Abh. Math. Sem. Univ. Hamburg **69** (1999), 59–65

A Note on Almost Kähler Manifolds

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Abstract. For any $n \geq 2$, we give examples of almost Kähler conformally flat manifolds M^{2n} which are not Kähler. We discuss the meaning of these examples in the context of the Goldberg conjecture on almost Kähler manifolds.

1 Introduction

Every Kähler manifold is obviously also an almost Kähler manifold. However, the converse statement does not hold in general, not even for compact manifolds. For examples of (compact) almost Kähler manifolds which are not Kähler manifolds, see e.g. [5]. For precise definitions of the concepts we refer to Section 2, where also appropriate references will be given.

Concerning the relation between Kähler and almost Kähler manifolds, GOLDBERG conjectured [3] that a compact almost Kähler Einstein manifold must be Kähler. This conjecture, which is still open, inspired a lot of work on almost Kähler manifolds.

OLSZAK proved that there are no properly almost Kähler manifolds of constant sectional curvature of dimension ≥ 8. His proof [13] is tensorial and bares an entirely local character. Using technics from Clifford analysis, BLAIR showed that in dimension 4 there are no almost Kähler manifolds of constant sectional curvature, besides the Kähler manifolds [1]. Finally, OGURO and SEKIGAWA found an argument which worked for all dimensions in the particular case of spaces with constant sectional curvature. They proved [11] that a complete almost Kähler manifold of constant sectional curvature is a flat Kähler manifold.

One may observe that on the subset of the Einstein manifolds which have constant sectional curvature a stronger version of Goldberg's conjecture has been proved. Indeed, besides in 6 dimensions, for spaces with constant sectional curvature the result has been shown to hold even entirely local, i.e. without any additional global assumption. Now, the Einstein manifolds generalize the spaces of constant sectional curvature, but in a certain sense staying 'close' to them. Therefore one might wonder if there could be any hope for Goldberg's conjecture to hold locally for all Einstein manifolds. Or, if not, to construct (locally) examples of almost

1991 *Mathematics Subject Classification.* 53A30, 53B35.
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Kähler Einstein spaces which are not Kähler. However, also the latter question does not seem to be so easy to solve immediately.

Looking for an answer, one may e.g. think of comparing this question with a closely related similar problem, in order to gain more insight. Indeed, the Einstein spaces are not the only possible way to generalize the spaces of constant sectional curvature. The conformally flat manifolds do also constitute a set of spaces generalizing the spaces of constant sectional curvature in a different direction, but equally staying 'close' to them.

In the light of the above mentioned discussion, we consider the question whether or not an almost Kähler conformally flat manifold must necessarilly be Kähler, and this from a strictly local point of view. We construct new explicit examples of almost Kähler conformally flat manifolds M^{2n} which are not Kähler, and this for every $n \geq 2$. After the construction, we discuss the differences between our new examples, and already existing ones.

2 Preliminaries

An almost complex manifold with a Hermitian metric is called [6], [4] an almost Hermitian manifold. For an almost Hermitian manifold (M, J, g) we thus have

$$J^2 = -1\,,$$
$$g(JX, JY) = g(X, Y)\,.$$

An almost complex stucture J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_J vanishes; with

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\,. \tag{1}$$

For an almost Hermitian manifold (M, J, g), we define the fundamental Kähler form Φ as:

$$\Phi(X, Y) = g(X, JY)\,. \tag{2}$$

(M, J, g) is then called almost Kähler if Φ is closed: $d\Phi = 0$. It can be shown that this condition for (M, J, g) to be almost Kähler is equivalent to

$$g\big((\nabla_X J)Y, Z\big) + g\big((\nabla_Y J)Z, X\big) + g\big((\nabla_Z J)X, Y\big) = 0\,.$$

An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $d\Phi = 0$ and $N = 0$. One can prove that these both conditions combined are equivalent with the single condition

$$\nabla J = 0\,.$$

Obviously, every Kähler manifold is also an almost Kähler manifold. For examples of manifolds with almost Kähler structures which are not Kähler, see e.g. the recent article by JELONEK [5], and references therein, in particular [2] and [14].

3 Conformally flat almost Kähler spaces

In the present section, we construct examples of conformally flat almost Kähler spaces which however are not Kähler. For every $n \geq 2$, we consider an n-dimensional Riemannian manifold (M_1, g_1). On the Cartesian product $M = M_1 \times \mathbb{R}^n$, we define a conformally flat metric g and an almost complex structure J. Manifold (M, g, J) will appear to be almost Kähler, but not Kähler. The construction goes as follows:

We define an n-dimensional Riemannian manifold (M_1, g_1). Let M_1 coincide with an open connected subset U of $\mathbb{R}^n$, equipped with coordinates $(x^1, \ldots, x^n)$. For a positive and nonconstant function f on U, consider the Riemannian metric g_1 defined on M_1 as follows

$$g_1\big(\tfrac{\partial}{\partial x^\alpha}, \tfrac{\partial}{\partial x^\beta}\big) = f^2 \delta_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n.$$

In what follows, we are interested in the situation when g_1 is a flat metric. By straightforward calculations, it can be proved that g_1 is flat when

(a) in case of $n = 2$, the function f satisfies the partial differential equation

$$\big(\tfrac{\partial^2}{\partial u^2} + \tfrac{\partial^2}{\partial v^2}\big) \log f = 0, \tag{3}$$

where for simplicity it is supposed $u = x^1$, $v = x^2$;

(b) in case of $n \geq 3$, the function f is of the form

$$f(x^1, \ldots, x^n) = \frac{a}{(x^1 - x_0^1)^2 + \cdots + (x^n - x_0^n)^2}, \tag{4}$$

where a is a positive constant and $(x_0^1, \ldots, x_0^n) \notin U$ is a fixed point; it is clear that $U \neq \mathbb{R}^n$ in this case.

We sketch briefly an outline for a proof showing that (4) is indeed necessary for the vanishing of the sectional curvature. For $(g_1)_{ij} = f^2 \delta_{ij}$, it will turn out convenient to calculate in terms of a function $h > 0$, defined by $f = \frac{1}{h}$. A standard calculation, involving the determination of the Christoffel symbols, gives the following expression for the components of the curvature tensor R

$$\begin{aligned} R_{ijk}{}^{l} &= \tfrac{1}{h}\big(\tfrac{\partial^2 h}{\partial x^j \partial x^k}\delta_i^l - \tfrac{\partial^2 h}{\partial x^i \partial x^k}\delta_j^l - \tfrac{\partial^2 h}{\partial x^j \partial x^l}\delta_{ik} + \tfrac{\partial^2 h}{\partial x^i \partial x^l}\delta_{jk}\big) \\ &\quad + \tfrac{1}{h^2}\sum_s \Big(\frac{\partial h}{\partial x^s}\Big)^2 \big(\delta_{ik}\delta_j^l - \delta_{jk}\delta_i^l\big), \end{aligned} \tag{5}$$

for the components of the Ricci tensor S

$$S_{jk} = \tfrac{1}{h}(n-2)\tfrac{\partial^2 h}{\partial x^j \partial x^k} + \tfrac{1}{h}\sum_s \tfrac{\partial^2 h}{(\partial x^s)^2}\delta_{jk} - (n-1)\tfrac{1}{h^2}\sum_s \big(\tfrac{\partial h}{\partial x^s}\big)^2, \tag{6}$$

and for the scalar curvature R

$$R = 2h(n-1)\sum_s \tfrac{\partial^2 h}{(\partial x^s)^2} - n(n-1)\sum_s \big(\tfrac{\partial h}{\partial x^s}\big)^2. \tag{7}$$

When we assume that (5) has to vanish, then also (6) and (7) should be zero; together this yields

$$\frac{\partial^2 h}{\partial x^j \partial x^k} = \frac{1}{2h} \sum_s \left(\frac{\partial h}{\partial x^s}\right)^2 \delta_{jk} \,. \tag{8}$$

For $j \neq k$, $\frac{\partial^2 h}{\partial x^j \partial x^k} = 0$, therefore

$$h(x^1, \cdots x^n) = k_1(x^1) + \cdots + k_n(x^n) \,. \tag{9}$$

From (8) and (9) there also follows that

$$\frac{\partial^2 h}{\partial x_1^2} = \frac{\partial^2 h}{\partial x_2^2} = \cdots = \frac{\partial^2 h}{\partial x_n^2} = \text{const.} =: 2c \,.$$

By (8) one now deduces that h is of the form

$$h = c \sum_i (x^i - x_0^i)^2 \,,$$

from where it is clear that $f = \frac{1}{h}$ takes the form (4).

In the sequel, we assume that f satisfies at least (3) or (4).

Now, let $M = M_1 \times \mathbb{R}^n$ and extend the function f to the whole of M so that f depends only on the first n coordinates $(x^1, \ldots, x^n)$. Define a Riemannian metric g on M in the following explicit way

$$g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = f \delta_{\alpha\beta} \,,$$
$$g\left(\frac{\partial}{\partial x^{n+\alpha}}, \frac{\partial}{\partial x^{n+\beta}}\right) = \frac{1}{f} \delta_{\alpha\beta} \,,$$
$$g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{n+\beta}}\right) = 0 \,.$$

The metric g can be viewed as a conformal deformation of a flat product metric

$$g = \frac{1}{f} \left(g_1 \times g_2\right),$$

where g_2 is the standard flat metric on $\mathbb{R}^n$. Therefore, g is conformally flat. We now also define an almost complex structure J on M as follows

$$J \frac{\partial}{\partial x^\alpha} = f \frac{\partial}{\partial x^{n+\alpha}} \,, \qquad J \frac{\partial}{\partial x^{n+\alpha}} = -\frac{1}{f} \frac{\partial}{\partial x^\alpha} \,. \tag{10}$$

We can verify that metric g is compatible with the almost complex structure J. Moreover, for the fundamental form Φ defined by formula (2), we have

$$\Phi\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = 0 \,,$$
$$\Phi\left(\frac{\partial}{\partial x^{n+\alpha}}, \frac{\partial}{\partial x^{n+\beta}}\right) = 0 \,,$$
$$\Phi\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{n+\beta}}\right) = -\delta_{\alpha\beta} \,.$$

Hence $d\Phi = 0$. Thus, the pair (J, g) realizes an almost Kähler structure on M.

We observe that the pair (J, g) cannot be Kählerian. Indeed, since f depends only of coordinates $x^1, \dots, x^n$, direct computation of the Nijenhuis tensor from (1) with using (10), for $X = \partial/\partial x^\alpha$ and $Y = \partial/\partial x^\beta$, gives

$$\begin{aligned} N_J\left(\tfrac{\partial}{\partial x^\alpha}, \tfrac{\partial}{\partial x^\beta}\right) &= \left[f\tfrac{\partial}{\partial x^{n+\alpha}}, f\tfrac{\partial}{\partial x^{n+\beta}}\right] - \left[\tfrac{\partial}{\partial x^\alpha}, \tfrac{\partial}{\partial x^\beta}\right] \\ &\quad - J\left[\tfrac{\partial}{\partial x^\alpha}, f\tfrac{\partial}{\partial x^{n+\beta}}\right] - J\left[f\tfrac{\partial}{\partial x^{n+\alpha}}, \tfrac{\partial}{\partial x^\beta}\right] \\ &= -J\tfrac{\partial f}{\partial x^\alpha}\tfrac{\partial}{\partial x^{n+\beta}} + J\tfrac{\partial f}{\partial x^\beta}\frac{\partial}{\partial x^{n+\alpha}} \\ &= -\tfrac{1}{f}\tfrac{\partial f}{\partial x^\beta}\tfrac{\partial}{\partial x^\alpha} + \tfrac{1}{f}\tfrac{\partial f}{\partial x^\alpha}\tfrac{\partial}{\partial x^\beta}. \end{aligned}$$

If now (M, g, J) were to be a Kähler manifold, then N_J should vanish. Consequently $\partial f/\partial x^\alpha = 0$ $(1 \le \alpha \le n)$, and f would have to be a constant. This is however excluded, since we started the construction with nonconstant f. This finishes the proof of the existence of almost Kähler conformally flat manifolds M^{2n} $(n \ge 2)$ which are not Kähler.

We now discuss the differences between our examples of conformally flat strictly almost Kähler manifolds, and already existing ones.

Paper [10] presents examples of almost Kähler non-Kähler structures on the products $\mathbb{H}^m \times \mathbb{R}^{2n-m}$ of an m-dimensional hyperbolic space of constant sectional curvature -1 and an $(2n-m)$-dimensional Euclidean space, which are also locally symmetric.

We remember that, when a manifold M^n is locally a product and conformally flat, then only one of the following two situations can occur, which are mutually exclusive:

(i) or $M^n = M^{n-1}(\kappa) \times \mathbb{R}$, with $M^{n-1}(\kappa)$ an $(n-1)$-dimensional space of constant sectional curvature κ (κ arbitrary),

(ii) or $M^n = M^p(\kappa) \times M^q(-\kappa)$, with $p + q = n$, and $p, q \ge 2$, and $\kappa > 0$.

The former case (i) occurs when the Ricci operator is algebraically degenerate, the latter case (ii) occurs when the Ricci operator is nondegenerate. This theorem may be deduced easily from material contained e.g. in [7]; see also [8].

Hence, the examples of [10] are also conformally flat for $2n - m = 1$, and they are of type (i) in reference to the above mentioned result. [12] gives also examples of non-Kähler almost Kähler $\mathbb{H}^3 \times \mathbb{R}$ which are locally symmetric and conformally flat ; they are also of type (i). So, all those examples of conformally flat non-Kähler almost Kähler manifolds are locally symmetric as well. We now consider first the 'general series' of our examples, for manifolds M^{2n}, $n \ge 2$, with f given by (4). This gives examples of conformally flat non-Kähler almost Kähler manifolds for any $n \ge 2$; the question is, are they also locally symmetric. Since they are conformally flat, the condition $\nabla R = 0$ is equivalent to the condition $\nabla S = 0$, where R and S denote the Riemann-Christoffel curvature tensor and the Ricci tensor, respectively. A direct calculation of the Christoffel symbols and the components of the Ricci tensor shows that they are locally symmetric indeed. However, a calculation of the determinant of S, gives

$$\det S = (-n)^{n+1}(n-2)^{n-1} .$$

Hence, for $n \geq 3$, the Ricci operator is nondegenerate and the examples are of type (ii), and therefore differ in type from those of [10]. For $n = 2$, the Ricci operator is degenerated and the examples are also of type (i), as those in [10] and [12].

We now consider our extra examples in the 4-dimensional case. In this particular dimension, our construction scheme gives many more examples of conformally flat non-Kähler almost Kähler manifolds, than the "general series" for $n = 2$. Indeed any f which solves (3) gives an example. When we write f in the following form

$$f = e^h ,$$

then (3) reduces to the following equation in terms of h,

$$\frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2} = 0 .$$

Hence, every harmonic function of two variables gives by our construction scheme, an example of a 4-dimensional conformally flat non-Kähler almost Kähler metric. The scalar curvature k of this metric, with h harmonic, takes the following form

$$k = -\frac{3}{2} e^h \left(\left(\frac{\partial h}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2 \right) .$$

Since the scalar curvature is not constant, in general, the corresponding metric g is not locally symmetric in general. For example: when $f = a \exp(bu)$, then g is not locally symmetric; when $f = \exp(u^2 - v^2)$, then g is not locally symmetric. When $f = (u^2+v^2)^p$, with p a parameter, the scalar curvature is $k = -6p^2(u^2+v^2)^{-p-1}$, and for $p \neq -1$ the metric is conformally flat but not locally symmetric. For $p = -1$, the metric is conformally flat and also locally symmetric, as it falls in this case in the "general series". We conclude by the observation that for $n = 4$ all examples of locally symmetric non-Kähler almost Kähler are indeed noncompact, in agreement with [9], where it was proved that a compact 4-dimensional locally symmetric almost Kähler manifold should be Kähler.

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Eingegangen am: 27. März 1998
in revidierter Fassung am: 22. April 1999

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