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ORIGINAL PAPER

On geodesic exponential maps of the Virasoro group

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Abstract We study the geodesic exponential maps corresponding to Sobolev type right-invariant (weak) Riemannian metrics $\mu^{(k)}$ $(k \geq 0)$ on the Virasoro group $\mathtt{Vir}$ and show that for $k \geq 2$, but *not* for $k = 0, 1$, each of them defines a smooth Fréchet chart of the unital element $e \in \mathtt{Vir}$. In particular, the geodesic exponential map corresponding to the Korteweg–de Vries (KdV) equation $(k = 0)$ is *not* a local diffeomorphism near the origin.

Keywords Geodesic exponential maps · Virasoro group

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1 Introduction

The aim of this paper is to contribute towards a theory of Riemannian geometry for infinite dimensional Lie groups which has attracted a lot of attention since Arnold's seminal paper [1] on hydrodynamics. As a case study, we consider the Virasoro group $\mathtt{Vir}$, a central extension $\mathcal{D} \times \mathbb{R}$ of the Fréchet Lie group $\mathcal{D} \equiv \mathcal{D}(\mathbb{T})$ of orientation preserving C^∞-diffeomorphisms of the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and thus a Fréchet Lie group itself. Its Lie algebra $\mathtt{vir}$ can be identified with the Fréchet space $C^\infty(\mathbb{T}) \times \mathbb{R}$. The Virasoro group and its algebra come up in string theory [14] as well as in hydrodynamics, playing the rôle of a configuration space for the celebrated Korteweg–de Vries equation [22, 32]. For $k \geq 0$ given, consider the scalar product $\langle\cdot,\cdot\rangle_k$: $\mathtt{vir} \times \mathtt{vir} \to \mathbb{R}$

$$\langle (u,a),(v,b)\rangle_k := \sum_{j=0}^{k} \int_0^1 \partial_x^j u \cdot \partial_x^j v \,\mathrm{d}x + ab.$$

It induces a weak right-invariant Riemannian metric $\mu^{(k)}$ on $\mathtt{Vir}$. The notion of a *weak* metric, introduced in [15], means that the topology induced by $\mu^{(k)}$ on any tangent space $T_\Phi\mathtt{Vir}$, $\Phi \in \mathtt{Vir}$, is weaker than the Fréchet topology on $T_\Phi\mathtt{Vir}$. The aim of this paper is to show that results of classical Riemannian geometry concerning the geodesic exponential map induced by the metric $\mu^{(k)}$ continue to hold in $\mathtt{Vir}$ if $k \geq 2$. Note that it has been shown by Kopell [23] (cf. also [17, 29]) that the Lie exponential map of the diffeomorphism group of the circle is *not* a local diffeomorphism near the origin. This fact can be used to show a similar result for the Virasoro group $\mathtt{Vir}$—see Sect. 5.

Theorem 1.1 *For any of the right-invariant metrics $\mu^{(k)}$, $k \geq 0$, there exists a neighborhood U_k of zero in $\mathtt{vir}$ such that for any initial vector $\xi \in U_k$ there exists a unique geodesic $\gamma(t;\xi)$ of $\mu^{(k)}$ with $\gamma|_{t=0} = e$ (e denotes the unital element in $\mathtt{Vir}$) and $\dot\gamma|_{t=0} = \xi$, defined on the interval $t \in (-2,2)$ and depending C^1_F-smoothly* on the initial data $\xi \in U_k$, i.e. $(-2,2) \times U_k \to \mathtt{Vir}$, $(t,\xi) \mapsto \gamma(t;\xi)$ is C^1_F-smooth.*

Theorem 1.1 allows to define, for any given $k \geq 0$, on $U_k \subseteq \mathtt{vir}$ the geodesic exponential map

$$\exp_k: U_k \to \mathtt{Vir}, \ \ \xi \mapsto \gamma(1;\xi).$$

The following two theorems show that there is a fundamental dichotomy between the exponential maps $\exp_k$ for $k = 0, 1$ and $k \geq 2$—see Remark 3.2 for an explanation of this dichotomy.

Theorem 1.2 *For any $k \geq 2$ there exist a neighborhood U_k of zero in $\mathtt{vir}$ and a neighborhood V_k of the unital element e in $\mathtt{Vir}$ such that the geodesic exponential map $\exp_k|_{U_k}$: $U_k \to V_k$ is a C^1_F-diffeomorphism.*

* A map is C^k_F-smooth if it is k-times continuously differentiable in the sense of Fréchet calculus—see Sect. 5.

Theorem 1.3 *For $k = 0$ and $k = 1$ there is* no *neighborhood W_k of zero in* vir *so that the geodesic exponential map* $\exp_k$ *is a C^1_F-diffeomorphism from W_k onto a neighborhood of the unital element e in* Vir.

Remark 1.4 Similar results as the one above have been established by Constantin and Kolev [11] for the Fréchet Lie group $\mathcal{D}$. Note however that the natural inclusion $\mathcal{D} \hookrightarrow \mathtt{Vir}$, $\phi \mapsto (\phi, 0)$ is *not* a subgroup of Vir and it turns out that the geodesic exponential map $\exp_k$ on Vir when projected to $\mathcal{D}$ is different from the corresponding geodesic exponential map $\exp^{\mathcal{D}}_k$ on $\mathcal{D}$. In fact it has been proved in [11] that $\exp^{\mathcal{D}}_k$ is a local C^1_F-diffeomorphism near the origin in $T_{\mathtt{id}}\mathcal{D}$ for any $k \geq 1$. According to [10], this is not true for $k = 0$. This fact reveals a difference between the Korteweg–de Vries equation which by [22, 32] is the Euler equation corresponding to the metric $\mu^{(0)}$ on Vir and the Camassa–Holm equation which by [22, 24, 30] is the Euler equation corresponding to the restriction of the metric $\mu^{(1)}$ to $\mathcal{D}$.

The paper is organized as follows: In Sect. 2 we fix notations and describe our set-up. Theorems 1.1 and 1.3 are shown in Sect. 4, whereas Theorem 1.2 is proved in Sect. 3. For the convenience of the reader, we have included at the end of the paper a section on the calculus in Fréchet spaces (Sect. 5), on the Euler equations on vir (Sect. 6), and on the Lie exponential map (Sect. 7).

2 Euler–Lagrange equations on the Virasoro group

Denote by $\mathcal{D} \equiv \mathcal{D}(\mathbb{T})$ the group of C^∞-smooth positively oriented diffeomorphisms of the 1-dimensional torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The topology on $\mathcal{D}$ is induced from the standard Fréchet topology on $C^\infty(\mathbb{T})$ corresponding to the countable system of H^k norms,

$$||u||_k^2 := \sum_{j=0}^{k} \int_0^1 (\partial_x^j u)^2 \, \mathrm{d}x, \tag{2.1}$$

$k \geq 0$ (cf. Sect. 5). The Fréchet manifold $\mathcal{D}$ is a Fréchet Lie group with multiplication $\circ: \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ given by the composition of diffeomorphisms, i.e. if $(\phi, \psi) \in \mathcal{D} \times \mathcal{D}$, then $(\phi \circ \psi)(x) := \phi(\psi(x))$ (cf. [17]).⋆

Definition 2.1 The *Virasoro group* Vir is the Fréchet manifold $\mathcal{D} \times \mathbb{R}$ with multiplication $\circ$: Vir × Vir → Vir given by the formula

$$(\phi, \alpha) \circ (\psi, \beta) := \Big(\phi \circ \psi, \alpha + \beta - \frac{1}{2} \int_0^1 \log(\phi(\psi(x)))_x \, \mathrm{d} \log \psi_x(x)\Big). \tag{2.2}$$

The map B, given by $B(\phi, \psi) := -\frac{1}{2} \int_0^1 \log(\phi \circ \psi)_x \, \mathrm{d} \log \psi_x$ is sometimes referred to as the Bott cocycle.

Remark 2.2 Passing to the universal cover $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ of the torus $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$ we identify a diffeomorphism $\phi \in \mathcal{D}$ with the set of its lifts $\tilde{\phi}: \mathbb{R} \to \mathbb{R}$, $\tilde{\phi} \in C^\infty(\mathbb{R}, \mathbb{R})$. Two

⋆ Note that the composition on $\mathcal{D}$ is C^∞_F-smooth.

lifts $\tilde{\phi}_1, \tilde{\phi}_2$ of ϕ are related by $\tilde{\phi}_2(x) = \tilde{\phi}_1(x+k) + l$ for some $k, l \in \mathbb{Z}$. It is readily seen that the expression

$$-\frac{1}{2}\int_0^1 \log(\tilde{\phi}(\tilde{\psi}(x)))_x \, \mathrm{d}\log\tilde{\psi}_x(x)$$

in formula (2.2) is independent of the choice of the lifts $\tilde{\phi}, \tilde{\psi} \in C^\infty(\mathbb{R}, \mathbb{R})$ of ϕ and ψ. Often we will choose a lift of an element ϕ in $\mathcal{D}$ of the form $\tilde{\phi}\colon \mathbb{R} \to \mathbb{R}, \tilde{\phi}(x) = x + v(x)$ with $0 \le v(0) < 1$ and v a smooth 1-periodic function. In the sequel we will not distinguish between ϕ and its lifts to $\mathbb{R}$.

One easily verifies that $\mathtt{Vir}$ is a Fréchet Lie group whose algebra $\mathtt{vir}$ can be identified with the Fréchet space $C^\infty \times \mathbb{R}$ with Lie bracket

$$[(u,a),(v,b)] = \Big(u_x v - v_x u, \int_0^1 u(x) v_{xxx}(x)\mathrm{d}x\Big). \tag{2.3}$$

The map C, given by $C(u,v) := \int_0^1 u(x)v_{xxx}(x)\mathrm{d}x$ is often referred to as Gelfand–Fuchs 2-cocycle. The unital element in $\mathtt{Vir}$ is $e := (\mathtt{id}, 0)$ where $\mathtt{id}$ denotes the identity in $\mathcal{D}$.

Remark 2.3 Usually the coefficient in front of the integral in (2.2) is taken to be equal to 1 instead of $-\frac{1}{2}$ (cf. [2, 22]). In this case, one has to insert a factor -2 in front of the integral in formula (2.3) for the Lie bracket.

For a given $k \ge 0$ consider on $\mathtt{vir} = C^\infty \times \mathbb{R}$ the Sobolev type scalar product $\langle\cdot,\cdot\rangle_k\colon \mathtt{vir} \times \mathtt{vir} \to \mathbb{R}$

$$\langle (u,a),(v,b)\rangle_k := \sum_{j=0}^k \int_0^1 \partial_x^j u \cdot \partial_x^j v \, \mathrm{d}x + ab, \quad \forall (u,a),(v,b) \in \mathtt{vir}. \tag{2.4}$$

This scalar product induces a right-invariant (weak) Riemannian metric* $\mu^{(k)}$ on $\mathtt{Vir}$. For any $\Phi \in \mathtt{Vir}$

$$\mu_\Phi^{(k)}(\xi,\eta) = \langle (\mathrm{d}_e R_\Phi)^{-1}\xi, (\mathrm{d}_e R_\Phi)^{-1}\eta\rangle_k, \quad \forall \xi,\eta \in T_\Phi\mathtt{Vir}, \tag{2.5}$$

where $R_\Phi\colon \mathtt{Vir} \to \mathtt{Vir}$ denotes the right translation $\Psi \mapsto \Psi \circ \Phi$ in $\mathtt{Vir}$. It follows from its definition that $\mu^{(k)}$ is a C_F^∞-smooth** weak Riemannian metric on $\mathtt{Vir}$.

We define the *geodesics* with respect to a smooth (weak) Riemannian metric μ on $\mathtt{Vir}$ in the classical way as the *stationary points* of the *action functional* corresponding to μ.‡ The following definition makes sense on an arbitrary Fréchet manifold.

* The word *weak* means that the topology induced by $\mu^{(k)}$ on any tangent space $T_\Phi\mathtt{Vir}$, $\Phi \in \mathtt{Vir}$, is weaker than the Fréchet topology on $T_\Phi\mathtt{Vir}$.

** The symbol C_F^k means that the corresponding map is k-times continuously differentiable in the sense of Fréchet calculus (see Sect. 5). We reserve the symbol C^k for the standard notion of continuous differentiability up to order k in Banach spaces.

‡ Another approach is to prove that there exists a Levi-Civita connection on $\mathtt{Vir}$ with respect to the metric $\mu^{(k)}$ and then to define the geodesics as the curves whose tangent vectors are parallel with respect to the connection (cf. [11, 15]).

Definition 2.4 A C_F^2-smooth curve $\gamma\colon [0,T] \to \mathtt{Vir}$, $T > 0$, is called a *geodesic* of the smooth (weak) Riemannian metric μ on $\mathtt{Vir}$ if for any C_F^2-smooth variation (with s denoting the variation parameter $-\epsilon < s < \epsilon$)

$$\gamma\colon (-\epsilon,\epsilon) \times [0,T] \to \mathtt{Vir}, \quad (s,t) \mapsto \gamma(s,t) \ \text{ with } \ \gamma(0,t) = \gamma(t) \tag{2.6}$$

such that $\gamma(s,0) = \gamma(0)$ and $\gamma(s,T) = \gamma(T)$ for any $-\epsilon < s < \epsilon$ one has

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} E_\mu(\gamma(s,\cdot)) = 0. \tag{2.7}$$

Here, E_μ denotes the *action functional*

$$E_\mu(\gamma(s,\cdot)) := \frac{1}{2}\int_0^T \mu(\dot\gamma(s,t),\dot\gamma(s,t))\mathrm{d}t$$

and $\dot\gamma(s,t) := \frac{\partial\gamma}{\partial t}(s,t)$. The variational equation (2.7) leads (cf. Sect. 6) to a partial differential equation for $\gamma(t)$, called the *Euler–Lagrange equation*. Note that the existence and the uniqueness of geodesics on Fréchet manifolds might not hold. Indeed, the corresponding Euler–Lagrange equation can be viewed as a dynamical system (ODE) on the tangent bundle which is a Fréchet manifold as well. But on Fréchet manifolds, smooth ODE's may have no or more than one solution (cf. [17], p. 129).

It turns out that a C_F^2-smooth curve $t \mapsto \Phi(t) = (\phi(t),\alpha(t)) \in \mathtt{Vir}$ with

$$\Phi|_{t=0} = e \quad \text{and} \quad \frac{\mathrm{d}\Phi}{\mathrm{d}t}|_{t=0} = (u_0,a_0) \in \mathtt{vir}$$

is a geodesic with respect to the metric $\mu^{(k)}$ if and only if $\phi(t)$ and $\alpha(t)$ are solutions of the ordinary differential equations

$$\dot\phi(t) = u(t,\phi(t)), \tag{2.8}$$

$$\phi|_{t=0} = \mathtt{id} \tag{2.9}$$

and

$$\dot\alpha(t) = a(t) - \frac{1}{2}\int_0^1 u_x(t,\phi(t,x))\ \mathrm{d}\log\phi_x(t,x), \tag{2.10}$$

$$\alpha(0) = 0, \tag{2.11}$$

where $(u(t),a(t)) \in \mathtt{vir}$ satisfies the so-called *Euler equation*

$$A_k u_t = -(2u_x A_k u + u A_k u_x) + a u_{xxx}, \tag{2.12}$$

$$\dot a = 0, \tag{2.13}$$

with $A_k := \sum_{j=0}^k (-1)^j \partial_x^{2j}$ and initial data

$$u(0,x) = u_0(x) \quad \text{and} \quad a(0) = a_0. \tag{2.14}$$

We will derive the above system (2.8)–(2.13) in Sect. 6. Let us point out that unlike in the case of the Lie group exponential map for $\mathtt{Vir}$ (see Sect. 7) the element u in (2.8) generically depends on time.

Let $t \mapsto (\phi(t;u_0,a_0),\alpha(t;u_0,a_0)) \in \mathtt{Vir}$ be a C_F^2-smooth solution of (2.8)–(2.9) and (2.10)–(2.11) where $u(t,x) \equiv u(t,x;u_0,a_0)$ is a solution of the Euler equations (2.12)–(2.14) that we assume is defined on an open set in $\mathbb{R}$ containing the interval $[-1,1]$.

Then, the geodesic exponential map at (u_0, a_0) is defined by the formula

$$\exp_k\colon (u_0, a_0) \mapsto (\phi(t, x; u_0, a_0), \alpha(t; u_0, a_0))|_{t=1}. \tag{2.15}$$

Theorem 1.1 stated in the introduction and proved in Sect. 4 says that for any $k \geq 0$ the geodesic exponential map $\exp_k$ is well-defined in a small open neighborhood of zero in $\mathtt{vir}$.

3 Proof of Theorem 1.2

We will prove Theorem 1.2 by applying, for any given $k \geq 2$, Proposition 5.5 in Sect. 5 to the *Hilbert approximation*⋆

$$\mathtt{vir}_{2k+1} \supseteq \mathtt{vir}_{2k+2} \supseteq \cdots \supseteq \mathtt{vir}$$

of the Fréchet space $\mathtt{vir} = C^\infty \times \mathbb{R}$ where $\mathtt{vir}_l := H^l \times \mathbb{R}$ and $H^l \equiv H^l(\mathbb{T})$ is the Sobloev space of real valued functions on $\mathbb{T}$. Let $\mathcal{D}^s$ $(s \geq 2)$ denote the Hilbert manifold, modeled on the Hilbert space H^s,

$$\mathcal{D}^s := \{\phi \in H^s(\mathbb{T}, \mathbb{T}) \mid \phi'(x) > 0 \;\; \forall x \in \mathbb{T}\}.$$

Representing an element $\phi \in \mathcal{D}^s$ in the form $\phi(x) = x + f(x)$ one can easily see that a neighborhood of the identity $\mathtt{id}$ in $\mathcal{D}^s$ can be identified with an open neighborhood of $0 \in H^s$ (cf. Sect. 5). The composition of mappings endows $\mathcal{D}^s$ with a topological Lie group structure.⋆⋆

The following result is the main ingredient in the proof of Theorem 1.2.

Proposition 3.1 *For any $k \geq 2$ given there exists a neighborhood U_{2k+1} of zero in $\mathtt{vir}_{2k+1}$ such that for any $l \geq 2k+1$ and any initial data $(u_0, a_0) \in U_l := U_{2k+1} \cap \mathtt{vir}_l$ there exists a unique solution $\Phi(t) = (\phi(t), \alpha(t)) \in C^1((-2, 2), \mathcal{D}^l \times \mathbb{R})$ of* (2.8)–(2.14)‡ *which depends C^1-smoothly on the initial data $(u_0, a_0) \in U_l$ in the sense that Φ belongs to $C^1((-2, 2) \times U_l, \mathcal{D}^l \times \mathbb{R})$.*

To prove Proposition 3.1, we need to establish first some auxiliary results. For a given $k \geq 2$ consider the pair of equations depending on the real parameter $a = a_0$,

$$\dot{\phi} = u(t, \phi(t)), \tag{3.1}$$

$$A_k u_t = -(2u_x A_k u + u A_k u_x) + a u_{xxx}, \tag{3.2}$$

with initial data $u|_{t=0} = u_0$ and $\phi|_{t=0} = \mathtt{id}$. Our first aim is to prove that for any $a \in \mathbb{R}$ and any $l \geq 2k+1$ there exist solutions $\phi \in C^1((-T, T), \mathcal{D}^l)$ and $u \in C^0((-T, T), H^l) \cap C^1((-T, T), H^{l-1})$ of the system (3.1)–(3.2) which are defined for some $T > 0$ (possibly depending on the initial data $u_0 \in H^l$ and $a \in \mathbb{R}$). To this end note that for any $l \geq 2k + 1$ the system (3.1)–(3.2) can be transformed by a change of variables to a

⋆ For any $k \geq 2$ given, we choose $\mathtt{vir}_{2k+1}$ as the first approximation space of $\mathtt{vir}$ to insure that all our calculations will take place in H^s with $s \geq 1$. Note that H^s is a Banach algebra for $s \geq 1$.

⋆⋆ Unfortunately, the composition $\circ\colon \mathcal{D}^s \times \mathcal{D}^s \to \mathcal{D}^s$ and the inverse operation $(\cdot)^{-1}\colon \mathcal{D}^s \to \mathcal{D}^s$ are not C^∞. However, for any $l \geq 0$, the composition $\circ\colon \mathcal{D}^{s+l} \times \mathcal{D}^{s+l} \to \mathcal{D}^s$ $(l \geq 0)$ is a C^l-smooth map (see e.g. [15]).

‡ with $u(t) = \phi_t(t) \circ \phi(t)^{-1}$.

parameter dependent ODE on the Hilbert manifold $\mathcal{D}^l \times H^l$. To see this note that for $u \in H^l$ one has

$$A_k(uu_x) = uA_k u_x + Q_k(u),$$

where $Q_k(u)$ is a polynomial in the variables $u, u_x, \ldots, \partial_x^{2k} u$. As H^{l-2k} $(l \geq 2k+1)$ is a Banach algebra with respect to multiplication of functions⋆ it follows that $Q_k \in C^\infty(H^l, H^{l-2k})$. Using the identity displayed above, (3.2) can be rewritten in the form $A_k(u_t + uu_x) = -2u_x A_k u + Q_k(u) + au_{xxx}$ or

$$u_t + uu_x = A_k^{-1} \circ B_k(u; a),$$

where for any given $a \in \mathbb{R}$,

$$u \mapsto B_k(u; a) := -2u_x A_k u + Q_k(u) + au_{xxx}$$

is an element in $C^\infty(H^l, H^{l-2k})$. Note that $B_k(0; a) = 0$. In the sequel, we will sometimes write $B_k(a)$ for $B_k(\cdot; a)$. Note also that B_k depends smoothly on the parameter $a \in \mathbb{R}$. More precisely

$$H^l \times \mathbb{R} \to H^{l-2k}, \ (u, a) \mapsto B_k(u; a),$$

is a C^∞-map.

Remark 3.2 The term au_{xxx} in the expression for $B_k(u; a)$ belongs to H^{l-3} which is contained in H^{l-2k} when $k \geq 2$. If $k = 0$ or $k = 1$ the latter inclusion does not hold.

Finally, the substitutions $v(t) = u(t) \circ \phi(t)$ and $\dot{\phi} = v$ lead to the equation

$$\dot{\phi} = v, \tag{3.3}$$

$$\dot{v} = F_k(\phi, v; a), \tag{3.4}$$

where $F_k(\phi, v; a) := (A_k^{-1} \circ B_k(v \circ \phi^{-1}; a)) \circ \phi$. The right-hand side of (3.3)–(3.4) is well defined for any $(\phi, v; a) \in \mathcal{D}^l \times H^l \times \mathbb{R}$ and belongs to the space $H^l \times H^l$. In particular, (3.3)–(3.4) defines a *dynamical system* (ODE) on $\mathcal{D}^l \times H^l$ which depends on the parameter $a \in \mathbb{R}$. For $s \geq 1$ given let $R_\phi \colon \mathcal{D}^s \to \mathcal{D}^s$ denote the right-translation in $\mathcal{D}^s$ by $\phi \in \mathcal{D}^s$ for $s \geq 1$. As

$$F_k(\phi, v; a) = R_\phi \circ A_k^{-1} \circ R_{\phi^{-1}} \circ R_\phi \circ B_k(a) \circ R_{\phi^{-1}} v,$$

the mapping

$$\mathcal{F}_k \colon (\phi, v; a) \mapsto (\phi, F_k(\phi, v; a))$$

can be written as a composition,

$$\mathcal{F}_k = \mathcal{A}_k \circ \mathcal{B}_k, \tag{3.5}$$

where

$$\mathcal{A}_k \colon (\phi, v) \mapsto (\phi, R_\phi \circ A_k^{-1} \circ R_{\phi^{-1}} v) \tag{3.6}$$

and

$$\mathcal{B}_k \colon (\phi, v; a) \mapsto (\phi, R_\phi \circ B_k(a) \circ R_{\phi^{-1}} v). \tag{3.7}$$

⋆ In particular, the multiplication $H^{l-2k} \times H^{l-2k} \to H^{l-2k}$, $(u, v) \mapsto u \cdot v$, is a continuous bilinear map.

The following result will allow us to prove Proposition 3.1 with the help of the local smoothness theorem for ODE's in Banach spaces (cf. [25, Chapter IV]).

Lemma 3.3 *Let $k \geq 2$. Then for any $l \geq 2k+1$,*

(i)
$$\mathcal{A}_k \in C^1(\mathcal{D}^l \times H^{l-2k}, \mathcal{D}^l \times H^l) \tag{3.8}$$

(ii)
$$\mathcal{B}_k \in C^1(\mathcal{D}^l \times H^l \times \mathbb{R}, \mathcal{D}^l \times H^{l-2k}). \tag{3.9}$$

As a consequence

(iii)
$$\mathcal{F}_k \in C^1(\mathcal{D}^l \times H^l \times \mathbb{R}, \mathcal{D}^l \times H^l)$$

and therefore the mapping $D^l \times H^l \times \mathbb{R} \to H^l \times H^l$, $(\phi, v; a) \mapsto (v, F_k(\phi, v; a))$ is C^1-smooth.

To prove Lemma 3.3, we first need to establish two auxiliary lemmas. Note that $\mathcal{A}_k$: $\mathcal{D}^l \times H^{l-2k} \to \mathcal{D}^l \times H^l$, defined by (3.6), is invertible and its inverse is given by

$$\mathcal{A}_k^{-1}: \mathcal{D}^l \times H^l \to \mathcal{D}^l \times H^{l-2k}, \ \ (\phi, v) \mapsto (\phi, R_\phi \circ A_k \circ R_{\phi^{-1}} v). \tag{3.10}$$

Lemma 3.4 *Let $k \geq 1$ and $l \geq 2k+1$. Then for any $\phi \in \mathcal{D}^l$ and $v \in H^l$ the following statements hold:*

(i) *For any $1 \leq s \leq 2k$,*

$$\partial_x^s (v \circ \phi^{-1}) = \sum_{j=1}^{s} P_{s,j}(\phi) \cdot (\partial_x^j v) \circ \phi^{-1}, \tag{3.11}$$

where $P_{s,j}$ is a polynomial in $\partial_x^m(\phi^{-1})$ $(1 \leq m \leq s)$ with integer coefficients.

(ii) *For any $1 \leq j \leq 2k$, $\partial_x^j(\phi^{-1}) = S_j(\phi)$ where $S_j(\phi)$ is a polynomial in $(\phi_x)^{-1} \circ \phi^{-1}$, $(\partial_x^m \phi) \circ \phi^{-1}$ $(1 \leq m \leq j)$ with integer coefficients.*

Proof of Lemma 3.4 As the proofs of items (i) and (ii) are similar we will prove here only (i). We argue by induction. For $s = 1$ one obtains by the chain rule that

$$(v \circ \phi^{-1})_x = (\phi^{-1})_x \cdot v_x \circ \phi^{-1}.$$

Assume that (3.11) is satisfied for s with $1 \leq s \leq 2k-1$. Differentiating both sides of (3.11) with respect to x and using that $((\partial_x^j v) \circ \phi^{-1})_x = (\partial_x^{j+1} v) \circ \phi^{-1} \cdot (\phi^{-1})_x$ one gets that (3.11) holds for $s+1$. This completes the proof of (i). The proof of (ii) is similar. □

The second lemma we need is the following one.

Lemma 3.5 *Let $s \geq 2$. Then the map, $\mathcal{D}^s \to H^{s-1}$, $\phi \mapsto 1/\phi_x$, is C^1-smooth.*

Proof of Lemma 3.5 Take $\phi \in \mathcal{D}^s$ and consider a neighborhood $U_\epsilon(\phi) = \{\phi - f \mid \|f\|_{H^s} < \epsilon\}$ of ϕ in $\mathcal{D}^s$ with $\epsilon > 0$ so small that

$$\|f_x/\phi_x\|_{H^{s-1}} < 1\,. \tag{3.12}$$

As H^{s-1} is a Banach algebra $\|f_x/\phi_x\|_{H^{s-1}} \leq C\|f_x\|_{H^{s-1}}\|1/\phi_x\|_{H^{s-1}}$ and, hence, (3.12) is satisfied for $0 < \epsilon \leq 1/(C\,\|1/\phi_x\|_{H^{s-1}})$. The lemma then follows from the expansion

$$\frac{1}{(\phi - f)_x} = \frac{1}{\phi_x}\left(1 + \frac{f_x}{\phi_x} + \left(\frac{f_x}{\phi_x}\right)^2 + \cdots\right)$$

which, by (3.12), converges in H^{s-1} uniformly in $U_\epsilon(\phi)$. □

Proof of Lemma 3.3 First we will show (3.8). Combining items (i) and (ii) of Lemma 3.4 one concludes that

$$R_\phi \circ A_k \circ R_{\phi^{-1}} v = \sum_{j=0}^{2k} P_j(\phi) \cdot \partial_x^j v, \tag{3.13}$$

where $P_j(\phi)$ is a polynomial in $(\phi_x)^{-1}$ and $\partial_x^s \phi$ $(1 \le s \le 2k)$. It suffices to show that $\mathcal{D}^l \times H^l \to \mathcal{D}^l \times H^{l-2k}$, $(\phi, v) \mapsto (\phi, R_\phi \circ A_k \circ R_{\phi^{-1}} v)$ is a local C^1-diffeomorphism. Note that it follows from Lemma 3.5 that the map

$$(\phi, v) \mapsto ((\phi_x)^{-1}, \phi_x, \ldots, \partial_x^{2k}\phi, v, v_x, \ldots, \partial_x^{2k} v), \;\; \mathcal{D}^l \times H^l \to (H^{l-2k})^{4k+2}$$

is C^1. Using that the multiplication $H^{l-2k} \times H^{l-2k} \to H^{l-2k}$, $(u, v) \mapsto u \cdot v$, is a bounded bilinear map, we conclude from (3.13) that the map (3.10) is C^1. For any $(\phi_0, v_0) \in \mathcal{D}^l \times H^l$, the differential $d_{(\phi_0,v_0)} \mathcal{A}_k^{-1}$: $H^l \times H^l \to H^l \times H^{l-2k}$ is of the form

$$\mathrm{d}_{(\phi_0,v_0)} \mathcal{A}_k^{-1}(\delta\phi, \delta v) = \begin{bmatrix} \delta\phi & 0 \\ S(\delta\phi) & R_{\phi_0} \circ A_k \circ R_{\phi_0^{-1}} \delta v \end{bmatrix}, \tag{3.14}$$

where S: $H^l \to H^{l-2k}$ and $R_{\phi_0} \circ A_k \circ R_{\phi_0^{-1}}$: $H^l \to H^{l-2k}$ are bounded linear maps. As $R_{\phi_0} \circ A_k \circ R_{\phi_0^{-1}}$: $H^l \to H^{l-2k}$ is invertible, the open mapping theorem implies that it is a linear isomorphism. Hence, $\mathrm{d}_{(\phi_0,v_0)} \mathcal{A}_k^{-1}$ is a linear isomorphism and by the inverse function theorem, the map

$$\mathcal{A}_k^{-1}: \mathcal{D}^l \times H^l \to \mathcal{D}^l \times H^{l-2k}, \;\; (\phi, v) \mapsto (\phi, R_\phi \circ A_k \circ R_{\phi^{-1}} v)$$

is a local C^1-diffeomorphism. This proves (i). Arguments similar to those used to prove that (3.10) is smooth, involving Lemma 3.4 and Lemma 3.5, prove (ii). Item (iii) is a direct consequence of (i), (ii), and (3.5). □

Fix $l \ge 2k+1$ and assume that the C^1-curve $(-2, 2) \to \mathcal{D}^l \times H^l$, $t \mapsto (\phi(t), v(t))$, is a solution of (3.3)–(3.4) with initial data (ϕ_0, v_0). Our next goal is to establish a relation between the regularity of the initial data (ϕ_0, v_0) and the regularity of $\phi(t)$—see (3.16) below. To this end note that the arguments in the proof of Lemma 6.2 in Sect. 6 show that

$$I_k(t) \equiv I_k(\phi(t), u(t)) := \phi_x(t)^2 \cdot \Big((A_k u(t)) \circ \phi(t)\Big) - aS(\phi(t)) \in H^{l-2k} \tag{3.15}$$

is independent of $t \in (-2, 2)$. Here $u = v \circ \phi^{-1}$ and $S(\phi(t))$ is the Schwarzian derivative $(\phi_x(t)\phi_{xxx}(t) - 3\phi_{xx}^2(t)/2)/\phi_x^2(t)$. As $u = \phi_t \circ \phi^{-1}$ one has $u_x = (\phi_{tx} \circ \phi^{-1}) \cdot (\phi^{-1})_x$. Using that $(\phi^{-1})_x = 1/(\phi_x \circ \phi^{-1})$ and hence $(\partial_x^{2k}\phi^{-1}) \circ \phi = -\partial_x^{2k}\phi/\phi_x^{2k+1} + \cdots$ one gets

$$(\partial_x^{2k} u) \circ \phi = (\partial_x^{2k}\phi_t)/\phi_x^{2k} - \phi_{tx} \cdot \partial_x^{2k}\phi/\phi_x^{2k+1} + \cdots,$$

where $\ldots$ stand for terms containing derivatives of ϕ or $\phi_t (=v)$ in x of order up to $2k-1$ and hence in H^{l-2k+1}. Together with (3.15) and the expression for A_k one then obtains

$$(-1)^k \phi_x^{2k-1} I_k(t) = \phi_x \cdot \partial_x^{2k}\phi_t - \phi_{tx} \cdot \partial_x^{2k}\phi + \cdots$$

Hence, we have for any $t \in (-2, 2)$

$$\phi_x(t) \cdot \partial_x^{2k}\phi_t(t) - \phi_{tx}(t) \cdot \partial_x^{2k}\phi(t) = (-1)^k \phi_x^2(t)\Big(\phi_x^{2k-3}(t) I_k(t) + J_k(\phi(t), v(t); a)\Big)$$

where $J_k(\phi, v; a) = P_k(\phi, v; a)/\phi_x^2$ and P_k is a polynomial in the variables $\phi, \partial_x\phi, \ldots, \partial_x^{2k-1}\phi$, $v, \partial_x v, \ldots, \partial_x^{2k-1}v$ and a. As $I_k(t) = I_k(0)$ we obtain that

$$\Big(\frac{\partial_x^{2k}\phi(t)}{\phi_x(t)}\Big)_t = (-1)^k\Big(\phi_x^{2k-3}(t) I_k(0) + J_k(\phi(t), v(t); a)\Big)$$

for any $t \in (-2, 2)$, hence upon integrating in t

$$\frac{\partial_x^{2k}\phi(t)}{\phi_x(t)} = \frac{\partial_x^{2k}\phi_0}{\phi_{0x}} + (-1)^k \int_0^t \Big(\phi_x^{2k-3}(s) I_k(0) + J_k(\phi(s), v(s); a)\Big)\mathrm{d}s. \tag{3.16}$$

Proof of Proposition 3.1 By Lemma 3.3 we can apply the existence and uniqueness theorem for ODE's in Banach spaces (with parameter) to conclude that for any $0 < M < \infty$ there exists a neighborhood $W_{2k+1} = W_{2k+1,M}$ of $(\mathtt{id}, 0)$ in $\mathcal{D}^{2k+1} \times H^{2k+1}$ such that for any initial data $(\phi_0, v_0) \in W_{2k+2}$ and any $a \in (-M, M)$ the ordinary differential equation (3.3)–(3.4) has a unique solution $\Psi(t) = (\phi(t), v(t))$ defined for $t \in (-2, 2)$.* Moreover, the solution $\Psi(t)$ depends C^1 smoothly on the parameter $a \in (-M, M)$ and the initial data $(\phi_0, v_0) \in W_{2k+1}$.

Define for $l \geq 2k+1$ the neighborhood $W_l := W_{2k+1} \cap (\mathcal{D}^l \times H^l)$ of the point $(\mathtt{id}, 0)$ in $\mathcal{D}^l \times H^l$. We will prove that for any initial data $(\phi_0, v_0) \in W_l$ and any $a \in (-M, M)$ there exists a solution of (3.3)–(3.4) in $\mathcal{D}^l \times H^l$ that is defined for $t \in (-2, 2)$ and depends C^1 smoothly on the initial data $(\phi_0, v_0) \in W_l$ and the parameter $a \in (-M, M)$. To this end we use induction in $l \geq 2k + 1$: For $l = 2k + 1$ the statement holds by the construction of W_{2k+1}. Now, take $a \in (-M, M)$ and $(\phi_0, v_0) \in W_{2k+2}$. Denote by $\Psi(t) = (\phi(t), v(t))$ the corresponding solution in $\mathcal{D}^{2k+1} \times H^{2k+1}$. As the right hand side of (3.3)–(3.4) is a C^1-smooth vector field on $\mathcal{D}^{2k+2} \times H^{2k+2}$ there exists a unique solution $\tilde{\Psi}(t) = \tilde{\Psi}(t; \phi_0, v_0, a)$ of (3.3)–(3.4) in $\mathcal{D}^{2k+2} \times H^{2k+2}$ defined on some maximal interval of existence $t \in (T_1, T_2)$ with $T_1 < 0 < T_2$ possibly depending on a and (ϕ_0, v_0). We claim that $T_2 \geq 2$ and $T_1 \leq -2$. As the two statements are proved similarly we concentrate on T_2 only. Arguing by contradiction suppose that $T_2 < 2$. Considered as a curve in $\mathcal{D}^{2k+1} \times H^{2k+1}$, the solution

$$\tilde{\Psi}(t) = (\tilde{\phi}(t), \tilde{v}(t)) \tag{3.17}$$

solves (3.3)–(3.4) in $\mathcal{D}^{2k+1} \times H^{2k+1}$ and therefore coincides with the solution Ψ: $(-2, 2) \to \mathcal{D}^{2k+1} \times H^{2k+1}$ of (3.3)–(3.4) on the interval $t \in (\max\{-2, T_1\}, T_2)$. For $t \in (\max\{-2, T_1\}, T_2)$ and $\tilde{\phi}$ defined by (3.17) equality (3.16) implies that

$$\partial_x^{2k}\tilde{\phi}(t) = \frac{\partial_x^{2k}\phi_0}{\partial_x\phi_0}\,\phi_x(t) + (-1)^k\phi_x(t)\int_0^t (\phi_x^{2k-3}(s) I_k(0) + J_k(\phi(s), v(s); a))\mathrm{d}s, \tag{3.18}$$

where $J_k(\phi, v; a) = P_k(\phi, v; a)/\phi_x^2$ and P_k is a polynomial in the variables $\phi, \partial_x\phi, \ldots, \partial_x^{2k-1}\phi$, $v, \partial_x v, \ldots, \partial_x^{2k-1}v$ and a. As $(\phi_0, v_0) \in W_{2k+2}$ we get from (3.15) that

* Note that for any $a \in \mathbb{R}$, $(\mathtt{id}, 0)$ is a zero of the vector field defined by the r.h.s. of (3.3)–(3.4).

$I_k(0) \in H^2$. Then equality (3.18) implies that $\partial_x^{2k}\tilde{\phi}(t) \in H^2$. Moreover, as $T_2 < 2$ by assumption one gets from (3.18) that the limit

$$\lim_{t\to T_2-0} \tilde{\phi}(t)$$

exists in $\mathcal{D}^{2k+2}$. As $\tilde{v} = \tilde{\phi}_t$ and $\tilde{v}_t = F_k(\tilde{\phi}, \tilde{v}; a)$ evolve both in H^{2k+2} for $T_1 < t < T_2$ and as $T_2 < 2$ by assumption one concludes, by taking the t-derivatives of both sides of the identity (3.18), that the limit

$$\lim_{t\to T_2-0} \tilde{v}(t)$$

exists in H^{2k+2}. Hence, there exists a limit

$$\lim_{t\to T_2-0} (\tilde{\phi}(t), \tilde{v}(t))$$

in $\mathcal{D}^{2k+2} \times H^{2k+2}$. This contradicts the fact that (T_1, T_2) is the maximal interval of existence of the solution $\tilde{\Psi}(t)$. Hence, $T_2 \geq 2$. In the same way one proves that $T_1 \leq -2$. As the vector field $\mathcal{F}$ is C^1 on $\mathcal{D}^{2k+2} \times H^{2k+2} \times \mathbb{R}$ the solution $t \mapsto \tilde{\Psi}(t; \phi_0, v_0, a)$, $(-2, 2) \to \mathcal{D}^{2k+2} \times H^{2k+2}$, depends C^1-smoothly on $(\phi_0, v_0, a) \in W_{2k+2} \times (-M, M)$.

The same arguments permit us to show that for any $l \geq 2k + 1$ and for any $(\phi_0, v_0) \in W_l$ and $a \in (-M, M)$ there exists a solution of (3.3)–(3.4) in $\mathcal{D}^l \times H^l$ which is defined for $t \in (-2, 2)$ and depends C^1 smoothly on the parameter $a \in (-M, M)$ and the initial data $(\phi_0, v_0) \in W_l$. Combining this with the formula (2.10) for $\dot{\alpha}(t)$ one gets that

$$\alpha(t) = a \cdot t - \frac{1}{2}\int_0^t \int_0^1 v_x(\tau)\, \frac{\phi_{xx}(\tau)}{\phi_x(\tau)^2}\, dx\, d\tau. \tag{3.19}$$

Finally, by (3.19) and Lemma 3.5, $\alpha \in C^1((-2, 2) \times W_l \times (-M, M), \mathbb{R})$. This completes the proof of Proposition 3.1. □

It follows from Proposition 3.1 that for any given $k \geq 2$ one can define the mapping

$$\mathtt{E}_k\colon U_{2k+1} \to \mathcal{D}^{2k+1} \times \mathbb{R}, \quad \mathtt{E}_k(u_0, a_0) := \Phi(t)|_{t=1},$$

where $U_{2k+1} \subseteq \mathtt{vir}_{2k+1}$ is given as in Proposition 3.1 and $\Phi(t) = (\phi(t), \alpha(t))$ is the solution of (2.8)–(2.14). According to Proposition 3.1 for any $l \geq 2k+1$ the restriction

$$\mathtt{E}_k|_{U_l}\colon U_l \to \mathcal{D}^l \times \mathbb{R}$$

of $\mathtt{E}_k$ to $U_l := U_{2k+1} \cap \mathtt{vir}_l$ is well-defined and C^1-smooth.

It follows from (2.8) to (2.14) that $\mathtt{E}_k(tu_0, ta_0) = \Phi(t)$. In particular, one gets that $d_{(\mathtt{id},0)}\mathtt{E}_k = \mathtt{id}_{\mathtt{vir}_{2k+1}}$ and by the inverse function theorem the neighborhood U_{2k+1} can be chosen so that

$$\mathtt{E}_k|_{U_{2k+1}}\colon U_{2k+1} \to V_{2k+1}$$

is a C^1-diffeomorphism where V_{2k+1} is a neighborhood of $(\mathtt{id}, 0)$ in $\mathcal{D}^{2k+1} \times \mathbb{R}$.

Lemma 3.6 *If* $(u_0, a_0) \in U_{2k+1}$ *and* $l \geq 2k + 1$, *then*

$$\mathtt{E}_k(u_0, a_0) \in \mathcal{D}^l \times \mathbb{R} \quad \textit{iff} \quad u_0 \in H^l.$$

Proof of Lemma 3.6 First we prove the "only if" part of the lemma. By the definition of U_{2k+1}, the statement is true for $l = 2k + 1$. Assume that it is true for any $2k + 1 \leq j < l$ and let $\mathtt{E}_k(u_0, a_0) := \Phi(1) = (\phi(1), \alpha(1)) \in H^l \times \mathbb{R}$. By Proposition 3.1 and the induction hypothesis we have that $(\phi(t), v(t))$ is in $C^1((-2, 2), \mathcal{D}^{l-1} \times H^{l-1})$. According to (3.16) one has

$$I_k(0, u_0, a_0) \int_0^1 \phi_x^{2k-3}(s)\mathrm{d}s = \frac{(-1)^k}{\phi_x(1)} \Big(\partial_x^{2k} \phi(1) - \int_0^1 J_k(\phi(s), v(s); a)\mathrm{d}s \Big). \quad (3.20)$$

By assumption

$$\frac{\partial_x^{2k} \phi(1)}{\phi_x(1)} \in H^{l-2k}, \quad \int_0^1 \phi_x^{2k-3}(s)\mathrm{d}s \in H^{l-2},$$

and

$$\frac{1}{\phi_x(1)} \int_0^1 J_k(\phi(s), v(s); a)\mathrm{d}s \in H^{l-2k}.$$

Thus, one gets from (3.20) that $A_k u_0 = I_k(0, u_0, a_0) \in H^{l-2k}$. Hence, $u_0 \in H^l$. The "if" statement of the lemma follows from Proposition 3.1. □

Lemma 3.7 *For any given* $(u_0, a_0) \in U_l$, $l \geq 2k + 1$,

$$(\mathrm{d}_{(u_0,a_0)}\mathtt{E}_k)(u, a) \in \mathtt{vir}_l \backslash \mathtt{vir}_{l+1}$$

for any $(u, a) \in \mathtt{vir}_l \backslash \mathtt{vir}_{l+1}$.

Proof of Lemma 3.7 The lemma is proved by passing to the variations of $(\phi(t), v(t))$ in (3.16) and then arguing as in the proof of the previous lemma. □

Proof of Theorem 1.2 Note that conditions (a), (b), and (c) of Proposition 5.5 in Sect. 5 hold in view of Lemma 3.6, Proposition 3.1, and Lemma 3.7, respectively. Hence, Proposition 5.5 can be applied and Theorem 1.2 is proved. □

4 Exponential maps corresponding to KdV and CH

If $k = 0$ the Euler equations (2.12) and (2.13) is the *Korteweg–de Vries equation* (KdV) with parameter $a_0 \in \mathbb{R}$,

$$u_t + 3uu_x - a_0 u_{xxx} = 0\,, \quad (4.1)$$

$$u|_{t=0} = u_0 \quad (4.2)$$

and if $k = 1$ we get the following variant of the *Camassa–Holm equation* (CH)

$$(1 - \partial_x^2)u_t = -2u_x(1 - \partial_x^2)u - u(1 - \partial_x^2)u_x + a_0 u_{xxx}, \quad (4.3)$$

$$u|_{t=0} = u_0, \quad (4.4)$$

with a_0 being again a real parameter. It is well known that both equations can be viewed as integrable Hamiltonian systems and both are bi-Hamiltonian [5, 9, 16, 19]. Of all the Hamiltonian vector fields induced by the H^k inner products ($k \geq 0$), only the cases $k = 0$ and $k = 1$ are bi-Hamiltonian relative to the canonical Lie–Poisson structure (cf. [12]).

The KdV and the CH equations are closely related. In fact, there is a correspondence between the so called KdV hierarchy and the one of (CH)—see [26, 27]. On the other hand, the two equations have quite different features with regard to (global) well-posedness—see, e.g., [6, 8, 13, 20, 31].

Denote by $\mathtt{vir}_l$ the space $H^l \times \mathbb{R}$.

Lemma 4.1 *There exist a neighborhood* U_3 *of the zero in* $\mathtt{vir}_3$ *and a time interval* $(-T,T)$, $T>0$, *such that for any* $l \geq 3$ *and any initial data* $(u_0,a_0) \in U_l := U_3 \cap \mathtt{vir}_l$ *Eqs.* (4.3) *and* (4.4) *have a unique solution* $u \in C^0((-T,T),H^l) \cap C^1((-T,T),H^{l-1})$ *which depends* C^1*-smoothly on the initial data* $(u_0,a_0) \in U_l$ *in the sense that* $u \in C^1((-T,T) \times U_l, H^{l-1})$.

Proof of Lemma 4.1 With the substitution

$$u(t,x) := v(t,x-3a_0t/2) + a_0/2, \tag{4.5}$$

Eqs. (4.3)–(4.4) transforms into the standard form of the Camassa–Holm shallow water equation (cf. [5, 7])

$$(1-\partial_x^2)v_t = -2v_x(1-\partial_x^2)v - v(1-\partial_x^2)v_x, \tag{4.6}$$

$$v|_{t=0} = u_0 - a_0/2 =: v_0. \tag{4.7}$$

Now, the statement of the lemma follows from the arguments used to prove Theorem 1.2. Indeed, according to [30] (see also [24]) the nonlinear equations (4.6) and (4.7) is the Euler equation of the geodesic equations corresponding to the right-invariant metric $\nu^{(1)}$ on the diffeomorphism group $\mathcal{D}$ generated by the scalar product on $T_{\mathtt{id}}\mathcal{D} \cong C^\infty$

$$\langle u,v\rangle_1 := \int_0^1 (uv + u_xv_x)\mathrm{d}x, \quad u,v \in C^\infty.$$

Using the same arguments as in the proof of Proposition 3.1, one shows that the geodesic equation on $T\mathcal{D}$ can be also considered as an ODE

$$(\dot\psi,\dot w) = (w, G(\psi,w)), \tag{4.8}$$

$$(\psi,w)|_{t=0} = (\psi_0,v_0) \tag{4.9}$$

on the tangent bundle $T\mathcal{D}^l$ of the Hilbert manifold $\mathcal{D}^l$ ($l \geq 3$) where the vector field $(\psi,w) \mapsto (w,G(\psi,w))$ is C^1-smooth in a neighborhood of $(\mathtt{id},0) \in T\mathcal{D}^l$ (cf. [10]). The local smoothness ODE theorem in Banach spaces [25, Chapter IV] then implies that there exist $T = T_l > 0$ and a neighborhood W_l of $(\mathtt{id},0)$ in $T\mathcal{D}^l$ such that for any $(\psi_0,v_0) \in W_l$ the nonlinear equations (4.8) and (4.9) has a unique solution $(\psi(t;\psi_0,v_0), w(t;\psi_0,v_0)) \in T\mathcal{D}^l$ for $t \in (-T,T)$ with $w(\cdot;\cdot,\cdot) \in C^1((-T,T) \times W_l, H^l)$ and $\psi(\cdot;\cdot,\cdot) \in C^1((-T,T) \times W_l, \mathcal{D}^l)$. Then

$$v(t;v_0) = w(t;\mathtt{id},v_0) \circ \psi(t;\mathtt{id},v_0)^{-1} \tag{4.10}$$

is the unique solution of (4.6)–(4.7) and has the property

$$v(\cdot;v_0) \in C^0((-T,T),H^l) \cap C^1((-T,T),H^{l-1}).$$

As the maps

$$\mathcal{D}^l \to \mathcal{D}^{l-1}, \quad \psi \mapsto \psi^{-1}$$

and

$$H^l \times \mathcal{D}^l \to H^{l-1}, \quad (w, \psi) \mapsto w \circ \psi$$

are C^1-smooth we conclude from (4.10) that

$$v \in C^1((-T, T) \times V_l, H^{l-1}) ,$$

where $V_l = W_l \cap T_{\mathtt{id}}\mathcal{D}$. Then, arguing as in the proof of Proposition 3.1 one proves that V_l can be taken of the form

$$V_l = V_3 \cap H^l$$

and T_l can be chosen to be T_3 and, hence, is independent of $l \geq 3$. Using these properties of v, the properties of u stated in Lemma 4.1 follow from formula (4.5). □

The following remark will be of use in the proof of Theorem 1.3.

Remark 4.2 As the vector field $(\psi, w) \mapsto (w, G_k(\psi, w))$ in (4.9) is of class C^1 in a neighborhood of $(\mathtt{id}, 0) \in T\mathcal{D}^l$, $l \geq 3$, the *local smoothness theorem* in [25, Chapter IV] implies that the partial derivatives $D_1D_3w(t; \psi_0, v_0)$ and $D_1D_3\psi(t; \psi_0, v_0)$ of $w(t; \psi_0, v_0)$ and $\psi(t; \psi_0, v_0)$ exist and, from the variational equation satisfied by $D_3w(t; \psi_0, v_0)$,

$$D_1D_3w(t; \psi_0, v_0) = D_3D_1w(t; \psi_0, v_0)$$

and

$$D_1D_3\psi(t; \psi_0, v_0) = D_3D_1\psi(t; \psi_0, v_0).^\star$$

In particular, the same is true for the solution

$$v(t; v_0) = w(t; \mathtt{id}, v_0) \circ \psi(t; \mathtt{id}, v_0)^{-1}$$

of the Camassa–Holm equations (4.6) and (4.7) as well as for its parametrized version (4.3)–(4.4).

Proof of Theorem 1.1 The case $k = 1$ follows from Lemma 4.1 and the existence of solutions of the ordinary differential equations (2.8)–(2.11) (cf. Remark 4.3 below). The case $k = 0$ follows for $\xi = (u_0, a_0)$ with $a_0 \neq 0$ from the well-posedness results of the KdV equation (cf., e.g., [3]) and for ξ with $a_0 = 0$ from the ones of the Burgers equation (cf., e.g., [4], [21]). The case $k \geq 2$ follows from Proposition 3.1 proved in Sect. 3. □

Proof of Theorem 1.3 As the statements for $k = 0$ and $k = 1$ are proved, similarly, we concentrate on $k = 1$ only. Taking $u_0 = c = constant$ one obtains from (4.3)–(4.4) that $u(t, x; c, a_0) \equiv c$. Solving (2.8)–(2.9) and (2.10)–(2.11) we then find that $\phi(t, x; c, a_0) = x + ct$ and $\alpha(t; c, a_0) = a_0t$. Hence,

$$\exp_1((c, a_0)) = (\tau_c, a_0) \in \mathtt{Vir},$$

where τ_c denotes the translation $x \mapsto x + c$ on $\mathbb{T}$.

* Let $1 \leq j \leq n$. We denote by $D_jf(x_1, \ldots, x_n)$ the partial derivative of f with respect to the jth variable at the point $(x_1, \ldots, x_n)$.

Our aim is to compute the Fréchet differential of the exponential map

$$(D\exp_1)|_{(c,a_0)}\colon \mathtt{vir} \to T_{(\tau_c,a_0)}\mathtt{Vir}.$$

To this end denote by $\xi := D_2u(t;u_0,a_0)(w)$ the partial directional derivative of the solution $u(t;u_0,a_0)$ with respect to the second variable u_0 in the direction $w \in C^\infty$ at the point $(t;u_0,a_0)$ (cf. Sect. 5), i.e. $\xi = \lim_{s\to 0}(u(t;u_0+sw,a_0) - u(t;u_0,a_0))/s$. Since the derivative $D_2u(t;u_0,a_0)$ is the restriction of the directional derivative of $u(t,\cdot,a_0)\colon H^l \to H^l$ to C^∞ where $l \geq 3$ we compute ξ by working in the Hilbert space H^l. As $u(t;u_0+sw,a_0)$ satisfies (4.3) with initial data $u|_{t=0} = u_0+sw$ one obtains from Remark 4.2 and a differentiation with respect to s that $\xi(t,x)$ satisfies the linear PDE

$$A_1\xi_t = -2u_xA_1\xi - 2\xi_xA_1u - uA_1\xi_x - \xi A_1u_x + a_0\xi_{xxx}, \tag{4.11}$$
$$\xi|_{t=0} = w. \tag{4.12}$$

Taking $u_0 = c$ and using that $u(t,x;c,a_0) \equiv c$, we obtain from (4.11) to (4.12) that $\xi = \xi(t,x;c,a_0,w)$ satisfies the linear PDE

$$A_1\xi_t = -2c\xi_x - cA_1\xi_x + a_0\xi_{xxx}, \tag{4.13}$$
$$\xi|_{t=0} = w. \tag{4.14}$$

If $a_0 = 2c$ the equation above becomes $A_1(\xi_t + 3c\xi_x) = 0$. As the operator $A_1 = 1-\partial_x^2\colon C^\infty \to C^\infty$ is a continuous bijection we obtain that $\xi_t+3c\xi_x = 0$ and $\xi|_{t=0} = w$. Solving the latter equation one gets $\xi(t,x) = w(x-3ct)$, i.e.

$$D_2u(t;c,2c)(w) = w(x-3ct). \tag{4.15}$$

Our next goal is to compute the directional derivatives $D_2\phi(t;c,a_0)$ (w) and $D_2\alpha(t;c,a_0)(w)$. Proceeding as above we linearize equations (2.8)–(2.11) at $u_0 = c$ and then find the directional derivatives by solving the corresponding linear equations.

Remark 4.3 Note that for any $l \geq 1$ Eqs. (2.8) and (2.9) can be regarded as a dynamical system (ODE) in the Hilbert space H^l depending on a parameter u from the Banach space $X := C^1([-T,T],H^{l+1})$

$$\dot\phi = F(t,\phi;u), \tag{4.16}$$
$$\phi|_{t=0} = \mathtt{id}, \tag{4.17}$$

where $F(t,\phi;u) := u(t)\circ\phi$. Since the composition $H^{l+1}\times H^l \to H^l$, $(u,\phi) \to u\circ\phi$ is C^1, it follows that $F \in C^1([-T,T]\times H^l\times X, H^l)$ and, hence, (4.16)–(4.17) has a (unique) solution $\phi(t;u)$ in H^l that belongs to the space $C^1([-T,T]\times X, H^l)$ (cf. [15, Sect. 3]).

Let $u(t;u_0,a_0)$ be the solution of (4.3)–(4.4). It follows from Lemma 4.1 and Remark 4.3 that the directional derivative $\eta := D_2\phi(t;u_0,a_0)(w)$ satisfies the variational equation

$$\eta_t = u_x\circ\phi\cdot\eta + D_2u(t;u_0,a_0)(w)\circ\phi, \tag{4.18}$$
$$\eta|_{t=0} = 0, \tag{4.19}$$

where $\phi = \phi(t;u_0,a_0)$ is the solution of (2.8)–(2.9) with $u = u(t;u_0,a_0)$. Taking $u_0 = c$ and $a_0 = 2c$ we have $u \equiv c$ and $\phi(t;x) = x+ct$ and obtain from (4.15) that

$\eta = \eta|_{u_0=c,a_0=2c}$ satisfies $\eta_t = w((x+ct) - 3ct) = w(x-2ct)$. Hence,

$$\eta = D_2\phi(t;c,2c)(w) = \int_0^t w(x-2c\tau)\mathrm{d}\tau. \tag{4.20}$$

As $\alpha(t;u_0,a_0) = a_0t - \frac{1}{2}\int_0^t\int_0^1 u_x(\tau,\phi(\tau,x))\ \mathrm{d}\log\phi_x(\tau,x)\mathrm{d}\tau$ one concludes from $u_x = 0$ *and* $\phi_x = 1$ that

$$D_2\alpha(t;c,a_0) = 0\ . \tag{4.21}$$

Taking c sufficiently small so that $(u_0,a_0) = (c,2c)$ is in the domain of definition of $\exp_1$ we obtain from (4.15), (4.20) and (4.21) that

$$(D\exp_1)|_{(c,2c)}(w,0) = \Big(\int_0^1 w(x-2c\tau)\mathrm{d}\tau, 0\Big). \tag{4.22}$$

Finally, taking $c = c_n := \frac{1}{n}$, $w(x) = w_n(x) := \sin n\pi x$ and $N > 0$ sufficiently large one obtains from the formula above that for any $n \geq N$

$$(D\exp_1)|_{(c_n,2c_n)}(w_n,0) = \Big(\int_0^1 w_n(x-2c_n\tau)\mathrm{d}\tau, 0\Big) = 0. \tag{4.23}$$

As $(c_n,2c_n) = (1/n,2/n) \to (0,0)$ for $n \to 0$ one then obtains from Remark 5.4 that there is *no* neighborhood U of zero in `vir` so that $\exp_1$ is a C^1_F-diffeomorphism from U onto a neighborhood of the unital element e in `Vir`. □

5 Appendix A: Calculus on Fréchet spaces

In this appendix we collect some definitions and notions from the calculus in Fréchet spaces. For more details we refer the reader to [17] (cf. also [24a]).

Fréchet spaces: Consider the pair $(X, \{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}})$ where X is a real vector space and $\{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is a countable collection of seminorms. We define a topology on X in the usual way using the collection of seminorms as follows: A basis of open neighborhoods of $0 \in X$ is given by the sets

$$U_{\epsilon,k_1,\ldots,k_s} := \{x \in X \mid ||x||_{k_j} < \epsilon\ \forall 1 \leq j \leq s\}$$

where $s, k_1, \ldots, k_s \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$. Then the topology on X is defined as the set of open sets generated by the sets $x + U_{\epsilon,k_1,\ldots,k_s}$ with $x \in X, s, k_1, \ldots, k_s \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$ arbitrary. In this way X becomes a topological vector space. Note that a sequence x_k converges to x in X iff for any $n \geq 0$, $||x_k - x||_n \to 0$ as $k \to \infty$.

Let X be a topological vector space whose topology is induced from the countable system of seminorms $\{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}}$. Then X is *Hausdorff* iff for any $x \in X$, $\|x\|_n = 0$ for any $n \in \mathbb{Z}_{\geq 0}$ implies $x = 0$. A sequence $(x_k)_{k\in\mathbb{N}}$ is called *Cauchy* iff it is a Cauchy sequence with respect to any of the seminorms $||\cdot||_n$, $n \in \mathbb{Z}_{\geq 0}$. By definition, X is complete iff every Cauchy sequence converges in X.

Definition 5.1 A pair $(X, \{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}})$ consisting of a topological vector space X and a countable system of seminorms $\{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is called a *Fréchet space*⋆ iff the topology of X is the one induced by $\{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}}$ and X is Hausdorff and complete.

⋆ Unlike for the standard notion of a Fréchet space, in this definition the countable system of seminorms defining the topology of X is a part of the structure of the space.

C^1_F-differentiability: Let f: $U \subseteq X \to Y$ be a map from an open set U of a Fréchet space X to a Fréchet space Y.

Definition 5.2 The *(directional) derivative* of f at the point $x \in U$ in the direction $h \in X$ is

$$D_x f(h) := \lim_{\epsilon \to 0} (f(x + \epsilon h) - f(x))/\epsilon \in Y \tag{5.1}$$

where the limit is taken with respect to the Fréchet topology of Y.

If the directional derivative $D_x f(h)$ exists then we say that *f is differentiable at x in the direction h.*

Definition 5.3 If the directional derivative $D_x f(h)$ exists for any $x \in U$ and any $h \in X$ and the map

$$(x, h) \mapsto D_x f(h), \ U \times X \to Y$$

is continuous with respect to the Fréchet topology on $U \times X$ and Y then f is called *continuously differentiable on U* or *C^1_F-smooth*. The space of all such maps is denoted by $C^1_F(U, Y)$.⋆ A map f: $U \to V$ from an open set $U \subseteq X$ onto an open set $V \subseteq Y$ is called *a C^1_F-diffeomorphism* if f is a homeomorphism and f as well as f^{-1} are C^1_F-smooth.

Remark 5.4 Using the chain rule one easily obtains that for any $x \in U$ the directional derivative $D_x f$: $X \to Y$ of a C^1_F-diffeomorphism f: $U \to V$ is a linear isomorphism.

We refer to [17] for the definitions of the higher derivatives ($k \geq 2$)

$$D^k_\bullet f\colon U \times \underbrace{X \times \cdots \times X}_{k} \to Y, \ (x, h_1, \ldots, h_k) \mapsto D^k_x f(h_1, \ldots, h_k)$$

and the definition of the space $C^k_F(U, Y)$. We only remark that as in the classical calculus in Banach spaces, $f \in C^k_F(U, Y)$ implies that the kth derivative $D^k_x f\colon \underbrace{X \times \cdots \times X}_{k} \to Y$ is a symmetric, k-linear continuous map for any $x \in U$.

In this paper we consider mainly the following spaces:

Fréchet space C^∞. The space $C^\infty \equiv C^\infty(\mathbb{T}, \mathbb{R})$ denotes the real vector space of real-valued C^∞-smooth, 1-periodic functions u: $\mathbb{R} \to \mathbb{R}$. The topology on C^∞ is induced by the countable system of Sobolev norms: $||u||_n := \left(\sum_{j=0}^n \int_0^1 u^{(j)}(x)^2 \, dx \right)^{1/2}$ with $n \geq 0$.

Fréchet manifold $\mathcal{D}$. By definition, $\mathcal{D}$ denotes the group of C^∞-smooth positively oriented diffeomorphisms of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A Fréchet manifold structure on $\mathcal{D}$ can be introduced as follows: Passing in domain and target to the universal cover $\mathbb{R} \to \mathbb{T}$, any element ϕ of $\mathcal{D}$ gives rise to a smooth diffeomorphism of $\mathbb{R}$ in $C^\infty(\mathbb{R}, \mathbb{R})$, again denoted by ϕ, satisfying the *normalization condition*

$$-1/2 < \phi(0) < 1/2 \tag{5.2}$$

⋆ Note that even in the case where X and Y are Banach spaces this definition of continuous differentiability is weaker than the usual one (cf. [17]). In order to distinguish it from the classical one we write C^1_F instead of C^1. We refer to [17] for a discussion of the reasons to introduce the notion of C^1_F-differentiability.

or

$$0 < \phi(0) < 1. \tag{5.3}$$

The function $f(x) := \phi(x) - x$ is 1-periodic and therefore lies in C^∞. Moreover, $f'(x) > -1$ for any $x \in \mathbb{R}$. The normalizations (5.2) and (5.3) give rise to two charts U_1, U_2 of $\mathcal{D}$ with $U_1 \cup U_2 = \mathcal{D}$

$$J_j\colon V_j \to U_j, f \mapsto \phi := \mathtt{id} + f,$$

where

$$V_1 := \{f \in C^\infty \mid |f(0)| < 1/2 \text{ and } f' > -1\} \subseteq C^\infty,$$
$$V_2 := \{f \in C^\infty \mid 0 < f(0) < 1 \text{ and } f' > -1\} \subseteq C^\infty.$$

As V_1, V_2 are both open sets in the Fréchet space C^∞, the construction above gives an atlas of Fréchet charts of $\mathcal{D}$. In this way, $\mathcal{D}$ is a Fréchet manifold modeled on C^∞.

Hilbert manifold $\mathcal{D}^s$ ($s \geq 2$). $\mathcal{D}^s$ denotes the group of positively oriented bijective transformations of $\mathbb{T}$ of class H^s. By definition, a bijective transformation ϕ of $\mathbb{T}$ is of class H^s iff the lift $\tilde{\phi}\colon \mathbb{R} \to \mathbb{R}$ of ϕ, determined by the normalization, $0 \leq \tilde{\phi}(0) < 1$, and its inverse $\tilde{\phi}^{-1}$ both lie in $H^s_{\text{loc}}(\mathbb{R}, \mathbb{R})$. As for $\mathcal{D}$ one can introduce an atlas for $\mathcal{D}^s$ with two charts in H^s, making $\mathcal{D}^s$ a Hilbert manifold modeled on H^s.

Hilbert approximations. Assume that for a given Fréchet space X there is a sequence of Hilbert spaces $\{(X_n, ||\cdot||_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ such that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X \quad \text{and} \quad X = \cap_{n=0}^\infty X_n,$$

where $\{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of norms inducing the topology on X so that $||x||_0 \leq ||x||_1 \leq ||x||_2 \leq \cdots \; \forall x \in X$. Such a sequence of Hilbert spaces $\{(X_n, ||\cdot||_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ is called a *Hilbert approximation* of the Fréchet space X. For Fréchet spaces admitting Hilbert approximations one can prove the following version of the inverse function theorem.

Proposition 5.5 *Let X and Y be Fréchet spaces admitting the Hilbert approximations $\{(X_n, ||\cdot||_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ and $\{(Y_n, |\cdot|_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ respectively. Assume that $f\colon U_0 \to V_0$ is a C^1-diffeomorphism between the open sets $U_0 \subseteq X_0$ and $V_0 \subseteq Y_0$ of the Hilbert spaces X_0 and Y_0 respectively. Define the sets $U_n := U_0 \cap X_n$ and $V_n := V_0 \cap Y_n$ and assume that the following properties are satisfied for any $n \geq 0$:*

(a) if $x \in U_0$ then $f(x) \in V_n$ iff $x \in X_n$;
(b) the restriction $f|_{U_n}\colon U_n \to Y_n$ is C^1-smooth;
(c) for any $x \in U_n$, $d_x f(X_n \setminus X_{n+1}) \subseteq Y_n \setminus Y_{n+1}$.

Then $U := U_0 \cap X$ and $V := V_0 \cap Y$ are open sets in X and Y respectively with $f(U) \subseteq V$ and the mapping $f_\infty := f|_U\colon U \to V$ is a C^1_F-diffeomorphism.

Remark 5.6 The same results hold for approximations of X and Y by Banach spaces instead of Hilbert spaces.

Proof of Proposition 5.5 Note first that for any $n \geq 0$, the set U_n is open in X_n and the set V_n is open in Y_n. As $f\colon U_0 \to V_0$ is bijective, (a) implies that $f|_{U_n}\colon U_n \to V_n$ is well defined and bijective as well. Indeed, the injectivity of $f|_{U_n}$ follows from the injectivity of f. As f is bijective, for any $y \in V_n \subset V_0$ there exists a unique element $x \in U_0$

such that $f(x) = y \in V_n$. Then according to (a), $x \in X_n$, hence $x \in U_n = U_0 \cap X_n$. Thus for any $n \geq 0$, $f|_{U_n}: U_n \to V_n$, and therefore $f_\infty: U \to V$, are bijective.

For any $n \in \mathbb{Z}_{\geq 0}$, let $f_n := f|_{U_n}$. According to (b), $f_n: U_n \to V_n$ is then a C^1-smooth bijective map. In order to prove that $f_n^{-1}: V_n \to U_n$ is C^1-smooth as well we will use the inverse function theorem in Hilbert spaces. Take $x \in U_n$ and consider the differential $\mathrm{d}_x f_n: X_n \to Y_n$. As

$$(\mathrm{d}_x f_0)|_{X_n} = \mathrm{d}_x f_n$$

and $f_0: U_0 \to V_0$ is a C^1-diffeomorphism one concludes that $\mathrm{d}_x f_n$ is injective. We prove by induction that $\mathrm{d}_x f_n: X_n \to Y_n$ is onto for any $x \in U_n$. For $n = 0$ the statement is true by assumption. Assume that it holds for any $k \leq n-1$. As for any $x \in U_n$, $\mathrm{d}_x f_{n-1}$ is onto, it follows that for any $\eta \in Y_n \subset Y_{n-1}$ there exists $\xi \in X_{n-1}$ such that $\mathrm{d}_x f_n(\xi) = \eta \in Y_n$. Then ($c$) implies that $\xi \in X_n$. Hence, we have shown that $\mathrm{d}_x f_n: X_n \to Y_n$ is bijective for any $x \in U_n$. By the inverse function theorem for Hilbert spaces, $f_n: U_n \to V_n$ is a C^1-diffeomorphism. In particular, for any $n \geq 0$ the maps

$$U_n \times X_n \to Y_n,\ (x, \xi) \mapsto \mathrm{d}_x f_n(\xi) \tag{5.4}$$

and

$$V_n \times Y_n \to X_n,\ (y, \eta) \mapsto \mathrm{d}_y(f_n^{-1})(\eta) \tag{5.5}$$

are continuous. As for any $x \in U$ and $n \geq 0$,

$$D_x f_\infty = (\mathrm{d}_x f_n)|_X,$$

one gets from (5.4)–(5.5) that

$$U \times X \to Y,\ (x, \xi) \mapsto D_x f_\infty(\xi) \quad \text{and} \quad V \times Y \to X,\ (y, \eta) \mapsto D_y(f_\infty^{-1})(\eta)$$

are continuous. In particular one concludes that

$$f_\infty: U \to V$$

is a C_F^1-diffeomorphism. □

6 Appendix B: Euler equation on `vir`

In this appendix we derive the Euler–Lagrange equations of geodesics of the right-invariant weak Riemannian metrics $\mu^{(k)}$ (cf. (2.5)) on the Virasoro group `Vir` given by the action principle. The cases $k = 0, 1$ were considered in [22, 24, 30] in a somewhat formal way using a purely algebraic approach (cf. [1, 15, 18, 28]). At the end of the appendix we derive for any $k \geq 0$ a conservation law for the geodesic flow of the metric $\mu^{(k)}$.

Let $\gamma(s,t) = (\phi(s,t), \alpha(s,t)) \in \mathtt{Vir}$ be a C_F^2-smooth variation ($-\epsilon < s < \epsilon, 0 \leq t \leq T$)

$$\gamma: (-\epsilon, \epsilon) \times [0, T] \to \mathtt{Vir} \tag{6.1}$$

of the C_F^2-smooth curve $\gamma(t) \equiv \gamma(0,t): [0,T] \to \mathtt{Vir}$ such that for any $-\epsilon < s < \epsilon$

$$\gamma(s, 0) \equiv e \quad \text{and} \quad \gamma(s, T) \equiv \gamma(T) \tag{6.2}$$

where e denotes the unital element in $\texttt{Vir}$, $e = (\texttt{id}, 0)$. It follows from the multiplication (2.2) on the Virasoro group $\texttt{Vir}$ that the derivative $\mathrm{d}_e R_{(\phi,\alpha)}$ of the right-translation $R_{(\phi,\alpha)}\colon \texttt{Vir} \to \texttt{Vir}$, $(\psi,\beta) \mapsto (\psi,\beta)\circ(\phi,\alpha)$ at $(\psi,\beta) = e$, $\mathrm{d}_e R_{(\phi,\alpha)}\colon T_e\texttt{Vir} \to T_{(\phi,\alpha)}\texttt{Vir}$, is given by

$$\mathrm{d}_e R_{(\phi,\alpha)}(u,a) = \Big(u\circ\phi,\ a - \frac{1}{2}\int_0^1 u_x(\phi(x))\mathrm{d}\log\phi_x(x)\Big),$$

where $(\phi,\alpha)\in\texttt{Vir}$ and $(u,a)\in\texttt{vir}(\cong T_e\texttt{Vir})$. In particular,

$$(\mathrm{d}_e R_{\gamma(s,t)})^{-1}(\dot\gamma(s,t)) = \Big(\phi_t(s,t)\circ\phi(s,t)^{-1},\ \dot\alpha(s,t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}(s,t)}{\phi_x(s,t)}\mathrm{d}\log\phi_x(s,t)\Big),$$

where $\dot\gamma(s,t) = \frac{\partial\gamma(s,t)}{\partial t}$, $\partial_t \equiv \partial/\partial t$ and $\partial_x \equiv \partial/\partial x$. Hence, for $k \geq 0$

$$\mu^{(k)}_{\gamma(s,t)}(\dot\gamma(s,t),\dot\gamma(s,t)) = \sum_{j=0}^{k}\int_0^1 (\partial_x^j(\phi_t(s,t)\circ\phi(s,t)^{-1}))^2\,\mathrm{d}x$$
$$+\Big(\dot\alpha(s,t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}(s,t)}{\phi_x(s,t)}\,\mathrm{d}\log\phi_x(s,t)\Big)^2$$

and the *action functional* is given by

$$E_{\mu^{(k)}}(\gamma(s,\cdot)) := \sum_{j=0}^{k} E_j(\gamma(s,\cdot)) + A(\gamma(s,\cdot)), \tag{6.3}$$

where for $0 \leq j \leq k$

$$E_j(\gamma(s,\cdot)) := \frac{1}{2}\int_0^T\int_0^1 (\partial_x^j(\phi_t(s,t)\circ\phi(s,t)^{-1}))^2\,\mathrm{d}x\,\mathrm{d}t$$

and

$$A(\gamma(s,\cdot)) := \frac{1}{2}\int_0^T\Big(\dot\alpha(s,t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}(s,t)}{\phi_x(s,t)}\,\mathrm{d}\log\phi_x(s,t)\Big)^2\,\mathrm{d}t.$$

Denoting $u(t) := \phi_t(t)\circ\phi^{-1}(t)$ where $\phi(t) = \phi(0,t)$ we obtain

$$\phi_t(t) = u(t)\circ\phi(t). \tag{6.4}$$

Introduce the variations $\delta E_j(\gamma) := \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}E_j(\gamma(s,\cdot))$, $\delta\phi := \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\phi(s,t)$, and $\delta\alpha := \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\alpha(s,t)$.

Remark 6.1 As $\gamma \in C^2_F((-\epsilon,\epsilon)\times[-T,T],\texttt{Vir})$ it follows from the definition of the space $C^2_F((-\epsilon,\epsilon)\times[-T,T],\texttt{Vir})$ that $\alpha = \alpha(s,t)$ lies in $C^2((-\epsilon,\epsilon)\times[-T,T])$ and $\phi(s,t) = \phi(s,t,x)$ considered as a $\mathbb{R}$-valued function of s, t and x is continuous and for any $j \geq 0$ the partial derivatives $\partial_s\partial_x^j\phi(s,t,x)$, $\partial_t\partial_x^j\phi(s,t,x)$, $\partial_s^2\partial_x^j\phi(s,t,x)$, $\partial_t^2\partial_x^j\phi(s,t,x)$ and $\partial_s\partial_t\partial_x^j\phi(s,t,x)$ are continuous. Moreover, in the expressions above all the partial derivative operators ∂_s, ∂_t, and ∂_x commute.

Using (6.4), $\phi(s,t) \circ \phi^{-1}(s,t) = \mathtt{id}$ and the change of variables $y = \phi(t)^{-1}(x)$ one obtains

$$\begin{aligned}
\delta E_j(\gamma) &= \int_0^T \int_0^1 \partial_x^j u \cdot \partial_x^j \Big((\delta\phi)_t \circ \phi^{-1} - \phi_{tx} \circ \phi^{-1} \cdot \delta\phi^{-1} \Big) \mathrm{d}x\mathrm{d}t \\
&= (-1)^j \int_0^T \int_0^1 \partial_x^{2j} u \cdot \Big((\delta\phi)_t \circ \phi^{-1} - \frac{\phi_{tx} \circ \phi^{-1}}{\phi_x \circ \phi^{-1}} \cdot (\delta\phi) \circ \phi^{-1} \Big) \mathrm{d}x\mathrm{d}t \\
&= (-1)^j \int_0^T \int_0^1 (\partial_x^{2j} u) \circ \phi \cdot \phi_x \cdot (\delta\phi)_t \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + (-1)^{j+1} \int_0^T \int_0^1 (\partial_x^{2j} u) \circ \phi \cdot \phi_{tx} \cdot \delta\phi \, \mathrm{d}x \, \mathrm{d}t \\
&= (-1)^{j+1} \int_0^T \int_0^1 \frac{\delta E_j}{\delta \phi} \cdot \delta\phi \, \mathrm{d}x \, \mathrm{d}t, \qquad (6.5)
\end{aligned}$$

where

$$\begin{aligned}
\frac{\delta E_j}{\delta\phi} &:= ((\partial_x^{2j} u) \circ \phi \cdot \phi_x)_t + (\partial_x^{2j} u) \circ \phi \cdot \phi_{tx} \\
&= (\partial_x^{2j} u_t) \circ \phi \cdot \phi_x + 2(\partial_x^{2j} u) \circ \phi \cdot \phi_{tx} \\
&\quad + (\partial_x^{2j} u_x) \circ \phi \cdot \phi_t \cdot \phi_x \\
&= \Big((\phi_x \circ \phi^{-1}) \cdot (\partial_x^{2j} u_t + 2u_x \partial_x^{2j} u + u \partial_x^{2j} u_x) \Big) \circ \phi \, . \qquad (6.6)
\end{aligned}$$

Here we have used that $\phi_t = u \circ \phi$ and thus $u_x \circ \phi = \frac{\phi_{tx}}{\phi_x}$. Analogously, one has

$$\begin{aligned}
\delta A(\gamma) &= \int_0^T \Big(\dot\alpha(t) + \frac{1}{2} \int_0^1 \frac{\phi_{tx}}{\phi_x} \, \mathrm{d}\log\phi_x \Big) \cdot \Big((\delta\alpha)_t + \frac{1}{2} \delta \int_0^1 \frac{\phi_{tx}}{\phi_x} \, \mathrm{d}\log\phi_x \Big) \mathrm{d}t \\
&= - \int_0^T \Big(\dot\alpha(t) + \frac{1}{2} \int_0^1 \frac{\phi_{tx}}{\phi_x} \, \mathrm{d}\log\phi_x \Big)_t \cdot \delta\alpha \, \mathrm{d}t \qquad (6.7) \\
&\quad + \frac{1}{2} \int_0^T \Big(\dot\alpha(t) + \frac{1}{2} \int_0^1 \frac{\phi_{tx}}{\phi_x} \, \mathrm{d}\log\phi_x \Big) \Big(\delta \int_0^1 \frac{\phi_{tx}}{\phi_x} \mathrm{d}\log\phi_x \Big) \mathrm{d}t. \qquad (6.8)
\end{aligned}$$

It follows from (6.3), (6.5), and (6.7) that for $\delta\phi = 0$ and $\delta\alpha$ arbitrary

$$\dot\alpha(t) + \frac{1}{2} \int_0^1 \frac{\phi_{tx}}{\phi_x} \, d\log\phi_x = a = \text{const}, \qquad (6.9)$$

where $a = \dot\alpha(0)$. In particular,

$$\dot\alpha(t) = a - \frac{1}{2} \int_0^1 \frac{\phi_{tx}}{\phi_x} \, \mathrm{d}\log\phi_x \, . \qquad (6.10)$$

Provided (6.10) is satisfied

$$\begin{aligned}
\delta A(\gamma) &= \frac{a}{2}\delta\int_0^T\int_0^1 \frac{\phi_{tx}}{\phi_x}\ \mathrm{d}\log\phi_x\ \mathrm{d}t\\
&= \frac{a}{2}\delta\int_0^T\int_0^1 (\log\phi_x)_t(\log\phi_x)_x\ \mathrm{d}x\,\mathrm{d}t\\
&= \frac{a}{2}\int_0^T\int_0^1 \Big(\frac{(\delta\phi)_x}{\phi_x}\Big)_t(\log\phi_x)_x\ \mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{a}{2}\int_0^T\int_0^1 (\log\phi_x)_t\Big(\frac{(\delta\phi)_x}{\phi_x}\Big)_x\ \mathrm{d}x\,\mathrm{d}t\\
&= -a\int_0^T\int_0^1 (\log\phi_x)_{tx}\Big(\frac{(\delta\phi)_x}{\phi_x}\Big)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

As $u_x\circ\phi = \frac{\phi_{tx}}{\phi_x} = (\log\phi_x)_t$ we get that $u_{xx}\circ\phi\cdot\phi_x = (\log\phi_x)_{tx}$ and hence

$$\begin{aligned}
\delta A(\gamma) &= -a\int_0^T\int_0^1 u_{xx}\circ\phi\cdot(\delta\phi)_x\ \mathrm{d}x\,\mathrm{d}t\\
&= a\int_0^T\int_0^1 u_{xxx}\circ\phi\cdot\phi_x\cdot\delta\phi\ \mathrm{d}x\,\mathrm{d}t. \qquad (6.11)
\end{aligned}$$

Finally, (6.3), (6.5) and (6.11) show that $\delta E(\gamma)=0$ iff $\gamma\colon[-T,T]\to \mathtt{Vir}$ satisfies the equations:

$$\phi_t(t) = u(t)\circ\phi(t), \qquad (6.12)$$

$$\dot\alpha(t) = a - \frac{1}{2}\int_0^1 \frac{\phi_{tx}}{\phi_x}\ \mathrm{d}\log\phi_x, \qquad (6.13)$$

$$A_k u_t = -2u_x A_k u - uA_k u_x + au_{xxx}, \qquad (6.14)$$

where $A_k := \sum_{j=0}^k (-1)^j\partial_x^{2j}$ and $a=\dot\alpha(0)$. The system (6.12)–(6.14) can be divided into two parts: the *Euler equation* part which is the equation for the curve $(u(t),a(t)) := (\mathrm{d}_e R_{\gamma(t)})^{-1}(\dot\gamma(t))$ in the Lie algebra $\mathtt{vir}$,

$$\begin{aligned}
A_k u_t &= -2u_x A_k u - uA_k u_x + au_{xxx},\\
\dot a &= 0 \qquad (6.15)
\end{aligned}$$

and the translation part

$$\phi_t(t) = u(t)\circ\phi(t), \qquad (6.16)$$

$$\dot\alpha(t) = a - \frac{1}{2}\int_0^1 \frac{\phi_{tx}}{\phi_x}\ \mathrm{d}\log\phi_x, \qquad (6.17)$$

coming from the right-translation $\mathrm{d}_e R_{\gamma(t)}(u(t),a(t)) = \dot\gamma(t)$. Hence we have derived the system of Eqs. (2.8)–(2.13) stated in Sect. 2.

We end this section by deriving for any $k\geq 0$ a conservation law for the geodesic flow of the metric $\mu^{(k)}$ on *Vir*. This conservation law corresponds to the Noether symmetries of the right-invariant metric $\mu^{(k)}$ on $\mathtt{Vir}$ corresponding to the left-invariant vector fields on $\mathtt{Vir}$ and can be formally obtained as follows: Given any geodesic $\gamma_0\colon[0,T]\to\mathtt{Vir}$ and any element $\xi\in\mathtt{vir}$, consider the 1-parameter family of curves

$\gamma\colon (-\epsilon, \epsilon) \times [0, T] \times \mathbb{R} \to \mathtt{Vir}$ $(\epsilon > 0)$

$$\gamma(s,t) := \gamma_0(t) \circ \eta(s)$$

where $\eta(s) := \exp^{\mathtt{Vir}}_{\mathtt{Lie}}(s\xi)$ denotes the Lie group exponential map on $\mathtt{Vir}$ (cf. Sect. 7). Note that $\dot{\gamma}(s,t) = \frac{\mathrm{d}}{\mathrm{d}t} R_{\eta(s)}\gamma_0(t) = \mathrm{d}_{\gamma_0(t)} R_{\eta(s)}\dot{\gamma}_0(t)$. As $\mu^{(k)}$ is a right invariant metric it follows that

$$\begin{aligned}\mu^{(k)}_{\gamma(s,t)}(\dot{\gamma}(s,t), \dot{\gamma}(s,t)) &= \mu^{(k)}_{R_{\eta(s)}\gamma_0(t)}(\mathrm{d}_{\gamma_0(t)} R_{\eta(s)}\dot{\gamma}_0(t), \mathrm{d}_{\gamma_0(t)} R_{\eta(s)}\dot{\gamma}_0(t)) \\ &= \mu^{(k)}_{\gamma_0(t)}(\dot{\gamma}_0(t), \dot{\gamma}_0(t))\,.\end{aligned}$$

Hence the action functional

$$E_{\mu^{(k)}}(\gamma(s,\cdot)) := \int_0^T \mu^{(k)}_{\gamma(s,t)}(\dot{\gamma}(s,t), \dot{\gamma}(s,t))\mathrm{d}t$$

is independent of s. In particular,

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} E_{\mu^{(k)}}(\gamma(s,\cdot))$$

for any choice of $\xi = (u,a) \in \mathtt{vir}$. Computing the above variation explicitly as above but with varying endpoints (i.e. without assuming (6.2)) one formally obtains the function (6.18) defined below.

Lemma 6.2 *If $\gamma\colon [-T,T] \to \mathtt{Vir}$, $\gamma(t) = (\phi(t), \alpha(t))$ is a geodesic of the right-invariant Riemannian metric $\mu^{(k)}$ on the Virasoro group $\mathtt{Vir}$ (cf. Definition 2.4) then*

$$I_k(\dot{\gamma}(t)) = \Big(\phi_x(t)^2 \cdot (A_k u(t)) \circ \phi(t)\Big) - aS(\phi(t)) \tag{6.18}$$

is independent of t where $u(t) := \phi_t(t) \circ \phi^{-1}(t)$, $a := \dot{\alpha}(0)$ and $S(\phi(t))$ denotes the Schwarzian derivative $(\phi_x(t)\phi_{xxx}(t) - 3\phi^2_{xx}(t)/2)/\phi^2_x(t)$.

Proof Using Eq. (6.14) and the identity $\phi_t(t) = u(t) \circ \phi(t)$ one obtains

$$\begin{aligned}I_k(\dot{\gamma}(t))_t &= 2\phi_x \cdot \phi_{tx} \cdot (A_k u) \circ \phi + \phi_x^2 \cdot (A_k u_t) \circ \phi \\ &\quad + \phi_x^2 \cdot (A_k u_x) \circ \phi \cdot \phi_t - aS(\phi)_t \\ &= 2\phi_x^2 \cdot u_x \circ \phi \cdot (A_k u) \circ \phi + \phi_x^2 \cdot (-2u_x A_k u - u A_k u_x + a u_{xxx}) \circ \phi \\ &\quad + \phi_x^2 \cdot (A_k u_x) \circ \phi \cdot u \circ \phi - aS(\phi)_t \\ &= a(u_{xxx} \circ \phi \cdot \phi_x^2 - S(\phi)_t)\,.\end{aligned} \tag{6.19}$$

As $\phi_t = u \circ \phi$ one has $u_x \circ \phi = \frac{\phi_{tx}}{\phi_x} = (\log \phi_x)_t$. Hence,

$$u_{xx} \circ \phi \cdot \phi_x = \left(\frac{\phi_{xx}}{\phi_x}\right)_t \tag{6.20}$$

and

$$u_{xxx} \circ \phi \cdot \phi_x^2 + u_{xx} \circ \phi \cdot \phi_{xx} = \left(\frac{\phi_{xx}}{\phi_x}\right)_{xt}\,. \tag{6.21}$$

Finally, (6.21) and (6.20) give

$$\begin{aligned} u_{xxx} \circ \phi \cdot \phi_x^2 &= \left(\frac{\phi_{xx}}{\phi_x}\right)_{xt} - \frac{\phi_{xx}}{\phi_x}\left(\frac{\phi_{xx}}{\phi_x}\right)_t \\ &= \left(\left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2\right)_t \\ &= S(\phi)_t \end{aligned} \tag{6.22}$$

which together with (6.19) implies that $(I_k(\gamma(t)))_t = 0$. □

7 Appendix C: Lie group exponential map for Vir

In this appendix we prove that the Lie group exponential map of $\mathtt{Vir}$,

$$\exp_{\mathtt{Lie}}^{\mathtt{Vir}}: \mathtt{vir} \to \mathtt{Vir},$$

is *not* locally onto near the unital element e of $\mathtt{Vir}$, i.e. there are elements in $\mathtt{Vir}$ arbitrarily close to e which are not in the image of $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$. The value of $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$ at $(u,a) \in \mathtt{vir}$ is defined as the time 1-map of the flow $t \mapsto (\phi(t), \alpha(t))$ corresponding to the right invariant vector field induced by $(u,a) \in \mathtt{vir}$,

$$\begin{aligned} (\phi_t, \alpha_t) &= d_e R_{(\phi,\alpha)}(u,a) \\ &= \left(u \circ \phi, a - \frac{1}{2}\int_0^1 u_x \circ \phi \; d\log\phi_x\right), \end{aligned} \tag{7.1}$$

$$(\phi,\alpha)|_{t=0} = (\mathtt{id}, 0). \tag{7.2}$$

Using the Hilbert approximation $(\mathtt{vir}_l)_{l\geq 1}$ of $\mathtt{vir}$ and the fact that $\mathcal{D}$ is the diffeomorphism group of the *compact* manifold $\mathbb{T}$ one concludes that there exists a unique solution of the above initial value problem and that it is defined globally in time. Hence, $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$ is well defined and it turns out to be C_F^∞-smooth. Kopell [23] (see also [17] or [29]) proved that the Lie group exponential map $\exp_{\mathtt{Lie}}^{\mathcal{D}}$ of the diffeomorphism group is not locally onto near the unital element of $\mathcal{D}$. This result can be used to prove a similar result for $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$.

Proposition 7.1 *The map* $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}: \mathtt{vir} \to \mathtt{Vir}$ *is not locally onto near the unital element of* $\mathtt{Vir}$*, i.e. there are elements arbitrarily close to e which are not in the image of* $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$.

Proof The Lie group exponential map $\exp_{\mathtt{Lie}}^{\mathcal{D}}: T_{\mathtt{id}}\mathcal{D} \to \mathcal{D}$ for the diffeomorphism group $\mathcal{D}$ is defined to be the time 1-map of the flow given by ($u \in T_{\mathtt{id}}\mathcal{D} \cong C^\infty$)

$$\begin{aligned} \phi_t &= \mathrm{d}_{\mathtt{id}} R_\phi u = u \circ \phi, \\ \phi|_{t=0} &= \mathtt{id}, \end{aligned}$$

where here, $R_\phi: \mathcal{D} \to \mathcal{D}$ denotes the right translation on $\mathcal{D}$. It then follows that

$$\exp_{\mathtt{Lie}}^{\mathcal{D}} = \pi \circ \exp_{\mathtt{Lie}}^{\mathtt{Vir}}|_{T_{\mathtt{id}}\mathcal{D}\times\{0\}} \tag{7.3}$$

and

$$\pi \circ \exp_{\mathtt{Lie}}^{\mathtt{Vir}}(\mathtt{vir}) = \exp_{\mathtt{Lie}}^{\mathcal{D}}(T_{\mathtt{id}}\mathcal{D}), \tag{7.4}$$

where π: $\mathtt{Vir} = \mathcal{D} \times \mathbb{R} \to \mathcal{D}$ is the projection onto the first component. By [23] (see also [17, p. 123] or [29, p. 1018]) $\exp^{\mathcal{D}}_{\mathtt{Lie}}$ is *not* locally onto near the unital element of $\mathcal{D}$. We then conclude from (7.4) that $\exp^{\mathtt{Vir}}_{\mathrm{Lie}}$ is not locally onto near the unital element of $\mathtt{Vir}$ as well. □

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