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A flat strip theorem for ptolemaic spaces

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1 Main result and motivation

A metric space (X, d) is called *ptolemaic* or short a PT space, if for all quadruples of points $x, y, z, w \in X$ the Ptolemy inequality

$$|xy|\,|zw| \;\leq\; |xz|\,|yw| \;+\; |xw|\,|yz| \tag{1}$$

holds, where $|xy|$ denotes the distance $d(x, y)$.

We prove a flat strip theorem for geodesic ptolemaic spaces. Two unit speed geodesic lines $c_0, c_1 : \mathbb{R} \to X$ are called *parallel*, if their distance is sublinear, i.e. if $\lim_{t\to\infty} \frac{1}{t} d(c_0(t), c_1(t)) = \lim_{t\to-\infty} \frac{1}{t} d(c_0(t), c_1(t)) = 0$.

Theorem 1.1 *Let X be a geodesic PT space which is homeomorphic to $\mathbb{R} \times [0, 1]$, such that the boundary curves are parallel geodesic lines, then X is isometric to a flat strip $\mathbb{R} \times [0, a] \subset \mathbb{R}^2$ with its euclidean metric.*

We became interested in ptolemaic metric spaces because of their relation to the geometry of the boundary at infinity of CAT(-1) spaces (compare [3,6]). We therefore think that these spaces have the right to be investigated carefully.

Our paper is a contribution to the following question

Q: Are proper geodesic ptolemaic spaces CAT(0)-spaces?

We give a short discussion of this question at the end of the paper in Sect. 5. Main ingredients of our proof is a theorem of Hitzelberger and Lytchak [8] about isometric embeddings of geodesic spaces into Banach spaces and the Theorem of Schoenberg [10] characterizing inner product spaces by the PT inequality.

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2 Preliminaries

In this section we collect the most important basic facts about geodesic PT spaces which we will need in our arguments. If we do not provide proofs in this section, these can be found in [5,7].

Let X be a metric space. By $|xy|$ we denote the distance between points $x, y \in X$. We will always parametrize geodesics proportionally to arclength. Thus a geodesic in X is a map $c : I \to X$ with $|c(t)c(s)| = \lambda|t - s|$ for all $s, t \in I$ and some constant $\lambda \geq 0$. A metric space is called geodesic if every pair of points can be joined by a geodesic.

In addition we will use the following convention in this paper. If a geodesic is parametrized on $[0, \infty)$ or on $\mathbb{R}$, the parametrization is *always* by arclength. A geodesic $c : [0, \infty) \to X$ is called a *ray*, a geodesic $c : \mathbb{R} \to X$ is called a *line*.

In the sequel X will always denote a geodesic metric space. For $x, y \in X$ we denote by $m(x, y) = \{z \in X \mid |xz| = |zy| = \frac{1}{2}|xy|\}$ the set of midpoints of x and y. A subset $C \subset X$ is *convex*, if for $x, y \in C$ also $m(x, y) \subset C$.

A function $f : X \to \mathbb{R}$ is *convex* (resp. *affine*), if for all geodesics $c : I \to X$ the map $f \circ c : I \to \mathbb{R}$ is convex (resp. affine).

The space X is called *distance convex* if for all $p \in X$ the distance function $d_p = |\cdot\ p|$ to the point p is convex. It is called *strictly distance convex*, if the functions $t \mapsto (d_p \circ c)(t)$ are strictly convex whenever $c : I \to X$ is a geodesic with $|c(t)\,c(s)| > ||p\,c(t)| - |p\,c(s)||$ for all $s, t \in I$, i.e., neither $c(t)$ and $c(s)$ being on a geodesic from p to the other. This definition is natural, since the restriction of d_p to a geodesic segment containing p is never strictly convex. The Ptolemy property easily implies:

Lemma 2.1 *A geodesic PT space is distance convex.*

As a consequence, we obtain that for PT metric spaces local geodesics are geodesics. Here we call a map $c : I \to X$ a *local geodesic*, if for all $t \in I$ there exists a neighborhood $t \in I' \subset I$, such that $c_{|I'}$ is a geodesic.

Lemma 2.2 ([7]) *If X is distance convex, then every local geodesic is globally minimizing.*

In [5] we gave examples of PT spaces which are not strictly distance convex. However, if the space is proper, then the situation is completely different.

Theorem 2.3 ([7]) *A proper, geodesic PT space is strictly distance convex.*

Since we have a relatively short proof of this result, we present the proof in Sect. 4.

Corollary 2.4 ([5]) *Let X be a proper, geodesic PT space. Then for $x, y \in X$ there exists a unique midpoint $m(x, y) \in X$. The midpoint function $m : X \times X \to X$ is continuous.*

Corollary 2.5 ([7]) *Let X be a proper, geodesic PT space, and $A \subset X$ be a closed and convex subset. Then there exists a continuous projection $\pi_A : X \to A$.*

Remark 2.6 For CAT(0) spaces this projection is always 1-Lipschitz. We do not know if π_A is 1-Lipschitz for general proper geodesic PT spaces.

The strict convexity of the distance function together with the properness implies easily (cf. Corollary 2.4)

Corollary 2.7 *Let X be a proper, geodesic PT space and let $x, y \in X$. Then there exists a unique geodesic $c_{xy} : [0, 1] \to X$ from x to y and the map $X \times X \times [0, 1] \to X$, $(x, y, t) \mapsto c_{xy}(t)$ is continuous.*

We call two rays $c, c' : [0, \infty) \to X$ *asymptotic*, if $\lim_{t\to\infty} \frac{1}{t}|c(t)c'(t)| = 0$.

Corollary 2.8 *Let X be a proper geodesic PT space and $c_1, c_2 : [0, \infty) \to X$ asymptotic rays with the same initial point $c_1(0) = c_2(0) = p$. Then c_1=c_2.*

Proof Assume that there exists $t_0 > 0$ such that $x = c_1(t_0) \neq c_2(t_0) = y$. Let $m = m(x, y)$. By Theorem 2.3 we have $|pm| < t_0$. Let $\delta = t_0 - |pm| > 0$. For $t > t_0$ consider the points $x, y, x_t = c_1(t_0 + t), y_t = c_2(t_0 + t)$. Note that $\frac{1}{t}|x_t y_t| \to 0$ by assumption. We write $|xy_t| = t + \alpha_t$ with $0 \le \alpha_t$ and $|yx_t| = t + \beta_t$ with $0 \le \beta_t$. The PT inequality applied to the four points gives

$$(t + \alpha_t)(t + \beta_t) \le t^2 + |xy|\,|x_t y_t|$$

and thus $(\alpha_t + \beta_t) \le \frac{1}{t}|x_t y_t|\,|xy| \to 0$. Thus for t large enough $\alpha_t \le \delta$. Therefore $|y_t m| \le \frac{1}{2}(|y_t x| + |y_t y|) \le t + \delta/2$, which gives the contradiction $(t + t_0) = |py_t| \le |pm| + |my_t| \le (t + t_0 - \delta/2)$. □

We now collect some results on the Busemann functions of asymptotic rays and parallel line.

X denotes always a geodesic PT space. Let $c : [0, \infty) \to X$ be a geodesic ray. As usual we define the *Busemann function* $b_c(x) = \lim_{t\to\infty}(|xc(t)| - t)$. Since b_c is the limit of the convex functions $d_{c(t)} - t$, it is convex.

The following proposition implies that, in a PT space, asymptotic rays define (up to a constant) the same Busemann functions.

Proposition 2.9 ([7]) *Let X be a PT space, let $c_1, c_2 : [0, \infty) \to X$ be asymptotic rays with Busemann functions $b_i := b_{c_i}$. Then $(b_1 - b_2)$ is constant.*

Let now $c : \mathbb{R} \to X$ be a geodesic line parameterized by arclength. Let $c^\pm : [0, \infty) \to X$ be the rays $c^+(t) = c(t)$ and $c^-(t) = c(-t)$. Let further $b^\pm := b_{c^\pm}$.

Lemma 2.10 ([7]) *$(b^+ + b^-) \ge 0$ and $(b^+ + b^-) = 0$ on the line c.*

We now consider Busemann functions for parallel lines.

Proposition 2.11 ([7]) *Let $c_1, c_2 : \mathbb{R} \to X$ be parallel lines with with Busemann functions $b_1^\pm$ and $b_2^\pm$. Then $(b_1^+ + b_1^-) = (b_2^+ + b_2^-)$.*

Corollary 2.12 *If $c_1, c_2 : \mathbb{R} \to X$ are parallel lines. Then there are reparametrizations of c_1, c_2 such that $b_1^+ = b_2^+$ and $b_1^- = b_2^-$.*

Proof Since $b_1^+ - b_2^+$ is constant by Proposition 2.9 we can obviously shift the parametrization of c_2 such that $b_1^+ = b_2^+$. It follows now from Proposition 2.11 that then also $b_1^- = b_2^-$. □

Corollary 2.13 *Let X be a geodesic space which is covered by parallels to to a line $c : \mathbb{R} \to X$; i.e. for any point $x \in X$ there exists a line c_x parallel to c with $x = c_x(0)$. Then the Busemann functions $b^\pm$ of c are affine.*

Proof We show that $b^+ + b^- = 0$. Let therefore $x \in X$ and let $b_x^\pm$ be the Busemann functions of c_x. By Proposition 2.11 $b^+ + b^- = b_x^+ + b_x^-$. Now $(b_x^+ + b_x^-)(x) = 0$, hence $(b^+ + b^-)(x) = 0$. Thus the sum of the two convex functions b^+ and b^- is affine. It follows that b^+ and b^- are affine. □

More generally the following holds:

Corollary 2.14 *Let $c : \mathbb{R} \to X$ be a line, then the Busemann functions $b^{\pm}$ are affine on the convex hull of the union of all lines parallel to c.*

Proof Indeed the above argument shows that $b^+ + b^-$ is equal to 0 on all parallel lines. Since $b^+ + b^-$ is convex and ≥ 0 by Lemma 2.10, $b^+ + b^- = 0$ on the convex hull of all parallel lines. Thus b^+ and b^- are affine on this convex hull. □

3 Proof of the main result

We prove a slightly stronger version of the main Theorem, namely:

Theorem 3.1 *Let X be a geodesic PT space which is topologically a connected 2-dimensional manifold with boundary ∂X, such that the the boundary consists of two parallel geodesic lines. Then X is isometric to a flat strip $\mathbb{R} \times [0, a] \subset \mathbb{R}^2$ with its euclidean metric.*

Using Corollary 2.12 we can assume that $\partial X = c(\mathbb{R}) \cup c'(\mathbb{R})$, where $c, c' : \mathbb{R} \to X$ are parallel lines with the same Busemann functions $b^{\pm}$. In particular $b^+(c(t)) = b^+(c'(t)) = -t$ and $b^-(c(t)) = b^-(c'(t)) = t$. Let $a := |c(0)c'(0)|$ and for $t \in \mathbb{R}$ let $h_t : [0, a] \to X$ the geodesic from $c(t)$ to $c'(t)$. We emphasize here, that h_0 is parametrized by arclength, but we do not know, if h_t has unit speed for $t \neq 0$. We also define $c_0 := c$ and $c_a := c'$. Define $h : \mathbb{R} \times [0, a] \to X$ by $h(t, s) = h_t(s)$. With H_t we denote the set $h_t([0, a])$. By Corollary 2.14 the Busemann functions $b^{\pm}$ are affine on the image of h and thus $b^+(h(t, s)) = -t$ and $b^-(h(t, s)) = t$ on H_t.

We claim that h is a homeomorphism: Clearly h is continuous by Corollary 2.7. To show injectivity we note first that $H_t \cap H_{t'} = \emptyset$ for $t \neq t'$ since b^+ has different values on the sets and secondly that for fixed t the map h_t is clearly injective. Since c_0, c_a are parallel, i.e. the length of h_t is sublinear, we easily see that h is a proper map. Since ∂X is in the image of h and $\mathbb{R} \times (0, a)$, $X \setminus \partial X$ are 2-dimensional connected manifolds and h is injective and proper, we see that h is a homeomorphism.

Lemma 3.2 *For all $x \in X$ there exists a unique line $c_x : \mathbb{R} \to X$ with c_x parallel to c_0 and c_a and $c_x(0) = x$.*

Proof The Uniqueness follows from Corollary 2.8. To show the existence let $x \in H_{t_0}$. Consider for i large enough the unit speed geodesics $c_i^+ : [0, d_i] \to X$ from x to $c_0(i)$, where $d_i = |xc_0(i)|$. By local compactness a subsequence will converge to a limit ray $c_x^+ : [0, \infty) \to X$ with $c_x^+(0) = x$. For topological reasons c_x^+ intersects H_t for $t \geq t_0$. Let $c_x^+(\varphi(t)) \in H_t$, then the sublinearity of the length of H_t implies that $\varphi(t)/t \to 1$ and that c_x^+ is asymptotic to c_0. Furthermore the convex function b^+ has slope -1 on c_x^+, i.e. $b^+(c_x^+(t)) = -t_0 - t$.

In a similar way we obtain a ray $c_x^- : [0, \infty) \to X$ with $c_x^-(0) = x$, c_x^- asymptotic to c_0^- with $b^+(c_x^-(t)) = -t_0 + t$. Now define $c_x : \mathbb{R} \to X$ by $c_x(t) = c_x^+(t)$ for $t \geq 0$ and $c_x(t) = c_x^-(-t)$ for $t \leq 0$. Then c_x is a line since

$$2t \geq |c_x(t)c_x(-t)| \geq |b^+(c_x(t)) - b^+(c_x(-t))| = 2t,$$

and hence $|c_x(t)c_x(-t)| = 2t$. □

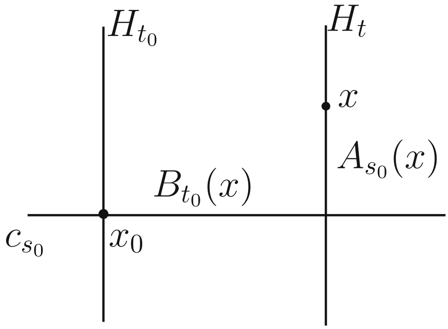
For $s \in [0, a]$ let $c_s := c_{h(0,s)}$ be the parallel line through $h(0, s)$. Consider $c : \mathbb{R} \times [0, a] \to X$, $c(t, s) = c_s(t)$. This is another parametrization of X. Note that $b^+(c_s(t)) = -t$.

Remark 3.3 We do not know at the moment whether $c(t,s) = h(t,s)$, our final result will imply that.

Since we have the foliation of X by the lines c_s, we have the property:

(A): If $t, t' \in \mathbb{R}$, $x \in H_t$, then there exist $x' \in H_{t'}$ with $|xx'| = |t - t'|$.

For $0 \le s \le a$ we define the *fibre distance* $A_s : X \to \mathbb{R}$ in the following way. Let $x \in X$, $x = c_{s'}(t')$, i.e. $x \in H_{t'}$. Then $A_s(x) = \pm|xc_s(t')|$, where the sign equals the sign of $(s' - s)$. Thus $A_s(x)$ is the distance in the fibre $H_{t'}$ from the point x to the intersection point $c_s(\mathbb{R}) \cap H_{t'}$. Note that by easy triangle inequality arguments A_s is a 2-Lipschitz function.



We also define for $t \in \mathbb{R}$ the function $B_t : X \to \mathbb{R}$ by $B_t(x) = (t' - t)$, when $x \in H_{t'}$. Note that B_t is 1-Lipschitz and affine, since b^+ is 1-Lipschitz and affine.

For fixed $x_0 = c_{s_0}(t_0) \in X \setminus \partial X$ consider the map $F_{x_o} : X \to \mathbb{R}^2$ defined by

$$F_{x_0}(x) = (B_{t_0}(x), A_{s_0}(x)).$$

Lemma 3.4 *F_{x_0} is a bilipschitz map, where $\mathbb{R}^2$ carries the standard euclidean metric d_{eu}, more precisely for all $x, y \in X$ we have*

$$\frac{1}{4}|xy| \le d_{\mathrm{eu}}(F_{x_0}(x), F_{x_0}(y)) \le 2|xy|.$$

Proof Since B_{t_0} is 1-Lipschitz and A_{s_0} is 2-Lipschitz, also F_{x_0} is 2-Lipschitz. Now assume $x \in H_t$, $y \in H_{t'}$. We claim that $|F_{x_0}(x) - F_{x_0}(y)| \ge \frac{1}{4}|xy|$. To prove this claim, we can assume that $|B_{t_0}(x) - B_{t_0}(y)| \le \frac{1}{4}|xy|$. By Property (A) there exists $x' \in H_{t'}$ with $|xx'| = |t - t'| \le \frac{1}{4}|xy|$. Thus $|x'y| \ge \frac{3}{4}|xy|$ and hence

$$|A_{s_0}(y) - A_{s_0}(x)| \ge |A_{s_o}(y) - A_{s_0}(x')| - |A_{s_0}(x') - A_{s_0}(x)|.$$

Note that

$$|A_{s_o}(y) - A_{s_0}(x')| = |yx'|$$

and

$$|A_{s_0}(x') - A_{s_0}(x)| \le 2|x'x|,$$

since A_{s_0} is 2-Lipschitz. Thus

$$|A_{s_0}(y) - A_{s_0}(x)| \ge |yx'| - 2|x'x| \ge \frac{1}{4}|xy|.$$

□

For $\lambda > 0$ we define $F^\lambda_{x_0} : X \to \mathbb{R}^2$ by $F^\lambda_{x_0}(x) = \lambda F_{x_0}(x)$. Then $F^\lambda_{x_0} : (X, \lambda d) \to (\mathbb{R}^2, d_{\text{eu}})$ is also a bilipschitz with the same constants $\frac{1}{4}$ and 2 for all $\lambda > 0$. Now consider an increasing sequence $\lambda_i \to \infty$ and let d_{λ_i} be the metric on $W_{\lambda_i} = \lambda_i \cdot (F_{x_0}(X)) \subset \mathbb{R}^2$ such that $F^{\lambda_i}_{x_0} : (X, \lambda_i d) \to (W_i, d_{\lambda_i})$ is an isometry. By the above we have $\frac{1}{2} d_{\text{eu}} \le d_{\lambda_i} \le 4 d_{\text{eu}}$.

Proposition 3.5 *If $\lambda_i \to \infty$ then d_{λ_i} converges uniformly on compact subsets to the standard euclidean distance d_{eu}.*

Proof Since x_0 is an inner point of X, $W_{\lambda_1} \subset W_{\lambda_2} \subset \cdots$ and $\bigcup W_{\lambda_i} = \mathbb{R}^2$. Since $\frac{1}{2} d_{\text{eu}} \le d_{\lambda_i} \le 4 d_{\text{eu}}$ any subsequence of the integers has itself a subsequence $i_j \to \infty$ with $d_{\lambda_{i_j}} \to d_\omega$ for some accumulation metric d_ω on $\mathbb{R}^2$. We show that $d_\omega = d_{\text{eu}}$ is always the the euclidean distance and hence d_{λ_i} will converge to d_{eu}.

To prove this we collect some properties of the accumulation metric d_ω:

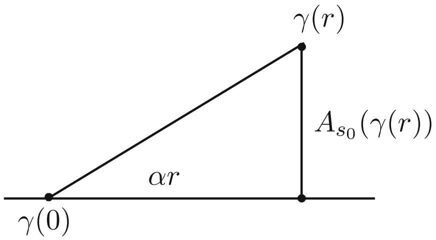
(a) $(\mathbb{R}^2, d_\omega)$ is a geodesic PT space.
(b) By construction $F^\lambda_{x_0}$ maps the geodesic c_{s_0} to the line $t \mapsto (t, 0)$ in $\mathbb{R}^2$ and the geodesic segment H_t to a part of the line $s \mapsto (\lambda(t - t_0), s)$. Therefore $t \mapsto (t, 0)$ is a geodesic parametrized by arclength in the metric $(\mathbb{R}^2, d_\omega)$ and $s \mapsto (t, s)$ is a geodesic parametrized by arclength for all s. Each of these vertical geodesics $s \mapsto (t, s)$ is contained in a level set of the Busemann function b_1 of the line $t \mapsto (t, 0)$. Thus $b_1(t, s) = -t$ and b_1 is affine as a limit of affine functions.
(c) The property (A) implies in the limit that for $x = (t, s)$ and $t' \in \mathbb{R}$ there exists $y = (t', s')$ with $|xy| = |t - t'|$. In particular the lines $s \mapsto (t, s)$ are all parallel. Thus if b_2 is the Busemann function of $s \mapsto (0, s)$, then this function is affine by Corollary 2.13. Note that $b_2(0, s) = -s$ and $b_2(t, s) = b_2(t, 0) - s$. Since b_2 is affine and $t \mapsto (t, 0)$ is a geodesic, we have $b_2(t, 0) = \alpha t$ for some $\alpha \in \mathbb{R}$ and hence $b_2(t, s) = \alpha t - s$.

Thus the two affine functions b_1 and b_2 separate the points in $(\mathbb{R}^2, d_\omega)$. It follows by the result of Hitzelsberger–Lytchak [8], that $(\mathbb{R}^2, d_\omega)$ is isometric to a normed vector space. It follows then from the theorem of Schoenberg [10], that $(\mathbb{R}^2, d_\omega)$ is isometric to an inner product space. We claim that the constant α equals 0: Since the line $s \mapsto (0, s)$ lies in some level set of the Busemannn function of the line $t \mapsto (t, 0)$ and the space is an inner product space, the two lines are orthogonal, i.e. $\alpha = 0$. It now follows easily that $d_\omega = d_{\text{eu}}$. □

Consider now a unit speed geodesic $\gamma : [0, d] \to X$ with $\gamma(0) = c_{s_0}(t_0) \in X \setminus \partial X$. Since B_{t_0} is affine, we have $B_{t_0}(\gamma(r)) = \alpha r$ for some $\alpha \in \mathbb{R}$.

Corollary 3.6 *With this notation we have*

$$\lim_{r \to 0} \frac{A^2_{s_0}(\gamma(r))}{r^2} = 1 - \alpha^2.$$



Proof Note that $F_{x_0}(\gamma(r)) = (\alpha r, A_{s_0}(\gamma(r)))$. By Proposition 3.5

$$d_{\mathrm{eu}}(0, F_{x_0}^{1/r}(\gamma(r))) \to \frac{1}{r}|x_0\gamma(r)| = 1.$$

Now

$$d_{\mathrm{eu}}^2(0, F_{x_0}^{1/r}(\gamma(r))) = \alpha^2 + \frac{A_{s_0}^2(\gamma(r))}{r^2}.$$

□

Let $\sigma = c_s(\mathbb{R})$ be one of the parallel lines with $0 \le s \le a$ considered as closed convex subset of X. We then have the projection $\pi_\sigma : X \to \sigma$ from Corollary 2.5. We show that the projection stays in the same fibre.

Lemma 3.7 $b^+(\pi_\sigma(x)) = b^+(x)$

Proof It suffices to prove the result for $\sigma = c_s(\mathbb{R})$, where $0 < s < a$, since for $s = 0, a$ it then follows by continuity.

Assume that $\pi_\sigma(x) = x_0 \in H_{t_0}$, while $x \in H_t$. Let $\gamma : [0, d] \to X$ be the unit speed geodesic from x_0 to x where $d = |x_0 x|$. Let $D : X \to [0, \infty)$ be the distance to σ, i.e. $D(x) = |x\pi_\sigma(x)|$. Note that $D(x) \le |A_s(x)|$ and that $D(\gamma(r)) = r$. Since b^+ is affine we have $b^+(\gamma(r)) = \alpha r - t_0$ for some $\alpha \in \mathbb{R}$.

We have to show that $\alpha = 0$. By Corollary 3.6

$$\lim_{r\to 0} \frac{A_s^2(\gamma(r))}{r^2} = 1 - \alpha^2.$$

If $|\alpha| \ne 0$ this would imply that for $r > 0$ small enough $|A_s(\gamma(r))| < r = D(\gamma(r))$, in contradiction to $D(x) \le |A_s(x)|$. □

Lemma 3.8 *For* $s_1, s_2 \in [0, 1]$ *the function* $t \mapsto |c_{s_1}(t)c_{s_2}(t)|$ *is constant.*

Proof Let $c = c_{s_1}$ and $c' = c_{s_2}$.

We put $\mu(t) = |c(t)c'(t)|$. By Lemma 3.7 $c'(t)$ is a closest to $c(t)$ point on $c'(\mathbb{R})$, and vice versa, $c(t)$ is a closest to $c'(t)$ point on $c(\mathbb{R})$ for every $t \in \mathbb{R}$. Thus $|c(t)c'(t')|, |c'(t)c(t')| \ge \max\{\mu(t), \mu(t')\}$ for each $t, t' \in \mathbb{R}$. Applying the Ptolemy inequality to the quadruple $(c(t), c(t'), c'(t'), c'(t))$, we obtain

$$\max\{\mu(t), \mu(t')\}^2 \le |c(t)c'(t')||c'(t)c(t')| \le \mu(t)\mu(t') + (t - t')^2.$$

We show that $\mu(a) = \mu(0)$ for every $a \in \mathbb{R}$. Assume W.L.G. that $a > 0$ and put $m = 1/\min_{0\le s\le a} \mu(s)$. Then $|\mu(t) - \mu(t')| \le m(t - t')^2$ for each $0 \le t, t' \le a$. Now

$$\mu(a) - \mu(0) = \mu(s) - \mu(0) + \mu(2s) - \mu(s) + \cdots + \mu(a) - \mu((k-1)s),$$

where $s = a/k$ for $k \in \mathbb{N}$. It follows $|\mu(a) - \mu(0)| \le mks^2 = ma^2/k \to 0$ as $k \to \infty$. Hence, $\mu(a) = \mu(0)$. □

As a consequence we have $|c(t, s)c(t, s')| = |s - s'|$ for all $t \in \mathbb{R}$ and of course we also have $|c(t, s)c(t', s)| = |t - t'|$ for all $s \in \mathbb{R}$. Note that Lemma 3.8 also implies the formula $A_{s_0}(c(t, s)) = s - s_0$.

We finally want to show that $|c(t, s)c(t', s')| = \sqrt{|t - t'|^2 + |s - s'|^2}$.

We assume for simplicity $t' \geq t$ and $s' \geq s$. Let $\gamma : [0, d] \to X$ be a unit speed geodesic from $c(t, s)$ to $c(t', s')$ with $d = |c(t, s)c(t', s')|$. We can write $\gamma(r) = c(\gamma_1(r), \gamma_2(r))$. By our assumption γ_1 and γ_2 are nondecreasing. Since $\gamma_1(r) = B_{t_0}(\gamma(r))$ is affine we have

$$\gamma_1(r) = t + \frac{t' - t}{d} r.$$

Note that by the above formula for A_s we have for $r_0, r_1 \in [0, d]$

$$A_{\gamma_2(r_0)}(\gamma(r_1)) = \gamma_2(r_1) - \gamma_2(r_0).$$

Therefore it follows from Corollary 3.6 that for every $r_0 \in [0, d)$

$$\lim_{r \to 0} \frac{(\gamma_2(r_0 + r) - \gamma_2(r_0))^2}{r^2} = 1 - \frac{(t' - t)^2}{d^2}.$$

This implies that γ_2 is differentiable with derivative

$$\gamma_2'(r_0) = \sqrt{1 - \frac{(t' - t)^2}{d^2}},$$

in particular the derivative is constant and therefore also γ_2 is affine and hence

$$\gamma_2(r) = s + \frac{s' - s}{d} r.$$

The formula for the derivative also implies

$$\frac{s' - s}{d} = \gamma_2'(r_0) = \sqrt{1 - \frac{(t' - t)^2}{d^2}}$$

which finally shows our claim $d^2 = (t' - t)^2 + (s' - s)^2$.

4 A short proof of strict convexity

In this section we give a short proof of Theorem 2.3.

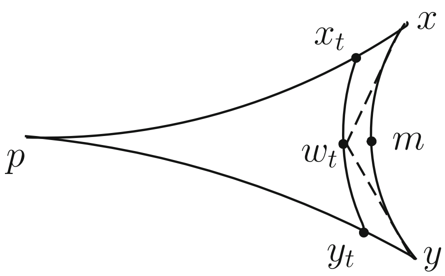
Theorem 4.1 *A proper, geodesic PT metric space is strictly distance convex.*

For the proof we need the following elementary

Lemma 4.2 *Let $f : [0, a] \to \mathbb{R}$ be a 1-Lipschitz convex function with $f(0) = 0$. For $t > 0$ define $g : (0, a] \to \mathbb{R}$ such that $f(t) = tg(t)$. Then $g(0) = \lim_{t \to 0} g(t)$ exists and $-1 \leq g(0) \leq 1$.*

Proof (of the Theorem) Since we already know that the distance function d_p is convex, it suffices to show that for $x, y \in X$ with $|xy| > ||px| - |py||$ there exits a midpoint $m \in m(x, y)$ such that for $|pm| < \frac{1}{2}(|px| + |py|)$. Using this, it is not hard to see that the midpoint is unique.

We choose a geodesic px from p to x and a geodesic py from p to y. For $t > 0$ small, let $x_t \in px$ and $y_t \in py$ be the points with $|x_t x| = t$ and $|y_t y| = t$. We choose geodesics $x_t y_t$ from x_t to y_t. For fixed t small enough there exists by continuity a point $w_t \in x_t y_t$ with $|xw_t| = |w_t y|$. By triangle inequality $|xw_t| = |w_t y| \geq a := \frac{1}{2}|xy|$. Using the properness of (X, d), it is elementary to show that there exists a sequence $t_i \to 0$, such that $\lim_{i \to \infty} w_{t_i} = m$ and $m \in m(x, y)$. Hence the function $\phi(t_i) = |w_{t_i} m| \to 0$ as $t_i \to 0$.



Let us assume to the contrary that

$$|pm| = \frac{1}{2}(|px| + |py|)$$

We have $a = \frac{1}{2}|xy| = |xm| = |my|$. Let $b = |px|, c = |pm|, d = |py|$ and we assume w.l.o.g that $b \leq c \leq d$. By assumption we have $2c = b + d$. We write

$$|mx_t| = a + ta_x(t), |my_t| = a + ta_y(t)$$

with the functions $a_x(t), a_y(t)$ according to the Lemma. The PT inequality applied to p, x_{t_i}, m, y_{t_i} gives

1. $(a + t_i a_x(t_i))(d - t_i) + (a + t_i a_y(t_i))(b - t_i) \geq c|x_{t_i} y_{t_i}|$. The sum of the PT inequalities for x, x_{t_i}, w_{t_i}, m and m, w_{t_i}, y_{t_i}, y give that
2. $a(a + t_i a_x(t_i)) + a(a + t_i a_y(t_i)) \leq a|x_{t_i} y_{t_i}| + 2t_i\phi(t_i)$. From (1) and (2) we obtain
3. $(a+t_i a_x(t_i))(d-t_i)+(a+t_i a_y(t_i))(b-t_i) \geq c((a+t_i a_x(t_i))+a+t_i a_y(t_i))-2\frac{c}{a}t_i\phi(t_i)$. Note that by the assumption $2c = b + d$. Thus
4. $(d - c)a_x(0) + (b - c)a_y(0) \geq 2a$. Since $0 \leq (d - c) \leq a$ and $0 \geq (b - c) \geq -a$ and $-1 \leq a_x(0), a_y(0) \leq 1$ this implies that
5. $a_x(0) = 1, a_y(0) = -1$ and $d - c = a, c - b = a$. Hence $|xy| = ||px| - |py||$ in contradiction to the assumption. □

5 4-Point curvature conditions

In this section we briefly discuss question (**Q**) stated in the introduction. We discuss it in the context of conditions for the distance between four points in a given metric space. We use the following notation. Let M^4 be the set of isometry classes of 4-point metric spaces. For a given metric space X let $M^4(X)$ the set of isometry classes of four point subspaces of X. We consider three inequalities between the distances of four points x, y, z, w.

The Ptolemaic inequality

$$|xy|\,|zw| \;\leq\; |xz|\,|yw| \;+\; |xw|\,|yz| \tag{2}$$

The inequality

$$|xy|^2 \;+\; |zw|^2 \;\leq\; |xz|^2 \;+\; |yw|^2 \;+\; |xw|^2 \;+\; |yz|^2 \tag{3}$$

which is called the *quadrilateral inequality* in [2] and is equivalent to the 2-roundness condition of Enflo [4].

We also consider the intermediate inequality

$$|xy|^2 \;+\; |zw|^2 \;\leq\; |xz|^2 \;+\; |yw|^2 \;+\; 2\,|xw|\,|yz| \tag{4}$$

With [2] we call it the *cosq* condition. Let us denote with $\mathcal{A}_{PT}, \mathcal{A}_{QI}, \mathcal{A}_{cosq}$ the isometry classes of spaces in M^4, such that for all relabeling of the points x, y, z, x the conditions (2),(3),(4) hold respectively. Since always $2ab \leq a^2 + b^2$ we clearly have $\mathcal{A}_{cosq} \subset \mathcal{A}_{QI}$, but no other inclusion holds: The space x, y, z, w with $|xy| = 2$ and all other distances equal to 1 shows that $\mathcal{A}_{PT} \not\subseteq \mathcal{A}_{QI}$ and the space x, y, z, w with $|xy| = |zw| = 2$, $|xz| = |xw| = 1$ and $|yz| = |yw| = a$ with $1 < a < 2$ and a very close to 2 shows $\mathcal{A}_{cosq} \not\subseteq \mathcal{A}_{PT}$.

A CAT(0)-space satisfies all conditions (2), (3), (4), i.e. $M^4(X) \subset \mathcal{A}_{cosq} \cap \mathcal{A}_{PT}$ (see [1,5]).

Berg and Nikolaev ([2], compare also [9]) proved a beautiful characterization of CAT(0) spaces:

A geodesic metric space X is CAT(0) if and only if all quadruples in X satisfy the quadrilateral condition (3).

This implies also the following characterization:

A geodesic metric space X is CAT(0) if and only if all quadruples in X satisfy the cosq condition (4).

Formally speaking [2] proves: if X is a geodesic metric space with $M^4(X) \subset \mathcal{A}_{QI}$, then X is CAT(0).

The question (Q) asks for a similar characterization in terms of the PT condition. In [5] we gave examples of geodesic PT spaces which are not CAT(0). Since these examples are not proper, they leave the question (Q) open. Actually in proper geodesic PT spaces the distance function to a point is strictly convex, see Theorem 2.3, thus there is some plausibility for a positive answer to the question. Our result is another indication in this direction.

Finally we remark that in [5] we characterized CAT(0) spaces by the property that they are geodesic PT spaces which are in addition Busemann convex.

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