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Interior Penalty Continuous and Discontinuous Finite Element Approximations of Hyperbolic Equations

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Abstract In this paper we present in a unified setting the continuous and discontinuous Galerkin methods for the numerical approximation of the scalar hyperbolic equation. Both methods are stabilized by the interior penalty method, more precisely by the jump of the gradient across element faces in the continuous case whereas in the discontinuous case the stabilization of the jump of the solution and optionally of its gradient is required to achieve optimal convergence. We prove that the solution in the case of the continuous Galerkin approach can be considered as a limit of the discontinuous one when the stabilization parameter associated with the penalization of the solution jump tends to infinity. As a consequence, the limit of the numerical flux of the discontinuous method yields a numerical flux for the continuous method as well. Numerical results will highlight the theoretical results that are proven in this paper.

Keywords Continuous and discontinuous Galerkin methods · Hyperbolic problems · Interior penalty

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1 Introduction

The discontinuous Galerkin finite element method (DGFEM) was introduced by Reed and Hill in 1973 for the neutron transport equation [28]. They compared the DGFEM with the continuous Galerkin finite element method (CGFEM) by means of numerical experiments. In their examples they highlighted the good stability properties of the DGFEM. The first analysis was performed a year later by Lesaint and Raviart [23]. A sharpened analysis using the stronger stability of the DG-method was proposed in 1986 by Johnson et al. [20]. More recently, Houston, Schwab and Süli [17] presented an hp-analysis for the upwind DGFEM applied to advection–diffusion–reaction equations, while Brezzi, Marini and Süli [5] generalized the upwind DGFEM by replacing the standard upwind flux by a consistency term and a jump stabilization term. Finally, Burman and Stamm [11] proved that optimal convergence still holds also for quadratic and higher polynomial degrees when only the jump of the tangential part of the gradient is penalized.

In parallel to this development for hyperbolic problems, Continuous Interior Penalty (CIP) finite element methods were introduced in the 1970s by Babuška and Zlámal [2] for the biharmonic operator and by Douglas and Dupont [13] for second-order elliptic and parabolic problems. The idea behind CIP consists in penalizing the jump of the gradient of the discrete solution at interfaces between elements, thus weakly imposing C^1-continuity. More recently, CIP-methods experienced a further development. A priori error estimates that are uniform with respect to the diffusion coefficient have been obtained for CIP linear finite element approximations to advection–diffusion equations by Burman and Hansbo [10]. A unified framework for the convergence analysis of both conforming and nonconforming linear finite elements with interior penalty (IP) has been proposed by Burman [7]. Finally, a CIP linear finite element method with a nonlinear shock-capturing term that rigorously guarantees a discrete maximum principle for advection–diffusion–reaction problems has been investigated by Burman and Ern [8].

In this paper we will show that the CIP-method for the transport equation can be seen as the asymptotic limit of the DG-method proposed in [5], provided the DG-formulation is augmented with the interior penalty term acting on the gradient jumps. Such a term was proposed as a stabilizing one for DG-methods in the approximation of elliptic problems by Romkes, Prudhomme and Oden [29] and by Brezzi, Cockburn, Marini and Süli [3] in a general framework focusing on stabilizing mechanisms for DG-methods. It does not downgrade the convergence order of the DG-method, rather it ensures more robustness with respect to variations in the stabilization parameter γ_0 acting on the solution jump. We prove that when γ_0 tends to infinity then the solution of the standard DG-method (without stabilization of the gradient jumps) converges to that of the unstabilized continuous Galerkin method. Two relevant properties follow. On the one hand a numerical flux can be defined for the continuous method as limit of the numerical flux of the discontinuous method as $\gamma_0 \to \infty$ and, on the other hand, the DG-method as proposed in [5] is not stable if overstabilized (that is when γ_0 becomes too large) for advection dominated problems. A similar phenomenon was observed by Brezzi, Houston, Marini and Süli [4] for the subgrid viscosity method of Guermond [15] and is true for the CIP-method using high order polynomials.

The asymptotic analysis is inspired by that for the elliptic case by Larson and Niklasson [22] and so is our discussion on the local fluxes in Sect. 3.

This paper is organized as follows. Section 2 introduces the two methods, the DGFEM and CGFEM, for the scalar hyperbolic equation. Special emphasis will be given to finding a uniform formalism for both methods. Further we recall h-convergence results for the continuous interior penalty method and for the augmented DG-method. In Sect. 3 we prove that the CIP-method can be considered as a limit of the DG-method if the jump stabilization parameter γ_0 tends to infinity. In Sect. 4 we discuss the local fluxes for the DG-method and the CG-method.

Some numerical examples for interior penalty stabilized finite element methods using continuous and discontinuous approximations are presented in Sect. 5, highlighting the theoretical results of Sects. 2 and 3. Section 6 is left for the conclusions.

2 Discontinuous and Continuous Finite Element Approximation with Interior Penalty

Let Ω be an open bounded and connected set in $\mathbb{R}^d$, $d = 2, 3$ with Lipschitz boundary $\partial\Omega$ and outer unit normal n. Moreover let $\beta \in [W^{1,\infty}(\Omega)]^d$ be a given vector field, $\mu \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$ two given functions and $\partial\Omega^\pm = \{x \in \partial\Omega : \pm\beta(x)\cdot n(x) > 0\}$ with $\partial\Omega^+$ and $\partial\Omega^-$ well separated, i.e. $\text{meas}_{d-1}(\overline{\partial\Omega^-} \cap \overline{\partial\Omega^+}) = 0$. Consider the problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \mu u + \beta\cdot\nabla u = f & \text{in } \Omega, \\ u|_{\partial\Omega^-} = 0. \end{cases} \tag{1}$$

Define $W = \{w \in L^2(\Omega) : \beta\cdot\nabla w \in L^2(\Omega)\}$ and observe that functions in W have traces in

$$L^2(\partial\Omega; \beta\cdot n) = \left\{ v \in L^2(\partial\Omega) : \int_{\partial\Omega} |\beta \cdot n|\, v^2 < \infty \right\}.$$

Consider the operator $A : W \ni w \mapsto \mu w + \beta\cdot\nabla w \in L^2(\Omega)$. Henceforth, it is assumed that there is $\mu_0 > 0$ such that

$$\mu - \frac{1}{2}\nabla\cdot\beta \geq \mu_0, \quad \text{a.e. in } \Omega. \tag{2}$$

Then, letting $V = \{w \in W : w|_{\partial\Omega^-} = 0\}$, $A : V \to L^2(\Omega)$ is an isomorphism, i.e., (1) is well-posed; see, e.g., [14, 27].

Let $\mathcal{K}$ be a finite element mesh of Ω into non-overlapping d-simplices. For $\kappa \in \mathcal{K}$, h_κ denotes its diameter and set $h = \max_{\kappa\in\mathcal{K}} h_\kappa$. Assume that (i) $\mathcal{K}$ covers $\overline{\Omega}$ exactly, (ii) $\mathcal{K}$ does not contain any hanging nodes, and (iii) $\mathcal{K}$ is locally quasi-uniform in the sense that there exists a constant $\rho > 0$, independent of h, such that

$$\rho h_\kappa \leq \min_{\kappa'\in\mathcal{N}(\kappa)} h_{\kappa'},$$

where $\mathcal{N}(\kappa)$ denotes the set of elements sharing at least one node with κ. Each $\kappa \in \mathcal{K}$ is an affine image of the unit simplex $\widehat{\kappa}$, i.e., $\kappa = F_\kappa(\widehat{\kappa})$. Let $\mathcal{F}_{int}$ denote the set of interior faces ($(d-1)$-manifolds) of the mesh, i.e., the set of faces that are not included in the boundary $\partial\Omega$. The sets $\mathcal{F}_\pm$ denote the faces that are included in $\partial\Omega^\pm$ respectively and denote $\mathcal{F} = \mathcal{F}_{int} \cup \mathcal{F}_+ \cup \mathcal{F}_-$. For $F \in \mathcal{F}$, h_F denotes its diameter.

Let $p \geq 1$ and let $\mathbb{P}_p(\widehat{\kappa})$ be the space of polynomials of total degree p. Introduce the continuous and discontinuous finite element spaces

$$V_h^p = \{ v_h \in C^0(\overline{\Omega}) : \forall\kappa \in \mathcal{K},\ v_h|_\kappa \circ F_\kappa \in \mathbb{P}_p(\widehat{\kappa})\}, \tag{3}$$

$$W_h^p = \{ w_h \in L^2(\Omega) : \forall\kappa \in \mathcal{K},\ w_h|_\kappa \circ F_\kappa \in \mathbb{P}_p(\widehat{\kappa})\}. \tag{4}$$

For a non-empty domain $R \subset \Omega$, $(\cdot,\cdot)_R$ denotes the $L^2(R)$-scalar product, $\|\cdot\|_R = (\cdot,\cdot)_R^{1/2}$ the associated norm, and $\|\cdot\|_{s,R}$ the $H^s(R)$-norm.

For $s \geq 1$, let $H^s(\mathcal{K})$ be the space of piecewise Sobolev H^s-functions. Let $\mathcal{S} \subset \mathcal{F}$ and define the scalar product $(\cdot,\cdot)_{\mathcal{S}} = \sum_{s\in\mathcal{S}}(\cdot,\cdot)_s$ and norm $\|\cdot\|_{\mathcal{S}} = (\cdot,\cdot)_{\mathcal{S}}^{1/2}$. For $v \in H^2(\mathcal{K})$ and an interior face $F = \kappa_1 \cap \kappa_2$, where κ_1 and κ_2 are two distinct elements of $\mathcal{K}$ with respective outer normals n_1 and n_2, introduce the jump $[\nabla v]_F = \nabla v|_{\kappa_1}\cdot n_1 + \nabla v|_{\kappa_2}\cdot n_2$ (the subscript F is dropped when there is no ambiguity). Similarly, for $v \in H^1(\mathcal{K})$, define the jump $[v]_F = v|_{\kappa_1} n_1 + v|_{\kappa_2} n_2$. The average is defined for all functions $v \in H^1(\mathcal{K})$ by $\{v\} = \frac{1}{2}(v|_{\kappa_1} + v|_{\kappa_2})$. On outer faces $F = \partial\kappa \cap \partial\Omega$ with outer normal n, the scalar-valued jump and the average are defined as $[v]_F = v|_\kappa n$ resp. $\{v\} = v|_\kappa$.

2.1 The Discontinuous Galerkin Approximation

On $W \times W$ define the discontinuous Galerkin bilinear form

$$a(v,w) = \big((\mu - \nabla\cdot\beta)v, w\big)_{\mathcal{K}} - (v, \beta\cdot\nabla w)_{\mathcal{K}} + (\{\beta v\}, [w])_{\mathcal{F}_{int}\cup\mathcal{F}_+}, \tag{5}$$

and on $H^q(\mathcal{K}) \times H^q(\mathcal{K})$ define the jump penalty and CIP bilinear form

$$b_0(v,w) = (\beta_n [v], [w])_{\mathcal{F}_{int}}, \quad \text{for } q > \frac{1}{2}, \tag{6}$$

$$b_1(v,w) = (h_F^2 \beta_n [\nabla v], [\nabla w])_{\mathcal{F}_{int}}, \quad \text{for } q > \frac{3}{2}, \tag{7}$$

where $\beta_n|_F = \|\beta\cdot n\|_{\infty,F} + \epsilon\|\beta \times n\|_{\infty,F}$, with $\epsilon \geq 0$ and where $\|\cdot\|_{\infty,F}$ denotes the L^∞-norm on the face $F \in \mathcal{F}$. For the asymptotic analysis of Sect. 3.2 we assume that either $\epsilon > 0$ and $\beta \neq 0$ or $\|\beta\cdot n\|_{\infty,F} > 0$ for all faces F of the mesh. Since $W^{1,\infty}(\Omega) \subset C^0(\overline{\Omega})$, the field β is continuous by assumption and, therefore, the quantity β_n is single-valued on all faces $F \in \mathcal{F}$.

The discontinuous finite element approximation of (1) consists of seeking $u_d \in W_h^p$ such that

$$a(u_d, w_h) + \gamma_0 b_0(u_d, w_h) + \gamma_1 b_1(u_d, w_h) = (f, w_h)_{\mathcal{K}}, \quad \forall w_h \in W_h^p \tag{8}$$

for $\gamma_0 > 0$ and $\gamma_1 \geq 0$.

Remark 2.1 If the parameters ϵ and γ_1 are set equal to zero, then this method coincides with the one proposed in [5].

2.2 The Continuous Galerkin Approximation

The continuous finite element approximation with weakly imposed boundary condition is obtained by replacing the discontinuous finite element space W_h^p by the continuous finite element space V_h^p. The problem becomes: find $u_c \in V_h^p$ such that

$$a(u_c, v_h) + \gamma_1 b_1(u_c, v_h) = (f, v_h)_{\mathcal{K}}, \quad \forall v_h \in V_h^p. \tag{9}$$

Remark that the bilinear form $a(\cdot,\cdot)$, defined in (5), simplifies to

$$a(u_c, v_h) = \big((\mu - \nabla\cdot\beta)u_c, v_h\big)_{\mathcal{K}} - (u_c, \beta\cdot\nabla v_h)_{\mathcal{K}} + (\beta\cdot n u_c, v_h)_{\mathcal{F}_+}$$

and that $b_0(u_c, v_h) = 0$ since u_h is continuous.

2.3 Basic Results

For $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, consider the norm

$$|||v|||^2 = \|\mu_0^{\frac{1}{2}} v\|_{\mathcal{K}}^2 + \frac{1}{2}\||\beta \cdot n|^{\frac{1}{2}} v\|_{\partial\Omega}^2 + \gamma_0 b_0(v, v) + \gamma_1 b_1(v, v). \tag{10}$$

The well-posedness of the approximate problems, (8) and (9), results from the following lemma.

Lemma 2.2 (Coercivity) *For all $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$,*

$$a(v, v) + \gamma_0 b_0(v, v) + \gamma_1 b_1(v, v) \geq |||v|||^2.$$

Proof It is a straightforward verification using integration by parts and condition (2). □

The next lemma shows the Galerkin orthogonality for both, the continuous and discontinuous, problems.

Lemma 2.3 (Consistency) *Let $v_h \in V_h^p$ and $w_h \in W_h^p$ and assume $u \in H^q(\Omega)$, for $q > \frac{3}{2}$, then*

$$a(u - u_c, v_h) + \gamma_1 b_1(u - u_c, v_h) = 0, \tag{11}$$

$$a(u - u_d, w_h) + \gamma_0 b_0(u - u_d, w_h) + \gamma_1 b_1(u - u_d, w_h) = 0, \tag{12}$$

where u, u_d and u_c denotes the solutions of (1), (8) *resp.* (9).

Proof For the first equality, let $v_h \in V_h^p$ and observe that

$$a(u_c, v_h) + \gamma_1 b_1(u_c, v_h) = (f, v_h)_{\mathcal{K}}$$

since u_c is the solution of (9). In addition note that

$$\begin{aligned} a(u, v_h) &= \big((\mu - \nabla\cdot\beta)u, v_h\big)_{\mathcal{K}} - (u, \beta\cdot\nabla v_h)_{\mathcal{K}} + (\beta\cdot n u, v_h)_{\mathcal{F}_+} \\ &= (\mu u + \beta\cdot\nabla u, v_h)_{\mathcal{K}} - (\beta\cdot n u, v_h)_{\partial\Omega} + (\beta\cdot n u, v_h)_{\mathcal{F}_+} \\ &= (\mu u + \beta\cdot\nabla u, v_h)_{\mathcal{K}} = (f, v_h)_{\mathcal{K}} \end{aligned}$$

having used integration by parts and the fact that $u_{|_{\partial\Omega^-}} = 0$. Moreover $b_1(u, v_h) = 0$ and consequently (11) holds. For the second equality (12), let $w_h \in W_h^p$ and thus

$$a(u_d, w_h) + \gamma_0 b_0(u_d, w_h) + \gamma_1 b_1(u_d, w_h) = (f, w_h)_{\mathcal{K}}.$$

Finally, using integration by parts on each element, we have

$$\begin{aligned} a(u, w_h) &= \big((\mu - \nabla\cdot\beta)u, w_h\big)_{\mathcal{K}} - (u, \beta\cdot\nabla w_h)_{\mathcal{K}} + (\{\beta u\}, [w_h])_{\mathcal{F}_{int}\cup\mathcal{F}_+} \\ &= (\mu u + \beta\cdot\nabla u, w_h)_{\mathcal{K}} - \sum_{\kappa\in\mathcal{K}} (\beta\cdot n u, w_h)_{\partial\kappa} + (\{\beta u\}, [w_h])_{\mathcal{F}_{int}\cup\mathcal{F}_+}. \end{aligned}$$

Observe that

$$\sum_{\kappa\in\mathcal{K}}(\beta\cdot nu, w_h)_{\partial\kappa} = (\{\beta u\}, [w_h])_{\mathcal{F}}$$

since u is continuous. Therefore still using $u_{|_{\partial\Omega^-}} = 0$ we obtain

$$a(u, w_h) = (\mu u + \beta\cdot\nabla u, w_h)_{\mathcal{K}} = (f, w_h)_{\mathcal{K}}.$$

As above, since $u \in H^q(\Omega)$, $q > \frac{3}{2}$,

$$b_0(u, w_h) = 0 \quad \text{and} \quad b_1(u, w_h) = 0.$$

Thus, we have consistency in both cases. □

The convergence analysis for the continuous and discontinuous method with weakly imposed boundary conditions and interior penalty gives the following result:

Theorem 2.4 (Convergence of CIP, [7]) *Let $u \in H^{p+1}(\Omega)$, with $p \geq 1$, solve (1) and let u_c solve (9). Further assume that $\beta \in [W^{1,\infty}(\Omega)]^d$. Then, there exists a constant $c > 0$, independent of h, such that*

$$|||u - u_c||| \leq ch^{p+\frac{1}{2}}\|u\|_{p+1,\mathcal{K}}.$$

Theorem 2.5 (Convergence of DGFEM, [5, 17, 20]) *Assume that $\gamma_0 > 0$, $\gamma_1 \geq 0$, let $u \in H^{p+1}(\Omega)$, with $p \geq 1$, solve (1) and let u_d solve (8). Further assume that $\beta \in [W^{1,\infty}(\Omega)]^d$. Then, there exists a constant $c > 0$, independent of h, such that*

$$|||u - u_d||| \leq ch^{p+\frac{1}{2}}\|u\|_{p+1,\mathcal{K}}.$$

Remark 2.6 The proof of Theorem 2.5 in the case of $\gamma_0 > 0$, $\gamma_1 = 0$ is given in [5, 17]. Adding the stabilization term $b_1(\cdot,\cdot)$ in their analysis is subject to some minor changes and yields optimal convergence.

Remark 2.7 For polynomial degrees $p \geq 2$ and $d = 2$, stability of the discontinuous Galerkin method can also be obtained by penalizing only the jump of the tangential part of the gradient, for more details see [11].

Remark 2.8 Using a more involved analysis, but similar techniques, we may prove an inf-sup condition in a norm containing the L^2-norms of both the jumps of the discrete solution over element boundaries and the elementwise streamline derivative.

2.4 Combining Continuous and Discontinuous Finite Element Spaces

The above theory is not only limited to continuous or discontinuous finite element spaces. Let $\{\Omega_i\}_{i=1}^N$ be a partition of Ω into subregions Ω_i, i.e. $\bigcup_{i=1}^N \overline{\Omega_i} = \overline{\Omega}$, and let $\mathcal{K}_i$ be a triangulation of Ω_i. Then, define on Ω_i the continuous finite element space

$$V_h^p(\Omega_i) = \{v_h \in C^0(\overline{\Omega}_i) \;:\; \forall\kappa \in \mathcal{K}_i,\, v_h|_\kappa \circ F_\kappa \in \mathbb{P}_p(\widehat{\kappa})\}$$

and match the subregions in a discontinuous manner

$$W_{h,N}^p = \{v_h \in L^2(\Omega) \;:\; \forall i = 1, \ldots, N,\; v_h|_{\Omega_i} \in V_h^p(\Omega_i)\}.$$

Observe that the bilinear form $a(\cdot,\cdot)$, defined in (5), simplifies to

$$a(v_h, w_h) = \big((\mu - \nabla\cdot\beta)v_h, w_h\big)_{\mathcal{K}} - (v_h, \beta\cdot\nabla w_h)_{\mathcal{K}} + (\{\beta v_h\}, [w_h])_{\mathcal{F}_{int}^N \cup \mathcal{F}_+}$$

for functions $v_h, w_h \in W_{h,N}^p$ having set

$$\mathcal{F}_{int}^N = \{F \in \mathcal{F}_{int} \;:\; F \subset \partial\Omega_j \cap \partial\Omega_k \text{ with } 1 \le j, k \le N,\, j \ne k\}.$$

The stabilizing terms are then defined by

$$b_0(v_h, w_h) = (\beta_n[v_h], [w_h])_{\mathcal{F}_{int}^N},$$

$$b_1(v_h, w_h) = (h_F^2\beta_n[\nabla v_h], [\nabla w_h])_{\mathcal{F}_{int}\setminus\mathcal{F}_{int}^N} + \delta\,(h_F^2\beta_n[\nabla v_h], [\nabla w_h])_{\mathcal{F}_{int}\cap\mathcal{F}_{int}^N},$$

with $\delta \ge 0$. A convergence analysis can be carried out combining the techniques of DG-methods and the CIP-method, see [12].

3 The Continuous Galerkin Method as a limit of the Discontinuous Galerkin Method

Hereafter $c > 0$ is considered a generic positive constant independent of h and γ_0. Its actual value can change at each occurrence.

3.1 Preliminaries

We first recall an interpolation operator between discrete spaces $\mathcal{I}_{\mathrm{Os}} : W_h^p \to V_h^p$ endowed with a local interpolation property.

Let $\kappa \in \mathcal{K}$. For a node ν in κ, set $\mathcal{K}_\nu = \{\kappa' \in \mathcal{K};\; \nu \in \kappa'\}$; then, for $w_h \in W_h^p$, define $\mathcal{I}_{\mathrm{Os}} w_h$ locally in κ by the value it takes at all the Lagrangian nodes of κ by setting

$$\mathcal{I}_{\mathrm{Os}} w_h(\nu) = \frac{1}{\mathrm{card}(\mathcal{K}_\nu)} \sum_{\kappa\in\mathcal{K}_\nu} w_h|_\kappa(\nu). \tag{13}$$

Clearly, $\mathcal{I}_{\mathrm{Os}} w_h \in V_h^p$. The operator $\mathcal{I}_{\mathrm{Os}}$ is sometimes referred to as the Oswald interpolation operator; it has been considered in [7, 16, 21]. The next lemma highlights some approximation results.

Lemma 3.1 *There exists c, independent of h_κ but not of the local mesh geometry, such that, for all $\kappa \in \mathcal{K}$, the following estimate holds*:

$$\forall w_h \in W_h^p, \quad \|w_h - \mathcal{I}_{\mathrm{Os}} w_h\|_\kappa \le c h_\kappa^{\frac{1}{2}} \|[w_h]\|_{\mathcal{F}(\kappa)}, \tag{14}$$

$$\forall w_h \in W_h^p, \quad \|\nabla(w_h - \mathcal{I}_{\mathrm{Os}} w_h)\|_\kappa \le c h_\kappa^{-\frac{1}{2}} \|[w_h]\|_{\mathcal{F}(\kappa)}, \tag{15}$$

where $\mathcal{F}(\kappa) = \{F \in \mathcal{F}_{int} \;:\; F \cap \kappa \ne \emptyset\}$.

3.2 Asymptotic Limit $\gamma_0 \to \infty$

Since we consider here consequences of an increasing γ_0, we may no longer use the triple norm defined in (10) since the parameter γ_0 is included in that definition. Instead a slightly modified norm is defined for this section

$$|||v|||_m^2 = \|\mu_0^{\frac{1}{2}} v\|_{\mathcal{K}}^2 + \frac{1}{2}\||\beta \cdot n|^{\frac{1}{2}} v\|_{\partial\Omega}^2 + b_0(v,v) + \gamma_1 b_1(v,v). \tag{16}$$

Observe that if $v_h \in V_h^p$, then $|||v_h||| = |||v_h|||_m$. One can easily show coercivity of the bilinear form $a(\cdot,\cdot) + b_0(\cdot,\cdot) + \gamma_1 b_1(\cdot,\cdot)$ with respect to this norm as well.

Lemma 3.2 (Coercivity) *For all $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$,*

$$a(v,v) + b_0(v,v) + \gamma_1 b_1(v,v) \geq |||v|||_m^2.$$

Proof The proof is similar to the one of Lemma 2.2. □

Theorem 3.3 *Let u_d and u_c be the solutions of the discontinuous resp. continuous problem (8) resp. (9). Assume that the mesh is globally quasi-uniform and that either $\epsilon > 0$ and $\beta \neq 0$ or $\|\beta \cdot n\|_{\infty,F} > 0$ for all faces F of the mesh. Let $u \in H^{p+1}(\Omega)$, with $p \geq 1$, solve (1). Then u_d converges to u_c as the parameter γ_0 tends to infinity. Precisely, there exists a constant $c > 0$, independent of γ_0 and h, such that*

$$|||u_c - u_d|||_m \leq \frac{c}{\gamma_0} h^{p-\frac{1}{2}} \|u\|_{p+1,\mathcal{K}}.$$

Proof Let us denote $\eta_d = u_c - u_d \in W_h^p$. Using coercivity, Lemma 3.2, and consistency leads to

$$\begin{aligned} |||\eta_d|||_m^2 &\leq a(\eta_d,\eta_d) + b_0(\eta_d,\eta_d) + \gamma_1 b_1(\eta_d,\eta_d) \\ &= a(\eta_d,\eta_d - v_h) + b_0(\eta_d,\eta_d - v_h) + \gamma_1 b_1(\eta_d,\eta_d - v_h) \end{aligned}$$

for all $v_h \in V_h^p$. Indeed, subtracting (12) from (11) leads to

$$a(u_c - u_d, v_h) + \gamma_1 b_1(u_c - u_d, v_h) = 0$$

since v_h is chosen to be continuous. For the same reason we have

$$b_0(u_c - u_d, v_h) = 0.$$

Define for simplicity

$$\begin{aligned} &\mathcal{I}_1 = ((\mu - \nabla\cdot\beta)\eta_d, \eta_d - v_h)_{\mathcal{K}}, && \mathcal{I}_4 = (\{\beta\eta_d\}, [\eta_d - v_h])_{\mathcal{F}_+}, \\ &\mathcal{I}_2 = -(\eta_d, \beta\cdot\nabla(\eta_d - v_h))_{\mathcal{K}}, && \mathcal{I}_5 = b_0(\eta_d, \eta_d - v_h), \\ &\mathcal{I}_3 = (\{\beta\eta_d\}, [\eta_d - v_h])_{\mathcal{F}_{int}}, && \mathcal{I}_6 = \gamma_1 b_1(\eta_d, \eta_d - v_h). \end{aligned}$$

Hence $|||\eta_d|||_m^2 \leq \sum_{i=1}^6 \mathcal{I}_i$. Set $v_h = \mathcal{I}_{\text{Os}}\eta_d \in V_h^p$. To upper bound the first four terms we use the Cauchy-Schwarz inequality, the trace inequality and Lemma 3.1:

$$\mathcal{I}_1 \leq c\,\|\mu_0^{\frac{1}{2}}\eta_d\|_{\mathcal{K}}\|\eta_d - \mathcal{I}_{\text{Os}}\eta_d\|_{\mathcal{K}} \leq ch^{\frac{1}{2}}\,|||\eta_d|||_m\,\|[\eta_d]\|_{\mathcal{F}_{int}},$$

$$\mathcal{I}_2 \leq c\,\|\mu_0^{\frac{1}{2}}\eta_d\|_{\mathcal{K}}\|\nabla(\eta_d - \mathcal{I}_{\mathrm{Os}}\eta_d)\|_{\mathcal{K}} \leq ch^{-\frac{1}{2}}|||\eta_d|||_m\|[\eta_d]\|_{\mathcal{F}_{int}},$$

$$\mathcal{I}_3 \leq ch^{-\frac{1}{2}}\|h_F^{\frac{1}{2}}\eta_d\|_{\mathcal{F}_{int}}\|[\eta_d]\|_{\mathcal{F}_{int}} \leq ch^{-\frac{1}{2}}\|\eta_d\|_{\mathcal{K}}\|[\eta_d]\|_{\mathcal{F}_{int}} \leq ch^{-\frac{1}{2}}|||\eta_d|||_m\|[\eta_d]\|_{\mathcal{F}_{int}},$$

$$\mathcal{I}_4 = (\beta\cdot n\,\eta_d, \eta_d - \mathcal{I}_{\mathrm{Os}}\eta_d)_{\mathcal{F}_+} \leq ch^{-\frac{1}{2}}\||\beta\cdot n|^{\frac{1}{2}}\eta_d\|_{\mathcal{F}_+}\|\eta_d - \mathcal{I}_{\mathrm{Os}}\eta_d\|_{\mathcal{K}}$$
$$\leq c\;|||\eta_d|||_m\|[\eta_d]\|_{\mathcal{F}_{int}}.$$

In a similar way we obtain

$$\mathcal{I}_5 = \|\beta_n^{\frac{1}{2}}[\eta_d]\|^2_{\mathcal{F}_{int}} \leq c\;|||\eta_d|||_m\|[\eta_d]\|_{\mathcal{F}_{int}}.$$

For $\mathcal{I}_6$, the trace inequality and Lemma 3.1 is used:

$$\mathcal{I}_6 \leq cb_1(\eta_d,\eta_d)^{\frac{1}{2}}\|h_F[\nabla(\eta_d - \mathcal{I}_{\mathrm{Os}}\eta_d)]\|_{\mathcal{F}_{int}}$$
$$\leq ch^{\frac{1}{2}}b_1(\eta_d,\eta_d)^{\frac{1}{2}}\|\nabla(\eta_d - \mathcal{I}_{\mathrm{Os}}\eta_d)\|_{\mathcal{K}} \leq c\;|||\eta_d|||_m\|[\eta_d]\|_{\mathcal{F}_{int}}.$$

Respecting all six bounds yields

$$|||\eta_d|||_m \leq c(h^{\frac{1}{2}} + 2h^{-\frac{1}{2}} + 3)\|[\eta_d]\|_{\mathcal{F}_{int}} \leq ch^{-\frac{1}{2}}\|[\eta_d]\|_{\mathcal{F}_{int}}, \tag{17}$$

since $h < 1$. Then, observe that by the assumption on β_n we get $\|\beta_n[\eta_d]\|^2_{\mathcal{F}_{int}} \geq c\,\|[\eta_d]\|^2_{\mathcal{F}_{int}}$ and using coercivity, Lemma 2.2, and consistency leads to

$$c\gamma_0\|[\eta_d]\|^2_{\mathcal{F}_{int}} \leq a(\eta_d,\eta_d) + \gamma_0 b_0(\eta_d,\eta_d) + \gamma_1 b_1(\eta_d,\eta_d)$$
$$= a(u_c - u, \eta_d) + \gamma_0 b_0(u_c - u, \eta_d) + \gamma_1 b_1(u_c - u, \eta_d)$$
$$= a(u_c - u, \eta_d) + \gamma_1 b_1(u_c - u, \eta_d)$$
$$= a(u_c - u, \eta_d - \mathcal{I}_{\mathrm{Os}}\eta_d) + \gamma_1 b_1(u_c - u, \eta_d - \mathcal{I}_{\mathrm{Os}}\eta_d)$$

since $u_c - u$ is continuous. Using analogous arguments as those for bounding $|||\eta_d|||_m$, we conclude that

$$\gamma_0\|[\eta_d]\|^2_{\mathcal{F}_{int}} \leq c(h^{\frac{1}{2}} + 2h^{-\frac{1}{2}} + 2)|||u_c - u|||\,\|[\eta_d]\|_{\mathcal{F}_{int}}$$

and hence

$$\|[\eta_d]\|_{\mathcal{F}_{int}} \leq \frac{c}{\gamma_0}h^{-\frac{1}{2}}|||u_c - u|||.$$

The convergence of the continuous approximation, Theorem 2.4, leads to the bound

$$\|[\eta_d]\|_{\mathcal{F}_{int}} \leq \frac{c}{\gamma_0}h^p\|u\|_{p+1,\mathcal{K}}. \tag{18}$$

Combining (17) and (18) yields

$$|||\eta_d|||_m \leq \frac{c}{\gamma_0}h^{p-\frac{1}{2}}\|u\|_{p+1,\mathcal{K}}.$$ □

Remark 3.4 Observe that the global quasi-uniformity assumption is uniquely due to the global character of the estimate of $|||u_c - u|||$.

4 Local Flux Conservation

In this section, we will study the behavior of the numerical flux of the DG-method in the asymptotic limit and show how this may be used to define a conservative numerical flux also for the continuous Galerkin method [18, 22].

We assume in this section that the mesh is globally quasi-uniform and that either $\epsilon > 0$ and $\beta \neq 0$ or $\|\beta \cdot n\|_{\infty,F} > 0$ for all faces F of the mesh. Consider problem (1) with $\mu = 0$ and $\nabla \cdot \beta = 0$, i.e. the pure transport problem, and let $\Lambda \subset \Omega$ be a subdomain of Ω. We associate to Λ its outer normal n_Λ. Further denote χ_Λ the characteristic function on Λ defined by $\chi_\Lambda = 1$ on Λ and $\chi_\Lambda = 0$ on $\Omega \backslash \Lambda$. Multiplying the first line of (1) by χ_Λ and integrating by parts on Λ yields

$$\int_{\partial \Lambda} \sigma_\Lambda(u) \cdot n_\Lambda = \int_\Lambda f$$

since β is divergence free, $\nabla \chi_\Lambda|_\Lambda = 0$ and where $\sigma_\Lambda(u) = \beta u$ denotes the problem flux. For the discontinuous Galerkin method the same relation is true on each element κ for a numerical flux $\Sigma^d_{\kappa,\gamma_0}$ defined by

$$\Sigma^d_{\kappa,\gamma_0}(w_h) = \begin{cases} \sigma_\kappa(\{w_h\}) + \gamma_0 \beta_n [w_h] & \text{on } \mathcal{F}_{int} \cap \partial\kappa, \\ \sigma_\kappa(w_h) & \text{on } \mathcal{F}_+ \cap \partial\kappa, \\ 0 & \text{on } \mathcal{F}_- \cap \partial\kappa, \end{cases} \tag{19}$$

for all $w_h \in W_h^p$. Then, replacing the test function in (8) by the characteristic function χ_κ where $\kappa \in \mathcal{K}$, leads to

$$\int_{\partial\kappa} \Sigma^d_{\kappa,\gamma_0}(u_d) \cdot n_\kappa = \int_\kappa f.$$

Hence there is a local flux conservation for the discontinuous Galerkin method. Now since the continuous Galerkin method can be considered as the limit of the discontinuous Galerkin method, we define a numerical flux for the continuous Galerkin method by

$$\Sigma^c_\kappa(u_c) = \begin{cases} \sigma_\kappa(u_c) + \beta_n[\rho_h] & \text{on } \mathcal{F}_{int} \cap \partial\kappa, \\ \sigma_\kappa(u_c) & \text{on } \mathcal{F}_+ \cap \partial\kappa, \\ 0 & \text{on } \mathcal{F}_- \cap \partial\kappa, \end{cases} \tag{20}$$

where ρ_h is defined by the problem: find $\rho_h \in (V_h^p)^\perp$ such that

$$b_0(\rho_h, w_h) = (f, w_h)_\mathcal{K} - a(u_c, w_h) - \gamma_1 b_1(u_c, w_h) \quad \forall w_h \in W_h^p, \tag{21}$$

where $(V_h^p)^\perp$ denotes the orthogonal component of V_h^p in W_h^p with respect to the $L^2(\Omega)$-scalar product defined by

$$(V_h^p)^\perp = \left\{\rho_h \in W_h^p \;:\; (\rho_h, v_h)_\mathcal{K} = 0,\ \forall v_h \in V_h^p\right\}.$$

Lemma 4.1 *The problem* (21) *admits a unique solution.*

Proof Consider the following auxiliary problem: find $\rho_h \in (V_h^p)^\perp$ such that

$$b_0(\rho_h, \bar{w}_h) = (f, \bar{w}_h)_\mathcal{K} - a(u_c, \bar{w}_h) - \gamma_1 b_1(u_c, \bar{w}_h) \quad \forall \bar{w}_h \in (V_h^p)^\perp.$$

Since the trial and test space are equal and since the kernel of $b_0(\cdot,\cdot)$ in $(V_h^p)^\perp$ is zero, one can apply the standard theory to show the result. Remember that we assume that either $\varepsilon > 0$ and $\beta \neq 0$ in the definition of β_n or $\|\beta\cdot n\|_{\infty,F} > 0$ on all faces of the mesh. Observe that one can decompose every function $w_h \in W_h^p$ in $w_h = \bar{w}_h + v_h$ with $\bar{w}_h \in (V_h^p)^\perp$ and $v_h \in V_h^p$. Then

$$b_0(\rho_h, v_h) = 0 \quad \text{and} \quad (f, v_h)_{\mathcal{K}} - a(u_c, v_h) - \gamma_1 b_1(u_c, v_h) = 0,$$

owing to the fact that v_h is continuous and to the consistency of the continuous Galerkin method. Therefore this auxiliary problem is equivalent to the original one and this implies the uniqueness of ρ_h. □

Lemma 4.2 *Let $u_d = u_d(\gamma_0)$ be the solution of (8) corresponding to a given value of γ_0; then*

$$\lim_{\gamma_0\to\infty} \gamma_0 b_0(u_d, w_h) = b_0(\rho_h, w_h)$$

for all $w_h \in W_h^p$.

Proof Using the discrete formulation of the discontinuous Galerkin method and the fact that the bilinear forms $a(\cdot,\cdot)$ and $b_1(\cdot,\cdot)$ are continuous with respect to both variables yields

$$\begin{aligned}\lim_{\gamma_0\to\infty} \gamma_0 b_0(u_d, w_h) &= (f, w_h)_{\mathcal{K}} - \lim_{\gamma_0\to\infty} a(u_d, w_h) - \lim_{\gamma_0\to\infty} \gamma_1 b_1(u_d, w_h) \\ &= (f, w_h)_{\mathcal{K}} - a(u_c, w_h) - \gamma_1 b_1(u_c, w_h) = b_0(\rho_h, w_h)\end{aligned}$$

for all $w_h \in W_h^p$ since $\|u_d - u_c\|_{\mathcal{K}} \to 0$ as $\gamma_0 \to \infty$. □

Proposition 4.3 *Let $F \in \mathcal{F}$ be an arbitrary face of an arbitrary element $\kappa \in \mathcal{K}$. Then the numerical flux $\Sigma^d_{\kappa,\gamma_0}(u_d)$ converges to $\Sigma^c_\kappa(u_c)$, i.e.*

$$\lim_{\gamma_0\to\infty} (\Sigma^d_{\kappa,\gamma_0}(u_d), [w_h])_F = (\Sigma^c_\kappa(u_c), [w_h])_F$$

for all $w_h \in W_h^p$.

Proof On faces contained in $\mathcal{F}_-$ the limit is obvious since both fluxes are zero. Since the exact flux $\sigma_\kappa(\cdot)$ is continuous we have on faces contained in $\mathcal{F}_+$ that $\sigma_\kappa(\{u_d\}) \to \sigma_\kappa(\{u_c\}) = \sigma_\kappa(u_c)$. On interior faces we use the same argument and Lemma 4.2. □

Corollary 4.4 *For the continuous Galerkin method, we still have the local conservation property, i.e.*

$$\int_{\partial\kappa} \Sigma^c_\kappa(u_c)\cdot n_\kappa = \int_\kappa f.$$

Proof Chose $w = \chi_\kappa$ in Proposition 4.3 and sum over all faces of κ. □

4.1 Behavior of the Numerical Flux as $h \to 0$

Since all fluxes are equal on the boundary of an element, i.e.

$$\int_{\partial\kappa} \Sigma_\kappa^c(u_c) \cdot n_\kappa = \int_{\partial\kappa} \Sigma_{\kappa,\gamma_0}^d(u_d) \cdot n_\kappa = \int_{\partial\kappa} \sigma_\kappa(u) \cdot n_\kappa = \int_\kappa f \tag{22}$$

it is evident that

$$\int_{\partial\kappa} \left(\sigma_\kappa(u) - \Sigma_\kappa^c(u_c)\right) \cdot n_\kappa = 0 \quad \text{and} \quad \int_{\partial\kappa} \left(\sigma_\kappa(u) - \Sigma_{\kappa,\gamma_0}^d(u_d)\right) \cdot n_\kappa = 0$$

for all $h > 0$. For what concerns the exact flux of the difference of the solutions consider the following two lemmas.

Lemma 4.5 *Assume that $u \in H^{p+1}(\Omega)$. Then, the difference of the flux of the exact solution u and the flux of the numerical solution u_d converges to zero as $h \to 0$ with a convergence rate of p, i.e.*

$$\sum_{\kappa\in\mathcal{K}} \left| \int_{\partial\kappa} (\sigma_\kappa(u) - \sigma_\kappa(u_d)) \cdot n_\kappa \right| \leq ch^p \|u\|_{p+1,\mathcal{K}}.$$

Proof Applying equality (22) and the Cauchy-Schwarz inequality yields

$$\begin{aligned}\mathcal{I} &:= \sum_{\kappa\in\mathcal{K}} \left| \int_{\partial\kappa} (\sigma_\kappa(u) - \sigma_\kappa(u_d)) \cdot n_\kappa \right| = \sum_{\kappa\in\mathcal{K}} \left| \int_{\partial\kappa} \left(\Sigma_{\kappa,\gamma_0}^d(u_d) - \sigma_\kappa(u_d)\right) \cdot n_\kappa \right| \\ &\leq \left(\sum_{\kappa\in\mathcal{K}} h_\kappa^{-1} \|\Sigma_{\kappa,\gamma_0}^d(u_d) - \sigma_\kappa(u_d)\|_{\partial\kappa}^2\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{K}} \int_{\partial\kappa} h_\kappa\right)^{\frac{1}{2}}.\end{aligned}$$

Observe that $\sum_{\kappa\in\mathcal{K}} \int_{\partial\kappa} h_\kappa \leq c$ using the shape regularity of the mesh. By the definition of the numerical flux $\Sigma_{\kappa,\gamma_0}^d$, the error estimate of Theorem 2.5 and since

$$\beta \cdot n_\kappa(\{u_d\} - u_d)|_{\partial\kappa} = -\frac{1}{2}\beta \cdot [u_d]|_{\partial\kappa} \quad \forall\kappa \in \mathcal{K}$$

we get

$$\begin{aligned}\mathcal{I} &\leq c\left(\sum_{\kappa\in\mathcal{K}} \|h_\kappa^{-\frac{1}{2}} \beta_n^{\frac{1}{2}} u_d\|_{\partial\kappa\cap\partial\Omega^-}^2 + \sum_{\kappa\in\mathcal{K}} \|h_\kappa^{-\frac{1}{2}} \beta_n^{\frac{1}{2}} [u_d]\|_{\partial\kappa\setminus\partial\Omega}^2\right)^{\frac{1}{2}} \leq c\, |||h^{-\frac{1}{2}}(u - u_d)||| \\ &\leq ch^p \|u\|_{p+1,\mathcal{K}}.\end{aligned}$$

□

For the next lemma assume for simplicity that $\beta \in \mathbb{R}^d$ and $\beta \neq 0$.

Lemma 4.6 *Assume that $u \in H^{p+1}(\Omega)$, $f \in H^p(\Omega)$. Then, the difference of the flux of the exact solution u and the flux of the numerical solution u_c converges to zero as $h \to 0$ with a convergence rate of p, i.e.*

$$\sum_{\kappa\in\mathcal{K}} \left| \int_{\partial\kappa} (\sigma_\kappa(u) - \sigma_\kappa(u_c)) \cdot n_\kappa \right| \leq ch^p \left(\|f\|_{p,\mathcal{K}} + \|u\|_{p+1,\mathcal{K}}\right).$$

Proof Applying again equality (22) and the Cauchy-Schwarz inequality yields

$$\mathcal{I} := \sum_{\kappa\in\mathcal{K}} \left| \int_{\partial\kappa} (\sigma_\kappa(u) - \sigma_\kappa(u_c)) \cdot n_\kappa \right| = \sum_{\kappa\in\mathcal{K}} \left| \int_{\partial\kappa} \left(\Sigma_\kappa^c(u_c) - \sigma_\kappa(u_c)\right) \cdot n_\kappa \right|$$
$$\leq \left(\sum_{\kappa\in\mathcal{K}} h_\kappa^{-1} \| \Sigma_\kappa^c(u_c) - \sigma_\kappa(u_c) \|_{\partial\kappa}^2 \right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{K}} \int_{\partial\kappa} h_\kappa \right)^{\frac{1}{2}}.$$

Observe that $\sum_{\kappa\in\mathcal{K}} \int_{\partial\kappa} h_\kappa \leq c$ using the shape regularity of the mesh. Using the definition of the numerical flux Σ_κ^c yields

$$\mathcal{I} \leq c \left(\sum_{\kappa\in\mathcal{K}} \| h_\kappa^{-\frac{1}{2}} \beta_n^{\frac{1}{2}} u_c \|_{\partial\kappa\cap\partial\Omega^-}^2 + \sum_{\kappa\in\mathcal{K}} \| h_\kappa^{-\frac{1}{2}} \beta_n^{\frac{1}{2}} [\rho_h] \|_{\partial\kappa\setminus\partial\Omega}^2 \right)^{\frac{1}{2}}$$
$$\leq c \left(c h^{2p} \|u\|_{p+1,\mathcal{K}}^2 + h^{-1} b_0(\rho_h, \rho_h) \right)^{\frac{1}{2}}. \tag{23}$$

Now, let $\pi_h \rho_h \in V_h^p$ denote the L^2-projection of ρ_h onto the continuous space. By (14) and (15) it follows that

$$\|\rho_h - \pi_h\rho_h\|_{\mathcal{K}}^2 + h\|\rho_h - \pi_h\rho_h\|_{\mathcal{F}_-}^2 + \gamma_1 h b_1(\rho_h - \pi_h\rho_h, \rho_h - \pi_h\rho_h) \leq c h b_0(\rho_h, \rho_h). \tag{24}$$

On the other hand using integration by parts we have that

$$a(u_c, \rho_h) = (\beta \cdot \nabla u_c, \rho_h)_{\mathcal{K}} - (\beta \cdot n u_c, \rho_h)_{\partial\Omega^-}.$$

By the definition of ρ_h, the Galerkin orthogonality and the orthogonality of the L^2-projection we deduce that

$$\begin{aligned}
& b_0(\rho_h, \rho_h) \\
&\quad = (f, \rho_h)_{\mathcal{K}} - a(u_c, \rho_h) - \gamma_1 b_1(u_c, \rho_h) \\
&\quad = (f, \rho_h - \pi_h\rho_h)_{\mathcal{K}} - a(u_c, \rho_h - \pi_h\rho_h) - \gamma_1 b_1(u_c, \rho_h - \pi_h\rho_h) \\
&\quad = (f - \pi_h f, \rho_h - \pi_h\rho_h)_{\mathcal{K}} - (\beta\cdot\nabla u_c - \mathcal{I}_{\mathrm{Os}}(\beta\cdot\nabla u_c), \rho_h - \pi_h\rho_h)_{\mathcal{K}} \\
&\qquad + (\beta\cdot n u_c, \rho_h - \pi_h\rho_h)_{\mathcal{F}_-} - \gamma_1 b_1(u_c, \rho_h - \pi_h\rho_h) \\
&\quad \leq c \big(\|f - \pi_h f\|_{\mathcal{K}} + \|\beta\cdot\nabla u_c - \mathcal{I}_{\mathrm{Os}}(\beta\cdot\nabla u_c)\|_{\mathcal{K}} + h^{-\frac{1}{2}} \|\beta_n^{\frac{1}{2}} u_c\|_{\mathcal{F}_-} \\
&\qquad + \gamma_1^{\frac{1}{2}} h^{-\frac{1}{2}} b_1(u_c, u_c)^{\frac{1}{2}} \big) \\
&\qquad \times \big(\|\rho_h - \pi_h\rho_h\|_{\mathcal{K}} + h^{\frac{1}{2}} \|\rho_h - \pi_h\rho_h\|_{\mathcal{F}_-} + \gamma_1^{\frac{1}{2}} h^{\frac{1}{2}} b_1(\rho_h - \pi_h\rho_h, \rho_h - \pi_h\rho_h)^{\frac{1}{2}} \big).
\end{aligned}$$

The inequality (24) leads to the following bound

$$\begin{aligned}
b_0(\rho_h, \rho_h) &\leq h^{\frac{1}{2}} b_0(\rho_h, \rho_h)^{\frac{1}{2}} \big(h^p \|f\|_{p,\mathcal{K}} + (1 + \gamma_1^{\frac{1}{2}}) h^{-\frac{1}{2}} b_1(u_c, u_c)^{\frac{1}{2}} + h^{-\frac{1}{2}} \|\beta_n^{\frac{1}{2}} u_c\|_{\mathcal{F}_-} \big) \\
&\leq h^{\frac{1}{2}} b_0(\rho_h, \rho_h)^{\frac{1}{2}} \big(h^p \|f\|_{p,\mathcal{K}} + h^{-\frac{1}{2}} |||u - u_c||| \big)
\end{aligned}$$

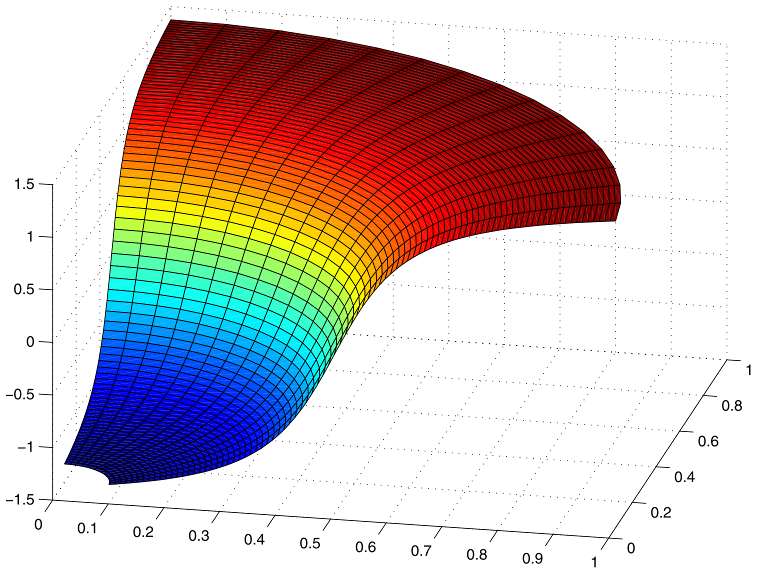


Fig. 1 The exact solution

and therefore

$$h^{-\frac{1}{2}} b_0(\rho_h, \rho_h)^{\frac{1}{2}} \leq h^p \left(\|f\|_{p,\mathcal{K}} + \|u\|_{p+1,\mathcal{K}} \right) \tag{25}$$

by the estimate of Theorem 2.4. Inserting (25) into (23) yields to the desired result. □

5 Numerical Results

The following transport problem is considered. Let $\Omega \subset \mathbb{R}^2$ be the domain defined by $\Omega = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0.1 \leq \sqrt{x^2 + y^2} \leq 1\}$. The problem consists of seeking u such that

$$\begin{cases} \mu u + \beta \cdot \nabla u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega^-} = g(y) \end{cases}$$

where

$$\beta(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix} \frac{1}{\sqrt{x^2 + y^2}} \quad \text{and} \quad g(y) = \arctan\left(\frac{y - 0.5}{0.1}\right).$$

Then, the solution writes

$$u(x, y) = e^{\mu\sqrt{x^2+y^2} \arcsin(\frac{y}{\sqrt{x^2+y^2}})} \arctan\left(\frac{\sqrt{x^2 + y^2} - 0.5}{0.1}\right).$$

The reaction coefficient, $\mu = 0.01$, is chosen sufficiently small such that the transport is dominating the reaction. Figure 1 shows the exact solution u. We consider sequences of unstructured triangular meshes for polynomial degrees $p = 1, \ldots, 5$. For the computations the C++ library *life* is used, see [25, 26].

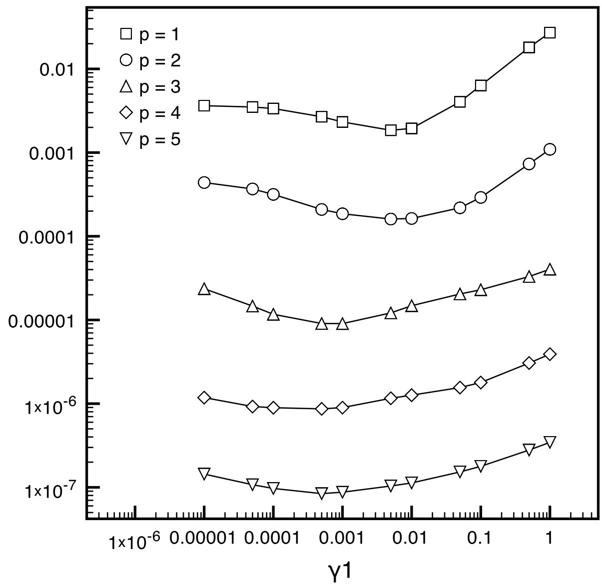


Fig. 2 Behavior of the error of the continuous interior penalty method with respect to the stabilization parameter γ_1 for different polynomial degrees and fixed h measured in the L^2-norm

5.1 Optimal Choice of the Stabilization Parameter of Continuous Interior Penalty Method

For the continuous interior penalty method on rectangular meshes the optimal choice of the stabilization parameter γ_1 with respect to the polynomial degree is carried out yielding that $\gamma_1 \sim p^{-3.5}$, see [9] for more details. Figure 2 shows the L^2-error depending of γ_1 for a fixed triangular mesh with size $h = 0.05$ and for each polynomial degree. The optimal choice for this example is illustrated in the following table:

p	1	2	3	4	5
$\gamma_{1,opt}$	0.005	0.005	0.001	0.0005	0.0005

These values will be the reference values for the following computations.

5.2 Convergence with Respect to h and p

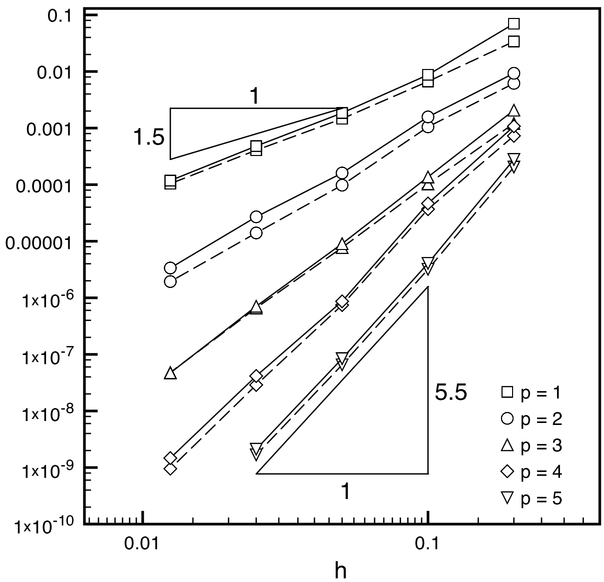
Since $u \in C^\infty(\overline{\Omega})$, $u \in H^r(\Omega)$ for all $r \geq 0$. Hence the solution of the continuous method satisfies

$$|||u - u_c||| \leq ch^{p+\frac{1}{2}} \|u\|_{p+1,\mathcal{K}}.$$

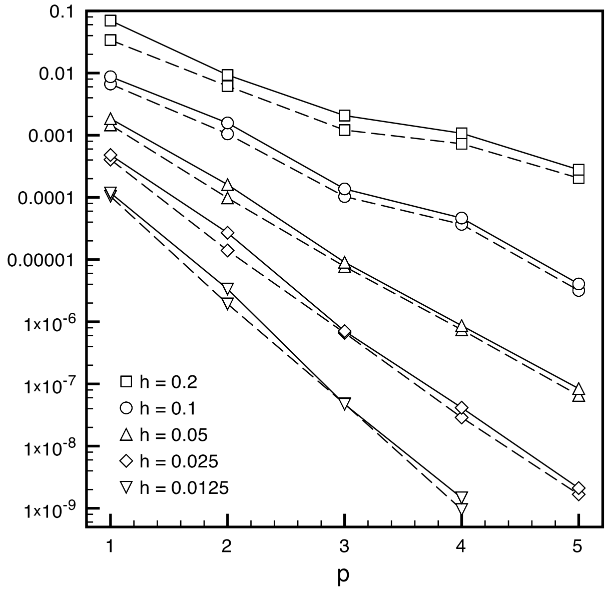
Similarly, for the discontinuous method, we get

$$|||u - u_d||| \leq ch^{p+\frac{1}{2}} \|u\|_{p+1,\mathcal{K}}.$$

Observe that the L^2-norm is controlled by the triple norm, i.e. $\|v\|_{\mathcal{K}} \leq |||v|||$. Note that the hp-analysis carried out in [9] for the continuous interior penalty method and in [17] for the DG-method only holds on rectangular meshes, whereas an h-analysis can be carried out for any polynomial degree p on triangular meshes for both methods. Figure 3 shows the L^2-norm of the error of the upwind discontinuous method, i.e. $\gamma_0 = 0.5$, $\gamma_1 = 0$, in



(a)



(b)

Fig. 3 Convergence behavior with respect to h (**a**) and p (**b**) of the upwind discontinuous method ($\gamma_0 = 0.5$, $\gamma_1 = 0$, *dashed line*) and the continuous interior penalty method with optimal parameter γ_1 (*solid line*) measured in the L^2-norm dashed line and the continuous interior penalty method with optimal stabilization parameter γ_1 according to Sect. 5.1 in solid line.

Observe the optimal convergence with respect to h and the exponential convergence with respect to p.

5.3 CG-method as Limit of the DG-method

Here we test the case when the stabilization parameter γ_0 of the DG-method increases to infinity for fixed mesh size $h = 0.05$. The theoretical result states us that the L^2-norm of the difference between the solutions of the discontinuous and continuous methods converges to zero with order one. The order of convergence is predicted as one. That is exactly what can be observed for a sufficiently large γ_0 in Fig. 4(a) for both cases $\gamma_1 = 0$ (solid line) and $\gamma_1 > 0$ (dashed line). The parameter $\gamma_1 > 0$ is chosen according to the optimal criterion for the continuous method as illustrated in Sect. 5.1.

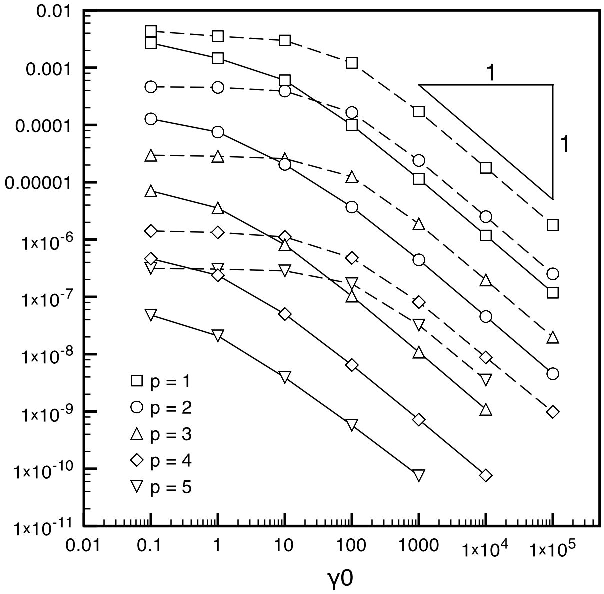
Figure 4(b) shows the L^2-norm of the difference between the exact solution u and the DG-approximation when γ_0 tends to infinity. We see that the parameter $\gamma_0 = 0.5$ corresponding to upwind stabilization is a good choice for all polynomial orders. Although it does not always correspond to the optimal choice for the error in the L^2-norm, the difference is very small.

Figure 4(b) also shows that for a fixed h there exists a γ_0 such that the DG-method is more precise than the continuous method without interior penalty. On the other hand if the DG-method is augmented with the gradient jump stabilization the solution is robust to over-stabilization (that is when γ_0 becomes too large), especially for high order approximations. Finally the results reported in Fig. 3 and 4(b) show that the CIP-method yields similar accuracy as the upwind DG-method. Indeed, in this numerical example, the CIP-method with the optimal parameter γ_1 leads to an approximation with an accuracy very similar to that of the upwind DG-method, but using much fewer degrees of freedom.

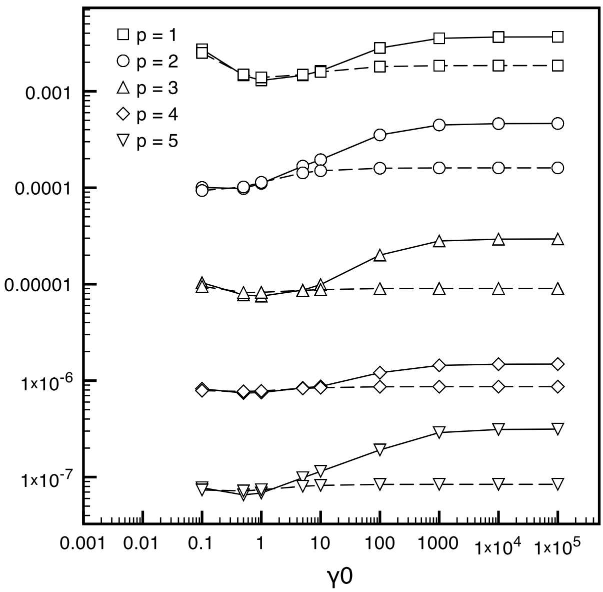
6 Conclusions

In this paper we have compared theoretically and numerically two methods which are suitable for the approximation of the scalar hyperbolic equation: the continuous Galerkin method stabilized by interior penalty on the jumps of the gradients over interelement faces and the discontinuous Galerkin method with parametrized interior penalty stabilization both of the jumps of the function itself and of its gradients over interior faces. We have reviewed the h-convergence analysis for the continuous method with interior penalty and the augmented discontinuous method. We proved that the solution of the discontinuous method converges to the solution of the continuous method as the stabilization parameter of the interelement solution jump increases to infinity. This is also showed numerically together with some comparisons of the behavior of the interior penalty method using continuous and discontinuous approximations.

The techniques that we have advocated here for the stabilization of scalar hyperbolic equations can be regarded as efficient alternatives to the more classical upwind-based finite element approximations dated back to the pioneering work by Mitchell and Griffiths [24], the generalization and analysis by Baba and Tabata [1] or the fully consistent SUPG- or GLS-methods, see the pioneering work [6, 19] or the books [27, 30].



(a)



(b)

Fig. 4 Difference between the solutions of the discontinuous method and of the continuous method (**a**) and between the exact solution and the solution of the discontinuous method (**b**) for variable γ_0 and fixed h. The *solid line* corresponds to the choice $\gamma_1 = 0$ and the *dashed line* to the optimal choice of $\gamma_1 > 0$ according to Fig. 2

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