\lambda)$ has two complex conjugate roots $\lambda_\pm$ such that $|\lambda_\pm| = \omega - 1$. If $\tilde{\gamma} > 0$ and $\omega \in (0; \omega_0(\tilde{\gamma}))$, $p(\lambda)$ has two real roots $\lambda_\pm$ given by

$$\lambda_\pm = \frac{\omega^2\tilde{\gamma}^2}{2} - \omega + 1 \pm \frac{\omega\tilde{\gamma}}{2}\sqrt{\omega^2\tilde{\gamma}^2 - 4\omega + 4}.$$

If $\tilde{\gamma} = 0$, $p(\lambda)$ has the only double root $\lambda = 1 - \omega$. Identity $\rho(B) = \rho(\tilde{\gamma}, \omega)$ is thus proved in the two-dimensional case.

Let us consider now the norm of operator B that can be written as

$$\|B\|^2 = \max_{x\in\mathbb{R}^2, x\neq 0} \frac{x^T\mathbf{B}^T\Gamma\mathbf{B}x}{x^T\Gamma x}, \tag{31}$$

where Γ is the Gramm matrix of the basis $\{v_1, v_2\}$,

$$\Gamma = \begin{pmatrix} 1 & \tilde{\gamma} \\ \tilde{\gamma} & 1 \end{pmatrix}.$$

By making the substitution $y = \Gamma^{1/2}x$, we can rewrite (31) as

$$\|B\|^2 = \max_{y\in\mathbb{R}^2, y\neq 0} \frac{y^T\Gamma^{-1/2}\mathbf{B}^T\Gamma\mathbf{B}\Gamma^{-1/2}y}{y^Ty}. \tag{32}$$

Since the matrix $\mathbf{C} = \Gamma^{-1/2}\mathbf{B}^T\Gamma\mathbf{B}\Gamma^{-1/2}$ is symmetric positive definite, (32) implies that $\|B\|^2$ is equal to the spectral radius of $\mathbf{C}$. Let μ^2 be an eigenvalue of $\mathbf{C}$, then

$$\det(\mathbf{C} - \mu^2\mathbf{I}) = 0. \tag{33}$$

But

$$\begin{aligned}
&\det(\mathbf{C} - \mu^2\mathbf{I}) \\
&\quad = \det(\mathbf{B}^T\Gamma\mathbf{B}\Gamma^{-1} - \mu^2\mathbf{I}) \\
&\quad = \mu^4 - \mu^2\mathrm{tr}(\mathbf{B}^T\Gamma\mathbf{B}\Gamma^{-1}) + \det(\mathbf{B}^T\Gamma\mathbf{B}\Gamma^{-1}) \\
&\quad = \mu^4 - \mu^2[(2-\omega)^2\omega^2\tilde{\gamma}^2 + 2(\omega-1)^2] + (\omega-1)^4 \\
&\quad = \left(\mu^2 - \omega\,(2-\omega)\,\tilde{\gamma}\mu - (\omega-1)^2\right)\left(\mu^2 + \omega\,(2-\omega)\,\tilde{\gamma}\mu - (\omega-1)^2\right).
\end{aligned}$$

The roots of (33) are thus given by

$$\mu = \pm\frac{1}{2}\omega\,(2-\omega)\,\tilde{\gamma} \pm \sqrt{\frac{1}{4}\omega^2\,(2-\omega)^2\,\tilde{\gamma}^2 + (\omega-1)^2},$$

and the largest among them gives $\|B\|$, i.e. identity $\|B\| = N(\tilde{\gamma}, \omega)$ is proved in the two-dimensional case.

Let us turn now to the general case. According to Proposition 4, V can be decomposed into the direct sum (19) where all the summands are invariant subspaces of projectors P_1 and P_2, and hence of B. Hence the spectrum of B is given by the set of all eigenvalues of the operators $B_0 = B|_{V_0}$, $B_{12} = B|_{V_1^\perp \cap V_2}$, $B_{21} = B|_{V_1 \cap V_2^\perp}$ and $B_m = B|_{L_m}$, $m = 1, 2, \dots, p$, where here $B|_W$ is the restriction of B to W. We verify easily that $\rho(B_0) = (1-\omega)^2$, $\rho(B_{12}) = \rho(B_{21}) = |1-\omega|$, and concerning the two-dimensional spaces $L_m, m = 1, 2, \dots, p$, we have proved just above that $\rho(B_m) = \rho(\gamma_m, \omega)$ where $\rho(\gamma, \omega)$ is defined by (29). Hence

$$\rho(B) = \max\left((1-\omega)^2, |1-\omega|, \rho(\gamma_1, \omega), \dots, \rho(\gamma_p, \omega)\right).$$

It is easy to verify that $\omega_0(\gamma)$ is an increasing function and for fixed ω, $\rho(\gamma, \omega)$ is a non-decreasing function. It follows that we have $\rho(\gamma_1, \omega) \geq \cdots \geq \rho(\gamma_p, \omega) > \rho(0, \omega) = |1-\omega|$. Since $\tilde{\gamma} = \gamma_1$ if $p > 0$ and $\tilde{\gamma} = 0$ if $p = 0$, we conclude that $\rho(B) = \rho(\tilde{\gamma}, \omega)$. Analogously, since all the subspaces in (19) are mutually orthogonal, Pythagore's Theorem implies

$$\|B\| = \max\left((1-\omega)^2, |1-\omega|, N(\gamma_1, \omega), \dots, N(\gamma_p, \omega)\right),$$

where $N(\gamma, \omega)$ is defined by (30). Noting that $N(0, \omega) = |1-\omega|$, we conclude that $\|B\| = N(\tilde{\gamma}, \omega)$. □

Finally, let us observe that:

1. The spectral radius $\rho(B)$ is less than one for $\omega \in (0; 2)$ and, for $\tilde{\gamma}$ given by (12), attains the minimum value $\rho(B) = \omega_0(\tilde{\gamma}) - 1$ at $\omega = \omega_0(\tilde{\gamma}) \in [1; 2)$. We have $\rho(B) = \tilde{\gamma}^2$ at $\omega = 1$.
2. The norm $||B||$ is less than one for $\omega \in (0; 2)$ and, for $\tilde{\gamma}$ given by (12), attains the minimum value $||B|| = \tilde{\gamma}$ at $\omega = 1$. This last result is given by Blaheta in [12].
3. The functions $\rho(\tilde{\gamma}, \omega)$ and $N(\tilde{\gamma}, \omega)$ are non-decreasing with respect to $\tilde{\gamma}$ for any fixed value of $\omega \in (0; 2)$.
4. Both formulas (29) and (30) can be rewritten in the case $V_1 \neq V_2$ as the functions only of C_0^{opt} and ω due to the relation (14).

3.3 Proof of Proposition 2

The above abstract analysis enables us to prove Proposition 2. This is readily done by applying Proposition 6 to $V = V_{Hh}$, $V_1 = V_h$ and $V_2 = V_H$ using the form $a(\cdot, \cdot)$ as scalar product. □

4 Estimates for γ

Estimates and upper bounds for the constant from the C.B.S. inequality are abundant in the literature as it is the main tool in the convergence analysis of many methods. The C.B.S. inequality has been used in two-level methods by Axelsson [4], Axelsson and Gustavson [7], Braess [13,14], Maître and Musy [27]. A survey of the role of this constant is reported by Axelsson and Vassilevski [8,9] and by Eijkhout and Vassilevski [21]. The constant is also used in local refinement preconditioning methods, e.g., by McCormick [29] and Bramble et al. [15]. The latest papers present estimates of γ depending generally on the bilinear form a, i.e. on the problem coefficients, and the type and shape of the finite element used. In some cases it is possible to have universal bounds [6]. Margenov [28] gives estimates of the 2D elasticity problem on a triangular mesh. More recently Achchab and Maître [1] and Axelsson [5] proved that the constant γ^2 is bounded from above by 3/4 for the 2D elasticity problem on a triangular mesh. Numerical experiments by Jung and Maître [25] generalize the latter to more choices of finite elements.

Let $a_{ij} \in W^{1,\infty}(\Omega)$, $1 \le i, j \le 2$, verifying $a_{ij} = a_{ji}$ and the hypothesis of strong ellipticity,

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \ge \alpha \sum_{i=1}^{2} \xi_i^2, \quad \forall(\xi_1,\xi_2) \in \mathbb{R}^2, \text{ a.e. in } \Omega, \tag{34}$$

where α is a positive constant. If $\mathcal{L}$ is the elliptic operator given by

$$\mathcal{L}(u) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right),$$

the associated bilinear form is given by

$$a(u,v) = \sum_{i,j=1}^{2} \int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}\mathbf{x}.$$

We consider the case with $\overline{\Lambda} \subset K$, for $K \in \mathcal{T}_H$. Let $\tilde{\Lambda} \supseteq \Lambda$ be a rectangle with dimensions L_1 and L_2 and define

$$\tilde{\lambda} = \min_{v\in H_0^1(\tilde{\Lambda}), v\neq 0} ||\nabla v||^2_{L^2(\tilde{\Lambda})}/||v||^2_{L^2(\tilde{\Lambda})}.$$

We have $\tilde{\lambda} = \pi^2(1/L_1^2 + 1/L_2^2)$ and we introduce $d = \sqrt{1/\tilde{\lambda}}$. We set

$$\beta = \left[\sum_{j=1}^{2}\left(\sum_{i=1}^{2}\left\|\frac{\partial a_{ij}}{\partial x_i}\right\|_{L^\infty(\Lambda)}\right)^2\right]^{\frac{1}{2}}.$$

Proposition 7 *If (34) is satisfied and if there exists $K \in \mathcal{T}_H$ such that $\overline{\Lambda} \subset K$ and if $r = 1$, then $\gamma \le \frac{\beta d}{\alpha}$. If furthermore the a_{ij}'s are constant over Λ, $1 \le i, j \le 2$, the Algorithm 1 converges in only one iteration when $\omega = 1$.*

Proof We shall first prove that $\gamma \leq \frac{\beta d}{\alpha}$. For any $u_H \in V_H, v_h \in V_h$, we have

$$|a(u_H, v_h)| = \left| \sum_{i,j=1}^{2} \int_{\Lambda} a_{ij} \frac{\partial u_H}{\partial x_j} \frac{\partial v_h}{\partial x_i} \, \mathrm{d}\mathbf{x} \right|,$$

as $v_h = 0$ in $\Omega \setminus \overline{\Lambda}$. Since $\overline{\Lambda} \subset K \in \mathcal{T}_H$, $\frac{\partial u_H}{\partial x_j}$ is constant over $\overline{\Lambda}$ so that

$$|a(u_H, v_h)| = \left| \sum_{i,j=1}^{2} \frac{\partial u_H}{\partial x_j}\bigg|_K \int_{\Lambda} \frac{\partial a_{ij}}{\partial x_i} v_h \, \mathrm{d}\mathbf{x} \right|,$$

where we have applied the divergence theorem taking into account that $v_h = 0$ on $\partial\Lambda$. By the Cauchy-Schwarz inequality we have

$$\begin{aligned} |a(u_H, v_h)| &\leq \sum_{i,j=1}^{2} \left\| \frac{\partial a_{ij}}{\partial x_i} \right\|_{L^\infty(\Lambda)} \left| \frac{\partial u_H}{\partial x_j} \right|_K \int_{\Lambda} |v_h| \, \mathrm{d}\mathbf{x} \\ &\leq \beta \left(\sum_{j=1}^{2} \left\| \frac{\partial u_H}{\partial x_j} \right\|_{L^2(\Lambda)}^2 ||v_h||_{L^2(\Lambda)}^2 \right)^{\frac{1}{2}} \\ &= \beta ||\nabla u_H||_{L^2(\Lambda)} ||v_h||_{L^2(\Lambda)}. \end{aligned}$$

At this point we need to bound $||v_h||_{L^2(\Lambda)}$ from above with $||\nabla v_h||_{L^2(\Lambda)}$. We introduce $\lambda = \min_{v \in H_0^1(\Lambda), v \neq 0} ||\nabla v||_{L^2(\Lambda)}^2 / ||v||_{L^2(\Lambda)}^2$, the smallest value of the Rayleigh quotient. In order to estimate λ, we consider the rectangle $\tilde{\Lambda}$ and $\tilde{\lambda}$ as introduced above. As $\Lambda \subseteq \tilde{\Lambda}$ we have $\lambda \geq \tilde{\lambda} = 1/d^2$, i.e. we get $||v_h||_{L^2(\Lambda)} \leq d||\nabla v_h||_{L^2(\Lambda)}$. Hence combining the previous results,

$$|a(u_H, v_h)| \leq \beta d ||\nabla u_H||_{L^2(\Lambda)} ||\nabla v_h||_{L^2(\Lambda)}.$$

The hypothesis of strong ellipticity (34) implies that, $\forall u \in H_0^1(\Omega)$,

$$\begin{aligned} a(u, u) &= \int_{\Omega} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \, \mathrm{d}\mathbf{x} \\ &\geq \alpha ||\nabla u||_{L^2(\Omega)}^2, \end{aligned}$$

i.e. $\alpha ||\nabla u||_{L^2(\Lambda)}^2 \leq \alpha ||\nabla u||_{L^2(\Omega)}^2 \leq a(u, u) = ||u||^2$. Applying this inequality to u_H and v_h, we obtain $|a(u_H, v_h)| \leq \frac{\beta d}{\alpha} ||u_H|| ||v_h||$, i.e. $\gamma \leq \frac{\beta d}{\alpha}$.

If the a_{ij}'s are constant over Λ, $1 \leq i, j \leq 2$, we clearly have $\beta = 0$, thus $\gamma = 0$ and $C_0 = 1$. In this case V_H and V_h are orthogonal and, since $B = 0$ for $\omega = 1$, the algorithm converges in only one iteration. □

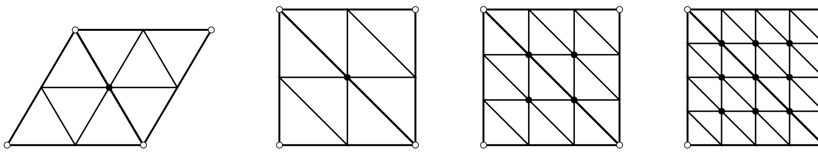


Fig. 2 Illustration of the triangulations of Λ considered in Table 1. White dots refer to the degrees of freedom of $r_H v$, black dots refer to those of $v - r_H v$

Table 1 Upper bounds for γ

Triangles	H/h	Upper bound for γ
equilateral	2	$\sqrt{3}/3 \approx 0.577$
right isosceles	2	$\sqrt{2}/2 \approx 0.707$
right isosceles	3	$2/3 \approx 0.667$
right isosceles	4	$\sqrt{2}/2 \approx 0.707$

In the case where $\Lambda \subset K_1 \cup K_2$, with $K_1, K_2 \in \mathcal{T}_H$, the analysis gets more complicated. In the sequel we present some upper bounds for γ in the case where $a_{ij} = \delta_{ij}$, i.e. $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x}$, and with Λ the union of two triangles K_1 and K_2 of $\mathcal{T}_H$, $\mathcal{T}_h$ conforming with $\mathcal{T}_H$ and $r = s = 1$. We consider the situations as illustrated in Figure 2 by the triangulations of the patch Λ. Estimates can be obtained by splitting $v \in V_{Hh}$ into $v = v_h + v_H$, where $v_H = r_H v$ is the interpolant of v in V_H and $v_h = v - r_H v \in V_h$. The degrees of freedom of v_h and v_H in $\overline{\Lambda}$ are depicted in Figure 2. Using the fact that $v_h = 0$ in $\Omega \setminus \overline{\Lambda}$ and the divergence theorem, we have that

$$a(v_H, v_h) \leq \left| \left[\frac{\partial v_H}{\partial n} \right]_\Gamma \right| \int_\Gamma |v_h| \, \mathrm{d}s, \tag{35}$$

where $\Gamma = \partial K_1 \cap \partial K_2$, $[\cdot]_\Gamma$ denotes the jump on Γ in the direction of a normal unit vector $\mathbf{n}$ on Γ. The first factor of the right-hand side of (35) can be bounded by

$$\left| \left[\frac{\partial v_H}{\partial n} \right]_\Gamma \right| \leq \sum_{i=1}^{2} \frac{||\nabla v_H||_{L^2(K_i)}}{\sqrt{\mathrm{area}(K_i)}} \leq \frac{\sqrt{2}}{\min_{i=1,2} \sqrt{\mathrm{area}(K_i)}} ||\nabla v_H||_{L^2(\Lambda)},$$

and $||\nabla v_H||_{L^2(\Lambda)} \leq ||\nabla v_H||_{L^2(\Omega)}$. As the dimension of V_h is small in our cases, we evaluate $\int_\Gamma |v_h| \, \mathrm{d}s$ explicitely, and express it in relation to $||\nabla v_h||_{L^2(\Lambda)} = ||\nabla v_h||_{L^2(\Omega)}$. Applying the above procedure to our situations, we get $a(v_H, v_h) \leq C||v_H||||v_h||$ and hence we have $\gamma \leq C$. The upper bounds found for γ are reported in Table 1. Note that the bound for γ on right isosceles triangles with $H/h = 2$ is reported by Axelsson and Gustafsson in [7].

The result of Proposition 6 with (29) gives an algebraic relationship of the spectral radius ρ of the operator B as function of $\tilde{\gamma}$ and ω. This leads to a very convenient application to determine numerically a good approximation for $\tilde{\gamma}$. Running Algorithm 1 for given ω we can evaluate numerically an estimate of ρ and hence find an estimate of the parameter $\tilde{\gamma}$. A study of $\tilde{\gamma}$ for various spaces $V_1 = V_h$ and

Table 2 Numerical estimates for γ

Triangles	H/h	Numerical estimate for γ
right isosceles	2	0.426
right isosceles	3	0.464
right isosceles	4	0.476

Table 3 Estimates for $\tilde{\gamma}$ for some patches and grids

H/h	2	4	8	16
$\tilde{\gamma}$	0.273	0.306	0.313	0.315

(a) Estimates for $\tilde{\gamma}$ for varying H/h in the structured nested case ($k = 0$) with $N = 8$.

N	8	16	32
$\tilde{\gamma}$	0.306	0.461	0.50

(b) Estimates for $\tilde{\gamma}$ for varying N in the structured nested case ($k = 0$) with $H/h = 4$.

k	0	0.5	1	1.5	2	2.5	3	3.5	4
$\tilde{\gamma}$	0.306	0.915	0.908	0.812	0.785	0.812	0.908	0.915	0.474

(c) Estimates for $\tilde{\gamma}$ for different non-nested cases of structured grids with $N = 8$ and $H/h = 4$.

H/h	2	4	8	16
$\tilde{\gamma}$	0.920	0.944	0.947	0.970

(d) Estimates for $\tilde{\gamma}$ for varying H/h in the unstructured case with $N = 8$

$V_2 = V_H$ is presented in §6.1. Estimates for the parameter γ corresponding to the situations of Figure 2 with right isosceles triangles are given in Table 2.

5 Generalization to multiple spaces

The objective of this section is to generalize the two-scale algorithm presented in Section 2 to more than one level of refinement.

We consider again problem (1). We approximate u by a finite element method. Introduce a triangulation $\mathcal{T}_H$ of $\overline{\Omega}$, define V_H by (2) and calculate $u_H \in V_H$ satisfying equation (3).

Consider now $\Lambda_j \subset \Omega$, $j = 1, \dots, N-1$, other polygonal domains wherein we would like to obtain a better precision on the solution u than the one given by u_H. Take note that $\overline{\Lambda}_j$ is not necessarily the union of several triangles K of $\mathcal{T}_H$. Let $\mathcal{T}_{h_j}$, $j = 1, \dots, N-1$, be a triangulation of $\overline{\Lambda}_j$ and consider for $j = 1, \dots, N-1$,

$$V_{h_j} = \{g : \overline{\Omega} \to \mathbb{R} \text{ continuous such that } g|_K \in \mathbb{P}_{s_j}(K), \forall K \in \mathcal{T}_{h_j} \\ \text{and } g = 0 \text{ on } \overline{\Omega} \setminus \Lambda_j\}.$$

Call $\mathcal{T}_{h_N} = \mathcal{T}_H$ and $V_{h_N} = V_H$. Setting $V_{Hh} = \sum_{j=1}^{N} V_{h_j}$ we search as approximation for u the function $u_{Hh} \in V_{Hh}$ satisfying the equation (4).

The intersection $V_i \cap V_j$, for any $1 \leq i, j \leq N$, does not necessarily reduce to the element zero, making it impossible to explicit a finite element basis of the space V_{Hh}. The generalization of Algorithm 1 to compute u_{Hh} is the following:

Algorithm 2

1. Set $u^0 = u_H \in V_H$ and choose $\omega \in (0; 2)$.
2. For $n = 1, 2, 3, \ldots$ find
   for $j = 1, 2, \ldots, N$,
   $w_{h_j} \in V_{h_j}$ such that
   $$a(w_{h_j}, v) = \langle f|v\rangle - a(u^{n-1+\frac{j-1}{N}}, v), \quad \forall v \in V_{h_j} \ ;$$
   $u^{n-1+\frac{j}{N}} = u^{n-1+\frac{j-1}{N}} + \omega w_{h_j}$.

If $P_{h_j} : V_{Hh} \to V_{h_j}$, $j = 1, 2, \ldots, N$, are orthogonal projectors from V_{Hh} to V_{h_j} with respect to the scalar product $a(\cdot, \cdot)$, it is easy to verify that $u_{Hh} - u^n = (I-\omega P_{h_N})(I-\omega P_{h_{N-1}}) \cdots (I-\omega P_{h_1})(u_{Hh}-u^{n-1})$, where I denotes the identity operator in V_{Hh}. Setting $B = (I - \omega P_{h_N})(I - \omega P_{h_{N-1}}) \cdots (I - \omega P_{h_1})$, we obtain that $u_{Hh} - u^n = B^n(u_{Hh} - u_H)$.

Proposition 8 *If $\omega \in (0; 2)$, then the algorithm converges, i.e.*

$$\lim_{n\to\infty} ||u^n - u_{Hh}|| = 0.$$

In order to prove Proposition 8 let us establish first a more general result.

Let V be a Hilbert space with scalar product $(\cdot, \cdot)$ and denote by $||\cdot||$ the induced norm. Consider $V_1, V_2, \ldots, V_N$ closed subspaces of V not reduced to zero. We call $P_j : V \to V_j \subset V$ the orthogonal projectors from V onto V_j, $j = 1, 2, \ldots, N$. If I denotes the identity operator in V and ω is a real parameter, we define the operator $B \in \mathcal{L}(V)$ by

$$B = (I - \omega P_N)(I - \omega P_{N-1}) \cdots (I - \omega P_1). \tag{36}$$

Introduce the numbers $\gamma_{ij} = \sup_{\substack{v_i \in V_i, v_i \neq 0 \\ v_j \in V_j, v_j \neq 0}} \frac{(v_i, v_j)}{||v_i|| ||v_j||} = \gamma_{ji} \leq 1$, which are the constants from the corresponding C.B.S. inequalities. Note that $\gamma_{jj} = 1$ for $j = 1, 2, \ldots, N$. Consider the following hypothesis:

Hypothesis ($\bar{\mathbf{H}}$) There exists a constant $\overline{C}_0$ such that for all $v \in V$ there exist $v_j \in V_j$, $j = 1, 2, \ldots, N$, satisfying $v = \sum_{j=1}^{N} v_j$ and $\sum_{j=1}^{N} ||v_j||^2 \leq \overline{C}_0^2 ||v||^2$.

Applying recursively Proposition 3 it is easy to see that ($\bar{\text{H}}$) is satisfied if and only if $V = V_1 + V_2 + \ldots + V_N$.

Proposition 9 *If Hypothesis ($\bar{H}$) is satisfied and if $0 < \omega < 2$, then the norm of the operator B given by (36) verifies*

$$||B|| \leq \left(1 - \frac{(2-\omega)\omega}{\overline{C}_0^2 (1 + \omega\overline{\gamma})^2}\right)^{\frac{1}{2}} < 1,$$

where $\overline{\gamma} = \max_{1\leq j\leq N} \sum_{1\leq i\leq N, i\neq j} \gamma_{ij}$, with $0 \leq \overline{\gamma} \leq N$.

Proof The proof can be adapted from [16] in a same way as the one in Proposition 5. □

The proof of Proposition 8 now follows easily:

Proof (Proposition 8) This Proposition is proved by applying Proposition 9 to $V = V_{Hh}$ and $V_j = V_{h_j}$, $j = 1, 2, \ldots, N$. □

Remark that a generalization of Proposition 6 to the case $N > 2$ is not straightforward.

6 Numerical results

We illustrate the above presented algorithm with the following example: Consider the Poisson-Dirichlet problem

$$-\Delta u = f,$$

in the domain $\Omega = (-1; 1)^2$, $u = 0$ on its boundary $\partial\Omega$. Take

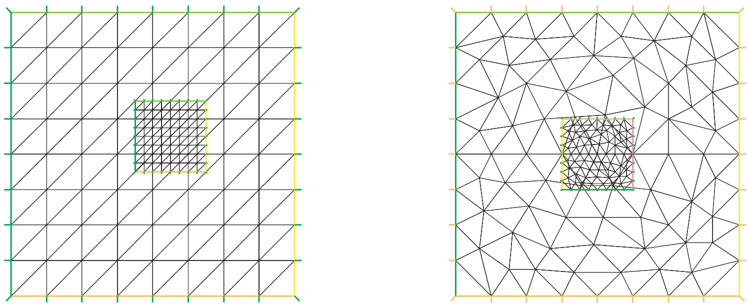
$$f = -4\eta\chi(R)\frac{R^2 + R^4 - \epsilon^4}{|\epsilon^2 - R^2|^4} \exp\left(\frac{1}{\epsilon^2}\right) \exp\left(\frac{-1}{|\epsilon^2 - R^2|}\right),$$

where $R = \sqrt{x_1^2 + x_2^2}$ and $\chi(R) = 1$ if $R \leq \epsilon$, $\chi(R) = 0$ if $R > \epsilon$; η and ϵ are parameters. The exact solution to the problem is given by $u = \eta\chi(R)\exp\left(\frac{1}{\epsilon^2}\right)$ $\exp\left(\frac{-1}{|\epsilon^2 - R^2|}\right)$. We choose $\eta = 10$ and $\epsilon = 0.5$.

Away from the origin (0, 0) the solution is smooth. In a region close to (0, 0) where the solution has a peak, we need to apply a patch with a finer mesh. For the triangulation of $\overline{\Omega}$, we use a coarse uniform grid with mesh size H and $r = 1$. We consider a patch Λ with a fine uniform triangulation of size h and $s = 1$.

We consider two cases of patches and grids. In a first constellation we take both grids structured. We choose $\Lambda = (-\frac{\epsilon}{2} + kh; \frac{\epsilon}{2} + kh)^2$ and the mesh sizes H and h such that the fine triangulation is nested in the coarse one for $k = 0$. We characterize the grids by N where $4N$ corresponds to the number of nodes chosen on $\partial\Omega$. Varying the parameter k induces a translation of the patch Λ hence leading to situations with non-nested grids. In Figure 3(a) we illustrate the case with $H = 1/4$, i.e. $N = 8$, $H/h = 4$ and $k = 2$. A second constellation of interest is where both grids are unstructured and $\Lambda = (-\frac{\epsilon}{2}; \frac{\epsilon}{2})^2$. Figure 3(b) illustrates this case with $N = 8$ and $H/h = 4$.

For numerical quadratures and calculating the errors, we introduce a global fine uniform structured triangulation wherein the fine grid, if structured and $k = 0$, is nested. In the nested case, it is an extension of the fine triangulation to the domain Ω, taken two times finer in order to minimize the projection errors introduced when comparing the results. We use the software `Freefem++` [24] to generate the grids and implement the algorithm.



(a) Non-nested structured grids with parameter $k = 2$.

(b) Unstructured grids.

Fig. 3 Illustrations with $N = 8$ and $H/h = 4$ of some patches and grids considered

6.1 Numerical evaluation of $\tilde{\gamma}$

As mentioned at the end of Section 4 we can evaluate the parameter $\tilde{\gamma}$ of two spaces V_H and V_h by running Algorithm 1, estimating the spectral radius ρ of the iteration operator B and using the result of Proposition 6 linking ρ to $\tilde{\gamma}$ for a given ω. Recall that, if $\omega = 1$ we have $\tilde{\gamma} = \sqrt{\rho(B)}$, and that $\rho(B)$ is the absolute value of the largest eigenvalue. A convenient numerical evaluation of the spectral radius is done by setting $f = 0$, starting in practice with any initial condition u^0 non-zero, and evaluating $\rho(B)$ as $\sqrt[n]{||u^n||}$ for large n. Indeed, we can prove that

$$\lim_{n\to\infty} \sqrt[n]{||u^n||} = \lim_{n\to\infty} \sqrt[n]{||B^n u^0||} = \rho, \tag{37}$$

if u^0 has a non-zero component in the direction of the eigenvector(s) corresponding to the eigenvalue(s) giving the spectral radius. Note that we do not use the standard power method as it does not apply when $\rho(B)$ corresponds to a complex eigenvalue.

For the case of right isosceles triangles as presented in Figure 2 we find the estimates of Table 2 for $\gamma = \tilde{\gamma}$. These results are to be compared with the estimated upper bounds presented earlier in Table 1: we remark that they are not sharp. Note that due to the relation (14) we also have an estimate for the optimal constant C_0 of (13).

Before presenting the estimates for $\tilde{\gamma}$ in the cases of the above introduced grids, it is worth illustrating the fitting of formula (29) for ρ with the numerical estimates. In the case of nested grids with $N = 8$, $H/h = 4$ we obtain $\tilde{\gamma} = 0.306$ for $\omega = 1$. In Figure 4 we plot $\rho(\tilde{\gamma} = 0.306, \omega)$ and the corresponding numerical results for ρ using (37) for $\omega \in (0; 2)$.

In order to get an idea of the convergence behavior of the algorithm for the different grid structures presented at the beginning of this Section, it is interesting to evaluate $\tilde{\gamma}$. The convergence factor of the algorithm being bounded by $||B||$ (see Prop. 2), with $||B|| = \tilde{\gamma}$ for $\omega = 1$, $\tilde{\gamma}$ is a bound for the convergence factor. In Table 3 we present estimates for $\tilde{\gamma}$ in the case of structured and unstructured

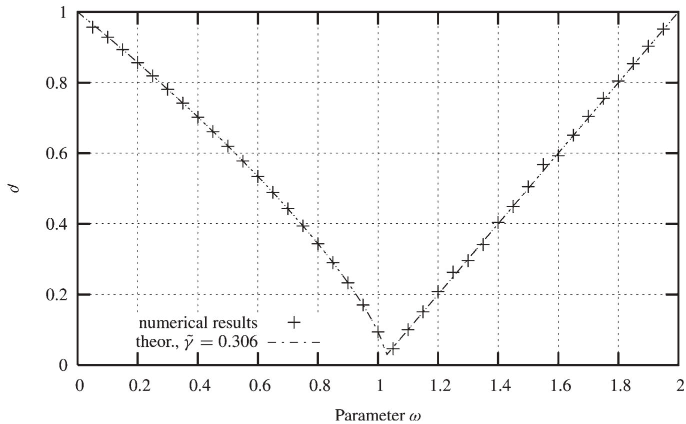


Fig. 4 Comparison of numerical estimates and theoretical results for ρ for different parameter ω in the case of nested grids, $N = 8$ and $H/h = 4$

grids. The parameter $\tilde{\gamma}$ is given in the case of structured nested grids ($k = 0$), first for fixed $N = 8$ and different ratios H/h (Table 4(a)), then for fixed $H/h = 4$ and variable N (Table 4(b)). Table 4(c) gives estimates for $\tilde{\gamma}$ for some non-nested constellations with structured grids in the case $N = 8$ and $H/h = 4$. Finally we give some estimates in the case with unstructured grids for variable ratios H/h (Table 4(d)).

6.2 Convergence of the algorithm

The objective of this paragraph is to illustrate the convergence of the algorithm on the chosen example. In the sequel we consider $\omega = 1$. We introduce the stopping criterion $||u^n - u^{n-1}||/||u^n|| < \epsilon_1$ where n, $n = 1, 2, \ldots,$ is the iteration number. If this criterion yields true at iteration n_c, we define $u_{Hh} = u^{n_c}$. To verify that the algorithm has well converged, we check that u_{Hh} satisfies a second criterion, namely $||\overline{u}_{Hh} - u_{Hh}||/||\overline{u}_{Hh}|| < \epsilon_2$, where $\overline{u}_{Hh} = u^{n_c+p}$, $p = 20$. We choose $\epsilon_1 = 10^{-4}$ and $\epsilon_2 = 10\epsilon_1$. We define the relative error at iteration n, $n = 0, 1, \ldots, n_c$, by $e^n = ||\overline{u}_{Hh} - u^n||/||\overline{u}_{Hh}||$. The evolution of this error through the iterative process gives information about the speed of convergence of the algorithm.

The type of grids we use is an important element for the convergence rate of the algorithm. This can be readily seen through the $\tilde{\gamma}$-estimates presented in Table 3. The curves of Figure 5 illustrate the influence of the chosen grids. We compare the behavior for the cases of nested ($k = 0$ and $k = 4$), non-nested ($k = 2$) and unstructured grids with $N = 8$ and $H/h = 4$. In the nested case, the algorithm converges very fast, in only a couple of steps. The results correspond to the behavior of the estimated bound $\tilde{\gamma}$ for the convergence factor. The rate of convergence can be well foreseen by the estimate of $\tilde{\gamma}$. From Table 3 we have that $\tilde{\gamma} = 0.306$ resp. 0.474 in the nested cases with $k = 0$ and 4, $\tilde{\gamma} = 0.785$ in the considered non-nested case and $\tilde{\gamma} = 0.944$ in the case of unstructured and completely uncorrelated grids. Note that even in the latter most general case the algorithm converges steadily. For the

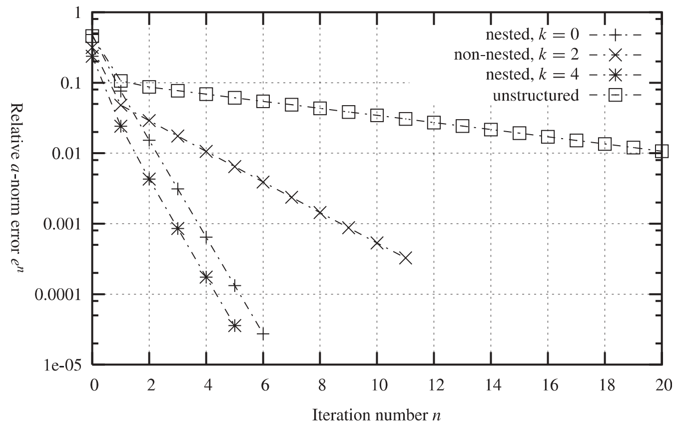


Fig. 5 Convergence of the algorithm for different cases of patches and grids

readibility of the graphic, we show only the evolution of the error through the first 20 iterations.

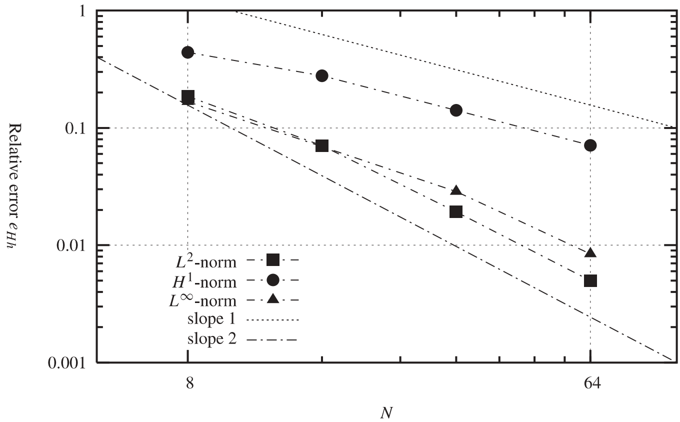
Similarly, the convergence behavior of the algorithm follows the convergence rate bound $\tilde{\gamma}$ (Table 3) in the cases with fixed N and variable H/h, respectively with fixed H/h and variable N. The convergence is slightly slower with increasing H/h for fixed N. The same holds for increasing N and fixed H/h.

6.3 Convergence in the grid size

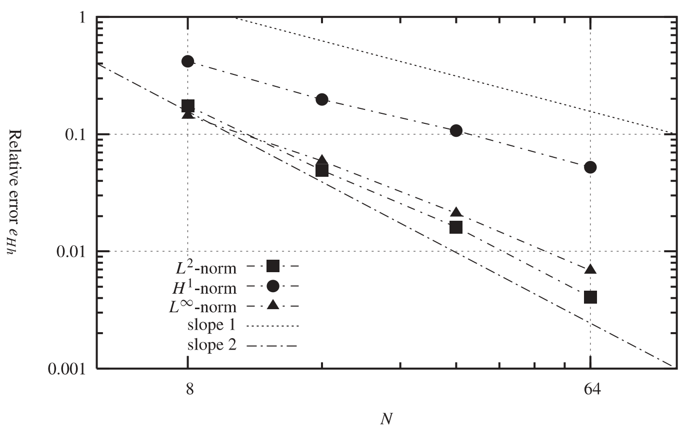
Up to now, we have only considered illustrations for the convergence of the algorithm, i.e. obtaining the approximation u_{Hh} to the exact solution u. To assess the convergence of u_{Hh} in H and h given by the *a priori* estimate of Proposition 1, we introduce the relative error $e_{Hh} = ||u - u_{Hh}||/||u||$. This error is evaluated by interpolating u and u_{Hh} on the fine uniform structured grid. The result u_{Hh} is obtained here by requiring a-norm convergence of the algorithm. On the graphics of Figure 6 we show the relative L^2-, H^1- and L^∞-norm errors e_{Hh} for increasing N, $N = 8, 16, 32, 64$, with $H/h = 2$ fixed, in the case of nested ($k = 0$) and unstructured grids. In both cases, we observe optimal convergence in the mesh size $H = 2/N$: we observe h^2-accuracy for the L^2-norm and rate of convergence one for the H^1-norm.

7 Concluding remarks

We have presented and analyzed a method for solving numerically problems with multi-scale data. The method uses patches whose triangulations are not necessarily nested. In Section 6 we have only presented results in the two-level case. Remark that the implementation of the generalization to multiple levels, as given in



(a) Nested grids.



(b) Unstructured grids.

Fig. 6 Convergence in the mesh size with $H/h = 2$

Section 5, leads to efficient programs, in particular if a telescopic set of well-adapted patches is chosen. At each level of refinement the correction in all non-overlapping patches can be parallelized. Using patches to connect problems in adjacent domains with nonconforming grids leads to efficient coupling techniques. Large domains can be split into non-overlapping subdomains for parallel treatment and patches are to be used at a second level to connect the subdomains. Applications of the method include all problems where local refinement is necessary, as, for example, in boundary layer problems. We are looking forward to presenting results illustrating different types of implementations and applications of the algorithm.

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