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Point Configurations in *d*-Space without Large Subsets in Convex Position*

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Abstract. In this paper we give a lower bound for the Erdős–Szekeres number in higher dimensions. Namely, in two different ways we construct, for every $n > d \geq 2$, a configuration of n points in general position in $\mathbb{R}^d$ containing at most $c_d(\log n)^{d-1}$ points in convex position. (Points in $\mathbb{R}^d$ are in convex position if none of them lies in the convex hull of the others.)

1. Introduction

A set of points in d-dimensional Euclidean space $\mathbb{R}^d$ is said to be in general position if any $\leq d+1$ of the points are affinely independent. In their seminal paper written in 1935, Erdős and Szekeres [3] proved that, for any integer $n \geq 3$, there is a smallest integer $f(n)$ such that any set of at least $f(n)$ points, in general position in the plane $\mathbb{R}^2$, contains the vertex set of a convex n-gon. In fact, they proved the following quantitative result.

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Theorem 1 ([3], [4]).

$$2^{n-2} + 1 \le f(n) \le \binom{2n-4}{n-2} + 1.$$

Various extensions of this result and its relation to Ramsey theory are explored, e.g., in [1], [9], and [11]. The lower bound is conjectured to be sharp [3], [4]. The best upper bound so far is

$$f(n) \le \binom{2n-5}{n-2} + 2,$$

see [10].

Much less is known about the situation in higher dimensions. We say that a set of points in $\mathbb{R}^d$ is *in convex position* if none of the points lies in the convex hull of the others. Let, for $n > d \ge 2$, $f_d(n)$ denote the smallest integer such that any set of at least $f_d(n)$ points, in general position in $\mathbb{R}^d$, contains n points in convex position. Thus, $f(n) = f_2(n)$. For $d \ge 3$, the only known values of $f_d(n)$ are $f_d(n) = 2n - d - 1$ for $d + 1 \le n \le \lfloor 3d/2 \rfloor + 1$ (see [2] for the upper bound and [9] for the lower bound) and $f_3(6) = 9$ [2].

The study of $f_d(n)$ was initiated by Grünbaum in [6] who also established its existence for every $n > d$ via Ramsey's theorem. A more effective general upper bound $f_d(n) \le f(n)$ follows from a simple projective argument (see [12]) and is slightly improved to

$$f_d(n) \le f(n - d + 2) + d - 2 \le \binom{2n-2d-1}{n-d} + d$$

in [8]. The aim of the present paper is to obtain the following general lower bound.

Theorem 2. *For every $d \ge 2$, there is a constant $c = c_d > 1$ such that*

$$f_d(n) = \Omega(c^{n^{1/(d-1)}}).$$

Equivalently, for every $N > d \ge 2$, there exists a configuration of N points in general position in Euclidean d-space which does not contain more than $c'(\log N)^{d-1}$ points in convex position, where the constant $c' = c'_d$ only depends on the dimension d.

The first proof of Theorem 2 gives constants $c_d \approx 2^{0.37d}$ and $c'_d \approx 2/(d-1)!$ (see the Appendix). The second proof gives somewhat worse constants. We include both proofs, since they are essentially different and their knowledge might help in attempts to close the gap between the lower bound in Theorem 2 and the above-mentioned exponential upper bound.

It has been conjectured by Füredi [5] that the bound in Theorem 2 is best possible apart from the value of the constant c. On the other hand, Morris and Soltan [9] contemplate about a possible recursive relation $f_d(n) = 4f_d(n-d) - 3$ that would imply the exponential lower bound $f_d(n) = \Omega(4^{n/d})$. We present two proofs for Theorem 2 based on two different constructions. Both constructions are in a sense recursive, but we think they are essentially different and may provide a support to Füredi's conjecture. After we introduce some notation in Section 2, the first proof is presented in Section 3. The second proof is based on the notion of so-called d-Horton sets which generalize Horton's construction of planar point sets that do not contain empty convex 7-gons, see [7] and [12]. This notion is explained in Section 4 and is used in Section 5 for the second proof of Theorem 2. The Appendix contains calculations giving the constants $c_d \approx 2^{0.37d}$ and $c'_d \approx 2/(d-1)!$.

2. Preliminaries

Fix the dimension $d \geq 2$. Identify, for every $1 \leq e \leq d$, $\mathbb{R}^e$ with the unique e-dimensional subspace of $\mathbb{R}^d$ spanned by the first e coordinate axes. This way $\mathbb{R}^f$ is identified with a subspace of $\mathbb{R}^e$, for every $f \leq e \leq d$.

For any $e \leq d$, denote by π_e the orthogonal projection from $\mathbb{R}^d$ onto $\mathbb{R}^e$. Thus, for the point $a = (a^1, \ldots, a^d) \in \mathbb{R}^d$, $\pi_e(a) = (a^1, \ldots, a^e) \in \mathbb{R}^e$. We also use the same symbol to denote the restriction of π_e to any $\mathbb{R}^f$, $e \leq f \leq d$. If it is not a cause for ambiguity we will denote the projection from $\mathbb{R}^e$ to $\mathbb{R}^{e-1}$ simply by π.

We say that a set P of points in $\mathbb{R}^e$ is in *strongly general position* if it is in general position and, for $f = 1, \ldots, e-1$, any $f+1$ points of P determine an f-dimensional affine subspace which is not parallel to the $(e-f)$-dimensional subspace of $\mathbb{R}^e$ spanned by the last $e-f$ coordinate axes of $\mathbb{R}^e$, which we denote by $(\mathbb{R}^f)^{\perp_e}$.

Lemma 3.

(i) *If P is in strongly general position in $\mathbb{R}^e$, then so is every subset of P.*
(ii) *P is in strongly general position in $\mathbb{R}^e$ if and only if Q is in strongly general position in $\mathbb{R}^e$ for every $Q \subseteq P$ of cardinality $\leq e+1$.*
(iii) *If P is in strongly general position in $\mathbb{R}^e$, then there is a point $p \in \mathbb{R}^e$, $p \notin P$, such that $P \cup \{p\}$ is in strongly general position in $\mathbb{R}^e$.*
(iv) *If P is in strongly general position in $\mathbb{R}^e$, then $|\pi_f(P)| = |P|$ and $\pi_f(P)$ is in strongly general position in $\mathbb{R}^f$ for every $1 \leq f \leq e$.*

Proof. The first two assertions are immediate consequences of the definition. To see the third one, observe that if $P \cup \{p\}$ is not in strongly general position, then p is contained in the union of finitely many proper affine subspaces of $\mathbb{R}^e$ that clearly cannot cover the whole space $\mathbb{R}^e$. Obviously, it is enough to prove the last assertion in the case $f = e-1$. We may assume, based on the first three assertions, that $|P| = e$. Thus let $P = \{p_1, \ldots, p_e\}$. Write $A = \mathrm{Aff}(p_1, \ldots, p_e)$ for the affine hull of P. We claim that $\pi(A) = \mathrm{Aff}(\pi(p_1), \ldots, \pi(p_e)) = \mathbb{R}^f = \mathbb{R}^{e-1}$. Indeed, $A \subseteq \pi(A) + (\mathbb{R}^f)^{\perp_e}$, indicating that $f = \dim A \leq \dim \pi(A) + 1$. Were $\dim \pi(A) \leq f-1$, it would follow that $A = \pi(A) + (\mathbb{R}^f)^{\perp_e}$. In particular, A would be parallel to $(\mathbb{R}^f)^{\perp_e}$, a contradiction. An immediate consequence of this claim is that $|\pi_f(P)| = |P|$, and also that $\dim \mathrm{Aff}(\pi(q_1), \ldots, \pi(q_{g+1})) = g$ for every $1 \leq g \leq f-1$ and every subset $Q = \{q_1, \ldots, q_{g+1}\}$ of P. If $\mathrm{Aff}(\pi(Q))$ was parallel to $(\mathbb{R}^g)^{\perp_f}$ in $\mathbb{R}^f$, then $\mathrm{Aff}(Q) \subseteq \mathrm{Aff}(\pi(Q)) + (\mathbb{R}^f)^{\perp_e}$ would be parallel to $(\mathbb{R}^g)^{\perp_f} + (\mathbb{R}^f)^{\perp_e} = (\mathbb{R}^g)^{\perp_e}$ in $\mathbb{R}^e$, a contradiction. This completes the proof of the last assertion. $\square$

Next, we will need the notion of *order type* that we only introduce for finite point sets in general position. We say that two finite point sets of equal size, in general position in $\mathbb{R}^e$, are *of the same order type* if there is a one-to-one correspondence between them that preserves the orientation of each $(e+1)$-tuple. It is clear that small perturbations do not affect the order type. More precisely, a routine compactness argument yields

Proposition 4. *For every finite set of points $X = \{x_1, x_2, \ldots, x_t\}$, in general position in $\mathbb{R}^e$, $e \le d$, there is a (largest) $\delta = \delta_d(X) > 0$ such that the following holds. Whenever $Y = \{y_1, y_2, \ldots, y_t\} \subset \mathbb{R}^e$ satisfies $|y_i^j - x_i^j| < \delta$ for every $1 \le i \le t$ and every $1 \le j \le e$, then Y is also in general position in $\mathbb{R}^e$, and has the same order type as X. In particular, X is in convex position if and only if so is Y.*

Finally, we denote by $\mathrm{mc}(P)$ the maximum size of any subset of P which is in convex position.

3. Recursive Construction

We will need the following general construction. Suppose that a set $X = \{x_1, \ldots, x_t\}$ is in strongly general position in $\mathbb{R}^e$. Let $0 < \varepsilon \le \varepsilon_e(X) = \min\{\delta_f(\pi_f(X)) \mid 1 \le f \le e\}$. Choose, for every $x \in X$, a vector $v(x) = (v^1(x), \ldots, v^e(x))$ such that $0 < v^1(x) < v^2(x) < \cdots < v^e(x) < \varepsilon$ and $v^f(x) < \varepsilon v^{f+1}(x)$ for every $1 \le f < e$. These vectors can be chosen in such a way that the set $X' = \{x \pm v(x) \mid x \in X\}$ of size $2|X|$ is in strongly general position, in which case X' is called an *ε-double of X* (see Fig. 1).

The following properties are immediate consequences of Lemma 3 and Proposition 4.

Proposition 5.

(i) *If X' is an ε-double of X, then $\pi_f(X')$ is an ε-double of $\pi_f(X)$ for any $1 \le f \le e$.*

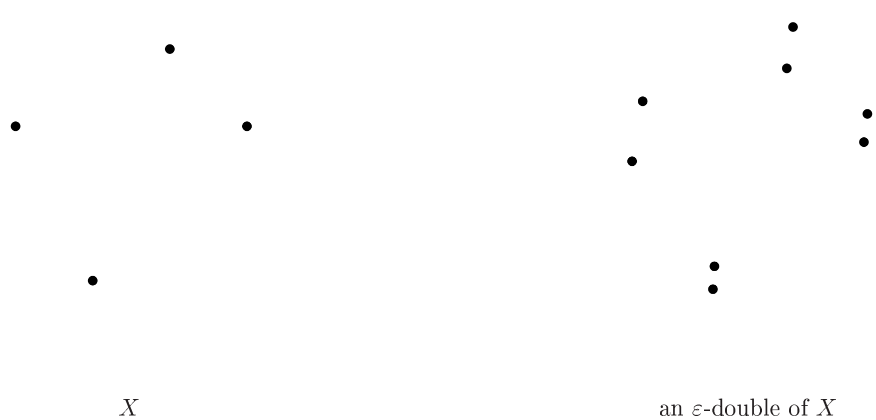

Fig. 1. A four-point planar set and its ε-double.

(ii) *If* $y_i^j \in \{x_i^j - v^j(x_i), x_i^j, x_i^j + v^j(x_i)\}$ *for* $1 \le i \le t$ *and* $1 \le j \le e$, *then, for every* $1 \le f \le e$, *the sequence* $\pi_f(y_1), \ldots, \pi_f(y_t)$ *has the same order type in* $\mathbb{R}^f$ *as the sequence* $\pi_f(x_1), \ldots, \pi_f(x_t)$.

The key observation is compressed in the following lemma.

Lemma 6. *Let* $X \subset \mathbb{R}^e$ *be in strongly general position. If* $0 < \varepsilon \le \varepsilon_e(X)$ *is small enough*, *then for any* ε*-double* X' *of* X,

$$\mathrm{mc}(X') \le \mathrm{mc}(X) + \mathrm{mc}(\pi(X)).$$

Proof. Suppose that $C \subseteq X'$ is in convex position. Consider first $C_1 = \{x \in X \mid x - v(x) \in C \text{ or } x + v(x) \in C\}$. It follows from Proposition 5(ii) that C_1 is also in convex position. Thus, $|C_1| \le \mathrm{mc}(X)$. Next, consider $C_2 = \{x \in X \mid x - v(x) \in C \text{ and } x + v(x) \in C\}$. If ε is small enough, then the vectors $v(x)$ $(x \in X)$ are almost parallel to the eth coordinate axis, and therefore, due to the strongly general position condition and Proposition 5(ii), $\pi(C_2)$ is in convex position. Thus, $|C_2| = |\pi(C_2)| \le \mathrm{mc}(\pi(X))$. Since $|C| = |C_1| + |C_2|$, the result follows. □

Proof of Theorem 2. Fix any one-point set X_0 in $\mathbb{R}^d$. Suppose that, for some integer $i \ge 0$, a set X_i of points in strongly general position in $\mathbb{R}^d$ has already been defined. Choose a very small $\varepsilon_i > 0$ and consider an ε_i-double X_i' of X_i; then it follows from Proposition 5 that $\pi_e(X_i')$ is an ε_i-double of $\pi_e(X_i)$ for every $1 \le e \le d$. Applying Lemma 6 to the sets $\pi_e(X_i)$ for $d \ge e \ge 2$, we obtain that if ε_i is small enough, then $\mathrm{mc}(\pi_e(X_i')) \le \mathrm{mc}(\pi_e(X_i)) + \mathrm{mc}(\pi_{e-1}(X_i))$, for $2 \le e \le d$. Choose such a small ε_i, and set $X_{i+1} = X_i'$.

This way an infinite sequence $X_0, X_1, X_2, \ldots$ of sets, in strongly general position in $\mathbb{R}^d$, is constructed such that $|X_i| = 2^i$ (see Fig. 2).

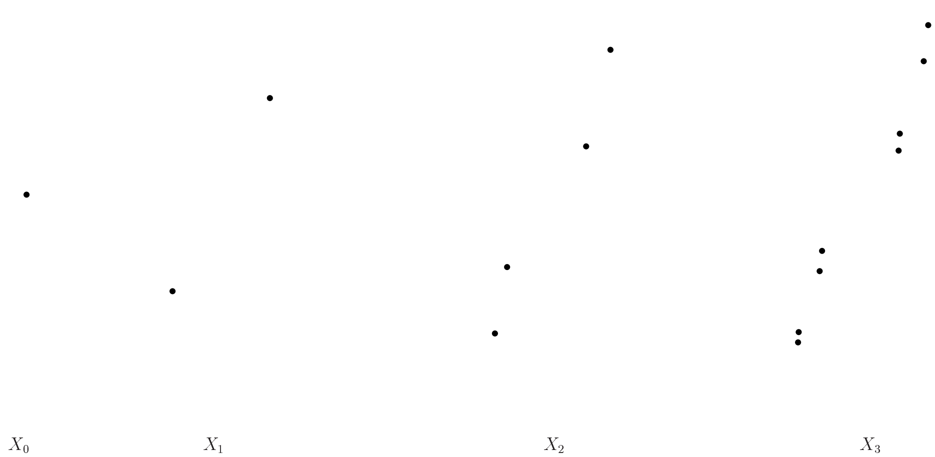

Fig. 2. The first four sets X_0, X_1, X_2, X_3 (the case $d = 2$).

Theorem 2 follows immediately from the following lemma.

Lemma 7. $\mathrm{mc}(\pi_e(X_i)) \le 2i^{e-1}$ *for every* $1 \le e \le d$ *and* $i \ge 1$.

Proof. The statement is clearly valid if $e = 1$ or $i = 1$. For double induction let $e \ge 2, i \ge 1$ and suppose that the assertion has already been proved for the pairs e, i and $e-1, i$. Then, according to the construction of X_{i+1},

$$\begin{aligned}\mathrm{mc}(\pi_e(X_{i+1})) &\le \mathrm{mc}(\pi_e(X_i)) + \mathrm{mc}(\pi_{e-1}(X_i)) \\ &\le 2i^{e-1} + 2i^{e-2} \\ &\le 2(i+1)^{e-1},\end{aligned}$$

as stated. □

A more careful calculation in fact yields that $\mathrm{mc}(\pi_e(X_i)) \le (2/(e-1)!)i^{e-1} + O(i^{e-2})$, see the Appendix. Thus, for large n and N, Theorem 2 is valid with $c_d \approx 2^{0.37d}$ and $c'_d \approx 2/(d-1)!$, respectively.

4. *d*-Horton Sets

Before we define d-Horton sets, we need to introduce some other notions.

We say that a point $a = (a^1, a^2 \dots, a^{d-1}, a^d)$ lies *below* a hyperplane h if $(a^1, a^2, \dots, a^{d-1}, a^d + c)$ lies on h for a unique $c > 0$. Similarly, a lies *above* h if $(a^1, a^2, \dots, a^{d-1}, a^d + c)$ lies on h for a unique $c < 0$.

Let A, B be two finite sets of points in strongly general position in $\mathbb{R}^d$. We say that A lies *deep below* B and B lies *high above* A if there are two sets $A' \supseteq A$, $B' \supseteq B$ in strongly general position, each of size at least d, such that the following holds: any point of A' lies below any hyperplane determined by d points of B' and any point of B' lies above any hyperplane determined by d points of A'.

We denote the $(d-1)$th prime number by p_d (thus, $p_2 = 2$, $p_3 = 3, \dots$). Let $H = \{h_0, h_1, ..., h_k\}$ be a set of points in strongly general position in $\mathbb{R}^d$, $d \ge 2$, ordered according to the first coordinate (i.e., if $i < j$, then h_i has a smaller first coordinate than h_j). (Note that the definition of strongly general position implies that any two points of H differ in the first coordinate.) We define p_d sets $H_z, z = 0, \dots, p_d - 1$, forming a partition of H as follows:

$$H_z := \{h_i \in H\colon\ i \equiv z \pmod{p_d}\}, \qquad z = 0, \dots, p_d - 1.$$

We now define the so-called d-Horton sets introduced in [12]. A finite set of points in strongly general position in $\mathbb{R}^d$, $d \ge 1$, is said to be a *d-Horton set* if either $d = 1$ or $|H| \le 1$ or if it satisfies the following three recursive conditions:

(a) $\pi(H)$ is $(d-1)$-Horton,
(b) each of the sets $H_z, z = 0, 1, \dots, p_d - 1$, is d-Horton,
(c) any index set $I, I \subseteq \{0, 1, \dots, p_d - 1\}, |I| \ge 2$, can be partitioned into nonempty sets J and $I - J$ in such a way that the set $\bigcup_{z\in J} H_z$ lies deep below the set $\bigcup_{z\in(I-J)} H_z$.

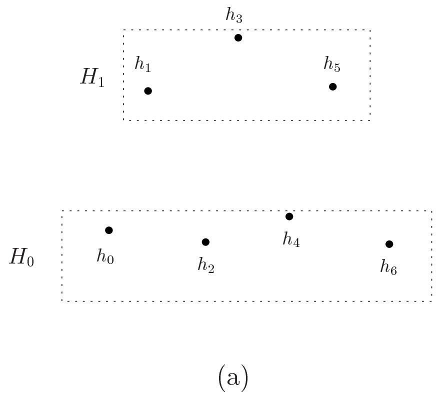

Fig. 3. Two examples of 2-Horton sets.

In dimension $d = 2$, condition (a) is void and condition (c) asserts that the subset H_0 lies deep below or high above the subset H_1. Two examples of 2-Horton sets are given in Fig. 3. An example of a 3-Horton set can be obtained from each of them by lifting the points h_1, h_4 in the third dimension by 1 and the points h_2, h_5 by 10 (say).

5. Construction Using *d*-Horton Sets

A recursive construction of d-Horton sets of arbitrary size was given in [12]. Thus, to give an alternative proof of Theorem 2, it suffices to prove the following theorem:

Theorem 8. *No d-Horton set of cardinality $n \geq 2$ contains a subset in convex position of size $\geq c'' \log^{d-1} n$, where the constant $c'' = c''_d$ only depends on d.*

The proof of Theorem 8 relies on the following lemma.

Lemma 9. *Let A_1, A_2, A_3 be three finite sets of points in strongly general position in $\mathbb{R}^d$ such that A_i lies deep below A_j for all $1 \leq i < j \leq 3$. If C is a set in convex position intersecting both A_1 and A_3, then $\pi(C \cap A_2) \subset \mathbb{R}^{d-1}$ is in convex position.*

Proof. Suppose that the set $\pi(C \cap A_2)$ is not in convex position. Then, by Carathéodory's theorem, there are $d+1$ points $t, t_1, \ldots, t_d \in C \cap A_2$ such that $\pi(t)$ lies in the convex hull of $\pi(t_1), \ldots, \pi(t_d)$.

Let $l(t)$ be the vertical (i.e., parallel to the last axis) line through t, and let K be the convex hull of $t_1, \ldots, t_d$ (K is a $(d-1)$-dimensional simplex). Since C is in convex position, t cannot lie in K. We may suppose without loss of generality that t lies strictly below (the hyperplane through) K. Let a be any point in $C \cap A_1$. Since C is in convex position, t lies also outside of $\text{conv}(K \cup \{a\}) = \text{conv}\{a, t_1, \ldots, t_d\}$.

Suppose that the bottommost point of $l(t) \cap \mathrm{conv}(K \cup \{a\})$ lies on that face of the simplex $\mathrm{conv}(K \cup \{a\})$ which is spanned by $a, t_1, t_2, \dots, t_{d-1}$ (say). Then a lies above the hyperplane determined by the points $t, t_1, t_2, \dots, t_{d-1}$—a contradiction. $\square$

Proof of Theorem 8. We proceed by induction on d. The statement is trivially true for $d = 1$, since no three points in $\mathbb{R}^1$ are in convex position. Now, let $d > 1$, let H be a d-Horton set of size $n \geq 2$, and let $C \subseteq H$ be in convex position.

We inductively choose d-Horton sets $H = H^{(0)} \supseteq H^{(1)} \supseteq H^{(2)} \supseteq \cdots$ so that, for each $s \geq 0$, $H^{(s+1)}$ is one of those sets $H_i^{(s)}$, $i = 0, \dots, p_d - 1$, which intersect C, and all other sets $H_i^{(s)}$ intersecting C lie high above it. The existence of such a set $H^{(s+1)} = H_i^{(s)}$ follows from condition (c) in the definition of d-Horton sets (if $|H^{(s)}| \geq p_d \cdot d$, then the relation "to lie high above" is a linear order on the sets $H_i^{(s)}$, $i = 0, \dots, p_d - 1$, and the set $H^{(s+1)}$ is uniquely determined).

Similarly, we inductively choose d-Horton sets $H = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$ so that, for each $s \geq 0$, $G^{(s+1)}$ is one of the sets $G_i^{(s)}$ intersecting C, and all other sets $G_i^{(s)}$ intersecting C lie deep below it.

If possible, we choose different sets $H^{(1)}$ and $G^{(1)}$. We may then assume that $H^{(1)} \neq G^{(1)}$, since otherwise $C \subseteq H^{(1)}$ and the proof for the smaller set $H^{(1)}$ gives the statement also for the set H.

We have $|H^{(s+1)}| \in \{\lfloor |H^{(s)}|/p_d \rfloor, \lceil |H^{(s)}|/p_d \rceil\}$. It follows that $|H^{(w)}| = 1$, where $w = \lceil \log_{p_d} n \rceil$. Similarly, $|G^{(w)}| = 1$.

We consider the decomposition of H into sets $H \backslash (H^{(1)} \cup G^{(1)})$, $H^{(1)} \backslash H^{(2)}, \dots, H^{(w-1)} \backslash H^{(w)}$, $H^{(w)}$, $G^{(1)} \backslash G^{(2)}, \dots, G^{(w-1)} \backslash G^{(w)}$, $G^{(w)}$. We will show that each of these $2w + 1$ sets contains at most $(p_d - 1) c''_{d-1} \log^{d-1} n$ points of C. Then the size of C is at most $(2w + 1)(p_d - 1) c''_{d-1} \log^{d-1} n < c''_d \log^d n$, where $c''_d = 10(p_d - 1) c''_{d-1}$ (say), and the theorem follows.

For each $s = 1, \dots, w - 1$, the set $H^{(s)} \backslash H^{(s+1)}$ is a disjoint union of the $p_d - 1$ sets $H_i^{(s)}$ different from $H^{(s+1)}$. If we intersect any such set $H_i^{(s)}$ with C and make the π-projection, the resulting set is a subset of $\pi(H)$ in convex position by Lemma 9 (applied on $A_1 := H^{(s+1)}$, $A_2 := H_i^{(s)}$, $A_3 := G^{(1)}$—here we use that $H^{(1)} \neq G^{(1)}$). Since $\pi(H)$ is a $(d - 1)$-Horton set of size n, it follows from the inductive hypothesis that $H^{(s)} \backslash H^{(s+1)}$ contains at most $(p_d - 1) c''_{d-1} \log^{d-1} n$ points of C. Analogously, the same estimate holds for each of the sets $G^{(s)} \backslash G^{(s+1)}$ and also for the set $H \backslash (H^{(1)} \cup G^{(1)})$. It certainly also holds for the one-point sets $H^{(w)}$ and $G^{(w)}$. $\square$

Appendix

Here we prove that, for every fixed $e \geq 1$, $\mathrm{mc}(\pi_e(X_i)) \leq (2/(e - 1)!) i^{e-1} + O(i^{e-2})$.

Notice first that $\mathrm{mc}(\pi_e(X_1)) = \mathrm{mc}(\pi_1(X_i)) = 2$ for every $e, i \geq 1$. Based on the inequality

$$\mathrm{mc}(\pi_e(X_{i+1})) \leq \mathrm{mc}(\pi_e(X_i)) + \mathrm{mc}(\pi_{e-1}(X_i)) \tag{1}$$

one can readily check that

$$\mathrm{mc}(\pi_e(X_i)) \leq 2^i \tag{2}$$

holds for every $1 \leq i \leq e$.

Consider the unique polynomial p_e of degree $e-1$ that satisfies $p_e(i) = 2^i$ for every $1 \le i \le e$. Thus, $p_1 \equiv 2$ and

$$\mathrm{mc}(\pi_1(X_i)) \le p_1(i) \tag{3}$$

for every $i \ge 1$. The sequence (a_i) defined by the recursion $a_1 = 2$, $a_{i+1} = a_i + p_e(i)$ satisfies $a_i = q(i)$ for some polynomial q of degree e. It is easy to check that $a_i = 2^i$ holds for every $1 \le i \le e+1$, and thus $q = p_{e+1}$. By induction it follows from inequalities (1)–(3) that $\mathrm{mc}(\pi_e(X_i)) \le p_e(i)$ for every $e, i \ge 1$. The coefficients of the polynomial $p_e(x) = a_1x^{e-1} + a_2x^{e-2} + \cdots + a_e$ can be obtained from the solution of the system of linear equations

$$\sum_{j=1}^{e} a_j i^{e-j} = 2^i$$

for $1 \le i \le e$. By Cramer's rule we can express a_1 as the ratio of the determinants of two $e \times e$ matrices $B = (b_{ij})$ and $D = (d_{ij})$, where $b_{ij} = 2^i$ if $j = 1$, $b_{ij} = i^{e-j}$ if $1 < j$, and $d_{ij} = i^{e-j}$. Given that D can be obtained from a Vandermonde matrix by exchanging certain columns we obtain that $\det D = (-1)^{\binom{e}{2}}(e-1)!\,(e-2)!\cdots 1!$. Similarly, if we expand $\det B$ according to its first column we obtain that $\det B = \sum_{i=1}^{e}(-1)^{i+1}2^i \det B_i$, where B_i is again a Vandermonde matrix. In fact we have that

$$\det B_i = (-1)^{\binom{e-1}{2}} \frac{(e-1)!\,(e-2)!\cdots 1!}{(e-i)!\,(i-1)!},$$

and therefore the leading coefficient of p_e is

$$a_i = (-1)^{\binom{e-1}{2}-\binom{e}{2}} \frac{2}{(e-1)!} \sum_{i=0}^{e-1}(-2)^i \binom{e-1}{i} = \frac{2}{(e-1)!}.$$

Therefore $p_e(x) = (2/(e-1)!)x^{e-1} + O(x^{e-2})$, proving our claim.

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