ORIGINAL ARTICLE

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Important aspects in the formulation of solid–fluid debris-flow models. Part II. Constitutive modelling

Abstract This article is the continuation of Part I: 'Thermodynamic Implications' of a article with the same title. Knowledge of the content/results of Part I, Hutter and Schneider (Continuum Mech. Thermodyn., 2009) or Schneider and Hutter (Solid–Fluid Mixtures of Frictional Materials in Geophysical and Geotechnical Context, 2009), is assumed. The intention is to see whether (i) well-known formulations of binary mixture models can be derived from the thermodynamic model, (ii) classical hypo-plasticity is deducible from the frictional evolution equation and (iii) the popular assumption of pressure equilibrium is justified. To this end, we ignore mass and volume fraction interaction rate densities, restrict considerations to isothermal processes, ignore higher order non-linearities in the constitutive relations and use the principle of phase separation. These assumptions transform the equilibrium stresses, heat flux and interaction forces to considerably simplified forms. Furthermore, the analysis shows that classical hypo-plasticity can be reconstructed with the introduction of a new objective time derivative for the stress-like variable. Non-equilibrium contributions to the stresses and interaction forces are also briefly discussed. It is, finally, shown that the assumption of pressure equilibrium precludes the application of frictional stresses in equilibrium. This unphysical assumption is, therefore, replaced by a thermodynamic closure condition that is more flexible and less restrictive. It allows for frictional stresses in thermodynamic equilibrium and, therefore, is sufficiently general for applications to mixture theories.

Keywords Debris flow · Second law of thermodynamics · Solid–fluid mixture · Frictional material · Pressure equilibrium

1 Introduction

In Hutter and Schneider [21], a continuum thermodynamic formulation of solid–fluid debris-flow models was presented. This article, henceforth referred to as 'Part I', presented the thermodynamic implications, which were deduced from an entropy principle due to Müller. Results, which are explicitly derived and extensively discussed in book form by Schneider and Hutter [38], were only collected. The 'credo' in Part I is that postulation of closure conditions for constitutive quantities must be done by a full exploitation of the second law of thermodynamics even when at last only a mechanical theory is pursued. In this article, we continue this analysis with the presentation of explicit constitutive relations for stress and interaction force parameterizations which

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are capable to reproduce the entire range of dynamical processes from quasi-static creep to catastrophic rapid motion of a saturated soil mass. In the following, we assume the reader to be familiar either with Part I or the content of the first seven chapters of Schneider and Hutter [38].

In Part I, we developed a theory for an isotropic visco-elasto-plastic heat conducting mixture of n constituents, (i) in which mass interactions between the constituents may occur, (ii) some or all of the constituents are density preserving in the sense that they possess constant constituent mass densities, (iii) which is saturated in the sense that no void spaces are present in the mixture, (iv) that ignores constituent energy interactions, (v) which is capable of measuring the distribution and evolution of submacroscopic structures by means of internal variables and corresponding balance laws and (vi) allows in the linearized case for a hyperbolic equation for the temperature distribution.

In the sequel, we aim to reduce the above theory to a model that is sufficiently simple to be numerically solvable but, equally, allows for the description of the main properties of debris flows, namely, (i) fluidisation in a thin shear band close to the bed, (ii) particle size segregation,[1] (iii) shear stresses present in thermodynamic equilibrium and (iv) velocity differences of the fluid and the solid grains.

To this end, we commence in Sect. 2 with the presentation of the basic physical assumptions, e.g. the binary mixture postulate, no mass interaction processes, etc, followed in Sect. 3 by the parameterization of the constituent equilibrium stresses, separately for the elastic and frictional parts of the solid stress. Section 4 deals with the non-equilibrium stresses as non-linear viscous solid and fluid sub-bodies. Here, we show that knowledge of viscometry is helpful in the parameterization of the viscous stress contributions. In Section 5, we state the final constitutive relations. Section 6 addresses the popular 'assumption of pressure equilibrium'; we point out its weaknesses and propose an alternative, which is free of its inconsistencies. We close in Sect. 7 with a discussion and conclusions.

2 Physical assumptions

We model debris flows here as saturated mixtures of two constituents, where we interpret the first constituent as solid grains and the second as a fluid. Thus, the Greek indices take the identifiers s, for the solid and f, for the fluid. As a consequence of the saturation condition, the volume fraction for the fluid, v_f, is replaced by $(1 - v_s)$ and, as we have seen in Part I, an independent constraint field s arises for which the field equations (to be specified) have to be solved. We also assume that both constituents are density-preserving,

$$\rho_s = \text{const.}, \qquad \rho_f = \text{const.}, \tag{1}$$

i.e., no constituent is compressible ($m = 0$, Part I). The binary mixture concept, in which the solid constituent is not split into a number of separate components, implies that different characterizations of the solid component by the grain size or differences in resilience, etc., are not accounted for. Furthermore, we shall also exclude melting of the solid particles in the moving process. This would in most situations require a three constituent or even more detailed mixture concept. This excludes very large landslides in which the frictional heat will melt the rock and—after solidification of the molten rock—generate so-called frictionites. Thus,

$$c_s = c_f = 0 . \tag{2}$$

As a consequence of this, the volume fraction production rate densities vanish,

$$n_s = n_f = 0 , \tag{3}$$

and the mass- and volume fraction balance equations turn into

$$\partial v_s + \nabla \cdot (v_s \mathbf{v}_s) = 0 ,$$
$$\partial v_s - \nabla \cdot (\mathbf{v}_f) + \nabla \cdot (v_s \mathbf{v}_f) = 0 . \tag{4}$$

By subtracting these two equations one obtains

$$\nabla \cdot (v_s \mathbf{v}_s + (1 - v_s)\mathbf{v}_f) = 0 , \tag{5}$$

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[1] Despite some claims in the literature, stating that particle size segregation can only be accounted for by discrete mechanics, we state that a mixture model with several solid constituents is one possible approach to model such separations.

which may replace one of the equations (4). If we now define by

$$\mathbf{v}_{\mathrm{vol}} = \nu_s \mathbf{v}_s + (1 - \nu_s)\mathbf{v}_f \tag{6}$$

the *volume-weighted mixture velocity*, (5) states that

$$\nabla \cdot (\mathbf{v}_{\mathrm{vol}}) = 0 . \tag{7}$$

This result is sufficiently significant to state it in words: *The volume-weighted mixture velocity is solenoidal.* We emphasize this property, because in the literature (primarily of fluvial hydraulics) it is often used without explicitly mentioning that the mixture velocity is volume-weighted rather than mass weighted (=barycentric). Its simplicity also yields modelling and computational advantages. For a formal comparison of volume and mass weighted mixture concepts, see Chen and Tai [8].

In Part I, we were concerned, among other things, with thermodynamic processes, involving the temperature, θ, its gradient, $\nabla\theta$, and its material time derivative, $\dot{\theta}$. However, for the sake of simplicity we shall restrict considerations in this Part II to *isothermal* processes, i.e., each material element of the mixture is thought to exhibit the same temperature for all times. Therefore, all temperature related quantities in the constitutive relations can be omitted and the energy equation as an evolution equation for the temperature can be dispensed with. As a consequence, the considered debris flow problem is *purely mechanical*. This may exclude some pyroclastic gravity currents and certainly lava flows.

Finally, we ignore frictional behaviour for the fluid constituent and set $\bar{\mathbf{Z}}_f = \mathbf{0}$.

When incorporating the above assumptions into the constitutive relations, these relations are of the class

$$\mathsf{C} = \hat{\mathsf{C}}\left(\nu_s, \ \nabla\nu_s, \ \mathbf{v}_{fs}, \ \mathbf{B}_s, \ \mathbf{B}_f, \ \mathbf{D}_s, \ \mathbf{D}_f, \ \mathbf{W}_{fs}, \ \bar{\mathbf{Z}}_s\right),$$
$$\text{for} \quad \mathsf{C} := \left\{\bar{\mathbf{T}}_s, \ \bar{\mathbf{T}}_f, \ \bar{\mathbf{m}}_s^i\right\} . \tag{8}$$

in which for reasons of material objectivity we have introduced [2]

$$\mathbf{v}_{fs} := \mathbf{v}_f - \mathbf{v}_s, \qquad \mathbf{W}_{fs} := \mathbf{W}_f - \mathbf{W}_s \tag{9}$$

as an objective difference velocity of the solid and the fluid and the difference of the solid and fluid vorticity tensors, respectively.

For the 'inner' part of the Helmholtz free energy, we have the following dependencies

$$\Psi_I^G = \hat{\Psi}_I^G\left(\nu_s, \ \mathbf{B}_s, \ \mathbf{B}_f, \ \bar{\mathbf{Z}}_s\right) . \tag{10}$$

As Ψ^G is independent of ρ_α, see (1), the true thermodynamic pressures, p_s^G and p_f^G, are not present in this model. We further remark, that Ψ_I^G only depends on equilibrium quantities, and thus, the identifier $(\cdot)|_E$ can be omitted for all quantities derived from Ψ_I^G, i.e. β_s^G, ζ_α ($\alpha = s, f$) and $\Psi_{I,\mathbf{x}^J}^G$, where $\mathbf{x}^J \in \left\{\mathbf{B}_s, \ \mathbf{B}_f, \ \bar{\mathbf{Z}}_s\right\}$. For details see Part I.

With all these simplifications, the constitutive laws for the equilibrium quantities $\mathbf{q}|_E$, $\bar{\mathbf{T}}_\beta|_E$ ($\beta = s, f$) and $\bar{\mathbf{m}}_s^i|_E$ (see relations in Part I) take the forms

$$\mathbf{q}|_E = \mathbf{q}^N|_E , \tag{11}$$

$$\bar{\mathbf{T}}_s|_E = -\bar{\varpi}_s \mathbf{I} + 2\rho \, \mathrm{sym}(\Psi_{I,\mathbf{B}_s}^G)\mathbf{B}_s + \rho\Psi_{I,\bar{\mathbf{Z}}_s}^G \, (\bar{\mathbf{\Phi}}_s),_{\mathbf{D}_s}\big|_E + \bar{\mathbf{T}}_s^N|_E , \tag{12}$$

$$\bar{\mathbf{T}}_f|_E = -\bar{\varpi}_f \mathbf{I} + 2\rho \, \mathrm{sym}(\Psi_{I,\mathbf{B}_f}^G)\mathbf{B}_f + \bar{\mathbf{T}}_f^N|_E , \tag{13}$$

$$\begin{aligned}
\bar{\mathbf{m}}_s^i|_E &= \left\{\left(\zeta_s - \bar{\xi}_s\zeta_s\right) + \bar{\xi}_s\zeta_f - \theta(k_s^v),_{\nu_s}\right\}\nabla\nu_s \\
&\quad + \rho\left(\Psi_I^G\right),_{\bar{\mathbf{Z}}_s}(\bar{\mathbf{\Phi}}_s),_{\mathbf{v}_s}\big|_E + \bar{\mathbf{m}}_s^N|_E \\
&= \left\{\zeta_s - \bar{\xi}_s\left(\zeta_s - \zeta_f\right) - \theta(k_s^v),_{\nu_s}\right\}\nabla\nu_s \\
&\quad + \rho\left(\Psi_I^G\right),_{\bar{\mathbf{Z}}_s}(\bar{\mathbf{\Phi}}_s),_{\mathbf{v}_s}\big|_E + \bar{\mathbf{m}}_s^N|_E \\
&= \left\{\beta_s^G\left(1 - \bar{\xi}_s\right) - \rho_f\Psi_I^G + \varsigma - \theta(k_s^v),_{\nu_s}\right\}\nabla\nu_s \\
&\quad + \rho\left(\Psi_I^G\right),_{\bar{\mathbf{Z}}_s}(\bar{\mathbf{\Phi}}_s),_{\mathbf{v}_s}\big|_E + \bar{\mathbf{m}}_s^N|_E ,
\end{aligned} \tag{14}$$

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[2] Analogously, $\mathbf{v}_{sf} = \mathbf{v}_s - \mathbf{v}_f = -\mathbf{v}_{fs}$. Therefore, either $\mathbf{v}_{fs}$ or $\mathbf{v}_{sf}$ is the generic variable and both are equivalent to one another. For a similar reason, we omit in (8) $\bar{\mathbf{m}}_f = -\bar{\mathbf{m}}_s$.

in which

$$\bar{\omega}_s = \nu_s \left(\beta_s^G - \rho_f \Psi_I^G + \varsigma \right) - \theta k_s^v \,, \tag{15}$$

$$\bar{\omega}_f = (1 - \nu_s) \left(-\rho_f \Psi_I^G + \varsigma \right) - \theta k_f^v \,. \tag{16}$$

Notations and variables are defined in Part I.

We know from hydrostatics, that fluids in thermodynamic equilibrium can only sustain spherical stresses, i.e. pressures, and thus the second and third term in (13) can only have the form

$$2\rho \, \mathrm{sym}(\Psi_{I,\mathbf{B}_f}^G)\mathbf{B}_f + \bar{\mathbf{T}}_f^N|_E =: \pi_f \mathbf{I} \,, \tag{17}$$

where π_f is a scalar which depends only on equilibrium variables. In addition, $\bar{\omega}_f$ contains the independent saturation pressure, ς. Thus, and if one so desires, $\bar{\omega}_f$ itself rather than ς or ζ_f could be regarded as an independent quantity which is not determined by constitutive relations but from the solution of the field equations. Without loss of generality, it is, therefore, permissible to incorporate π_f into $\bar{\omega}_f$, and we are left with

$$\bar{\mathbf{T}}_f|_E = -\bar{\omega}_f \mathbf{I} \,. \tag{18}$$

In the above expressions (12), (13) and (14), $\bar{\mathbf{T}}_s^N|_E$, $\bar{\mathbf{T}}_f^N|_E$ and $\bar{\mathbf{m}}_s^{iN}|_E$ which are nonlinear of higher order in the variables Y (see Part I) are formally accounted for. In the sequel, these will now be ignored as they can be shown to be of minor influence (see Svendsen and Hutter [39] and Schneider and Hutter [38]).

In summary, we, therefore, have the following equilibrium quantities

$$\bar{\mathbf{T}}_s|_E = -\bar{\omega}_s \mathbf{I} + 2\rho \, \mathrm{sym}(\Psi_{I,\mathbf{B}_s}^G)\mathbf{B}_s + \rho \Psi_{I,\bar{\mathbf{Z}}_s}^G (\bar{\boldsymbol{\Phi}}_s), \mathbf{D}_s |_E \,, \tag{19}$$

$$\bar{\omega}_s = \nu_s \left(\beta_s^G - \rho_f \Psi_I^G + \varsigma \right) - \theta k_s^v \,,$$

$$\bar{\mathbf{T}}_f|_E = -\bar{\omega}_f \mathbf{I} \,, \tag{20}$$

$$\bar{\omega}_f = (1 - \nu_s) \left(-\rho_f \Psi_I^G + \varsigma \right) - \theta k_f^v - \pi_f \,,$$

$$\bar{\mathbf{m}}_s^i|_E = \left\{ \beta_s^G \left(1 - \bar{\xi}_s \right) - \rho_f \Psi_I^G + \varsigma - \theta (k_s^v),_{\nu_s} \right\} \nabla \nu_s$$

$$+ \rho \left(\Psi_I^G \right),_{\bar{\mathbf{Z}}_s} \left(\bar{\boldsymbol{\Phi}}_s \right),_{\mathbf{v}_s} |_E \,, \tag{21}$$

where, in particular, the elastic and hypo-plastic parts of (19) still have to be discussed in greater detail.

3 Parameterization of the constituent equilibrium stresses

We have pointed out in Part I that all constitutive quantities can be decomposed into equilibrium and non-equilibrium parts, i.e.

$$\bar{\mathbf{T}}_s = \bar{\mathbf{T}}_s|_E + \bar{\mathbf{T}}_s|_N \,, \qquad \bar{\mathbf{T}}_f = \bar{\mathbf{T}}_f|_E + \bar{\mathbf{T}}_f|_N \,, \qquad \bar{\mathbf{m}}_s^i = \bar{\mathbf{m}}_s^i|_E + \bar{\mathbf{m}}_s^i|_N \,. \tag{22}$$

In the last section, we have found representations for the equilibrium parts of the constitutive quantities, but in particular the equilibrium solid stress tensor, $\bar{\mathbf{T}}_s|_E$, which consists of a constraint-configuration (cs), an elastic (es) and a frictional (fric) (hypo-plastic) part, i.e.,

$$\bar{\mathbf{T}}_s|_E = \underbrace{-\bar{\omega}_s \mathbf{I}}_{\bar{\mathbf{T}}_{cs}} + \underbrace{2\rho \, \mathrm{sym}(\Psi_{I,\mathbf{B}_s}^G)\mathbf{B}_s}_{\bar{\mathbf{T}}_{es}} + \underbrace{\rho \Psi_{I,\bar{\mathbf{Z}}_s}^G (\bar{\boldsymbol{\Phi}}_s),_{\mathbf{D}_s} |_E}_{\bar{\mathbf{T}}_{fric}} \tag{23}$$

requires further modelling. Let us make the constitutive relations for $\bar{\mathbf{T}}_{es}$ and $\bar{\mathbf{T}}_{fric}$ more specific. To this end, we assume the 'inner' free energy to have the form

$$\Psi_I^G = \sum \Psi_\alpha^G = \tilde{\Psi}_s^G \left(\nu_s, \, \mathbf{B}_s, \, \bar{\mathbf{Z}}_s \right) + \tilde{\Psi}_f^G ((1 - \nu_s), \, \mathbf{B}_f) \,. \tag{24}$$

By prescribing this representation for Ψ_s^G and Ψ_f^G we have used the principle of phase separation. [3] We remark that for single-material bodies the 'principle of phase separation' reduces to the well known *principle of equi-presence*, Truesdell and Noll [42] . We further notice that the principle must likely be wrong when exchange processes between the constituents take place.

In order to specify the elastic parts, of the constituent Cauchy stress tensors we isolate the elastic and frictional effects in Ψ_I^G separately by assuming (cf. Hutter et al. [19])

$$\Psi_I^G = \hat{\Psi}_{\text{fric}}^G \left(\nu_s, \, \bar{\mathbf{Z}}_s \right) + \hat{\Psi}_{\text{es}}^G \left(\mathbf{B}_s \right) + \hat{\Psi}_{\text{ef}}^G \left(\mathbf{B}_f \right) \; . \tag{25}$$

Here, the indices 'fric', 'es' and 'ef' stand for 'friction', 'elastic-solid' and 'elastic-fluid', respectively. The last two terms in (25) are thought to account for the elastic contributions of the solid and fluid, respectively. In $\hat{\Psi}_{\text{fric}}^G$, on the other hand, we have subsumed all other dependencies of Ψ_I^G. It is believed that the representation of $\hat{\Psi}_{\text{fric}}^G$ in (25) is able to describe all effects of the visco-elasto-plastic binary mixture, except those of elasticity.

3.1 Elastic stress for the solid (and the fluid)

We know from the representation theory of isotropic functions, that isotropic scalar-valued functions of a single symmetric tensor, such as $\hat{\Psi}_{\text{es}}^G$ and $\hat{\Psi}_{\text{ef}}^G$, can only depend on the invariants of this tensor (cf. Ogden [33]). Consequently, those two functions exhibit the following dependencies

$$\Psi_{\text{es}}^G = \check{\Psi}_{\text{es}}^G \left(\text{I}_{\mathbf{B}_s}, \, \text{II}_{\mathbf{B}_s}, \, \text{III}_{\mathbf{B}_s} \right), \qquad \Psi_{\text{ef}}^G = \check{\Psi}_{\text{ef}}^G \left(\text{I}_{\mathbf{B}_f}, \, \text{II}_{\mathbf{B}_f}, \, \text{III}_{\mathbf{B}_f} \right), \tag{26}$$

where the invariants for a general symmetric second-order tensor, $\mathbf{A}$, are defined as

$$\text{I}_{\mathbf{A}} = \text{tr}(\mathbf{A}), \qquad \text{II}_{\mathbf{A}} = \tfrac{1}{2} \left(\, (\text{I}_{\mathbf{A}})^2 - \text{I}_{\mathbf{A}^2} \right), \qquad \text{III}_{\mathbf{A}} = \det(\mathbf{A}) \; . \tag{27}$$

With these results in mind, we can now turn the attention to the contributions of Ψ_I^G in the elastic parts of the constituent Cauchy stress tensors. If we, first, ignore the arguments for π_f (see (17)) for a moment, the elastic part of the fluid Cauchy stress tensor, $\left(\bar{\mathbf{T}}_f \right)_{\text{ef}}$, can be written in the forms

$$\begin{aligned} \left(\bar{\mathbf{T}}_f \right)_{\text{ef}} &= 2\rho \, \text{sym} \left(\, (\Psi_I^G)_{,\mathbf{B}_f} \right) \mathbf{B}_f \overset{(25)}{=} 2\rho \, \text{sym} \left(\, (\hat{\Psi}_{\text{ef}}^G)_{,\mathbf{B}_f} \right) \mathbf{B}_f \\ &= 2\rho \, \text{sym} \left(\frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{I}_{\mathbf{B}_f}} \frac{\partial \text{I}_{\mathbf{B}_f}}{\partial \mathbf{B}_f} + \frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{II}_{\mathbf{B}_f}} \frac{\partial \text{II}_{\mathbf{B}_f}}{\partial \mathbf{B}_f} + \frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{III}_{\mathbf{B}_f}} \frac{\partial \text{III}_{\mathbf{B}_f}}{\partial \mathbf{B}_f} \right) \mathbf{B}_f \; , \\ &= 2\rho \left(\frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{I}_{\mathbf{B}_f}} \mathbf{I} + \frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{II}_{\mathbf{B}_f}} \left(\text{I}_{\mathbf{B}_f} \mathbf{I} - \mathbf{B}_f \right) + \frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{III}_{\mathbf{B}_f}} \text{III}_{\mathbf{B}_f} \mathbf{B}_f^{-1} \right) \mathbf{B}_f \; , \end{aligned} \tag{28}$$

where the chain rule of differentiation has been used. For a general second-rank tensor $\mathbf{A}$, the above derivatives take the forms (cf. Hutter and Jöhnk [18])

$$\frac{\partial \text{I}_{\mathbf{A}}}{\partial \mathbf{A}} = \mathbf{I}, \qquad \frac{\partial \text{II}_{\mathbf{A}}}{\partial \mathbf{A}} = (\text{I}_{\mathbf{A}} \mathbf{I} - \mathbf{A}), \qquad \frac{\partial \text{III}_{\mathbf{A}}}{\partial \mathbf{A}} = \text{III}_{\mathbf{A}} \mathbf{A}^{-1} \; . \tag{29}$$

If we apply these results with $\mathbf{A} = \mathbf{B}_f$ in (28), we see that the term in parentheses is already symmetric, which justifies the last line in (28).

In thermodynamic equilibrium, fluids can only sustain spherical stresses. Consequently, only those derivatives of the invariants are to be considered which allow $(\bar{\mathbf{T}}_f)_{\text{ef}}$ to become proportional to the unit tensor, $\mathbf{I}$. This situation can only be reached if we require

$$\frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{I}_{\mathbf{B}_f}} = \frac{\partial \check{\Psi}_{\text{ef}}^G}{\partial \text{II}_{\mathbf{B}_f}} = 0 \; . \tag{30}$$

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[3] The '*Principle of phase separation*' introduced by Passman et al. [35], requires the '*material-specific*' constitutive quantities for constituent K_α, to depend only on those constitutive variables that belong to the same constituent. This principle does not apply to the remaining quantities, e.g. those for the whole mixture or those describing interactions between the constituents. The principle was introduced into the literature much earlier by Morland [28], however, by not declaring it a 'principle'. In Morland [29] and subsequent articles [30], [31] it was re-iterated on and the poor terminology 'effective' was changed to 'intrinsic' which we call 'true'.

It follows that $\check{\Psi}_{\mathrm{ef}}^{G}$ cannot depend on $\mathrm{I}_{\mathbf{B}_{\mathrm{f}}}$ and $\mathrm{II}_{\mathbf{B}_{\mathrm{f}}}$. Furthermore, only for mixtures with non-vanishing mass interactions, i.e. $c_{\alpha} \neq 0$, the variables $\mathbf{B}_{\alpha}$ and ρ_{α} ($\alpha = 1, \ldots, m$) are independent of one another (see Schneider and Hutter [38]). Thus, in the present model for which $c_{\alpha} = 0$ the assumption of constant true mass density allows the conclusion [4]

$$\det(\mathbf{B}_{\mathrm{f}}) = \mathrm{III}_{\mathbf{B}_{\mathrm{f}}} = \text{const.} , \qquad \det(\mathbf{B}_{\mathrm{s}}) = \mathrm{III}_{\mathbf{B}_{\mathrm{s}}} = \text{const.} \tag{31}$$

and, therefore, $\check{\Psi}_{\mathrm{ef}}^{G}$ cannot depend on $\mathrm{III}_{\mathbf{B}_{\mathrm{f}}}$ either, i.e.,

$$\frac{\partial \check{\Psi}_{\mathrm{ef}}^{G}}{\partial \mathrm{III}_{\mathbf{B}_{\mathrm{f}}}} = 0 . \tag{32}$$

We obtain from (30) and (32) that

$$\frac{\partial \hat{\Psi}_{\mathrm{ef}}^{G}}{\partial \mathbf{B}_{\mathrm{f}}} \overset{(25)}{=} \frac{\partial \Psi_{I}^{G}}{\partial \mathbf{B}_{\mathrm{f}}} = \mathbf{0} . \tag{33}$$

Thus, Ψ_{I}^{G} cannot be a function of $\mathbf{B}_{\mathrm{f}}$. This implies that π_{f} vanishes, see (17).

The elastic part of the solid Cauchy stress tensor, on the other hand, becomes[4]

$$\bar{\mathbf{T}}_{\mathrm{es}} = 2\rho \left(\frac{\partial \check{\Psi}_{\mathrm{es}}^{G}}{\partial \mathrm{I}_{\mathbf{B}_{\mathrm{s}}}} + \mathrm{I}_{\mathbf{B}_{\mathrm{s}}} \frac{\partial \check{\Psi}_{\mathrm{es}}^{G}}{\partial \mathrm{II}_{\mathbf{B}_{\mathrm{s}}}} \right) \mathbf{B}_{\mathrm{s}} - 2\rho \frac{\partial \check{\Psi}_{\mathrm{es}}^{G}}{\partial \mathrm{II}_{\mathbf{B}_{\mathrm{s}}}} \mathbf{B}_{\mathrm{s}}^{2} . \tag{34}$$

With this relation we have reached the point, where, except for the postulate of an explicit representation for the elastic part of the solid free energy, Ψ_{es}^{G}, no other simplification can be performed. The simple choice

$$\begin{aligned}
\rho \check{\Psi}_{\mathrm{es}}^{G} &= C_{1} \left(\mathrm{I}_{\mathbf{B}_{\mathrm{s}}} - 3 \right) + C_{2} \left(\mathrm{II}_{\mathbf{B}_{\mathrm{s}}} - 3 \right) , \\
C_{1} &= \tfrac{1}{2}\mu \left(\tfrac{1}{2} + \beta \right) = \text{const.}, \quad C_{2} = \tfrac{1}{2}\mu \left(\tfrac{1}{2} - \beta \right) = \text{const.} ,
\end{aligned} \tag{35}$$

which is attributed to Mooney and Rivlin (cf. Rivlin and Saunders [37]), leads to

$$\bar{\mathbf{T}}_{\mathrm{es}} = 2 \left(C_{1} + C_{2}\mathrm{I}_{\mathbf{B}_{\mathrm{s}}} \right) \mathbf{B}_{\mathrm{s}} - 2C_{2}\mathbf{B}_{\mathrm{s}}^{2} . \tag{36}$$

In (35), μ can be interpreted as the *shear modulus of the solid grains* and β as a modelling parameter. For the special case of $\beta = \tfrac{1}{2}$, we attain a fairly simple representation, namely

$$\bar{\mathbf{T}}_{\mathrm{es}} = \mu \mathbf{B}_{\mathrm{s}} , \tag{37}$$

which is denoted *Neo-Hookean* behaviour. An alternative, physically linear relation would be

$$\bar{\mathbf{T}}_{\mathrm{es}} = \check{\mu} \mathbf{E}_{\mathrm{s}} , \qquad \mathbf{E}_{\mathrm{s}} := \tfrac{1}{2} \left(\mathbf{B}_{\mathrm{s}} - \mathbf{I} \right) . \tag{38}$$

This choice has the advantage that $\bar{\mathbf{T}}_{\mathrm{es}} = \mathbf{0}$ when the elastic strain $\mathbf{E}_{\mathrm{s}}$ vanishes.

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[4] The density-preserving assumption for a constituent K_{α} whose mass production rate is not present, $c_{\alpha} = 0$, implies that it also preserves its volume along its own trajectory. Hence, $c_{\alpha} = 0$ also means $\det \mathbf{F}_{\alpha} = $ constant. Otherwise stated, the constituent motion is isochoric, and $\mathbf{F}_{\alpha}$ and $\mathbf{B}_{\alpha}$ are unimodular. Therefore, the elastic stress $\bar{\mathbf{T}}_{\mathrm{es}}$ cannot depend on $\mathrm{III}_{\mathbf{B}_{\mathrm{s}}}$.
If c_{α} were not zero, then $\mathbf{B}_{\alpha}$ would not be unimodular and density-preserving could not imply volume-preserving of constituent K_{α}. Insensitivity of $\bar{\mathbf{T}}_{\mathrm{es}}$ to solid volume changes would then require that (34) holds true if $\mathbf{B}_{\mathrm{s}}$ is replaced by

$$\mathbf{B}_{\mathrm{s}}^{\mathrm{unimod}} := (\det \mathbf{B}_{\mathrm{s}})^{-1/3} \mathbf{B}_{\mathrm{s}} .$$

This then simply would mean that there is no bulk elastic response, for details see Schneider and Hutter [38].

3.2 Frictional part of the solid stress

The frictional part of the solid stress (23),

$$\bar{\mathbf{T}}_{\text{fric}} = \rho \Psi^G_{I,\bar{\mathbf{Z}}_s}(\bar{\boldsymbol{\Phi}}_s),_{\mathbf{D}_s}|_E = \rho \Psi^G_{\text{fric},\bar{\mathbf{Z}}_s}(\bar{\boldsymbol{\Phi}}_s),_{\mathbf{D}_s}|_E, \tag{39}$$

is still an unknown function of the equilibrium quantities, since so far no representation has been given for Ψ^G_{fric} and $\bar{\boldsymbol{\Phi}}_s$. We follow the argumentation of Teufel [41] who formulated the following postulate: the frictional stress $\bar{\mathbf{T}}_{\text{fric}}$ is collinear to $\bar{\mathbf{Z}}_s$, i.e.

$$\bar{\mathbf{T}}_{\text{fric}} = \rho \delta \bar{\mathbf{Z}}_s, \quad \delta = \text{constant} . \tag{40}$$

With the choice (40) a special functional relation has been chosen for the solid frictional stress. Substituting (40) into (39) allows by way of integration an explicit determination of Ψ^G_{fric}; therefore, (40) is not a genuine assumption but rather a convenient choice by which hypo-plastic behaviour can be demonstrated.[5] Using assumption (40), i.e. substituting $\bar{\mathbf{Z}}_s = \bar{\mathbf{T}}_{\text{fric}}/(\rho\delta)$ into the evolution equation for $\bar{\mathbf{Z}}_s$,

$$\overset{\circ}{\bar{\mathbf{Z}}}_s := \acute{\bar{\mathbf{Z}}}_s - [\boldsymbol{\Omega}_s, \bar{\mathbf{Z}}_s] = \bar{\boldsymbol{\Phi}}_s, \qquad \boldsymbol{\Omega}_s = -\boldsymbol{\Omega}_s^{\text{T}}, \qquad (\acute{\cdot}) := \frac{\mathrm{d}^s(\cdot)}{\mathrm{d}t} \tag{42}$$

yields

$$\frac{1}{\rho\delta}\overset{\circ}{\bar{\mathbf{T}}}_{\text{fric}} - \frac{1}{\rho^2\delta}\frac{\mathrm{d}^s\rho}{\mathrm{d}t}\bar{\mathbf{T}}_{\text{fric}}$$
$$= \frac{1}{\rho\delta}\left\{ \frac{\mathrm{d}^s\bar{\mathbf{T}}_{\text{fric}}}{\mathrm{d}t} - [\boldsymbol{\Omega}_s, \bar{\mathbf{T}}_{\text{fric}}] - \frac{1}{\rho}\frac{\mathrm{d}^s\rho}{\mathrm{d}t}\bar{\mathbf{T}}_{\text{fric}} \right\}$$
$$= \bar{\boldsymbol{\Phi}}_s\left(\frac{1}{\rho\delta}\bar{\mathbf{T}}_{\text{fric}}, \cdot \right) , \tag{43}$$

where the dot in the argument indicates additional dependencies, say on ν_s and $\mathbf{D}_s$. The choice of $\boldsymbol{\Omega}_s$ specifies the objective time derivative $(\overset{\circ}{\cdot})$. No unique selection is possible, but the most obvious choice is $\boldsymbol{\Omega}_s = \mathbf{W}_s$, where $\mathbf{W}_s$ is the vorticity tensor $\mathbf{W}_s = \text{skw}\,\mathbf{L}_s$. If we use in addition

$$\frac{\mathrm{d}^s\rho}{\mathrm{d}t} = \rho_s\frac{\mathrm{d}^s\nu_s}{\mathrm{d}t} + \rho_f\frac{\mathrm{d}^s(1-\nu_s)}{\mathrm{d}t}$$
$$= (\rho_s - \rho_f)\left(\frac{\partial\nu_s}{\partial t} + \nabla(\nu_s)\mathbf{v}_s \right)$$
$$= -\nu_s(\rho_s - \rho_f)\nabla \cdot \mathbf{v}_s , \tag{44}$$

which is obtained from (4) and the saturation condition, (43) reduces to

$$\frac{1}{\rho\delta}\left\{ \overset{\circ}{\bar{\mathbf{T}}}_{\text{fric}} + \nu_s\frac{\rho_s - \rho_f}{\rho}\,(\nabla \cdot \mathbf{v}_s)\,\bar{\mathbf{T}}_{\text{fric}} \right\} = \bar{\boldsymbol{\Phi}}_s\left(\frac{1}{\rho\delta}\bar{\mathbf{T}}_{\text{fric}}, \cdot \right) . \tag{45}$$

So far, we are dealing with a general constitutive quantity, $\bar{\boldsymbol{\Phi}}_s$, for which the hypo-plastic behaviour has not been, but could be explicitly described. We note that to model hypo-elastic behaviour, $\bar{\boldsymbol{\Phi}}_s$ must be linear in $\mathbf{D}_s$; however, proposed hypo-elastic constitutive proposals cannot capture the fact that the material behaviour of debris in slow or rapid flows is, in general, different in extension from compression (cf. Kolymbas [24]). In

---

[5] This substitution yields the differential equation

$$\delta\bar{\mathbf{Z}}_s = \left(\Psi^G_{\text{fric},\bar{\mathbf{Z}}_s} \right)(\bar{\boldsymbol{\Phi}}_s),_{\mathbf{D}_s}|_E \tag{41}$$

which for given $\bar{\boldsymbol{\Phi}}_s$ may be regarded as a differential equation for Ψ^G_{fric}. The existence of a solution is tacitly assumed. This may not be the case so the assumption of this existence is quite a 'courageous' statement. Yet, it is no more courageous than the assumption that the field equations (i.e., the union of the balance laws and the constitutive equations together) possess a solution. Such an assumption is needed when exploiting the entropy principle.

order to incorporate this property, we prescribe $\bar{\boldsymbol{\Phi}}_s$ to have a hypo-plastic structure. Such a procedure promises success, so we require

$$\bar{\boldsymbol{\Phi}}_s = \hat{\boldsymbol{\Phi}}_s \left(\cdot, \bar{\mathbf{Z}}_s, \mathbf{D}_s \right) \tag{46}$$

to be positively homogenous of the first degree in $\bar{\mathbf{Z}}_s$ and $\mathbf{D}_s$. We also decompose $\bar{\boldsymbol{\Phi}}_s$ into an operator which is linear in $\mathbf{D}_s$ and another one which is non-linear in $\mathbf{D}_s$, i.e.

$$\bar{\boldsymbol{\Phi}}_s = \mathcal{L} \left(\cdot, \bar{\mathbf{Z}}_s, \mathbf{D}_s \right) + \mathcal{N} \left(\cdot, \bar{\mathbf{Z}}_s, \mathbf{D}_s \right) \ . \tag{47}$$

Following standard hypoplasticity, we now assume the representation

$$\bar{\boldsymbol{\Phi}}_s = \mathrm{f}_1(\cdot) \left(\mathbf{L} \left(\bar{\mathbf{Z}}_s \right) \mathbf{D}_s + \mathrm{f}_2(\cdot) \mathbf{N} \left(\bar{\mathbf{Z}}_s \right) |\mathbf{D}_s| \right) \ , \tag{48}$$

where the norm of $\mathbf{D}_s$ is defined as

$$|\mathbf{D}_s| := \sqrt{\mathrm{tr}(\mathbf{D}_s^2)} \ , \tag{49}$$

and f_1 and f_2 are the coefficients of *barotropy* and *pyknotropy*, which may depend on the variables $\mathbb{S}$. The tensors $\mathbf{L}$ and $\mathbf{N}$ are of fourth and second order, respectively. Representation (48) satisfies automatically the requirement of positive homogeneity in $\mathbf{D}_s$. If we require homogeneity of $\bar{\boldsymbol{\Phi}}_s$ with respect to $\bar{\mathbf{Z}}_s$, (45) can be reduced to the form

$$\overset{\circ}{\bar{\mathbf{T}}}_{\mathrm{fric}} + v_s \frac{\rho_s - \rho_f}{\rho} \ (\nabla \cdot \mathbf{v}_s) \ \bar{\mathbf{T}}_{\mathrm{fric}}$$
$$= \mathrm{f}_1(\cdot) \left(\mathbf{L} \left(\bar{\mathbf{T}}_{\mathrm{fric}} \right) \mathbf{D}_s + \mathrm{f}_2(\cdot) \mathbf{N} \left(\bar{\mathbf{T}}_{\mathrm{fric}} \right) |\mathbf{D}_s| \right). \tag{50}$$

This representation of our hypo-plastic evolution law is close to that postulated by Wu and Kolymbas [46]. The differences are due to the binary mixture, and an additional term bilinear in $\nabla \cdot \mathbf{v}_s$ and $\bar{\mathbf{T}}_{\mathrm{fric}}$. However, for $\rho_f = 0$, agreeing with the dry granular case, (50) reduces to the form previously derived by Svendsen et al. [40]. With the result (50), we could be satisfied and state that thermodynamic considerations suggest that the evolution equation for the frictional stress of classical hypoplasticity must be changed by adding on the left-hand side the second term on the left-hand side of (50). However, this is not the only way to resolve the difference with classical hypoplasticity. By using an idea of Teufel [41], we define the new objective time derivative

$$\overset{\diamond}{\bar{\mathbf{Z}}}_s := \frac{\mathrm{d}^s \bar{\mathbf{Z}}_s}{\mathrm{d}t} - \left[\boldsymbol{\Omega} \ , \ \bar{\mathbf{Z}}_s \right] - v_s \frac{(\rho_s - \rho_f)}{\rho} \ (\nabla \cdot \mathbf{v}_s) \bar{\mathbf{Z}}_s \ , \tag{51}$$

which in view of (40) immediately leads to a form of the hypo-plastic stress evolution equation agreeing with that of Wu and Kolymbas [46], i.e.

$$\overset{\circ}{\bar{\mathbf{T}}}_{\mathrm{fric}} = \mathrm{f}_1(\cdot) \left(\mathbf{L} \left(\bar{\mathbf{T}}_{\mathrm{fric}} \right) \mathbf{D}_s + \mathrm{f}_2(\cdot) \mathbf{N} \left(\bar{\mathbf{T}}_{\mathrm{fric}} \right) |\mathbf{D}_s| \right) \ . \tag{52}$$

Obviously, and importantly to recognize, *the new objective time derivative does change the above thermodynamic analysis, but only the result for the solid equilibrium* Cauchy *stress tensor is affected by these changes*. The incorporation of

$$\overset{\diamond}{\bar{\mathbf{Z}}}_s = \bar{\boldsymbol{\Phi}}_s \ , \tag{53}$$

instead of (42), into the inequality (63) of Part I leads to

$$\bar{\mathbf{T}}_s|_E = -\bar{\varpi}_s \mathbf{I} + 2\rho \, \mathrm{sym}(\Psi^G_{I,\mathbf{B}_s}) \mathbf{B}_s + \rho \Psi^G_{I,\bar{\mathbf{Z}}_s} (\bar{\boldsymbol{\Phi}}_s),_{\mathbf{D}_s} |_E + v_s(\rho_s - \rho_f) \left(\Psi^G_{I,\bar{\mathbf{Z}}_s} \cdot \bar{\mathbf{Z}}_s \right) \mathbf{I} \tag{54}$$

rather than (19). The fact that the last term in (54) is spherical allows its incorporation into $\bar{\varpi}_s$, which, therefore, has the form

$$\bar{\varpi}_s = v_s \left(\beta_s - \rho_f \Psi^G_I + \varsigma \right) - v_s(\rho_s - \rho_f) \left(\Psi^G_{I,\bar{\mathbf{Z}}_s} \cdot \bar{\mathbf{Z}}_s \right) - \theta k^v_s \ . \tag{55}$$

Thus, by changing $\overset{\circ}{\bar{\mathbf{Z}}}_{\mathrm{s}}$ to $\overset{\circ}{\bar{\mathbf{Z}}}_{\mathrm{s}}$ an additional contribution to the solid pressure arises. This pressure contains contributions from the configration pressure, saturation pressure, the free energy Ψ_I^G and extra entropy flux $\mathbf{k}$.

The obvious drawback of this hypo-plastic approach is the lack of differentiability of $\bar{\boldsymbol{\Phi}}_{\mathrm{s}}$ at $\mathbf{D}_{\mathrm{s}} = \mathbf{0}$ and, therefore, the singularity of $\bar{\mathbf{T}}_{\mathrm{fric}}$ in thermodynamic equilibrium (see (23)). To circumvent this situation, Svendsen et al. [40] proposed a so-called *non-standard analysis* which for the purpose here is too complicated. We may try to regularize the problem by replacing $(\mathbf{D}_{\mathrm{s}}/|\mathbf{D}_{\mathrm{s}}|)$ which arises in $(\bar{\boldsymbol{\Phi}}_{\mathrm{s}})_{,\mathbf{D}_{\mathrm{s}}}$ by[6]

$$\frac{\mathbf{D}_{\mathrm{s}}}{\varepsilon + |\mathbf{D}_{\mathrm{s}}|}, \qquad 0 < \varepsilon \ll 1 \,. \tag{56}$$

Regularizing the problem in such a way has the advantage that the limit

$$\lim_{\mathbf{D}_{\mathrm{s}} \to 0} \frac{\mathbf{D}_{\mathrm{s}}}{\varepsilon + |\mathbf{D}_{\mathrm{s}}|} = \mathbf{0} \tag{57}$$

is finite, in fact zero, but, on the other hand, this procedure contradicts the requirement that $\bar{\boldsymbol{\Phi}}_{\mathrm{s}}$ is positively homogenous of first order in $\mathbf{D}_{\mathrm{s}}$ and consequently, $\bar{\mathbf{T}}_{\mathrm{fric}}$ does not have a rate-independent part. Nevertheless, we are convinced that for very small values of ε the term $\{\mathbf{D}_{\mathrm{s}}/(\varepsilon + |\mathbf{D}_{\mathrm{s}}|)\}$ is only affected by ε in the vicinity of $\mathbf{D}_{\mathrm{s}} = \mathbf{0}$. For rapid motions, i.e. steep velocity gradients and thus large values of $|\mathbf{D}_{\mathrm{s}}|$, ε is negligibly small. However, with the introduction of (56) the equilibrium stress (54) of the solid no longer contains the frictional contribution, because this term now vanishes in equilibrium. This means that the equilibrium stresses will now have to be carried by the pressure like contributions and, above these, the elastic stresses. This may be somewhat unrealistic, but it is so only in a very small regime.

As an alternative method of regularization, we may apply the following approach: when starting from a state of rest, at which the strains and stresses must first be determined, an initial value problem of the quasi-static equations using the stress representation (54) without the original frictional term replaced by the frictional term with the regularization (56) is integrated in time. As soon as $|\mathbf{D}_{\mathrm{s}}|$ has reached the value $10^n \times \varepsilon$, where n can be selected $(1 < n \leqslant 2)$, the actual value of $(\mathbf{D}_{\mathrm{s}}/|\mathbf{D}_{\mathrm{s}}|)$ is assigned to the equilibrium frictional stress in (54). Computations are then continued with the classical hypo-plastic equations. On the other hand, for a decelerating phase of the motion, the regularization (56) does not need to be introduced at all. If $|\mathbf{D}_{\mathrm{s}}|$ reaches the value ε from above, we then may simply maintain this limiting value $(\mathbf{D}_{\mathrm{s}}/|\mathbf{D}_{\mathrm{s}}|)$ also for smaller values of $|\mathbf{D}_{\mathrm{s}}|$ down to $|\mathbf{D}_{\mathrm{s}}| = 0$ (essentially locking it to the equilibrium). This then defines the equilibrium value for the stress according to (54).[7] Reloading phases of a dynamical process can then be started from this 'frozen' equilibrium state. This procedure corresponds to the approach of non-standard analysis. Another method is based on convex analysis and the calculus of variations as briefly outlined by Svendsen in Schneider and Hutter [38].

For the modelling of $\bar{\mathbf{T}}_{\mathrm{fric}}$ there still remains the specification of $\mathbf{L}$ and $\mathbf{N}$. In general, both tensors are allowed to depend on the following set of constitutive variables

$$\left\{ \nu_{\mathrm{s}}, \ \nabla\nu_{\mathrm{s}}, \ \mathbf{v}_{\mathrm{fs}}, \ \mathbf{B}_{\mathrm{s}}, \ \mathbf{B}_{\mathrm{f}}, \ \mathbf{W}_{\mathrm{fs}}, \ \mathbf{D}_{\mathrm{f}}, \ \bar{\mathbf{Z}}_{\mathrm{s}} \right\}\,, \tag{58}$$

but considering all these variables leads to very complex isotropic representations of $\mathbf{L}$ and $\mathbf{N}$. Therefore, we here adopt the 'principle of phase separation', and abandon those quantities which are related, (i) to the interaction of the constituents, (ii) to the mixture and (iii) to the fluid constituent, i.e. $\mathbf{v}_{\mathrm{fs}}$, $\mathbf{W}_{\mathrm{fs}}$, $\mathbf{B}_{\mathrm{f}}$ and $\mathbf{D}_{\mathrm{f}}$. In the hypo-plastic single-material theory of Svendsen et al. [40], $\boldsymbol{\Phi}$ is assumed to depend only on the set $\{\mathbf{B}, \ \mathbf{Z}, \ \mathbf{D}\}$. In the present model we are left with the equivalent quantities $\mathbf{B}_{\mathrm{s}}$, $\bar{\mathbf{Z}}_{\mathrm{s}}$ and $\mathbf{D}_{\mathrm{s}}$, but owing to the mixture character of the model the quantities ν_{s} and $\nabla\nu_{\mathrm{s}}$ should also arise. To disregard the latter contributions is hardly feasible and, therefore, if we want to use the representations for $\mathbf{L}$ and $\mathbf{N}$ proposed in the literature for single-body hypo-plasticity,[8] their adaption is necessary. The existing recent literature on hypo-plastic constitutive modelling and parameter identification for special choices of the operators $\mathbf{L}$ and $\mathbf{N}$ clearly point at a dominant role played by the void ratio $e = (1 - \nu_{\mathrm{s}})/\nu_{\mathrm{s}}$.

As we are presently not dealing with specific problems, we leave the choice of $\mathbf{L}$ and $\mathbf{N}$ open, but draw the reader's attention to the footnote below.

---

[6] Cf. Fang, Wang, Hutter [15,14], however, the regularization proposal (56) is well known in the rheological literature.

[7] This procedure is obviously 'mesh' dependent, the mesh being given by ε.

[8] Cf. Svendsen et al. [40], Kolymbas [23]–[24], Niemunis [32], von Wolffersdorff [43], Bauer [3,4], Masin [26], Wu [45], Wu and Kolymbas [46], Chambon [7]–[6], Darve [9,10].

3.3 Final equilibrium constitutive laws

We now substitute the representations found for the solid elastic stress (37), for the frictional stress (40) and the expressions (15), (16) for the solid and fluid pressures into (12), (13) and (14). The result of this process may be written as follows:

$$
\begin{aligned}
\bar{\mathbf{T}}_s|_E &= -\bar{\varpi}_s \mathbf{I} + \mu \mathbf{B}_s + \rho \delta \bar{\mathbf{Z}}_s \,, \\
\bar{\mathbf{T}}_f|_E &= -\bar{\varpi}_f \mathbf{I} \,, \\
\bar{\mathbf{m}}_s^i|_E &= \left\{ \beta_s^G \left(1 - \bar{\xi}_s\right) - \rho_f \Psi_I^G + \varsigma - \theta \left(k_s^v\right),_{\nu_s} \right\} \nabla \nu_s \\
&\quad + \rho \left(\Psi_I^G\right),_{\bar{\mathbf{Z}}_s} \left(\bar{\boldsymbol{\Phi}}_s\right),_{\mathbf{v}_s}|_E \\
&= -\bar{\mathbf{m}}_f^i|_E \,,
\end{aligned}
\tag{59}
$$

with

$$
\begin{aligned}
\bar{\varpi}_s &= \nu_s \left(\beta_s^G - \rho_f \Psi_I^G + \varsigma\right) - \theta k_s^v - \nu_s(\rho_s - \rho_f) \left(\Psi_{I,\bar{\mathbf{Z}}_s}^G \cdot \bar{\mathbf{Z}}_s\right) \,, \\
\bar{\varpi}_f &= (1 - \nu_s) \left(-\rho_f \Psi_I^G + \varsigma\right) - \theta k_f^v \,.
\end{aligned}
\tag{60}
$$

Here, $\bar{\mathbf{Z}}_s$ satisfies the evolution equation (53). We remark that with (53), (59) and (60), classical hypoplasticity is still not exactly recovered, because of the additional frictional pressure in (60). Presently it is still not clear, which of the two slightly different versions of hypoplasticity are to be preferred. Formula (60) is interesting by the fact how friction contributes to the total pressure. If the solid and fluid densities are the same ($\rho_f = \rho_s$), then the last term of $(60)_1$ obviously vanishes. In this case the solid is completely buoyant in the fluid and friction is expected to be minimal—in the equation (60) zero. In a dry granular material ($\rho_f = 0$), rubbing friction operates and the frictional pressure contribution is proportional to the partial density $\bar{\rho}_s = \nu_s \rho_s$. This form is adequate for a solid body with voids, ($\nu_s < 1$), or without voids ($\nu_s = 1$). These results appear to be reasonable.

4 Non-equilibrium contributions

So far, the findings were based on rather strong assumptions. Unfortunately, only a few rational arguments exist that allow the construction of reasonable constitutive laws for the quantities [9] $\bar{\mathbf{T}}_s|_N$, $\bar{\mathbf{T}}_f|_N$ and $\bar{\mathbf{m}}_s^i|_N$.

4.1 Non-linear viscous solid and fluid stresses

We recall the dependencies of the constitutive quantities $\mathbf{C} = \{\bar{\mathbf{T}}_s, \bar{\mathbf{T}}_f, \bar{\mathbf{m}}_s^i\}$ as listed in (8). Applying, now, the 'principle of phase separation', to $\bar{\mathbf{T}}_s|_N$ and $\bar{\mathbf{T}}_f|_N$, these dependencies reduce to

$$
\bar{\mathbf{T}}_s|_N = \hat{\bar{\mathbf{T}}}_s|_N \left(\nu_s, \ \nabla \nu_s, \ \mathbf{B}_s, \ \mathbf{D}_s, \ \bar{\mathbf{Z}}_s\right),
\tag{61}
$$

$$
\bar{\mathbf{T}}_f|_N = \hat{\bar{\mathbf{T}}}_f|_N \left(\nu_f, \ \mathbf{B}_f, \ \mathbf{D}_f\right) \,.
\tag{62}
$$

The isotropic representations for $\bar{\mathbf{T}}_s|_N$ and $\bar{\mathbf{T}}_f|_N$ are still very complex and, therefore, we shall neglect dependencies on $\mathbf{B}_s$, $\bar{\mathbf{Z}}_s$ and $\mathbf{B}_f$. Doing so, we are indeed loosing information, but as $\mathbf{B}_s$ and $\bar{\mathbf{Z}}_s$ affect the 'equilibrium' constitutive laws, their information is automatically carried over to non-equilibrium processes. The 'principle of phase separation' makes only sense in connection with constituent-specific constitutive quantities. Interaction supply rate densities, such as that for the solid momentum interaction term $\bar{\mathbf{m}}_s^i$, are by definition excluded

---

[9] Note the subscript $(\cdot)|_N$ denotes non-equilibrium contributions, whilst the superscript $(\cdot)^N$ indicates a general non-linear expression.

from the application of this principle. Under these restrictive assumptions, the isotropic representations of the two non-equilibrium Cauchy stress tensors (61) and (62) take the forms

$$
\begin{aligned}
\bar{\mathbf{T}}_s|_N = \ & \kappa_1^s \mathrm{I}_{\mathbf{D}_s} \mathbf{I} + \kappa_2^s \mathbf{D}_s + \kappa_3^s \mathbf{D}_s^2 \\
& + \kappa_4^s \mathbf{M}_s + \kappa_5^s \, \mathrm{sym} \, (\mathbf{M}_s \mathbf{D}_s) + \kappa_6^s \, \mathrm{sym} \, (\mathbf{M}_s \mathbf{D}_s^2) \, ,
\end{aligned}
\tag{63}
$$

$$
\bar{\mathbf{T}}_f|_N = \ \kappa_1^f \mathrm{I}_{\mathbf{D}_f} \mathbf{I} + \kappa_2^f \mathbf{D}_f + \kappa_3^f \mathbf{D}_f^2 \, ,
\tag{64}
$$

where

$$
\mathbf{M}_s := \nabla \nu_s \otimes \nabla \nu_s \, .
\tag{65}
$$

The coefficients κ_{1-6}^s are functions of ν_s, $\mathrm{I}_{\mathbf{D}_s}$, $\mathrm{II}_{\mathbf{D}_s}$, $\mathrm{III}_{\mathbf{D}_s}$, $\mathrm{I}_{\mathbf{M}_s}$, $\mathrm{I}_{\mathbf{M}_s \mathbf{D}_s}$ and $\mathrm{I}_{\mathbf{M}_s \mathbf{D}_s^2}$. Moreover, (63), (64) are complete isotropic representations of (61) and (62); for a proof see Schneider and Hutter [38]. Now, from the definition of thermodynamic equilibrium we have

$$
\lim_{n \to 0} \bar{\mathbf{T}}_s|_N = \mathbf{0} \, ,
\tag{66}
$$

and the fact that $\nabla \nu_s$ is a quantity which does not necessarily vanish in equilibrium, implies

$$
\kappa_4^s|_E = 0 \qquad \rightarrow \qquad \kappa_4^s = \kappa_4^s|_N
\tag{67}
$$

The above representations (63–67) are due to Hutter and Rajagopal [20].

We now introduce simplifications:

Following Hutter et al. [19] we assume, that $\bar{\mathbf{m}}_s^i|_N$ and $\bar{\mathbf{T}}_f|_N$ can be adequately modelled by their strict linear forms and $\bar{\mathbf{T}}_s|_N$ by its quasi-linear form. In addition, we adopt the assumption that $\bar{\mathbf{T}}_s|_N$ and $\bar{\mathbf{m}}_s^i|_N$ are independent of the variable $\nabla \nu_s$. From a mathematical point of view there is no obvious reason for this assumption but at least for $\bar{\mathbf{m}}_s^i$ the information contained in $\nabla \nu_s$ is not entirely lost because its equilibrium part depends linearly on $\nabla \nu_s$. The equations that evolve from these considerations read as follows

$$
\bar{\mathbf{T}}_s|_N = \ \kappa_1^s \mathrm{I}_{\mathbf{D}_s} \mathbf{I} + \kappa_2^s \mathbf{D}_s \, , \quad \bar{\mathbf{T}}_f|_N = \ \kappa_1^f \mathrm{I}_{\mathbf{D}_f} \mathbf{I} + \kappa_2^f \mathbf{D}_f \, , \quad \bar{\mathbf{m}}_s^i|_N = \ m_{\mathbf{D}} \mathbf{v}_{fs} \, ,
\tag{68}
$$

where κ_1^s and κ_2^s are, in general, functions of ν_s, $\mathrm{I}_{\mathbf{D}_s}$, $\mathrm{II}_{\mathbf{D}_s}$ and $\mathrm{III}_{\mathbf{D}_s}$, whilst κ_1^f and κ_2^f depend on $\nu_f = (1 - \nu_s)$, and $m_{\mathbf{D}}$ is a function of ν_s. An explicit dependence on $\mathbf{M}_s$ has dropped out entirely from (68).

The quasi-linearity of the solid Cauchy stress tensor which is expressed through the non-linear dependence of κ_1^s and κ_2^s on $\mathbf{D}_s$ reflects the strong non-linear stress-stretching behaviour that arises during creep or rapid shear of the granular part of the debris flow (cf. Hutter et al. [19]). By excluding the dependence upon $\mathbf{D}^2$, the proposal (68) is not capable of modelling normal stress effects. These effects are not likely important in rapid granular flows and only come to bear when strong decelerations in the approach to the deposition are active. However, the above model properties for $\bar{\mathbf{T}}_s|_N$ are in agreement with Bagnold's experiments (cf. Hutter and Rajagopal [20]).

Rheologically, the solid and fluid stresses (68)$_{1,2}$ are better written as

$$
\begin{aligned}
\bar{\mathbf{T}}_f|_N = \ & \kappa_f \mathrm{I}_{\mathbf{D}_f} \mathbf{I} + 2\mu_f \mathbf{D}_f' \, , \\
& \kappa_f := \kappa_1^f + \tfrac{1}{3}\kappa_2^f \, , \quad \mu_f := \tfrac{1}{2}\kappa_2^f \, ,
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
\bar{\mathbf{T}}_s|_N = \ & \kappa_s \mathrm{I}_{\mathbf{D}_s} \mathbf{I} + 2\mu_s \mathbf{D}_s' \, , \\
& \kappa_s := \kappa_1^s + \tfrac{1}{3}\kappa_2^s \, , \quad \mu_s := \tfrac{1}{2}\kappa_2^s \, ,
\end{aligned}
\tag{70}
$$

where

$$
\mathbf{D}_f' = \left\{ \mathbf{D}_f - \tfrac{1}{3}\mathrm{I}_{\mathbf{D}_f} \mathbf{I} \right\} , \qquad \mathbf{D}_s' = \left\{ \mathbf{D}_s - \tfrac{1}{3}\mathrm{I}_{\mathbf{D}_s} \mathbf{I} \right\}
\tag{71}
$$

or alternatively for the solid

$$
\begin{aligned}
\mathbf{D}_s = \ & \mathrm{K}_s \mathrm{I}_{\bar{\mathbf{T}}_s|_N} \mathbf{I} + \mathrm{B}_s \left(\bar{\mathbf{T}}_s|_N \right)' \, , \\
& \text{with} \quad \left(\bar{\mathbf{T}}_s|_N \right)' = \left\{ \bar{\mathbf{T}}_s - \tfrac{1}{3}\mathrm{I}_{\bar{\mathbf{T}}_s} \mathbf{I} \right\} |_N \, .
\end{aligned}
\tag{72}
$$

The coefficients $\kappa_{f,s}$, $\mu_{f,s}$ are reminiscent of bulk and shear viscosities, K_s and B_s of bulk and shear fluidities. They are, in general functions of ν_s and the invariants $\mathrm{I}_{\mathbf{D}_{s,f}}$, $\mathrm{II}_{\mathbf{D}_{s,f}}$ and $\mathrm{III}_{\mathbf{D}_{s,f}}$ (respectively, $\mathrm{I}_{\bar{\mathbf{T}}_s|_N}$, $\mathrm{II}_{\bar{\mathbf{T}}_s|_N}$ and $\mathrm{III}_{\bar{\mathbf{T}}_s|_N}$). We now introduce

Postulate 1 The constitutive parameters in the viscous laws (69), (70), (72) obey the principle of phase separation. Moreover,

- the bulk quantities $\kappa_{f,s}$, K_s depend on $\nu_{f,s}$ and only the first invariants $I_{D_{f,s}}$, $I_{\bar{T}_{s|N}}$;
- the shearing coefficients $\mu_{f,s}$ and B_s depend on ν_s and the deviator invariants $II_{D'_{s,f}}$, $II_{(\bar{T}_{f,s|N})'}$, $III_{D'_{s,f}}$, $III_{(\bar{T}_{f,s|N})'}$, respectively.

Hence,

$$\kappa_{f,s} = \hat{\kappa}_{f,s}(\nu_s, I_{D_{f,s}}) \,,\; \mu_{f,s} = \hat{\mu}_{f,s}(\nu_s, II_{D'_{f,s}}, III_{D'_{f,s}}) \,,$$
$$K_s = \hat{K}_s(\nu_s, I_{T_{s|N}}) \,,\quad B_s = \hat{B}_s(\nu_s, II_{\bar{T}'_{s|N}}, III_{\bar{T}'_{s|N}}) \,,$$

(73)

□

It can be shown that full triaxial experiments are needed in order that a dependence of the parameters (73) on the second *and* third invariants can be experimentally identified. Such identifications have so far not been done for soil. Simpler propositions for the above viscosity coefficients can be made with the following viscometric gedanken experiments.

4.2 Viscous stress parameterization based on viscometric considerations

(i) *Isotropic extension-compression* In the laboratory such an experiment is not difficult to perform, but it may be very hard to deduce inferences for the identification of the bulk viscosities $\kappa_{s,f}$. Therefore, this case is rather treated as a Gedanken experiment. We shall treat the fluid as volume (and density) preserving and set $\kappa_f \equiv 0$. For the solid a drained compression experiment is thought to be conducted. With

$$\mathbf{D}_s = \dot{\varepsilon}\mathbf{1}, \quad \mathbf{D}'_s = \mathbf{0}, \quad I_{\mathbf{D}_s} = 3\dot{\varepsilon}, \quad II_{\mathbf{D}'_s} = 0, \quad III_{\mathbf{D}'_s} = 0 \,, \tag{74}$$

and $\bar{\mathbf{T}}_s|_N = \sigma\mathbf{1}$, one deduces from (70) that

$$\sigma = \kappa_s\,(\nu_s,\, 3\dot{\varepsilon})\, 3\dot{\varepsilon} \,, \tag{75}$$

or with $\dot{\varepsilon}_{vol} := 3\dot{\varepsilon}$,

$$\kappa_s\,(\nu_s,\, \dot{\varepsilon}_{vol}) = \frac{\sigma}{\dot{\varepsilon}_{vol}} \,. \tag{76}$$

It should be clear that in the performance of this isotropic compression experiment ν_s cannot be assumed to remain constant. As $(-\dot{\varepsilon}_{vol})$ increases in a compression experiment, the compaction of the grains will also increase. It follows that equation (76) is only meaningful as long as volumetric strains remain small. We now introduce[10]

Postulate 2 (i) κ_s does not depend on $\dot{\varepsilon}_{vol}$,
  (ii) at densest packing, $\nu_s = \nu_{s\,max}$,
 (iii) when $\nu_s \leqslant \nu_{s\,crit}$, $\kappa_s = \boldsymbol{\kappa}_s$.

□

With this postulate and the assumption that κ is monotonically decreasing with growing ν_s reading the zero value at $\nu_{s\,max}$, we may parameterize κ_s as follows:

$$\kappa_s = \boldsymbol{\kappa}_s f_s^{(\kappa_s)}(\nu_{scale}) \,, \quad \nu_{scale} = \left(\frac{\nu_s - \nu_{s\,crit}}{\nu_{s\,max} - \nu_s}\right) \,, \tag{77}$$

where the shape of the function $f_s^{(\kappa_s)}$ is given in Fig. 1. According to this graph, the bulk viscosity vanishes at densest packing and assumes the value $\boldsymbol{\kappa}_s$ at the critical packing and beyond, when $\nu_s \leqslant \nu_{s\,crit}$. It is further assumed that the value of κ_s stays constant for dilute concentrations. Apart from these assumptions the graph

---

[10] $\nu_{s\,crit}$ is the solid volume fraction at which the nominal particle distance is larger than, or equal to, the distance at which the particle contact ceases to exist. $\nu_s = \nu_{s\,max}$ is the maximum solid volume fraction
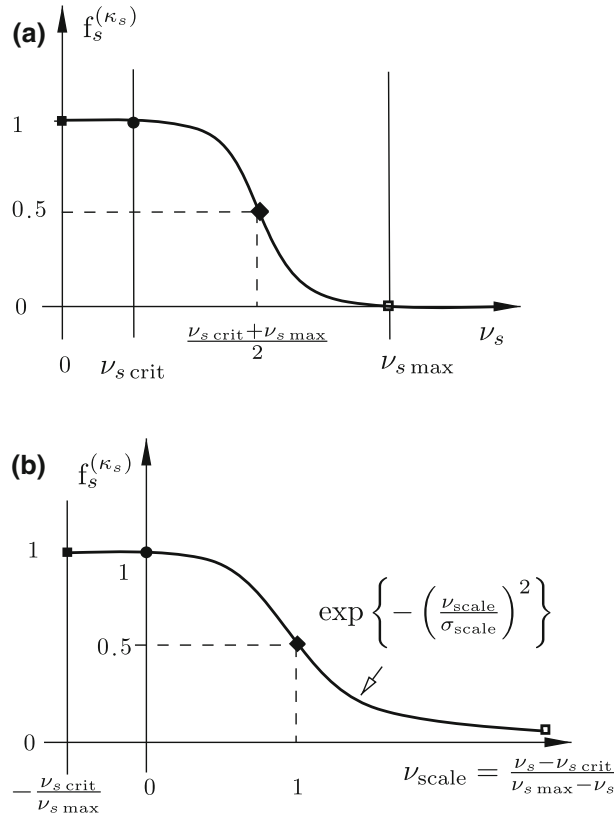
Fig. 1 a Dependence of the scaled solid bulk viscosity as a function of the solid volume fraction and **b** as parameterized in (78) with $\sigma_{\text{scale}} = (1/\ln 2)^{(1/2)} = 1.2011$

in Fig. 1a simply connects these limiting stages with a smooth curve. The function $f_s^{(\kappa_s)}$ is of sigmoidal type, and may, for instance, be written as

$$f_s^{(\kappa_s)} = \begin{cases} 1 , & -\frac{\nu_{s\,\text{crit}}}{\nu_{s\,\text{max}}} \leqslant \nu_{\text{scale}} \leqslant 0 , \\ \exp\left[-\left(\frac{\nu_{\text{scale}}}{\sigma_{\text{scale}}}\right)^2\right] , & 0 \qquad \leqslant \nu_{\text{scale}} \leqslant \infty. \end{cases} \tag{78}$$

If the value of $f_s^{(\kappa_s)}(\nu_{\text{scale}} = 1)$ is given by f_{mean}, then

$$\sigma_{\text{scale}} = \left(\frac{-1}{\ln(f_{\text{mean}})}\right)^{1/2} . \tag{79}$$

A concrete identification would consist in matching the graph of Fig. 1 with experimental results. As far as relations (77) and (78) are concerned, $\nu_{s\,\text{max}}$ and $\nu_{s\,\text{crit}}$ must be identified, which is not difficult, and κ_s must be determined, which may be more difficult and may require (semi) inverse modelling. The following first estimates are suggested:

$$\left.\begin{array}{l} \nu_{s\,\text{max}} = 0.75 \\ \nu_{s\,\text{crit}} = 0.20 \\ \kappa_s = 10^{-3} \text{ Pa s} \end{array}\right\} \text{ only first estimates.}$$
$$\sigma_{\text{scale}} = 1.20 \tag{80}$$

The reader may question the choice that $f_s^{(\kappa_s)}$ vanishes for $\nu_s = \nu_{s\,\text{max}}$. The idea behind this limiting value is the recognition that the deformation is strongly decelerating as ν_s is approaching $\nu_{s\,\text{max}}$ and frictional stress

is becoming dominant in the regime of maximum solid contact. Of course, the above parameterization leaves room also for e.g. a generalization

$$f_s^{(\kappa_s)} = \left\{ \exp\left[-\left(\frac{\nu_{\text{scale}}}{\sigma_{\text{scale}}} \right)^2 \right] + \kappa_0 \right\} \qquad \text{for} \qquad 0 \leq \nu_{\text{scale}} \leq \infty \tag{81}$$

(ii) *Simple shearing*   The rheologically most popular and probably simplest experiment is viscometric shearing e.g. in an axi-symmetric cone-plate viscometer. We consider an experiment being conducted for the fluid [11] and solid in isolation. With

$$\mathbf{D}_{\text{f,s}} = \mathbf{D}'_{\text{f,s}} = \begin{pmatrix} 0 & \frac{1}{2}\dot{\gamma} & 0 \\ \frac{1}{2}\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{82}$$

for the fluid and the solid, one deduces

$$\text{I}_{\mathbf{D}_{\text{f,s}}} = 0, \quad \text{II}_{\mathbf{D}'_{\text{f,s}}} = \frac{\dot{\gamma}^2}{4}, \quad \text{III}_{\mathbf{D}'_{\text{f,s}}} \equiv 0 , \tag{83}$$

and then obtains with

$$\bar{\mathbf{T}}_{\text{f,s}}|_{\text{N}} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{84}$$

and (69), (70)

$$\mu_{\text{f,s}}(\nu_s, \frac{\dot{\gamma}^2}{4}, 0) = \frac{\tau}{\dot{\gamma}} . \tag{85}$$

Monitoring τ and measuring $\dot{\gamma}$ allows identification of the functions $\mu_{\text{f,s}}$ in (85). It is obvious from the above formulae (83)$_3$ and (85) that simple shearing experiments cannot identify a III$_{\mathbf{D}_{\text{s,f}}}$-dependence of the viscosity functions $\mu_{\text{s,f}}(\cdot)$. Applied rheologists, therefore, generally omit the third variable, III$_{\mathbf{D}'_{\text{f,s}}}$, and also do not make the ν_s dependence explicit. Instead, we shall use the following

Postulate 3   (i)   The shear viscosity functions $\mu_{\text{f,s}}(\nu_s, \text{II}_{\mathbf{D}'_{\text{f,s}}}, \text{III}_{\mathbf{D}'_{\text{f,s}}})$ allow the product decomposition

$$\mu_{\text{f,s}} = \text{M}_{\text{f,s}}(\text{III}_{\mathbf{D}'_{\text{f,s}}}) \bar{\eta}_{\text{f,s}}(\nu_s, \text{II}_{\mathbf{D}'_{\text{f,s}}}) , \tag{86}$$

with a first estimate $\text{M}_{\text{f,s}}(\text{III}_{\mathbf{D}'_{\text{f,s}}}) = 1$.

(ii)   The solid volume fraction as a variable enters only the functions $\bar{\eta}_{\text{f,s}}(\cdot)$. This dependence may again be separated from that of $\text{II}_{\mathbf{D}'_{\text{f,s}}}$ as follows

$$\bar{\eta}_{\text{f,s}} = g_{\text{f,s}}(\nu_s) \bar{\bar{\eta}}_{\text{f,s}}(\text{II}_{\mathbf{D}'_{\text{f,s}}}) , \tag{87}$$

or the coefficients in the parameterizations may be assumed to be ν_s-dependent.

□

With this postulate, we may identify the functions $\eta_{\text{f,s}}(\cdot)$ for a fixed solid volume fraction, formally treated to have a reference value.

The literature dealing with shear viscosity functions (they will be called here simply 'viscosity functions') is abundant; justification for all the proposals is not possible. Therefore, we restrict here considerations to what is referred to as *viscometry of fluids with yield stress*. These have been in the past few years the concern of many rheologists. Our attention here is to propose formulae, which embrace possibly all cases that may occur

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[11] Most likely, the fluid in a debris flow will be loaded with silt to clay components of the debris that extends over a large range of particle diameters. Therefore, the fluid is not pure water, but a slurry with a certain concentration of the fine particles.

in debris flow modelling, so that only the identification of the parameters in specific situations is left to the user. With the above postulate, and in order to be in conformity with the rheological literature, we shall write

$$\tau_{f,s}^* = \bar{\bar{\eta}}_{f,s} \left(\frac{\dot{\gamma}^2}{4} \right) \dot{\gamma} = \eta_{f,s} \left(\dot{\gamma} \right) \dot{\gamma} \; , \tag{88}$$

where,

$$\tau_{f,s}^* = \frac{\tau}{M_{f,s} \left(III_{\mathbf{D}'_{f,s}} \right) g_{f,s}(\nu_s)} \tag{89}$$

is an appropriately scaled stress, i.e., $M_{f,s}$ and $g_{f,s}$ are dimensionless so that $\bar{\bar{\eta}}_{f,s}$ or $\eta_{f,s}$ has dimension [Pa s]. Popular viscous shear parameterizations are by Bingham [5] or Herschel and Bulkley [16], viz.,

$$\begin{cases} \dot{\gamma} = 0, & \text{if} \quad \tau \leqslant \tau_0 \; , \\ \tau = \left(k + \frac{\tau_0}{\dot{\gamma}} \right) \dot{\gamma}, & \text{if} \quad \tau \geqslant \tau_0 \; , \end{cases} \qquad \text{Bingham} \tag{90a}$$

$$\begin{cases} \dot{\gamma} = 0, & \text{if} \quad \tau \leqslant \tau_0 \; , \\ \tau = \left(k \, \dot{\gamma}^{\lambda-1} + \frac{\tau_0}{\dot{\gamma}} \right) \dot{\gamma}, & \text{if} \quad \tau \geqslant \tau_0 \; , \end{cases} \qquad \text{Herschel and Bulkley} \tag{90b}$$

but they have singular behaviour at $\dot{\gamma} \to 0$. Other parameterizations are reviewed by Mendes and Dutra [27] and Ancey [1]. Luca et al [25] proposed the shear viscosity function

$$\eta(\dot{\gamma}) = \eta_1 \exp\left(-t_1 \dot{\gamma} \right) + \frac{2}{\pi} \eta_2 \left(\frac{\dot{\gamma}}{\dot{\gamma}_r} \right)^{\lambda-1} \arctan\left(t_2 \left(\frac{\dot{\gamma}}{\dot{\gamma}_r} \right)^{\beta} \right) \frac{\tau_0}{\dot{\gamma}} \left(1 - \exp(-m\dot{\gamma}) \right) \; , \tag{91}$$

in which η_1 and η_2 are constant reference viscosities [Pa s], $\dot{\gamma}_r$ is a constant stretching [s^{-1}], τ_0 is the yield stress [Pa], t_1 and m are reference times [s] and λ, β and t_2 are dimensionless constants, for which numbers must be given subject to the following constraints:

$$\begin{aligned} \eta_1 > 0, \quad \eta_2 > 0, \quad \tau_0 \geqslant 0, \quad \lambda \in (0, \, 1) \, , \\ t_1 > 0, \quad t_2, \, m > 0, \quad \beta + \lambda - 2 > 0. \end{aligned} \tag{92}$$

These guarantee the limits

$$\lim_{\dot{\gamma} \to 0} \eta(\dot{\gamma}) = \eta_1 + \tau_0 m \neq \infty \; ,$$

$$\lim_{\dot{\gamma} \to 0} \eta'(\dot{\gamma}) = -\eta_1 t_1 - \tfrac{1}{2} \tau_0 m^2 = \text{finite} \; ,$$

$$\lim_{\dot{\gamma} \to \infty} \eta(\dot{\gamma}) \approx \eta_2 \left(\frac{\dot{\gamma}}{\dot{\gamma}_r} \right)^{\lambda-1} \; , \tag{93}$$

and the shear thinning properties prevail for at least large stretchings.[12] Model (91) includes the Bingham–Papanastasiou fluid [34], if $t_0 = 0$, $\eta_2 = 0$, $\tau_0 \neq 0$ and the model introduced by Zhu et al. [47] ($\eta_2 = 0$) as an extension of the De Kee and Turcotte [11] proposal.

In the above, we started from the Bingham and Herschel–Bulkley models as two popular models describing the stress-deformation response of a large class of visco-plastic fluids. The behaviour in these formulae is described by the yield stress, but this feature led to stress-stretching relations with slope discontinuities at zero stretching. They become manifest in the formulae as an infinite viscosity at zero stretching. A similar singularity also arises for the power law fluid ($\tau_0 = 0$) when $\lambda \in (0, 1)$. For a fluid with yield stress this singularity becomes physically apparent as an abrupt transition from the viscous fluid to rigid solid behaviour, which generates mathematical-numerical complexities which one wishes to avoid. The intention in the

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[12] From a purely practical point of view, it could also be criticized that the regularized final formulae (91–93) are overly complicated, and identification of the many parameters by experiments must be very difficult, if not impossible. However, since many parameters in the model (91) are introduced for regularization purposes, they need not be 'accurately' determined. Values can be estimated such that regularity is established and the graphs of the functions $\eta(\dot{\gamma})$ still mimic the experiments, which anyhow never allows inferences without errors, reasonably well.

improved viscosity proposal (91) is to smooth-out these singularities. However, in doing so, the plastic, rate-independent response is formally replaced by a viscous, rate-dependent response. In the context of the model equations in these articles, Parts I and II, such a 'viscofaction' is even a mandatory smoothing operation, since the non-equilibrium stresses for which the above parameterizations are presented must necessarily vanish in thermodynamic equilibrium. As we know from the modelling of the hypo-plastic stress parameterization, the rate independent parts do not vanish in thermodynamic equilibrium.

In the above formulae, a dependence of the viscosity parameters on solid volume fraction has not been made explicit, but there are indications that such dependencies exist. For instance, Ancey [1] reports work of Husband et al. [17] and others, who identified clear yielding behaviour in suspensions with solid-volume fractions $\nu_s \leqslant 0.47$. They observed that this yield stress increased dramatically when the solid concentration approached its densest packing. Ancey [1] mentions other supporting evidences for such a yield stress and quotes Wildemuth and Williams' [44] yield stress formula

$$\tau_0 = \left[A \frac{\nu_s/\nu_0 - 1}{1 - \nu_s/\nu_\infty} \right]^{1/m} , \tag{94}$$

in which A, ν_0, ν_∞ and m are parameters fitting their data. This demonstrates that yielding is associated with a size range $\nu_s \in (\nu_0, \ \nu_\infty)$ of solid volume fraction.

In slurries at moderate to small solid volume fraction, theoretical models predict Newtonian behaviour with viscosities whose value depends on the viscosity of the pure fluid, η_f^{pure}, and the solid volume fraction ν_s; this function is increasing with growing ν_s. Two famous formulae are

$$\begin{cases} \eta_f = \eta_f^{\text{pure}}(1 - 2.5\nu_s), & \text{Einstein [12,13],} \\ \eta_f = \eta_f^{\text{pure}}(1 - 2.5\nu_s - 7.6\nu_s^2), & \text{Batchelor and Green [2]} . \end{cases} \tag{95}$$

Ancey [1] quotes a general formula supposed to be adequate beyond small ν_s^2-terms,

$$\eta_f = \eta_f^{\text{pure}} \left(1 - \frac{\nu_s}{\nu_{s\,\text{max}}} \right)^{-2.5\nu_{s\,\text{max}}} . \tag{96}$$

Relation (96) matches the Einstein [12,13] relation at small ν_s. It becomes singular when $\nu_s \to \nu_{s\,\text{max}}$ which is certainly unphysical. Therefore, it can be only valid for ν_s sufficiently below $\nu_{s\,\text{max}}$. For values of ν_s close to $\nu_{s\,\text{max}}$, the yield stress will become important and the parameterization (91) should be used. For the latter, however, dependencies on the solid volume fraction have to our knowledge not been suggested.

Finally, it is emphasized that the parameterizations as suggested by rheometry are based on the relatively simple formulae (69), (70) and the application of these formulae to only two very special processes of isotropic extension/compression and simple shear. This does not permit identification of a third invariant dependence, as we have seen. In order to identify such possible dependencies, compound deformations consisting of shear and normal strains are needed. To conduct such experiments for dynamic situations must be very difficult. Moreover, it is also clear that (69), (70) do not include dynamic normal stress effects for which the quadratic $\mathbf{D}_{f,s}$-dependences in (63), (64) must not be dropped. Such arguments explain that non-equilibrium stress parameterizations of the class (63), (64) will keep debris flow modellers busy for a long time until a complete satisfactory parameterization is known.

5 Final constitutive laws

The reduced forms of the constitutive laws for the solid and fluid Cauchy stress tensors and the interaction supply rate density for the solid momentum now read

$$\bar{\mathbf{T}}_s = -\bar{\varpi}_s \mathbf{I} + \bar{\mathbf{T}}_{es}(\mathbf{B}_s) + \rho \delta \bar{\mathbf{Z}}_s + \lambda_s I_{\mathbf{D}_s} \mathbf{I} + 2\mu_s \mathbf{D}_s' ,$$
$$\bar{\boldsymbol{\Phi}}_s = f_1 \left(\mathbf{L} \left(\bar{\mathbf{Z}}_s \right) \mathbf{D}_s + f_2 \mathbf{N} \left(\bar{\mathbf{Z}}_s \right) |\mathbf{D}_s| \right) ,$$
$$\bar{\mathbf{T}}_f = -\bar{\varpi}_f \mathbf{I} \quad + \kappa_f I_{\mathbf{D}_f} \mathbf{I} + 2\mu_f \mathbf{D}_f',$$
$$\bar{\mathbf{m}}_s^i = \left\{ \beta_s^G \left(1 - \bar{\xi}_s \right) - \rho_f \Psi_I^G + \varsigma - \theta \left(k_s^v \right)_{,\nu_s} \right\} \nabla \nu_s$$
$$+ \rho \left(\Psi_I^G \right)_{,\bar{\mathbf{Z}}_s} \left(\bar{\boldsymbol{\Phi}}_s \right)_{,\mathbf{v}_s} |_E + m_{\mathbf{D}} \mathbf{v}_{fs}$$
$$= -\bar{\mathbf{m}}_f^i , \tag{97}$$

with

$$\begin{aligned}
\bar{\varpi}_{\mathrm{s}} &= \nu_{\mathrm{s}}\left(\beta_{\mathrm{s}}^{G} - \rho_{\mathrm{f}}\Psi_{I}^{G} + \varsigma\right) - \theta k_{\mathrm{s}}^{v} - \nu_{\mathrm{s}}(\rho_{\mathrm{s}} - \rho_{\mathrm{f}})\left(\Psi_{I,\bar{\mathbf{Z}}_{\mathrm{s}}}^{G} \cdot \bar{\mathbf{Z}}_{\mathrm{s}}\right), \\
\bar{\varpi}_{\mathrm{f}} &= (1 - \nu_{\mathrm{s}})\left(-\rho_{\mathrm{f}}\Psi_{I}^{G} + \varsigma\right) - \theta k_{\mathrm{f}}^{v},
\end{aligned} \tag{98}$$

and

$$\varpi = \bar{\varpi}_{\mathrm{s}} + \bar{\varpi}_{\mathrm{f}} = \nu_{\mathrm{s}}\beta_{\mathrm{s}}^{G} + \left(-\rho_{\mathrm{f}}\Psi_{I}^{G} + \varsigma\right) - \nu_{\mathrm{s}}(\rho_{\mathrm{s}} - \rho_{\mathrm{f}})\left(\Psi_{I,\bar{\mathbf{Z}}_{\mathrm{s}}}^{G} \cdot \bar{\mathbf{Z}}_{\mathrm{s}}\right) \tag{99}$$

for the spherical contribution to the mixture Cauchy stress tensor. $\bar{\mathbf{T}}_{\mathrm{es}}$ follows from any elastic potential $\bar{\Psi}_{\mathrm{es}}^{G}$, see (35), ff.

6 An alternative to the assumption of 'Pressure Equilibrium'

In order to analyse the assumption of 'pressure equilibrium', we start with the collection of balance equations for the reduced model. For isothermal conditions, density-preserving constituents, saturation and vanishing interaction supply rate densities for mass, these equations are

$$\begin{aligned}
\partial \nu_{\mathrm{s}} + \nabla \cdot (\nu_{\mathrm{s}}\mathbf{v}_{\mathrm{s}}) &= 0, \\
\partial \nu_{\mathrm{s}} - \nabla \cdot \mathbf{v}_{\mathrm{f}} + \nabla \cdot (\nu_{\mathrm{s}}\mathbf{v}_{\mathrm{f}}) &= 0, \\
\nu_{\mathrm{s}}\rho_{\mathrm{s}}\left(\partial \mathbf{v}_{\mathrm{s}} + (\nabla \mathbf{v}_{\mathrm{s}})\mathbf{v}_{\mathrm{s}}\right) &= \nabla \cdot \bar{\mathbf{T}}_{\mathrm{s}} + \bar{\mathbf{b}}_{\mathrm{s}} + \bar{\mathbf{m}}_{\mathrm{s}}^{i}, \\
(1 - \nu_{\mathrm{s}})\rho_{\mathrm{f}}\left(\partial \mathbf{v}_{\mathrm{f}} + (\nabla \mathbf{v}_{\mathrm{f}})\mathbf{v}_{\mathrm{f}}\right) &= \nabla \cdot \bar{\mathbf{T}}_{\mathrm{f}} + \bar{\mathbf{b}}_{\mathrm{f}} + \bar{\mathbf{m}}_{\mathrm{f}}^{i}, \\
\overset{\diamond}{\bar{\mathbf{Z}}}_{\mathrm{s}} &= \bar{\boldsymbol{\Phi}}_{\mathrm{s}}.
\end{aligned} \tag{100}$$

Together with the constitutive laws for $\bar{\mathbf{T}}_{\mathrm{s}}, \bar{\mathbf{T}}_{\mathrm{f}}, \bar{\mathbf{m}}_{\mathrm{s}}^{i}, \bar{\mathbf{m}}_{\mathrm{f}}^{i}$ and $\bar{\boldsymbol{\Phi}}_{\mathrm{s}}$, given in (97) and (98), a set of field equations can be constructed. For the solvability of this set we have to ensure that the number of equations is in conformity with the number of unknown variables arising in the field equations.

6.1 Pressure equilibrium

From (97) and (98) we observe that, besides $m_{\mathbf{D}}, \kappa_{\mathrm{f}}, \mu_{\mathrm{f}}, \kappa_{\mathrm{s}}, \mu_{\mathrm{s}}, \mathbf{L}$ and $\mathbf{N}$, we are still missing explicit expressions for Ψ_{I}^{G} and k_{α}^{v} ($\alpha = s, f$) which are necessary for the description of the evolution of $\beta_{\mathrm{s}}^{G}, \Psi_{I,\bar{\mathbf{Z}}_{\mathrm{s}}}^{G}, \bar{\varpi}_{\mathrm{s}}$ and $\bar{\varpi}_{\mathrm{f}}$, respectively. To avoid postulating representations for Ψ_{I}^{G} and k_{α}^{v} ($\alpha = s, f$) and to facilitate the construction of solutions of the field equations, in the literature on multiphase mixtures (cf. [22] and [36]), the assumption of 'pressure equilibrium' is usually made. This assumption is not based on any physical principle, but rather on surmised 'feelings of adequacy'. For the model it reads

$$\varpi_{\mathrm{s}} = \varpi_{\mathrm{f}} = \pi. \tag{101}$$

Proposition 1 *In view of* (98) *the assumption of pressure equilibrium expressed as* (101) *can only hold if*

(i) *any hypo-plastic effect from the model is abandoned, i.e. if*

$$\Psi_{I,\bar{\mathbf{Z}}_{\mathrm{s}}}^{G} = \mathbf{0}, \tag{102}$$

(ii) *if*

$$k_{\alpha}^{v} = 0 \quad (\alpha = s, f), \tag{103}$$

which follows from (101) *and*

(iii) *if the Helmholtz free energy is such that*

$$\beta_{\mathrm{s}}^{G} = \rho(\Psi_{I}^{G}),_{\nu_{\mathrm{s}}} = 0, \tag{104}$$

implying that Ψ_{I}^{G} *cannot be a function of* ν_{s}. □

The assumption states 'physically' that the spherical contribution to the mixture Cauchy stress is distributed among the constituents according to their volume fractions. With (101–104) the constitutive laws (97) reduce to

$$
\begin{aligned}
\bar{\mathbf{T}}_s &= -\nu_s \pi \mathbf{I} + \bar{\mathbf{T}}_{es}(\mathbf{B}_s) + \bar{\mathbf{T}}_s|_N, \\
\bar{\mathbf{T}}_f &= -(1 - \nu_s)\pi \mathbf{I} + \bar{\mathbf{T}}_f|_N, \\
\bar{\mathbf{m}}_s^i &= -\bar{\mathbf{m}}_f^i = \pi \nabla \nu_s + m_{\mathbf{D}} \mathbf{v}_{fs}
\end{aligned}
\tag{105}
$$

where, obviously, no frictional stress, $\bar{\mathbf{T}}_{fric}$ and no implicit volume fraction dependence arise, because

$$
\Psi_I^G \neq \hat{\Psi}^G(\nu_s, \bar{\mathbf{Z}}_s) \ .
\tag{106}
$$

Therefore, the hypo-plastic balance law (100)$_5$ is no longer of interest, and the dependence of Ψ_I^G reduces to

$$
\Psi_I^G = \hat{\Psi}_I^G (\mathbf{B}_s) \ .
\tag{107}
$$

For 'pressure equilibrium' the field equations read

$$
\begin{aligned}
\partial \nu_s + \nabla \cdot (\nu_s \mathbf{v}_s) &= 0 \ , \\
\partial \nu_s - \nabla \cdot \mathbf{v}_f + \nabla \cdot (\nu_s \mathbf{v}_f) &= 0 \ , \\
\nu_s \rho_s (\partial \mathbf{v}_s + (\nabla \mathbf{v}_s)\mathbf{v}_s) &= -\nabla(\nu_s \pi) + \nabla \cdot (\bar{\mathbf{T}}_{es}(\mathbf{B}_s) + \bar{\mathbf{T}}_s|_N) \\
&\quad + \pi \nabla \nu_s + \bar{\mathbf{b}}_s + m_{\mathbf{D}} \mathbf{v}_{fs} \ , \\
(1 - \nu_s)\rho_f (\partial \mathbf{v}_f + (\nabla \mathbf{v}_f)\mathbf{v}_f) &= -\nabla((1 - \nu_s)\pi) + \nabla \cdot (\bar{\mathbf{T}}_f|_N) \\
&\quad - \pi \nabla \nu_s + \bar{\mathbf{b}}_f - m_{\mathbf{D}} \mathbf{v}_{fs} \ .
\end{aligned}
\tag{108}
$$

In order to solve these equations in a well-posed initial boundary value problem, they must be complemented by an evolution equation for $\mathbf{B}_s$. This equation follows from the definition of $\mathbf{B}_s = \mathbf{F}_s \mathbf{F}_s^T$ as[13]

$$
\mathbf{B}_s' := \partial \mathbf{B}_s + (\nabla \mathbf{B}_s)\mathbf{v}_s = \mathbf{L}_s \mathbf{B}_s + \mathbf{B}_s \mathbf{L}_s^T \ .
\tag{109}
$$

If adequate initial values for ν_s, $\mathbf{v}_{f,s}$, $\mathbf{B}_s$, π are prescribed and there exist boundary conditions such that the resulting initial boundary value problem (IBVP) is not ill-posed,[14] we can, in principle, solve (108) and (109) for ν_s, $\mathbf{v}_s$, $\mathbf{v}_f$, $\mathbf{B}_s$ and π. Although this procedure seems convenient, we observe (see items (i) to (iii)) that 'pressure equilibrium' is a rough ad hoc assumption which destroys the structure of hypo-plasticity and that of configuration pressures; moreover, it rules out the linear dependence of $\mathbf{k}$ on $\mathbf{v}_s$ and $\mathbf{v}_f$. We conclude that the assumption of 'pressure equilibrium' is based on unnecessary restrictions and thus in general not appropriate for the modelling of debris flows. These unnecessary restrictions prevent first, the description of frictional stresses in thermodynamic equilibrium by means of a hypo-plastic stress contribution and second, eliminate consideration of the configuration pressure, β_s^G, which represents the driving force between the grains and between the fluid and the grains (cf. Passman et al. [35]) and thus might be necessary for the description of particle size segregation in debris flows. Thus, we reject the pressure equilibrium assumption as a physically acceptable assumption; this is also confirmed by Passman et al. [35]. In spite of this, the assumption is still popular and often used, see e.g. Pitman and Le [36], Iverson and Denlinger [22]. It is pleasing, however, that the thermodynamic approach has proved the assumption to be superfluous, or replaceable by a more useful alternative.

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[13] If the elastic strain, used in $\bar{\mathbf{T}}_{es}$, is not $\mathbf{B}_s$ but another strain measure, then the evolution equation for that variable must be used, e.g. for $\mathbf{E}_s = \frac{1}{2}(\mathbf{B}_s - \mathbf{I})$, $\mathbf{E}_s' := \partial \mathbf{E}_s + (\nabla \mathbf{E}_s)\mathbf{v}_s = \mathbf{L}_s \mathbf{E}_s + \mathbf{E}_s \mathbf{L}_s^T + \mathbf{D}_s$.

[14] As pointed out by Passman et al. [35] well-posedness is not always the case.

6.2 Thermodynamic closure assumption

A less restrictive assumption which replaces 'pressure equilibrium' was proposed by Hutter et al. [19] who simply suppose that

$$k_\alpha^v = 0 \qquad (\alpha = s, f) . \tag{110}$$

This assumption fixes the extra entropy flux without making it collinear with the mixture heat flux. Obviously, it is an *ad hoc* assumption, too, but does not eliminate the possibility of modelling frictional effects by hypo-plasticity and maintains a possible dependence of Ψ_I^G on ν_s. With the above assumption we obtain

$$\varpi_f = -\rho_f \Psi_I^G + \varsigma , \qquad \varpi_s = \beta_s^G + \varpi_f - (\rho_s - \rho_f) \left(\Psi_{I,\bar{\mathbf{Z}}_s}^G \cdot \bar{\mathbf{Z}}_s \right) . \tag{111}$$

If we, therefore, regard ϖ_f as an independent field, rather than ς, we obtain the following set of field equations: The constitutive laws are expressed as

$$
\begin{aligned}
\bar{\mathbf{T}}_s &= -\nu_s \left\{ \beta_s^G + \varpi_f - (\rho_s - \rho_f) \left(\Psi_{I,\bar{\mathbf{Z}}_s}^G \cdot \bar{\mathbf{Z}}_s \right) \right\} \mathbf{I} + \bar{\mathbf{T}}_{es}(\mathbf{B}_s) + \rho\delta\bar{\mathbf{Z}}_s + \bar{\mathbf{T}}_s|_N , \\
\bar{\mathbf{T}}_f &= -(1 - \nu_s)\varpi_f \mathbf{I} + \bar{\mathbf{T}}_f|_N , \\
\bar{\mathbf{m}}_s^i &= \left\{ \beta_s^G(1 - \bar{\xi}_s) + \varpi_f \right\} \nabla \nu_s + \rho \left(\Psi_I^G \right)_{,\bar{\mathbf{Z}}_s} |_E \left(\bar{\boldsymbol{\Phi}}_s \right)_{,\mathbf{v}_s} |_E + m_D \mathbf{v}_{fs} \\
&= -\bar{\mathbf{m}}_f^i , \\
\bar{\boldsymbol{\Phi}}_s &= \mathrm{f}_1 \left(\mathbf{L} \left(\bar{\mathbf{Z}}_s \right) \mathbf{D}_s + \mathrm{f}_2 \mathbf{N} \left(\bar{\mathbf{Z}}_s \right) |\mathbf{D}_s| \right) ,
\end{aligned}
\tag{112}
$$

and the corresponding balance laws (see(100)) have the form

$$
\begin{aligned}
\partial\nu_s + \nabla \cdot (\nu_s\mathbf{v}_s) &= 0 , \\
\partial\nu_s - \nabla \cdot \mathbf{v}_f + \nabla \cdot (\nu_s\mathbf{v}_f) &= 0 , \\
\nu_s\rho_s \left(\partial\mathbf{v}_s + (\nabla\mathbf{v}_s)\mathbf{v}_s \right) &= \nabla \cdot \bar{\mathbf{T}}_s + \bar{\mathbf{b}}_s + \bar{\mathbf{m}}_s^i , \\
(1 - \nu_s)\rho_f \left(\partial\mathbf{v}_f + (\nabla\mathbf{v}_f)\mathbf{v}_f \right) &= \nabla \cdot \bar{\mathbf{T}}_f + \bar{\mathbf{b}}_f + \bar{\mathbf{m}}_f^i , \\
\frac{\mathrm{d}^s \bar{\mathbf{Z}}_s}{\mathrm{d}t} - [\boldsymbol{\Omega}_s, \bar{\mathbf{Z}}_s] - \nu_s\frac{(\rho_s - \rho_f)}{\rho}\bar{\mathbf{Z}}_s(\nabla \cdot \mathbf{v}_s) &= \bar{\boldsymbol{\Phi}}_s , \\
\mathbf{B}_s' := \partial\mathbf{B}_s + (\nabla\mathbf{B}_s)\mathbf{v}_s &= \mathbf{L}_s\mathbf{B}_s + \mathbf{B}_s\mathbf{L}_s^T ,
\end{aligned}
\tag{113}
$$

in which we have chosen $\boldsymbol{\Omega}_s = \mathbf{W}_s$. If $\bar{\mathbf{T}}_{es}$ is given as $\bar{\mathbf{T}}_{es}(\mathbf{E}_s)$, where $\mathbf{E}_s = \frac{1}{2}(\mathbf{B}_s - \mathbf{I})$, then the evolution equation for $\mathbf{E}_s$ is given by

$$\mathbf{E}_s' := \partial\mathbf{E}_s + (\nabla\mathbf{E}_s)\mathbf{v}_s = \mathbf{L}_s\mathbf{E}_s + \mathbf{E}_s\mathbf{L}_s^T + \mathbf{D}_s . \tag{114}$$

Remarks • One of the equations $(113)_{1,2}$ could be replaced by the mixture volume balance

$$\nabla \cdot \mathbf{v}_{vol} = \nabla \cdot (\nu_s\mathbf{v}_s + (1 - \nu_s)\mathbf{v}_f) = 0 . \tag{115}$$

Similarly, $(113)_3$ or $(113)_4$ could be replaced by the momentum balance relation for the mixture as a whole, but this equation does not offer computational advantages.

• The above field equations have to be complemented by appropriate functions for the non-elastic part of the 'inner' free energy, $\hat{\Psi}_{fric}^G(\nu_s, \bar{\mathbf{Z}}_s)$, for the tensors $\hat{\mathbf{L}}(\nu_s, \nabla\nu_s, \mathbf{B}_s, \bar{\mathbf{Z}}_s)$ and $\hat{\mathbf{N}}(\nu_s, \nabla\nu_s, \mathbf{B}_s, \bar{\mathbf{Z}}_s)$ and for the coefficients $m_D(\nu_s)$, $\lambda_s(\nu_s)$ and $\mu_s(\nu_s)$, $\kappa_f(\nu_s)$, $\mu_f(\nu_s)$, f_1 and f_2. Suppose that these functions are known and initial and boundary conditions are proposed such that the resulting IBVP is well-posed; then we are, in principle, in the position to solve the field equations for the variables $\{\nu_s, \mathbf{v}_s, \mathbf{v}_f\}$.

Obviously, (110) is a weaker assumption than that of 'pressure equilibrium', because it does not rule out hypo-plasticity and the configuration pressure β_s^G. The price we have to pay for the gain of sensitivity of the model, is the need of an additional postulate for Ψ_I^G and, presumably, the increased complexity of the resulting IBVP.

• According to $(112)_{1,2}$, 'pressure equilibrium' is recovered in this formulation only when
    (i) $\rho_s = \rho_f$ and $\beta_s^G = 0$,
    (ii) for $\rho_s \neq \rho_f$ when $\Psi_I^G \neq \hat{\Psi}^G(\nu_s, \bar{\mathbf{Z}}_s, \cdot)$.
The second case is equivalent to (105), the first is not realistic for soil.

7 Discussion and conclusions

In this article, a binary solid–fluid mixture model was presented which is thought to mathematically describe the deformation and motion of a soil mass, which is saturated with water, and moves down any arbitrary terrain topography. The basis for the model equations in the thermodynamic theory of Part I (Hutter and Schneider [21]) in which the material model assumptions were so constrained that the requirements posed by the second law of thermodynamics are fulfilled. The special feature of the mixture model in Part I is its validity for any number of solid and fluid constituents, the allowance of mass and volume fraction production rate densities as well as the consideration of an internal second order tensor variable by which frictional heat effects such as hypoplastic behaviour can be modelled. In this article, a restricted class of material behaviour was looked at, namely a density preserving elastic-hypoplastic-viscous solid–fluid compound, in which no mass and volume fraction production rate densities were considered and processes were assumed to be isothermal.

One particular feature of the thermodynamic model derived in Part I and employed here, is that the equilibrium stresses, pressures, interaction forces etc., are determined once and for all when the Helmholtz free energy is prescribed as a function of its variables, and when the production rate density of the frictional symmetric stress-like variable is equally given.

On the assumption that the frictional and the solid and fluid elasticity effects are describable additively by separate contributions to the Helmholtz free energy (see (25)), it was shown that only the solid stress exhibits elastic constitutive behaviour of which the physically linear behaviour was illustrated as a simple case. On the other hand, the frictional behaviour was shown to nearly reproduce hypoplastic behaviour, if the frictional stress was set proportional to the tensor valued internal variable $\bar{\mathbf{Z}}_s$ (see (40)) and its production rate density was identified with the classical hypoplastic constitutive proposal. The resulting frictional model agreed with the classical hypoplastic constitutive model only approximately, the difference essentially being an additional frictional pressure (see (55)). The solid and fluid pressures, given in (60), do not satisfy the assumption of pressure equilibrium; $(60)_{1,2}$ differ from this assumption and the configuration solid pressure. If pressure equilibrium is any useful concept at all within this theory, then it can only be applied to the saturation pressure.

Several forms of the non-equilibrium stress and interaction force parameterizations are of viscous nature are proposed, but eventually attention is restricted to the quasi-linear relations (68), and identification of solid bulk and shear viscosities (fluidities) by viscometric (gedanken) experiments. Solid bulk viscosity is suggested as a strongly non-linear function of the solid volume fraction (see (77) and (78)), and the solid shear viscosity is given by expression (91), which is regularized for of a viscous fluid, possibly with yield surface. The fluid constituent is treated as a slurry with volume fraction dependent viscosity, motivated by Einstein's formula.

We, finally, close with a suggestion of the replacement of the assumption of pressure equilibrium by the requirement that the extra entropy flux vector $\mathbf{k}$ is set to zero.

In summary, the general theory presented in Part I and its specialization to a density preserving binary fluid-solid mixture in this article, have clearly demonstrated that physically fairly complex granular fluid flows with elasto-visco-hypoplastic constitutive behaviour can describe many thermodynamic flows with the potential to cover from quasi-static to rapid deformations with closure conditions which are thermodynamically consistent with the second law.

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