Shock Waves (2012) 22:381–384
DOI 10.1007/s00193-012-0377-8

TECHNICAL NOTE

On impedance in shock-refraction problems

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Received: 30 January 2012 / Revised: 20 April 2012 / Accepted: 3 May 2012 / Published online: 22 May 2012

Abstract Publications on shock-refraction problems typically predict wave patterns resulting from the interaction from the acoustic-impedance ratio. In this note, an analysis based on the shock-impedance ratio is used to derive conditions under which the acoustic-impedance ratio predicts the incorrect type of reflected wave. The range of density ratios for which incorrect types of reflected waves are predicted is found to be quite narrow.

Keywords Shock-wave refraction · Shock impedance · Shock–bubble interaction

1 Motivation

The propagation of shock waves through inhomogeneous gases is an interesting and important problem that has been considered by many authors. In its simplest form, the problem consists of a planar shock wave propagating in a perfect gas of uniform density ρ_1 and pressure p_1 toward a single spherical inhomogeneity containing a perfect gas of density ρ_2 and pressure $p_2 = p_1$. The shock wave is refracted and diffracted on interacting with the inhomogeneity, leading to complicated wave patterns and large distortions of the inhomogeneity.

Communicated by R. Bonazza.

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Publications on shock-refraction problems typically use the ratio Z_2/Z_1, where $Z_i = \rho_i c_i$ is the acoustic impedance and c is the speed of sound, to determine the wave pattern (see Niederhaus et al. [1] and Ranjan et al. [2] for recent examples). It is stated that the reflected wave is a shock wave if $Z_2/Z_1 > 1$ and that the reflected wave is an expansion wave if $Z_2/Z_1 < 1$.

However, the acoustic-impedance ratio is applicable only in the limit of an infinitely weak shock wave. For a shock wave of finite strength, the shock-wave impedance Z^s_{ij} must be used instead. Following Henderson [3], the shock-wave impedance is defined as

$$Z^s_{ij} = \rho_i c_i \sqrt{1 + \frac{\gamma + 1}{2\gamma} \left(p_{ji} - 1\right)} = Z_i M_s = \rho_i u_s, \qquad (1)$$

where the subscripts i and j denote the states ahead of and behind the shock wave, γ is the ratio of specific heats, $p_{ji} = p_j/p_i$, M_s is the Mach number of the shock wave, and u_s is the speed of the shock wave. In the limit of an infinitely weak shock wave, $Z^s_{ij} \to Z_i$.

The objective of this note is to derive the conditions under which the use of the acoustic instead of the shock-wave impedance leads to incorrect prediction of the type of reflected wave. The analysis is based on the assumption of one-dimensional flow, see Fig. 1, where it is noted that the density inhomogeneity now occupies region 5. The results thus apply only to the very early stages of the interaction between the shock wave and the spherical inhomogeneity. More elaborate theories can be derived, but these theories are typically very complex. For example, Henderson [4] analyzed the interaction of a plane shock wave with a planar, but non-aligned, inhomogeneity. At first sight, this problem appears to be only marginally more complex, but it leads to

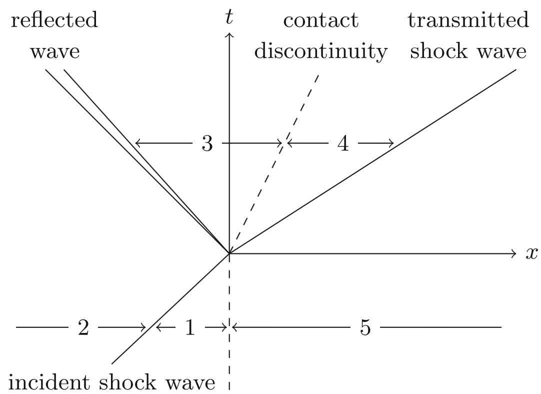

Fig. 1 Schematic illustration shock-refraction problem in the x–t plane to indicate notation used. Because this is only a schematic illustration, the slopes of the lines should not be interpreted as implying restrictions on the speeds of the incident and transmitted shock waves

a polynomial equation of degree 12 and exhibits regular and irregular refraction patterns.

2 Analysis

It is useful to briefly review the standard theory developed by Taub [5] and Paterson [6] because it serves as the foundation for the subsequent analysis, see also Glass and Sislian [7]. The theory states that the reflected wave is a shock or expansion wave depending on whether the ratio of impedances of the transmitted and incident shock waves is greater or smaller than unity,

$$\frac{Z^s_{54}}{Z^s_{12}} = \frac{\rho_5 c_5}{\rho_1 c_1}\sqrt{\frac{1+\frac{\gamma_5+1}{2\gamma_5}(p_{45}-1)}{1+\frac{\gamma_1+1}{2\gamma_1}(p_{21}-1)}} \gtrless 1. \tag{2}$$

Equality of pressure across the contact discontinuity gives $p_{45} = p_{31} = p_{21}p_{32}$. Following [6], examining the problem in the (p, u) plane indicates that the impedance ratio can be evaluated for $p_{32} = 1$. Then (2) can be expressed as

$$D(p_{21}-1) \gtrless N, \tag{3}$$

where

$$D = \frac{\gamma_5+1}{2\gamma_5} - \frac{\gamma_1+1}{2\gamma_1}\frac{\gamma_1\rho_1}{\gamma_5\rho_5}, \tag{4}$$

$$N = \frac{\gamma_1\rho_1}{\gamma_5\rho_5} - 1. \tag{5}$$

Four cases need to be considered.

1. $D > 0$ and $N > 0$, so $(\gamma_1+1)/(\gamma_5+1) < \rho_5/\rho_1 < \gamma_1/\gamma_5$. The inequalities can be satisfied only if $\gamma_5 < \gamma_1$. The reflected wave can be a shock wave or an expansion wave depending on

$$p_{21} - 1 \gtrless \left(\frac{\gamma_1\rho_1}{\gamma_5\rho_5} - 1\right) \Big/ \left(\frac{\gamma_5+1}{2\gamma_5} - \frac{\gamma_1+1}{2\gamma_1}\frac{\gamma_1\rho_1}{\gamma_5\rho_5}\right). \tag{6}$$

2. $D > 0$ and $N < 0$, so $\rho_5/\rho_1 > (\gamma_1+1)/(\gamma_5+1)$ if $\gamma_5 > \gamma_1$ and $\rho_5/\rho_1 > \gamma_1/\gamma_5$ if $\gamma_5 < \gamma_1$. The reflected wave is a shock wave.
3. $D < 0$ and $N > 0$, so $\rho_5/\rho_1 < \gamma_1/\gamma_5$ if $\gamma_5 > \gamma_1$ and $\rho_5/\rho_1 < (\gamma_1+1)/(\gamma_5+1)$ if $\gamma_5 < \gamma_1$. The reflected wave is an expansion wave.
4. $D < 0$ and $N < 0$, so $\gamma_1/\gamma_5 < \rho_5/\rho_1 < (\gamma_1+1)/(\gamma_5+1)$. The inequalities can be satisfied only if $\gamma_5 > \gamma_1$. The reflected wave can be a shock wave or expansion wave depending on

$$p_{21} - 1 \lessgtr \left(\frac{\gamma_1\rho_1}{\gamma_5\rho_5} - 1\right) \Big/ \left(\frac{\gamma_5+1}{2\gamma_5} - \frac{\gamma_1+1}{2\gamma_1}\frac{\gamma_1\rho_1}{\gamma_5\rho_5}\right). \tag{7}$$

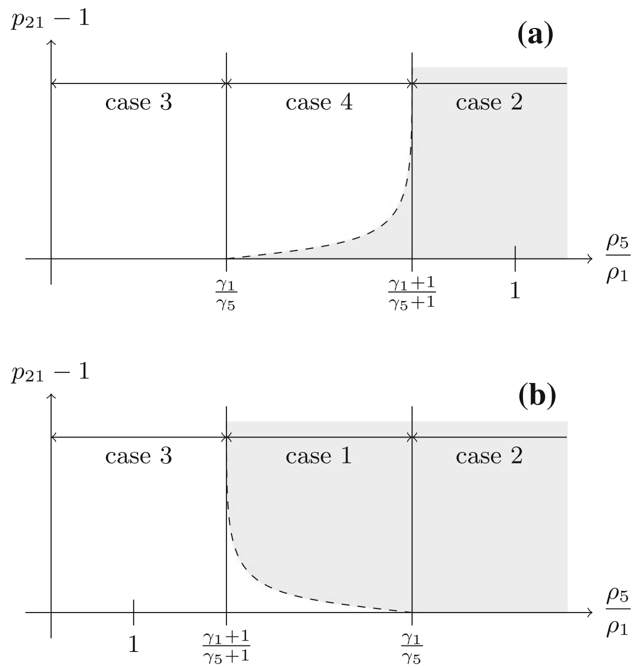

Fig. 2 Schematic illustration of types of reflected waves as determined by shock-impedance ratio as a function of ρ_5/ρ_1, γ_1, γ_5, and p_{21} for **a** $\gamma_1 < \gamma_5$ and **b** $\gamma_1 > \gamma_5$. The *shaded regions* indicate that the reflected wave is a shock wave. The lack of shading indicates that the reflected wave is an expansion wave. The *dashed lines* are given by (6) and (7)

The results of this analysis are depicted schematically in Fig. 2.

We now turn our attention to determine when the type of reflected wave is different when determined from the acoustic-impedance ratio compared to the shock-impedance ratio.

To this end, we write (1) as

$$\frac{Z_{54}^s}{Z_{12}^s} = \frac{Z_5}{Z_1}\frac{M_{s,t}}{M_{s,i}} \gtrless 1, \tag{8}$$

where $M_{s,t}$ and $M_{s,i}$ are the Mach numbers of the transmitted and incident shock waves, respectively. We need to consider two main cases.

1. $Z_5/Z_1 < 1$ and $M_{s,t}/M_{s,i} > Z_1/Z_5$ (the acoustic-impedance ratio incorrectly predicts a reflected expansion wave). Then

$$\frac{\gamma_1\left[(\gamma_5+1)p_{21}+(\gamma_5-1)\right]}{\gamma_5\left[(\gamma_1+1)p_{21}+(\gamma_1-1)\right]} > \frac{\gamma_1\rho_1}{\gamma_5\rho_5} > 1. \tag{9}$$

From the left inequality we obtain $D(p_{21}-1) > N$, where D and N are given by (4) and (5). The right inequality implies that $N > 0$ or

$$\frac{\rho_5}{\rho_1} < \frac{\gamma_1}{\gamma_5}. \tag{10}$$

We need to consider two subcases. If we assume that $D > 0$, then

$$\frac{\rho_5}{\rho_1} > \frac{\gamma_1+1}{\gamma_5+1}. \tag{11}$$

The bounds on ρ_5/ρ_1 are related through

$$\frac{\gamma_1+1}{\gamma_5+1} < \frac{\gamma_1}{\gamma_5} \quad \text{if} \quad \gamma_1 > \gamma_5, \tag{12}$$
$$\frac{\gamma_1}{\gamma_5} < \frac{\gamma_1+1}{\gamma_5+1} \quad \text{if} \quad \gamma_1 < \gamma_5. \tag{13}$$

It is impossible to satisfy simultaneously (10), (11), and (13). We are left with

$$\frac{\gamma_1+1}{\gamma_5+1} < \frac{\rho_5}{\rho_1} < \frac{\gamma_1}{\gamma_5} \quad \text{if} \quad \frac{\gamma_1}{\gamma_5} > 1, \tag{14}$$

and

$$p_{21}-1 > \left(\frac{\gamma_1\rho_1}{\gamma_5\rho_5}-1\right)\bigg/\left(\frac{\gamma_5+1}{2\gamma_5}-\frac{\gamma_1+1}{2\gamma_1}\frac{\gamma_1\rho_1}{\gamma_5\rho_5}\right). \tag{15}$$

On the other hand, if we assume that $D < 0$, then $p_{21} - 1 < N/D < 0$, which cannot be satisfied by any $p_{21} > 1$.

2. $Z_5/Z_1 > 1$ and $M_{s,t}/M_{s,i} < Z_1/Z_5$ (the acoustic-impedance ratio incorrectly predicts a reflected shock wave). Then

$$\frac{\gamma_1\left[(\gamma_5+1)p_{21}+(\gamma_5-1)\right]}{\gamma_5\left[(\gamma_1+1)p_{21}+(\gamma_1-1)\right]} < \frac{\gamma_1\rho_1}{\gamma_5\rho_5} < 1. \tag{16}$$

From the left inequality we obtain $D(p_{21}-1) < N$, where D and N are defined in (4) and (5). The right inequality implies that $N < 0$ or

$$\frac{\rho_5}{\rho_1} > \frac{\gamma_1}{\gamma_5}. \tag{17}$$

Again we need to consider two subcases. If we assume that $D > 0$, then $p_{21} - 1 < N/D < 0$, which cannot be satisfied by any $p_{21} > 1$. On the other hand, if we assume that $D < 0$, then

$$\frac{\rho_5}{\rho_1} < \frac{\gamma_1+1}{\gamma_5+1}. \tag{18}$$

The lower bound given by (17) and the upper bound given by (18) are related through (12) and (13). No value of ρ_5/ρ_1 that satisfies (17) and (18) can satisfy (12). Thus we are left with

$$\frac{\gamma_1}{\gamma_5} < \frac{\rho_5}{\rho_1} < \frac{\gamma_1+1}{\gamma_5+1} \quad \text{if} \quad \frac{\gamma_1}{\gamma_5} < 1, \tag{19}$$

and p_{21} is again given by (15).

These results are summarized in Fig. 3. By comparing with Fig. 2, we see that the use of the acoustic-impedance ratio leads to incorrect predictions for cases *(a)* and *(d)* of the standard theory in which the shock-impedance ratio predicts either a reflected shock wave or expansion wave depending on the pressure ratio of the incident shock wave.

3 Discussion

Having derived the conditions under which the acoustic-impedance ratio predicts the incorrect type of reflected wave in shock-refraction problems, it remains to be established how restrictive the conditions are. Defining

$$\Gamma = \frac{\gamma_5}{\gamma_1}, \tag{20}$$

the conditions relate γ_1, Γ, ρ_5/ρ_1, and p_{21}. Because $1 < \gamma \leq 5/3$, we have $3/5 < \Gamma \leq 5/3$. For $\gamma_1 = 1.4$, the restrictions can be represented by Fig. 4. The plot is to be interpreted in the following manner. The acoustic-impedance ratio predicts the incorrect type of reflected wave if the point $(\Gamma, \rho_5/\rho_1)$ lies within the regions bounded by the (dashed) lines $\rho_5/\rho_1 = 1/\Gamma$

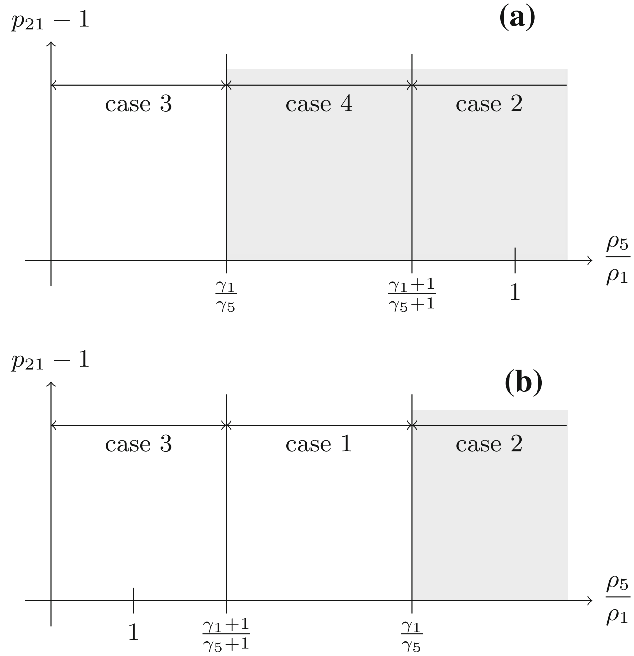

Fig. 3 Schematic illustration of types of reflected waves as determined by acoustic-impedance ratio as a function of ρ_5/ρ_1, γ_1, and γ_5 for **a** $\gamma_1 < \gamma_5$ and **b** $\gamma_1 > \gamma_5$. The *shaded regions* indicate that the reflected wave is a shock wave. The lack of shading indicates that the reflected wave is an expansion wave

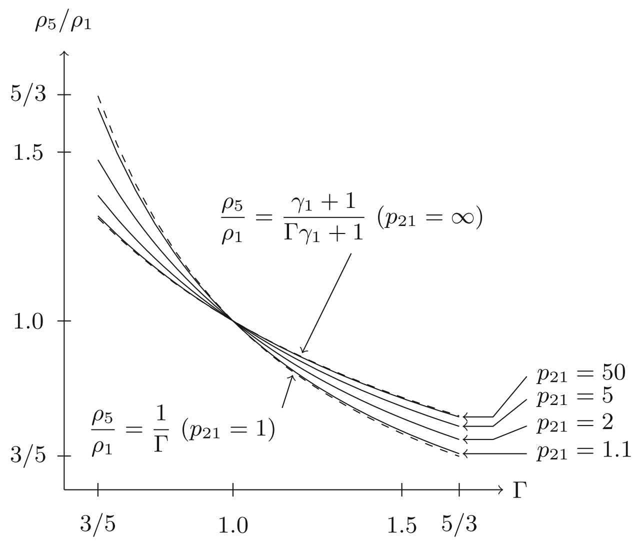

Fig. 4 Values of p_{21} above which the acoustic-impedance ratio predicts the incorrect type of reflected wave as a function of Γ and ρ_5/ρ_1 for $\gamma_1 = 1.4$

and $\rho_5/\rho_1 = (\gamma_1+1)/(\Gamma\gamma_1+1)$ and if p_{21} exceeds the values shown in the figure.

It is seen that the range of ρ_5/ρ_1 over which incorrect types of reflected waves are predicted is $O(1)$ and quite narrow. For obvious reasons, shock-refraction experiments usually consider density ratios that are significantly different from unity. As a result, none of the articles published on shock-refraction problems appears to include an experiment in which the acoustic-impedance ratio predicts the incorrect type of reflected wave. This may explain why the applicability of the acoustic-impedance ratio to shock-refraction problems has not been investigated before. Nevertheless, the conditions derived here may be of use for future studies.

It should be noted that along the line $\rho_5/\rho_1 = (\gamma_1 + 1)/(\Gamma\gamma_1 + 1)$, we have $D = 0$ and hence $p_{21} = \infty$ (unless $\Gamma = 1$, a case that is of no interest). At high values of p_{21}, i.e., for strong shock waves, of course, the assumption of perfect-gas behavior becomes questionable. Data presented by Owczarek [8, section 5.4] indicate that real-gas effects on the pressure ratio across a shock wave in air are negligible provided that $M_s \lessapprox 8 (p_{21} \lessapprox 75)$ if the state ahead of the shock wave is at standard conditions. As may be seen from Fig. 4, curves corresponding to such large values of p_{21} are essentially indistinguishable from the curve for $p_{21} = \infty$. For this reason, and because shock-refraction experiments are usually carried out at conditions that ensure perfect-gas behavior, Fig. 4 should be useful for a wide range of conditions.

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