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Minimal invariant varieties and first integrals for algebraic foliations

Philippe Bonnet

Abstract. Let X be an irreducible algebraic variety over $\mathbb{C}$, endowed with an algebraic foliation $\mathcal{F}$. In this paper, we introduce the notion of minimal invariant variety $V(\mathcal{F}, Y)$ with respect to $(\mathcal{F}, Y)$, where Y is a subvariety of X. If $Y = \{x\}$ is a smooth point where the foliation is regular, its minimal invariant variety is simply the Zariski closure of the leaf passing through x. First we prove that for very generic x, the varieties $V(\mathcal{F}, x)$ have the same dimension p. Second we generalize a result due to X. Gomez-Mont (see [G-M]). More precisely, we prove the existence of a dominant rational map $F : X \to Z$, where Z has dimension $(n - p)$, such that for very generic x, the Zariski closure of $F^{-1}(F(x))$ is one and only one minimal invariant variety of a point. We end up with an example illustrating both results.

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1 Introduction

Let X be an affine irreducible variety over $\mathbb{C}$, and $\mathcal{O}_X$ its ring of regular functions. Let $\mathcal{F}$ be an algebraic foliation, i.e. a collection of algebraic vector fields on X stable by Lie bracket. We consider the elements of $\mathcal{F}$ as $\mathbb{C}$-derivations on the ring $\mathcal{O}_X$. In this paper, we are going to extend the notion of algebraic solution for $\mathcal{F}$: this will be the minimal invariant varieties for $\mathcal{F}$. We will study some of their properties and relate them to the existence of rational first integrals for $\mathcal{F}$.

Recall that a subvariety Y of X is an algebraic solution of $\mathcal{F}$ if Y is the closure (for the metric topology) of a leaf of $\mathcal{F}$. A non-constant rational function f on X is a first integral if $\partial(f) = 0$ for any ∂ in $\mathcal{F}$. Since the works of Darboux, the existence of such varieties has been extensively studied in the case

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of codimension 1 foliations (see [Jou],[Gh],[Bru]). In particular, from these works, we know that only two cases may occur for codimension 1 foliations:

- $\mathcal{F}$ has finitely many algebraic solutions,
- $\mathcal{F}$ has infinitely many algebraic solutions, and a rational first integral.

So rational first integrals appear if and only if all leaves of $\mathcal{F}$ are algebraic solutions. In this case, the fibres of any rational first integral is a finite union of closures of leaves. This fact has been generalised by Gomez-Mont (see [G-M]) in the following way.

Theorem 1.1. *Let X be a projective variety and $\mathcal{F}$ an algebraic foliation on X such that all leaves are quasi-projective. Then there exists a rational map $F : X \to Y$ such that, for every generic point y of Y, the Zariski closure of $F^{-1}(y)$ is the closure of a leaf of $\mathcal{F}$.*

We would like to find a version of this result that does not need all leaves to be algebraic. To that purpose, we need to give a correct definition to the algebraic object closest to a leaf. A good candidate would be the Zariski closure of a leaf, but this choice may rise difficulties due to the singularities of both X and $\mathcal{F}$. We counterpass this problem by the following algebraic approach.

Let Y be an algebraic subvariety of X and I_Y the ideal of vanishing functions on Y. Let $\mathcal{J}$ be the set of ideals I in $\mathcal{O}_X$ satisfying the two conditions:

$$\text{(i)} \quad (0) \subseteq I \subseteq I_Y \qquad \text{and} \qquad \text{(ii)} \quad \forall\, \partial \in \mathcal{F}, \quad \partial(I) \subseteq I\,.$$

Since (0) belongs to $\mathcal{J}$, $\mathcal{J}$ is non-empty and it is partially ordered by the inclusion. Since it is obviously inductive, $\mathcal{J}$ admits a maximal element I. If J is any other ideal of $\mathcal{J}$, then $I + J$ enjoys the conditions (i) and (ii), hence it belongs to $\mathcal{J}$. By maximality, we have $I = I + J$ and J is contained in I. Therefore I is the unique maximal element of $\mathcal{J}$, which we denote by $I(\mathcal{F}, Y)$.

Definition 1.2. *The minimal invariant variety $V(\mathcal{F}, Y)$ is the zero set of $I(\mathcal{F}, Y)$ in X.*

From a geometric viewpoint, $V(\mathcal{F}, Y)$ can be seen as the smallest subvariety containing Y and invariant by the flows of all elements of $\mathcal{F}$. In particular, if x is a smooth point of X where the foliation is regular, then $V(\mathcal{F}, x)$ is the Zariski closure of the leaf passing through x. In section 2, we show that $V(\mathcal{F}, Y)$ is irreducible if Y is itself irreducible.

In this paper, we would like to study the behaviour of these invariant varieties, and relate it to the existence of first integrals. We analyze some properties of the function:

$$n_{\mathcal{F}} : X \longrightarrow \mathbb{N}, \quad x \longmapsto \dim\ V(\mathcal{F}, x)$$

Let $\mathcal{M}$ be the σ-algebra generated by the Zariski topology on X. A function $f : X \to \mathbb{N}$ is *measurable for the Zariski topology* if $f^{-1}(p)$ belongs to $\mathcal{M}$ for any p. The space $\mathcal{M}$ contains in particular all countable intersections θ of Zariski open sets. A property $\mathcal{P}$ holds for *every very generic point* x *in* X if $\mathcal{P}(x)$ is true for any point x in such an intersection θ.

Theorem 1.3. *Let X be an affine irreductible variety over $\mathbb{C}$ and $\mathcal{F}$ an algebraic foliation on X. Then the function $n_{\mathcal{F}}$ is measurable for the Zariski topology. Moreover there exists an integer p such that* (1) *$n_{\mathcal{F}}(x) \leq p$ for any point x in X and* (2) *$n_{\mathcal{F}}(x) = p$ for any very generic point x in X.*

Set $p = \max\ \dim\ V(\mathcal{F}, x)$ and note that p is achieved for every generic point of X. In the last section, we will produce an example of a foliation on $\mathbb{C}^4$ where the function $n_{\mathcal{F}}$ is measurable but not constructible for the Zariski topology. In this sense, theorem 1.3 is the best result one can expect for any algebraic foliation.

Let $K_{\mathcal{F}}$ be the field generated by $\mathbb{C}$ and the rational first integrals of $\mathcal{F}$. By construction, the invariant varieties $V(\mathcal{F}, x)$ are defined set-theoretically, and they seem to appear randomly, i.e. with no link within each other. In fact there does exist some order among them, and we are going to see that they are "mostly" given as the fibres of a rational map. More precisely:

Theorem 1.4. *Let X be an affine irreducible variety over $\mathbb{C}$ of dimension n and $\mathcal{F}$ an algebraic foliation on X. Then there exists a dominant rational map $F : X \to Y$, where Y is irreducible of dimension $(n - p)$, such that for every very generic point x of X, the Zariski closure of $F^{-1}(F(x))$ is equal to $V(\mathcal{F}, x)$. In particular, the transcendence degree of $K_{\mathcal{F}}$ over $\mathbb{C}$ is equal to $(n - p)$.*

The idea of the proof is to construct enough rational first integrals. These will be the coordinate functions of the rational map F given above. The construction consists in choosing a codimension d irreducible variety H in X. We show there exists an integer $r > 0$ such that, for every very generic point x of X, $V(\mathcal{F}, x)$ intersects H in r distinct points $y_1, ..., y_r$. We then obtain a correspondence:

$$\mathcal{H} : x \longmapsto \{y_1, ..., y_r\}.$$

We can modify $\mathcal{H}$ so as to get a rational map F that represents every r-uple $\{y_1, ..., y_r\}$ by a single point. Since the image of x only depends on the intersection of $V(\mathcal{F}, x)$ with H, the map F will be invariant with respect to the elements of $\mathcal{F}$.

One question may arise after these two results. Does there exist an effective way of computing these minimal invariant varieties and detect the presence of rational first integrals? For instance, we may attempt to use the description of the ideals $I(\mathcal{F}, Y)$ given by lemma 2.1. Unfortunately we cannot hope to compute them in a finite number of steps bounded, for instance, by the degrees of the components of the vector fields of $\mathcal{F}$. Indeed, consider the well-known derivation ∂ on $\mathbb{C}^2$:

$$\partial = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} .$$

For any couple of non-zero coprime integers (p, q), this derivation will have $f(x, y) = x^q y^{-p}$ as a rational first integral, and we cannot find another one of smaller degree. The minimal invariant varieties of points will be given in general by the fibres of f. Therefore we cannot bound the degree of the generators of $I(\mathcal{F}, x)$ solely by the degree of ∂.

However, we may find them by an inductive process. For one derivation, an approach is given in the paper of J.V.Pereira via the notion of extatic curves (see [Pe]). The idea is to compute a series of Wronskians attached to the derivation. Then one of them vanishes identically if and only the derivation has a rational first integral.

Last thing to say is that the previous results carry over all algebraic irreducible varieties. Given an algebraic variety X with an algebraic foliation, we choose a covering of X by open affine sets U_i and work on the U_i. For any algebraic subvariety Y of X, we define the minimal invariant variety $V(\mathcal{F}, Y)$ by gluing together the Zariski closure of the varieties $V(\mathcal{F}, Y \cap U_i)$ in X.

2 The contact order with respect to $\mathcal{F}$

In this section, we are going to show that the minimal invariant variety $V(\mathcal{F}, Y)$ is irreducible if Y is irreducible. This result is already known when $\mathcal{F}$ consists of one derivation (see [Ka]). We could reproduce the proof given in [Ka] for any set of derivations, but we prefer to adopt another strategy. We will instead introduce a notion of contact order with respect to $\mathcal{F}$, and we will use it to show that $I(\mathcal{F}, Y)$ is prime if I_Y is prime. Denote by $M_{\mathcal{F}}$ the $\mathcal{O}_X$-module spanned by the elements of $\mathcal{F}$. We start by giving the following characterisation of $I(\mathcal{F}, Y)$.

Lemma 2.1. $I(\mathcal{F}, Y) = \left\{ f \in I_Y,\ \forall \partial_1, ..., \partial_k \in M_{\mathcal{F}},\ \partial_1 \circ ... \circ \partial_k(f) \in I_Y \right\}.$

Proof. Let f be an element of I_Y such that $\partial_1 \circ ... \circ \partial_k(f)$ belongs to I_Y for any $\partial_1, ..., \partial_k$ in $M_{\mathcal{F}}$. Then $\partial_1 \circ ... \circ \partial_k(f)$ belongs to I_Y for any elements $\partial_1, ..., \partial_k$ of

$\mathcal{F}$. Let I be the ideal generated by f and all the elements of the form $\partial_1 \circ \ldots \circ \partial_k(f)$, where every ∂_i lies in $\mathcal{F}$. By construction, this ideal is contained in I_Y, and is stable by every derivation of $\mathcal{F}$. Therefore I is contained in $I(\mathcal{F}, Y)$, and a fortiori f belongs to $I(\mathcal{F}, Y)$. We then have the inclusion:

$$\left\{ f \in I_Y,\ \forall \partial_1, \ldots, \partial_k \in M_{\mathcal{F}},\ \partial_1 \circ \ldots \circ \partial_k(f) \in I_Y \right\} \subseteq I(\mathcal{F}, Y)\,.$$

Conversely let f be an element of $I(\mathcal{F}, Y)$. Since $I(\mathcal{F}, Y)$ is contained in I_Y and is stable by every derivation of $\mathcal{F}$, $\partial_1 \circ \ldots \circ \partial_k(f)$ belongs to I_Y for any elements $\partial_1, \ldots, \partial_k$ of $\mathcal{F}$. Since $M_{\mathcal{F}}$ is spanned by $\mathcal{F}$, $\partial_1 \circ \ldots \circ \partial_k(f)$ belongs to I_Y for any $\partial_1, \ldots, \partial_k$ in $M_{\mathcal{F}}$. □

Since the space of $\mathbb{C}$-derivations on $\mathcal{O}_X$ is an $\mathcal{O}_X$-module of finite type and $\mathcal{O}_X$ is noetherian, $M_{\mathcal{F}}$ is finitely generated as an $\mathcal{O}_X$-module. Let $\{\partial_1, \ldots, \partial_r\}$ be a system of generators of $M_{\mathcal{F}}$. If $I = (i_1, \ldots, i_n)$ belongs to $\{1, \ldots, r\}^n$, we set $\partial_I = \partial_{i_1} \circ \ldots \circ \partial_{i_n}$ and $|I| = n$. By convention $\{1, \ldots, r\}^0 = \{\emptyset\}$, $|\emptyset| = 0$ and $\partial_\emptyset$ is the identity on $\mathcal{O}_X$. We introduce the following map:

$$\mathrm{ord}_{\mathcal{F},Y} : \mathcal{O}_X \longrightarrow \mathbb{N} \cup \{+\infty\}, \qquad f \longmapsto \inf\{|I|,\ \partial_I(f) \notin I_Y\}\ .$$

Definition 2.2. *The map* $\mathrm{ord}_{\mathcal{F},Y}$ *is the contact order with respect to* $(\mathcal{F}, Y)$.

By lemma 2.1, f belongs to $I(\mathcal{F}, Y)$ if and only if $\mathrm{ord}_{\mathcal{F},Y}(f) = +\infty$, and f does not belong to I_Y if and only if $\mathrm{ord}_{\mathcal{F},Y}(f) = 0$. A priori, the map $\mathrm{ord}_{\mathcal{F},Y}$ depends on the set of generators chosen for $M_{\mathcal{F}}$. We are going to see that it only depends on $\mathcal{F}$. Let $\{d_1, \ldots, d_s\}$ be another set of generators for $M_{\mathcal{F}}$, and define in an analogous way the map $\mathrm{ord}'_{\mathcal{F},Y}$ corresponding to this set. By assumption there exist some elements $a_{i,j}$ of $\mathcal{O}_X$ such that:

$$\partial_i = \sum_{j=1}^{s} a_{i,j} d_j\ .$$

By Leibniz rule, it is easy to check via an induction on $|I|$ that there exist some elements $a_{I,J}$ in $\mathcal{O}_X$ such that:

$$\partial_I = \sum_{|J| \le |I|} a_{I,J} d_J\ .$$

Let f be an element of $\mathcal{O}_X$ such that $\mathrm{ord}_{\mathcal{F},Y}(f) = n$. Then there exists an index I of length n such that:

$$\partial_I(f) = \sum_{|J| \le n} a_{I,J} d_J(f) \ \notin\ I_Y\ .$$

Since I_Y is an ideal, this means there exists an index J of length $\leq n$ such that $d_J(f)$ does not belong to I_Y. By definition we get that $\mathrm{ord}'_{\mathcal{F},Y}(f) \leq n = \mathrm{ord}_{\mathcal{F},Y}(f)$ for any f. By symmetry we find that $\mathrm{ord}'_{\mathcal{F},Y}(f) = \mathrm{ord}_{\mathcal{F},Y}(f)$ for any f, and the maps coincide.

Proposition 2.3. *If Y is irreducible, the contact order enjoys the following properties:*

- $\mathrm{ord}_{\mathcal{F},Y}(f+g) \geq \inf\{\mathrm{ord}_{\mathcal{F},Y}(f), \mathrm{ord}_{\mathcal{F},Y}(g)\}$ *with equality if* $\mathrm{ord}_{\mathcal{F},Y}(f) \neq \mathrm{ord}_{\mathcal{F},Y}(g)$,
- $\mathrm{ord}_{\mathcal{F},Y}(fg) = \mathrm{ord}_{\mathcal{F},Y}(f) + \mathrm{ord}_{\mathcal{F},Y}(g)$ *for all f, g in $\mathcal{O}_X$.*

Proof of the first assertion. If $\mathrm{ord}_{\mathcal{F},Y}(f) = \mathrm{ord}_{\mathcal{F},Y}(g) = +\infty$, then f, g both belong to $I(\mathcal{F}, Y)$, $f+g$ belongs to $I(\mathcal{F}, Y)$ and the result follows. So assume that $\mathrm{ord}_{\mathcal{F},Y}(f)$ is finite and for simplicity that $n = \mathrm{ord}_{\mathcal{F},Y}(f) \leq \mathrm{ord}_{\mathcal{F},Y}(g)$. For any index I of length $< n$, $\partial_I(f)$ and $\partial_I(g)$ both belong to I_Y. So $\partial_I(f+g)$ belong to I_Y for any I with $|I| < n$, and $\mathrm{ord}_{\mathcal{F},Y}(f+g) \geq n$. Therefore we have for all f, g:

$$\mathrm{ord}_{\mathcal{F},Y}(f+g) \geq \inf\{\mathrm{ord}_{\mathcal{F},Y}(f), \mathrm{ord}_{\mathcal{F},Y}(g)\}\,.$$

Assume now that $\mathrm{ord}_{\mathcal{F},Y}(f) < \mathrm{ord}_{\mathcal{F},Y}(g)$. Then there exists an index I of length n such that $\partial_I(f)$ does not belong to I_Y. Since $|I| < \mathrm{ord}_{\mathcal{F},Y}(g)$, $\partial_I(g)$ belongs to I_Y. Therefore $\partial_I(f+g)$ does not belong to I_Y and $\mathrm{ord}_{\mathcal{F},Y}(f+g) \leq n$, so that $\mathrm{ord}_{\mathcal{F},Y}(f+g) = n$. □

For the second assertion, we will need the following lemmas. The first one is easy to get via Leibniz rule, by an induction on the length of I.

Lemma 2.4. *Let $\partial_1, ..., \partial_r$ a system of generators of $M_{\mathcal{F}}$. Then there exist some elements α_{I_1,I_2} of $\mathbb{C}$, depending on I and such that for all f, g:*

$$\partial_I(fg) = \sum_{|I_1|+|I_2|=|I|} \alpha_{I_1,I_2} \partial_{I_1}(f) \partial_{I_2}(g)\,.$$

Lemma 2.5. *Let f be an element of $\mathcal{O}_X$ such that $\mathrm{ord}_{\mathcal{F},Y}(f) \geq n$. Let $I = (i_1, ..., i_n)$ be any index. For any rearrangement $J = (j_1, ..., j_n)$ of the i_k, $\partial_J(f) - \partial_I(f)$ belongs to I_Y.*

Proof. Every rearrangement of the i_k can be obtained after a composition of transpositions on two consecutive terms. So we only need to check the lemma in the case $J = (i_1, ..., i_{l+1}, i_l, ..., i_n)$. If we denote by I_1, I_2 the indices $I_1 = (i_1, ..., i_{l-1})$ and $I_2 = (i_{l+2}, ..., i_n)$, then we find:

$$\partial_J - \partial_I = \partial_{I_1} \circ [\partial_{i_l}, \partial_{i_{l+1}}] \circ \partial_{I_2} .$$

Since $M_{\mathcal{F}}$ is stable by Lie bracket, $d = [\partial_{i_l}, \partial_{i_{l+1}}]$ belongs to $M_{\mathcal{F}}$. Then $\partial_J - \partial_I$ is a composite of $(n-1)$ derivations that span $M_{\mathcal{F}}$. Since $\text{ord}_{\mathcal{F},Y}$ is independent of the set of generators and $\text{ord}_{\mathcal{F},Y}(f) = n$, $\partial_J(f) - \partial_I(f)$ belongs to I_Y. □

Proof of the second assertion of Proposition 2.3. Let f, g be a couple of elements of $\mathcal{O}_X$. If either f or g has infinite contact order, then one of them belongs to $I(\mathcal{F}, Y)$ and fg belongs to $I(\mathcal{F}, Y)$, so that $\text{ord}_{\mathcal{F},Y}(fg) = +\infty = \text{ord}_{\mathcal{F},Y}(f) + \text{ord}_{\mathcal{F},Y}(g)$. Assume now that $\text{ord}_{\mathcal{F},Y}(f) = n$ and $\text{ord}_{\mathcal{F},Y}(g) = m$ are finite. By lemma 2.4, we have:

$$\partial_I(fg) = \sum_{|I_1|+|I_2|=|I|} \alpha_{I_1,I_2} \partial_{I_1}(f) \partial_{I_2}(g) .$$

Since $|I_1| + |I_2| < n + m$, either $|I_1| < n$ or $|I_2| < m$, and $\partial_{I_1}(f)\partial_{I_2}(g)$ belongs to I_Y. So $\partial_I(fg)$ belongs to I_Y and we obtain:

$$\text{ord}_{\mathcal{F},Y}(fg) \geq n + m .$$

Conversely, consider the following polynomials P, Q in the indeterminates $x, t_1, ..., t_r$:

$$\begin{aligned} P(x, t_1, ..., t_r) &= (t_1\partial_1 + ... + t_r\partial_r)^n(f)(x) , \\ Q(x, t_1, ..., t_r) &= (t_1\partial_1 + ... + t_r\partial_r)^m(g)(x) . \end{aligned}$$

By lemma 2.5, we get that $\partial_I(f) \equiv \partial_J(f)\ [I_Y]$ for any rearrangement J of I if I has length n. Idem for $\partial_I(g)$ and $\partial_J(g)$ if I has length m. Therefore in the expressions of P, Q, everything happens modulo I_Y as if the derivations ∂_i commuted. We then obtain the following expansions modulo I_Y:

$$P \equiv \sum_{i_1+...+i_r=n} \frac{n!}{i_1!...i_r!} t_1^{i_1}...t_r^{i_r} \partial_1^{i_1} \circ ... \circ \partial_r^{i_r}(f)\ [I_Y] .$$

$$Q \equiv \sum_{i_1+...+i_r=m} \frac{m!}{i_1!...i_r!} t_1^{i_1}...t_r^{i_r} \partial_1^{i_1} \circ ... \circ \partial_r^{i_r}(g)\ [I_Y] .$$

Since $\text{ord}_{\mathcal{F},Y}(f) = n$ and $\text{ord}_{\mathcal{F},Y}(g) = m$, both P and Q have at least one coefficient that does not belong to I_Y by lemma 2.5. So neither of them belong to the ideal $I_Y[t_1, ..., t_r]$, which is prime because I_Y is prime. So PQ does not belong to $I_Y[t_1, ..., t_r]$. If $\partial = t_1\partial_1 + ... + t_r\partial_r$, then we have by Leibniz rule:

$$\partial^{n+m}(fg) = \sum_{k=0}^{n} C_{n+m}^k \partial^k(f)\partial^{n+m-k}(g)\,.$$

Since $\mathrm{ord}_{\mathcal{F},Y}(f) = n$ and $\mathrm{ord}_{\mathcal{F},Y}(g) = m$, $\partial^k(f)\partial^{n+m-k}(g)$ belongs to $I_Y[t_1, ..., t_r]$ except for $k = n$. So $\partial^{n+m}(fg) = C_{n+m}^n PQ$ does not belong to $I_Y[t_1, ..., t_r]$. Choose a point $(y, z_1, ..., z_r)$ in $Y \times \mathbb{C}^r$ such that $PQ(y, z_1, ..., z_r) \neq 0$ and set $d = z_1\partial_1 + ... + z_r\partial_r$. By construction we have:

$$d^{n+m}(fg)(y) = C_{n+m}^n PQ(y, z_1, ..., z_r) \neq 0\,.$$

So $d^{n+m}(fg)$ does not belong to I_Y and fg has contact order $\leq n + m$ with respect to the system of generators $\{\partial_1, ..., \partial_r, d\}$. Since the contact order does not depend on the system of generators, we find:

$$\mathrm{ord}_{\mathcal{F},Y}(fg) = n + m = \mathrm{ord}_{\mathcal{F},Y}(f) + \mathrm{ord}_{\mathcal{F},Y}(g)\,. \qquad \square$$

Corollary 2.6. *Let Y be an irreducible subvariety of X. Then the ideal $I(\mathcal{F}, Y)$ is prime. In particular, the minimal invariant variety $V(\mathcal{F}, Y)$ is irreducible.*

Proof. Let f, g be two elements of $\mathcal{O}_X$ such that fg belongs to $I(\mathcal{F}, Y)$. Then fg has infinite contact order. By proposition 2.3, either f or g has infinite contact order. So one of them belongs to $I(\mathcal{F}, Y)$, and this ideal is prime. $\square$

3 Behaviour of the function $n_{\mathcal{F}}$

In this section we are going to establish theorem 1.3 about the measurability of the function $n_{\mathcal{F}}$ for the Zariski topology. Recall that a function $f : X \to \mathbb{N}$ is lower semi-continuous for the Zariski topology if the set $f^{-1}([0, r])$ is closed for any r. Note that such a function is continuous for the constructible topology. We begin with the following lemma.

Lemma 3.1. *Let F be a finite dimensional vector subspace of $\mathcal{O}_X$. Then the map $\varphi_F : X \to \mathbb{N}$, $x \mapsto \dim_{\mathbb{C}} F - \dim_{\mathbb{C}} I(\mathcal{F}, x) \cap F$ is lower semi-continuous for the Zariski topology.*

Proof. For any fixed finite-dimensional vector space F, consider the affine algebraic set:

$$\Sigma_F = \left\{(x,\ f) \in X \times F,\ \forall d_1, ..., d_m \in M_{\mathcal{F}},\ d_1 \circ ... \circ d_m(f)(x) = 0\right\}\,.$$

together with the projection $\Pi : \Sigma_F \longrightarrow X, \ (x, f) \longmapsto x$. Since Σ_F is affine, there exists a finite collection of linear operators $\Delta_1, ..., \Delta_r$, obtained by composition of elements of $M_{\mathcal{F}}$, such that:

$$\Sigma_F = \{(x, f) \in X \times F, \ \Delta_1(f)(x) = ... = \Delta_r(f)(x) = 0\} .$$

By lemma 2.1, the fibre $\Pi^{-1}(x)$ is isomorphic to $I(\mathcal{F}, x) \cap F$ for any point x of X. Since every Δ_i is linear, Δ_i can be considered as a linear form on F with coefficients in $\mathcal{O}_X$. So the map $\Delta = (\Delta_1, ..., \Delta_r)$ is represented by a matrix with entries in $\mathcal{O}_X$. We therefore have the equivalence:

$$f \in I(\mathcal{F}, x) \cap F \quad \Longleftrightarrow \quad f \in ker \ \Delta(x) .$$

By the rank theorem, we have $\varphi_F(x) = rk \ \Delta(x)$. But the rank of this matrix is a lower semi-continuous function because it is given as the maximal size of the minors of Δ that do not vanish at x. Therefore φ_F is lower semi-continuous for the Zariski topology. □

Proof of theorem 1.3. Since X is affine, we may assume that X is embedded in $\mathbb{C}^k$ for some k. We provide $\mathbb{C}[x_1, ..., x_k]$ with the filtration $\{F_n\}$ given by the polynomials of homogeneous degree $\leq n$. By Hilbert-Samuel theorem (see [Ei]), for any ideal I of $\mathbb{C}[x_1, ..., x_k]$, the function:

$$h_I(n) = \dim_{\mathbb{C}} \ F_n - \dim_{\mathbb{C}} \ I \cap F_n .$$

is equal to a polynomial for n large enough, and the degree p of this polynomial coincides with the dimension of the variety $V(I)$. It is therefore easy to show that:

$$p = \lim_{n \to +\infty} \frac{\log(h_I(n))}{n} .$$

Let $\Pi : \mathbb{C}[x_1, ..., x_k] \to \mathcal{O}_X$ be the morphism induced by the inclusion $X \hookrightarrow \mathbb{C}^k$, and set $\widetilde{F_n} = \Pi(F_n)$. For any ideal I of $\mathcal{O}_X$, consider the function:

$$\widetilde{h_I}(n) = \dim_{\mathbb{C}} \ \widetilde{F_n} - \dim_{\mathbb{C}} \ I \cap \widetilde{F_n} .$$

Since Π is onto, we have $\widetilde{h_I}(n) = h_{\Pi^{-1}(I)}(n)$, so that $\widetilde{h_I}(n)$ coincides for n large enough with a polynomial of degree p equal to the dimension of $V(I)$. With the notation of lemma 3.1, we obtain for $I = I(\mathcal{F}, x)$:

$$p = n_{\mathcal{F}}(x) = \lim_{n \to +\infty} \frac{\log(\widetilde{h_I}(n))}{n} = \lim_{n \to +\infty} \frac{\log(\varphi_{\widetilde{F_n}}(x))}{n} .$$

By lemma 3.1, every $\varphi_{\widetilde{F_n}}$ is lower semi-continuous for the Zariski topology, hence measurable. Since a pointwise limit of measurable functions is measurable, the function $n_{\mathcal{F}}$ is measurable for the Zariski topology. Moreover since $\varphi_{\widetilde{F_n}}$ is lower semi-continuous, there exist a real number r_n and an open set U_n on X such that:

- $\dfrac{\log(\varphi_{\widetilde{F_n}}(x))}{n} \leq r_n$ for any x in X,
- $\dfrac{\log(\varphi_{\widetilde{F_n}}(x))}{n} = r_n$ for any x in U_n.

Denote by U the intersection of all U_n. Since this intersection is not empty, there exists an x in X for which $\log(\varphi_{\widetilde{F_n}}(x))/n = r_n$ for any n, so that r_n converges to a limit p. By passing to the limit, we obtain that:

- $n_{\mathcal{F}}(x) \leq p$ for any x in X,
- $n_{\mathcal{F}}(x) = p$ for any x in U.

Note that p has to be an integer. The theorem is proved. □

4 The family of minimal invariant varieties

In this section, we are going to study the set of minimal invariant varieties associated to the points of X. The result we will get will be the first step towards the proof of theorem 1.4. Let M be the following set:

$$M = \{(x, y) \in X \times X,\ \ y \in V(\mathcal{F}, x)\}$$

together with the projection $\Pi : M \longrightarrow X, (x, y) \longmapsto x$. Note that for any x, the preimage $\Pi^{-1}(x)$ is isomorphic to $V(\mathcal{F}, x)$, so that the couple (M, Π) parametrizes the set of all minimal invariant varieties. Our purpose is to show that:

Proposition 4.1. *The Zariski closure $\overline{M}$ is an irreducible affine set of dimension $dim\ X + p$, where p is the maximum of the function $n_{\mathcal{F}}$. Moreover, for every very generic point x in X, $\overline{M} \cap \Pi^{-1}(x)$ is equal to $\{x\} \times V(\mathcal{F}, x)$.*

The proof of this proposition is a direct consequence of the following lemmas.

Lemma 4.2. *The Zariski closure $\overline{M}$ is irreducible.*

Proof. For any ∂_i in $\mathcal{F}$, consider the new $\mathbb{C}$-derivation Δ_i on $\mathcal{O}_{X\times X} = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X$ given by the following formula:

$$\forall f, g \in \mathcal{O}_X, \quad \Delta_i(f(x) \otimes g(y)) = f(x) \otimes \partial_i(g)(y)\,.$$

It is easy to check that Δ_i is a well-defined derivation. Denote by $\mathcal{G}$ the collection of the Δ_i, by D the diagonal $\{(x, x),\ x \in X\}$ in $X \times X$ and set $M_0 = V(\mathcal{G}, D)$. By corollary 2.6, M_0 is irreducible. We are going to prove that $\overline{M} = M_0$.

First let us check that $M_0 \subseteq \overline{M}$. Let f be a regular function on $X \times X$ that vanishes on $\overline{M}$. Then $f(x, y) = 0$ for any couple (x, y) where y belongs to $V(\mathcal{F}, x)$. If $\varphi_t(y)$ is the flow of ∂_i at y, then $\psi_t(x, y) = (x, \varphi_t(y))$ is the flow of Δ_i at (x, y). Since y lies in $V(\mathcal{F}, x)$, $\varphi_t(y)$ belongs to $V(\mathcal{F}, x)$ for any small value of t, and we obtain:

$$f(\psi_t(x, y)) = f(x, \varphi_t(y)) = 0\,.$$

By derivation with respect to t, we get that $\Delta_i(f)(x, y) = 0$ for any (x, y) in M. So $\Delta_i(f)$ vanishes along $\overline{M}$, and the ideal $I(\overline{M})$ is stable by the family $\mathcal{G}$. Since it is contained in $I(D)$, we have the inclusion:

$$I(\overline{M}) \subseteq I(\mathcal{G}, D)\,.$$

which implies that $M_0 \subseteq \overline{M}$.

Second let us show that $\overline{M} \subseteq M_0$. Let f be a regular function that vanishes along M_0. Fix x in X and consider the function $f_x(y) = f(x, y)$ on X. Then for any Δ_1, ..., Δ_n in $\mathcal{G}$, we have:

$$\Delta_1 \circ \ldots \circ \Delta_n(f)(x, y) = \partial_1 \circ \ldots \circ \partial_n(f_x)(y)\,.$$

Since $M_0 = V(\mathcal{G}, D)$, D is contained in M_0 and $f_x(x) = 0$. So $f(x, x) = 0$ and for any ∂_1, ..., ∂_n in $\mathcal{F}$ and any x in X, we get that:

$$\partial_1 \circ \ldots \circ \partial_n(f_x)(x) = 0\,.$$

In particular, f_x belongs to $I(\mathcal{F}, x)$ and f_x vanishes along $V(\mathcal{F}, x)$. Thus f vanishes on $\{x\} \times V(\mathcal{F}, x) = \Pi^{-1}(x)$ for any x in X. This implies that f is equal to zero on M and on $\overline{M}$, so that $I(\mathcal{G}, D) \subseteq I(\overline{M})$. As a consequence, we find $\overline{M} \subseteq M_0$ and the result follows. □

Lemma 4.3. *The variety $\overline{M}$ has dimension* $\geq \dim\ X + p$.

Proof. Consider the projection $\Pi : \overline{M} \to X,\ (x, y) \mapsto x$. Since M contains the diagonal D, the map Π is onto. By the theorem on the dimension of fibres, there exists a non-empty Zariski open set U in X such that:

$$\forall x \in U, \quad \dim\ \overline{M} = \dim\ X + \dim\ \Pi^{-1}(x) \cap \overline{M}\,.$$

By theorem 1.3, there exists a countable intersection θ of Zariski open sets in X such that $n_{\mathcal{F}}(x) = p$ for all x in X. In particular, $U \cap \theta$ is non-empty. For any x in $U \cap \theta$, $\Pi^{-1}(x) \cap \overline{M}$ contains the variety $V(\mathcal{F}, x)$ whose dimension is p, and this yields:

$$\dim\ \overline{M} \geq \dim\ X + p\,. \qquad \square$$

Lemma 4.4. *The variety* $\overline{M}$ *has dimension* $\leq \dim\ X + p$.

Proof. Let $\{F_n\}$ be a filtration of $\mathcal{O}_X$ by finite-dimensional $\mathbb{C}$-vector spaces, and set:

$$M_n = \{(x, y) \in X \times X,\ \forall f \in I(\mathcal{F}, x) \cap F_n,\ \ f(y) = 0\}\ .$$

The sequence $\{M_n\}$ is decreasing for the inclusion, and $M = \cap_{n\in\mathbb{N}} M_n$. Moreover every M_n is constructible for the Zariski topology by Chevalley's theorem (see [Ei]). Indeed its complement in $X \times X$ is the image of the constructible set:

$$\begin{aligned}\Sigma_n \ = \ \Big\{(x, y,\ f) \in X \times X \times F_n,\ \forall \partial_1, ..., \partial_k \in \mathcal{F},\\ \partial_1 \circ ... \circ \partial_k(f)(y) \ = \ 0 \quad \text{and} \quad f(y) \neq 0\Big\}\,.\end{aligned}$$

under the projection $(x, y,\ f) \mapsto (x, y)$. Since D is contained in every M_n, the projection $\Pi : M_n \to X$ is onto. By the theorem on the dimension of fibres applied to the irreducible components of $\overline{M_n}$, there exists a non-empty Zariski open set U_n in X such that:

$$\forall x \in U_n, \quad \dim\ M_n \leq \dim\ X + \dim\ \Pi^{-1}(x) \cap M_n\ .$$

Since $\overline{M} \subseteq \overline{M_n}$ for any n, and $\Pi^{-1}(x) \cap M_n \simeq V(I(\mathcal{F}, x) \cap F_n)$, we obtain:

$$\forall x \in U_n, \quad \dim\ \overline{M} \leq \dim\ X + \dim\ V(I(\mathcal{F}, x) \cap F_n)\,.$$

Since every U_n is open, the intersection $\theta' = \cap_{n\in\mathbb{N}} U_n$ is non-empty. Let θ be an intersection of Zariski open sets of X such that $n_{\mathcal{F}}(x) = p$ for any x of θ. For any fixed x in $\theta \cap \theta'$, we have:

$$\forall n \in \mathbb{N}, \quad \dim\ \overline{M} \leq \dim\ X + \dim\ V(I(\mathcal{F}, x) \cap F_n)\,.$$

Since $\mathcal{O}_X$ is noetherian, there exists an order n_0 such that $I(\mathcal{F}, x)$ is generated by $I(\mathcal{F}, x) \cap F_n$ for any $n \geq n_0$. In this context, $V(\mathcal{F}, x) = V(I(\mathcal{F}, x) \cap F_n)$ for all $n \geq n_0$, and $V(\mathcal{F}, x)$ has dimension p, which implies that:

$$\dim\ \overline{M} \leq \dim\ X + p\,. \qquad \square$$

Lemma 4.5. *For every very generic point x in X, $\overline{M} \cap \Pi^{-1}(x)$ is equal to $\{x\} \times V(\mathcal{F}, x)$.*

Proof. Consider the constructible sets M_n introduced in lemma 4.4. By construction their intersection is equal to M. The $\{\overline{M_n}\}$ form a decreasing sequence which converges to $\overline{M}$. Since these are algebraic sets, there exists an index n_0 such that for any $n \geq n_0$, we have $\overline{M_n} = \overline{M}$. We consider the sequence $\{M_n\}_{n \geq n_0}$ and denote by G_n the Zariski closure of $\overline{M} - M_n$. By the theorem on the dimension of fibres, there exists a Zariski open set V_n on X such that for any x in V_n, either $\Pi^{-1}(x) \cap G_n$ is empty or has dimension $< p$. Since $\Pi^{-1}(x) \cap M = \{x\} \times V(\mathcal{F}, x)$ for any x in X, we have the following decomposition:

$$\Pi^{-1}(x) \cap \overline{M} = \{x\} \times V(\mathcal{F}, x) \cup \cup_{n \geq n_0} \Pi^{-1}(x) \cap G_n \,.$$

For all x in $\theta = \cap V_n$, the set $\Pi^{-1}(x) \cap G_n$ has dimension $< p$ for any $n \geq n_0$, hence its Hausdorff dimension is no greater than $(2p - 2)$ (see [Ch]). Consequently the countable union $\cup_{n \geq n_0} \Pi^{-1}(x) \cap G_n$ has an Hausdorff dimension $< 2p$. Let $H_{i,x}$ be the irreducible components of $\Pi^{-1}(x) \cap \overline{M}$ distinct from $\{x\} \times V(\mathcal{F}, x)$. These $H_{i,x}$ are covered by the union $\cup_{n \geq n_0} \Pi^{-1}(x) \cap G_n$, hence their Hausdorff dimension does not exceed $(2p - 2)$. Therefore the Krull dimension of $H_{i,x}$ is strictly less than p for any i and any x in θ. If H_x denotes the union of the $H_{i,x}$, then we have for any x in θ:

$$\Pi^{-1}(x) \cap \overline{M} = \{x\} \times V(\mathcal{F}, x) \cup H_x \quad \text{and} \quad \dim\ H_x < p \,.$$

Now by Stein factorization theorem (see [Ha]), the map $\Pi : \overline{M} \to X$ is a composite of a quasi-finite map with a map whose generic fibres are irreducible. In particular $\Pi^{-1}(x) \cap \overline{M}$ is equidimensionnal of dimension p for generic x in X. Therefore the variety H_x should be contained in $\{x\} \times V(\mathcal{F}, x)$, and we have for any x in θ:

$$\Pi^{-1}(x) \cap \overline{M} = \{x\} \times V(\mathcal{F}, x) \,.$$ □

5 Proof of theorem 1.4

Let X be an irreducible affine variety over $\mathbb{C}$ of dimension n, endowed with an algebraic foliation $\mathcal{F}$. Let p be the integer given by theorem 1.3. In this section we will establish theorem 1.4. We begin with a few lemmas.

Lemma 5.1. *Let $F : X \to Y$ be a dominant morphism of irreducible affine varieties. Then for any Zariski open set U in X, $F(U)$ is dense in Y.*

Proof. Suppose on the contrary that $F(U)$ is not dense in Y. Then there exists a non-zero regular function f on Y that vanishes along $\overline{F(U)}$. The function $f \circ F$

vanishes on U, hence on X by density. So $F(X)$ is contained in $f^{-1}(0)$, which is impossible since this set is dense in Y. □

Lemma 5.2. *Let $\overline{M}$ be the variety defined in section 4. Then there exists an irreducible variety H in X such that $\overline{M} \cap X \times H$ has dimension n and the morphism $\Pi : \overline{M} \cap X \times H \to X$ induced by the projection is dominant.*

Proof. Let (x, y) be a smooth point of $\overline{M}$ such that x is a smooth point of X. By the generic smoothness theorem, we may assume that $d\Pi_{(x,y)}$ is onto. Consider the second projection $\Psi(x, y) = y$. Since the map (Π, Ψ) defines an embedding of $\overline{M}$ into $X \times X$, and $d\Pi_{(x,y)}$ is onto, there exist some regular functions $g_1, ..., g_p$ on X such that $(d\Pi_{(x,y)}, dg_{1(y)}, ..., dg_{p(y)})$ is an isomorphism from $T_{(x,y)}\overline{M}$ to $T_x X \oplus \mathbb{C}^p$.

Let $G : \overline{M} \to \mathbb{C}^p$ be the map $(g_1, ..., g_p)$, and denote by E the set of points (x, y) in $\overline{M}$ where either $\overline{M}$ is singular or (Π, G) is not submersive. By construction E is a closed set distinct from $\overline{M}$. Since $dG_{(y)}$ has rank p on $T_{(x,y)}\overline{M}$, the map $G : \overline{M} \to \mathbb{C}^p$ is dominant. So its generic fibres have dimension n. Fix a fibre $G^{-1}(z)$ of dimension n that is not contained in E. Then there exists a smooth point (x, y) in $G^{-1}(z)$ such that $d(\Pi, G)_{(x,y)}$ is onto. The morphism $\Pi : G^{-1}(z) \to X$ is a submersion at (x, y), hence it is dominant. Moreover $G^{-1}(z)$ is of the form $X \times F^{-1}(z) \cap \overline{M}$, where $F : X \to \mathbb{C}^p$ is the map $(g_1, ..., g_p)$.

Choose an irreducible component H of $F^{-1}(z)$ such that $\Pi : X \times H \cap \overline{M} \to X$ is dominant. By construction $X \times H \cap \overline{M}$ has dimension $\leq n$. Since the latter map is dominant, its dimension is exactly equal to n. □

Proof of theorem 1.4. Let H be an irreducible variety of codimension p in X satisfying the conditions of lemma 5.2. Denote by N the union of irreducible components of $\overline{M} \cap X \times H$ that are mapped dominantly on X by Π. By construction N has dimension dim X and the morphism $\Pi : N \to X$ is quasi-finite. So there exists an open set U in X such that:

$$\widetilde{\Pi} : \Pi^{-1}(U) \cap N \longrightarrow U \, .$$

is a finite unramified morphism. Let r be the degree of this map. For any point x in U, there exist r points $y_1, ..., y_r$ in H such that $\widetilde{\Pi}^{-1}(x) = \{y_1, ..., y_r\}$. Let $\mathfrak{S}_r$ act on H^r by permutation of the coordinates, i.e $\sigma.(y_1, ..., y_r) = (y_{\sigma(1)}, ..., y_{\sigma(r)})$. Since this action is algebraic and $\mathfrak{S}_r$ is finite, the algebraic quotient $H^r//\mathfrak{S}_r$ exists and is an irreducible affine variety (see [Mu]). Let $Q : H \to H^r//\mathfrak{S}_r$ be the corresponding quotient morphism. Consider the mapping:

$$\varphi : U \longrightarrow H^r//\mathfrak{S}_r, \quad x \longmapsto Q(y_1, ..., y_r) \, .$$

Note that its graph is constructible in $U \times H^r//\mathfrak{S}_r$. Indeed it is given by the set:

$$\Sigma = \Big\{(x, y'),\ \exists(y_1, ..., y_r) \in H^r,\ \forall i \neq j,\ y_i \neq y_j,\\ (x, y_i) \in \overline{M} \text{ and } Q(y_1, ..., y_r) = y'\Big\}.$$

By Serre's theorem (see [Lo]), φ is a rational map on U. Since $\widetilde{\Pi}$ is unramified, φ is also holomorphic on U, hence it is regular on U. Denote by Y the Zariski closure of $\varphi(U)$ in $H^r//\mathfrak{S}_r$. Since U is irreducible, Y is itself irreducible.

By construction, for any x in U, $\{x\} \times \varphi^{-1}(\varphi(x))$ is equal to $\Pi^{-1}(x) \cap \overline{M}$. For every very generic point x in X, $\Pi^{-1}(x) \cap \overline{M}$ corresponds to $\{x\} \times V(\mathcal{F}, x)$ by proposition 4.1. So $\varphi^{-1}(\varphi(x)) = V(\mathcal{F}, x)$ for every generic point x in X, hence it has dimension p. By the theorem on the dimension of fibres, Y has dimension $(n - p)$.

Since $\varphi^{-1}(\varphi(x)) = V(\mathcal{F}, x)$ for every generic point x in X, this fibre is tangent to the foliation $\mathcal{F}$. Since tangency is a closed condition, all the fibres of φ are tangent to $\mathcal{F}$. Let f be a rational function on Y. In the neighborhood of any smooth point x where $\mathcal{F}$ is regular and $f \circ \varphi$ is well-defined, the function $f \circ \varphi$ is constant on the leaves of $\mathcal{F}$. So $f \circ \varphi$ is a rational first integral of $\mathcal{F}$. Via the morphism φ^* induced by φ, $K_{\mathcal{F}}$ is clearly isomorphic to $\mathbb{C}(Y)$ which has transcendence degree $(n - p)$ over $\mathbb{C}$. □

6 An example

In this last section, we introduce an example that illustrates both theorems 1.3 and 1.4. Consider the affine space $\mathbb{C}^4$ with coordinates (u, v, x, y), and the algebraic foliation $\mathcal{F}$ induced by the vector field:

$$\partial = ux\frac{\partial}{\partial x} + vy\frac{\partial}{\partial y}.$$

For any (λ, μ) in $\mathbb{C}^2$, the plane $V(u - \lambda, v - \mu)$ is tangent to $\mathcal{F}$. Denote by $\partial_{\lambda,\mu}$ the restriction of ∂ to that plane parametrized by (x, y). Then two cases may occur:

- If $[\lambda; \mu]$ does not belong to $\mathbb{P}^1(\mathbb{Q})$, then $\partial_{\lambda,\mu}$ has no rational first integrals. The only algebraic curves tangent to $\partial_{\lambda,\mu}$ are the lines $x = 0$ and $y = 0$. There is only one singular point, namely $(0, 0)$.

- If $[\lambda; \mu]$ belongs to $\mathbb{P}^1(\mathbb{Q})$, choose a couple of coprime integers $(p, q) \neq (0, 0)$ such that $p\lambda + q\mu = 0$. The function $f(x, y) = x^p y^q$ is a rational first integral for $\partial_{\lambda,\mu}$. The algebraic curves tangent to $\partial_{\lambda,\mu}$ are the lines $x = 0$, $y = 0$ and the fibres $f^{-1}(z)$ for $z \neq 0$. There is only one singular point, namely $(0, 0)$.

From those two cases, we can get the following values for the function $n_{\mathcal{F}}$:

- $n_{\mathcal{F}}(u, v, x, y) = 2$ if $[\lambda; \mu] \notin \mathbb{P}^1(\mathbb{Q})$ and $xy \neq 0$,
- $n_{\mathcal{F}}(u, v, x, y) = 0$ if $x = y = 0$,
- $n_{\mathcal{F}}(u, v, x, y) = 1$ otherwise.

In particular, this function is measurable but not constructible for the Zariski topology, as can be easily seen from its fibre $n_{\mathcal{F}}^{-1}(2)$. Moreover since $p = 2$, its field $K_{\mathcal{F}}$ has transcendence degree 2 over $\mathbb{C}$. In fact it is easy to check that $K_{\mathcal{F}} = \mathbb{C}(u, v)$.

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Philippe Bonnet
Mathematisches Institut, Universität Basel
Rheinsprung 21, Basel 4051
SWITZERLAND

E-mail: Philippe.bonnet@unibas.ch