Numer. Math. (2011) 118:737–764
DOI 10.1007/s00211-011-0373-4

Numerische Mathematik

Analysis of a finite volume element method for the Stokes problem

Alfio Quarteroni · Ricardo Ruiz-Baier

Received: 19 April 2010 / Revised: 12 February 2011 / Published online: 8 March 2011
© Springer-Verlag 2011

Abstract In this paper we propose a stabilized conforming finite volume element method for the Stokes equations. On stating the convergence of the method, optimal a priori error estimates in different norms are obtained by establishing the adequate connection between the finite volume and stabilized finite element formulations. A superconvergence result is also derived by using a postprocessing projection method. In particular, the stabilization of the continuous lowest equal order pair finite volume element discretization is achieved by enriching the velocity space with local functions that do not necessarily vanish on the element boundaries. Finally, some numerical experiments that confirm the predicted behavior of the method are provided.

Mathematics Subject Classification (2000) 65N30 · 35Q30 · 65N12 · 65N15

1 Introduction

Finite volume element methods (FVEM) [9,21], also known as marker and cell methods [20,27,29], generalized difference methods [36], finite volume methods [35,44], covolume methods [10,39], combined finite volume-finite element methods [3,5,22, 23,31] or box methods [4,16], are approximation methods that could be placed somehow in between classical finite volume schemes and standard finite element (FE)

A. Quarteroni · R. Ruiz-Baier (✉)
CMCS-MATHICSE-SB, Ecole Polytechnique Fédérale de Lausanne EPFL,
Station 8, 1015 Lausanne, Switzerland
e-mail: ricardo.ruiz@epfl.ch

A. Quarteroni
MOX, Dipartimento di Matematica "F. Brioschi", Politecnico di Milano,
via Bonardi 9, 20133 Milan, Italy
e-mail: alfio.quarteroni@epfl.ch

methods. Roughly speaking, the FVEM is able to keep the simplicity and local conservativity of finite volume methods, and at the same time permits a natural and systematic development of error analysis in the L^2-norm as in standard FE methods. This is basically achieved by introducing a transfer map which allows to rewrite the FE formulation as its classical finite-volume-like counterpart, i.e., using piecewise constant test functions. In the FVEM approach, a complementary dual (or adjoint) mesh is also constructed, and this is commonly done by connecting the barycenters of the triangles in the FE mesh, with the midpoints of the associated edges (see [10,23,35]), or the vertices of the triangles (see [14,44]). A usual difficulty in the analysis of classical finite volume methods arises from trial and test functions to lie in different spaces and to be associated with different meshes. In contrast, in the FVE approach, one of the most appealing features is that using the transfer operator mentioned above, an equivalent auxiliary problem can be formulated whose approximate solution is found in the same subspace used in the construction of the FE method. In fact, FVE methods might be regarded as a special class of Petrov–Galerkin methods where the trial function spaces are connected with the test functions' spaces associated with the dual partition induced by the control volumes [33,35]. Furthermore, the method used herein (based on the relation between finite volume and FE approximations) possesses the appealing feature of preserving the local conservation property in each control volume.

As for the numerical approximation of Stokes equations, numerous methods have been proposed, analyzed and tested (for an overview, the reader is referred to [28] and the references therein). In the framework of finite volume methods, recent contributions include the work by Gallouët et al. [26] which treat the nonlinear case based on Crouzeix-Raviart elements, Nicase and Djadel [38] prove different error estimates for a finite volume scheme by using nonconforming elements, Eymard et al. [19] obtained error estimates for a stabilized finite volume scheme based on the Brezzi-Pitkäranta method. Regarding FVE approximations for the Stokes problem, in his early paper, Chou [10] used nonconforming piecewise linear elements for velocity and piecewise constant for pressure. In the contribution by Ye [43], the analysis is carried out for both conforming and nonconforming elements on triangles and rectangles. We also mention the recent work of Li and Chen [35] who advanced a FVE method based on a stabilization method that uses the residual of two local Gauss integration formulae on each finite element.

In this paper we will devote ourselves to the study of a particular stabilized FVE method constructed on the basis of a conforming finite element formulation where the velocity and pressure fields are approximated by piecewise linear polynomials. Since the considered approximation of the Stokes equations is based on the pair $\mathbb{P}_1 - \mathbb{P}_1$ that does not satisfy the discrete inf–sup condition (see [28]), one of the most common remedies consists in including a stabilization technique, i.e., to add a mesh dependent term to the usual formulation. One of the motivations for keeping the unstable pair of lowest equal order elements, is that they allow a more efficient implementation, by achieving a reduction of the number of unknowns in the final systems. Among the wide class of stabilized FE formulations available from the literature, such as Streamline-Upwind/Petrov–Galerkin (SUPG), Galerkin-Least-Squares (GLS) and other methods (see for instance [41, Sect. 9.4] and the references therein), in this paper we include a stabilization technique similar to the one introduced by Franca et al. [25], in which a

Petrov–Galerkin approach is used to enrich the trial space with bubble functions being solutions to a local problem involving the residual of the momentum equation, which can be solved analytically. As recently proposed by Araya et al. [2], by enriching the velocity space using a *multiscale* approach combined with static condensation, the resulting FE method includes the classical GLS additional terms at the element level and a suitable jump term on the normal derivative of the velocity field at the element boundaries. For the latter, the stabilization parameter is known exactly.

For our method, the essential point is to appropriately connect the FE and FVE formulations. After establishing such relationship we deduce the corresponding optimal a priori error estimates for the new stabilized FVE method using a *usual* approach for classical FE methods. In contrast to classical finite volume schemes, the velocity fluxes will not be discretized in a finite-difference fashion. This fact plays an important role at the implementation stage as well, since all the information corresponding to the dual partition, needed for the derivation of the FVE formulation can be retrieved from the information on the edges of the primal mesh.

Another important novel ingredient of this paper is the superconvergence analysis of the approximate solution. The main goal is to improve the current accuracy of the approximation by applying a postprocessing technique constructed on the basis of a projection method similar to those presented in [30,34,37,42]. Super-convergence properties of FVE approximations in the nonconforming and conforming cases were first studied in the recent works by Cui and Ye [14] and Wang and Ye [42]. The technique consists in projecting the FVE space to another approximation space (possibly of higher order) related to a coarser mesh. A detailed study including the analysis of a posteriori error estimates for FVE methods in the spirit of [7,17], and adaptivity following [11] have been postponed for a forthcoming paper. Further efforts are also being made to extend the analysis herein presented to the transient Navier–Stokes equations, and coupled problems of multiphase flow [8].

The remainder of the paper is organized as follows. In the next section, a set up of some preliminary results and notations concerning the spaces involved in the analysis is followed by a detailed description of the model problem and the FE discretization used as reference. Further, some auxiliary lemmas are also provided in that section. Next, the stabilized finite volume formulation that we will employ and its corresponding link with the reference finite element method are provided in Sect. 3. The main results of the paper, namely the convergence analysis of the stabilized finite volume element approximation, are proved in Sect. 4, and additional superconvergence estimates are given in Sect. 5. Finally, Sect. 6 is devoted to the presentation of an illustrative numerical test which confirms the expected rates of convergence and superconvergence.

2 Preliminaries

The standard notation will be used for Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, $L_0^2(\Omega) = \{v{\in}L^2(\Omega)\colon \int_\Omega v{=}0\}$ and Sobolev functional spaces $H^m(\Omega)$, $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$, where Ω is an open, bounded and connected subset of $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. Further, let us denote $\boldsymbol{H}^m(\Omega) = H^m(\Omega)^2$, and in general $\boldsymbol{M}$ will denote the corresponding vectorial counterpart of the scalar space M. For a subset

$R \subset \Omega$, $(\cdot,\cdot)_R$ denotes the $L^2(R)$-inner product. In addition, $\mathbb{P}_r(R)$ will represent the space of polynomial functions of degree $s \leq r$ on R.

2.1 The boundary value problem

Let us consider the following steady state Stokes problem with Dirichlet boundary conditions: Find $\boldsymbol{u}, p$ such that

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$
$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.3}$$

This linear problem describes the steady motion of an incompressible viscous fluid. As usual, the sought quantities are the vectorial velocity field $\boldsymbol{u}$, the scalar pressure p, the prescribed external force $\boldsymbol{f}$ and the constant fluid viscosity $\nu > 0$. Multiplying (2.1) by a test function $\boldsymbol{v}$, (2.2) by a test function q, integrating by parts both equations over Ω and summing the result, one obtains the weak formulation of problem (2.1)–(2.3): Find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\nu\,(\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_\Omega - (p, \nabla\cdot\boldsymbol{v})_\Omega + (q, \nabla\cdot\boldsymbol{u})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \forall(\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega). \tag{2.4}$$

This model problem is well-posed (see e.g. [28] for details on the analysis).

Throughout the paper, $C > 0$ will denote a constant depending only on the data $(\nu, \Omega, \boldsymbol{f})$ and not on the discretization parameters.

2.2 Finite element approximation

Let $\mathcal{T}_h$ be a triangulation of Ω constructed by closed triangle elements K with boundary ∂K. We fix the numbering s_j, $j = 1, \ldots, N_h$ of all nodes or vertices of $\mathcal{T}_h$. With $\mathcal{E}_h$ we denote the set of edges of $\mathcal{T}_h$, while $\mathcal{E}_h^{\text{int}}$ will denote the edges of $\mathcal{T}_h$ that are not part of $\partial\Omega$. In addition, h_K denotes the diameter of the element K, and the mesh parameter is given by $h = \max_{K\in\mathcal{T}_h}\{h_K\}$. The partition $\mathcal{T}_h$ is assumed to be regular, that is, there exists $C > 0$ such that

$$\frac{h_K}{\varrho_K} \leq C, \quad \text{for all } K \in \mathcal{T}_h, \tag{2.5}$$

where ϱ_K denotes the diameter of the largest ball contained in K. By $\boldsymbol{\mathcal{V}}_h^r$ and Q_h^t, for $1 \leq r \leq 2$, $0 \leq t \leq 1$, we will denote the standard finite element spaces for the approximation of velocity and pressure on the triangulation $\mathcal{T}_h$, respectively. These spaces are defined as

$$\boldsymbol{\mathcal{V}}_h^r = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{C}^0(\bar{\Omega}) : \boldsymbol{v}|_K \in \mathbb{P}_r(K)^2 \text{ for all } K \in \mathcal{T}_h\}$$

provided with the basis $\{\phi_j\}_j$, and

$$Q_h^t = \{q \in L_0^2(\Omega)\, q|_K \in \mathbb{P}_t(K) \text{ for all } K \in \mathcal{T}_h\}.$$

It is well known that choosing for instance $r = t = 1$, the classic Galerkin formulation of the problem: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1$ such that

$$\nu\,(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_\Omega - (p_h, \nabla\cdot \boldsymbol{v}_h)_\Omega + (q_h, \nabla\cdot \boldsymbol{u}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \quad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1,$$

does not satisfy the discrete inf–sup condition. To overcome this difficulty, we include a stabilization correction similar to that introduced in [2]. In that paper, and differently than other stabilization techniques available, the stabilization parameter corresponding to the jump terms is known. Moreover, the trial velocity space is enriched with a space of functions that do not vanish on the element boundary, which is split into a bubble part and an harmonic extension of the boundary condition. An essential point in our analysis is based on one of the formulations presented in [2]. The main ingredients of that idea are included here for sake of completeness.

Let $\boldsymbol{H}^1(\mathcal{T}_h)$ denote the space of functions whose restriction to $K \in \mathcal{T}_h$ belongs to $\boldsymbol{H}^1(K)$, and $\boldsymbol{E}_h \subset \boldsymbol{H}^1(\mathcal{T}_h)$ be a finite dimensional space, called *multiscale space* such that $\boldsymbol{E}_h \cap \boldsymbol{\mathcal{V}}_h^r = \{\boldsymbol{0}\}$, and consider the following Petrov–Galerkin formulation: Find $(\boldsymbol{u}_h + \boldsymbol{u}_e, p_h) \in [\boldsymbol{\mathcal{V}}_h^r \oplus \boldsymbol{E}_h] \times Q_h^t$ such that

$$\nu\,(\nabla(\boldsymbol{u}_h + \boldsymbol{u}_e), \nabla \boldsymbol{v}_h)_\Omega - (p_h, \nabla\cdot \boldsymbol{v}_h)_\Omega + (q_h, \nabla\cdot(\boldsymbol{u}_h + \boldsymbol{u}_e))_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega,$$

for all $(\boldsymbol{v}_h, q_h) \in [\boldsymbol{\mathcal{V}}_h^r \oplus \boldsymbol{E}_h^0] \times Q_h^t$, where $\boldsymbol{E}_h^0$ denotes the space of functions in $\boldsymbol{H}^1(\mathcal{T}_h)$ whose restriction to $K \in \mathcal{T}_h$ belongs to $\boldsymbol{H}_0^1(K)$. Notice that trial and test function spaces do not coincide. The Petrov–Galerkin scheme above can be equivalently written as: Find $(\boldsymbol{u}_h + \boldsymbol{u}_e, p_h) \in [\boldsymbol{\mathcal{V}}_h^r \oplus \boldsymbol{E}_h] \times Q_h^t$ such that

$$\begin{aligned} &\nu\,(\nabla(\boldsymbol{u}_h + \boldsymbol{u}_e), \nabla \boldsymbol{v}_a)_\Omega - (p_h, \nabla\cdot \boldsymbol{v}_a)_\Omega + (q_h, \nabla\cdot(\boldsymbol{u}_h + \boldsymbol{u}_e))_\Omega = (\boldsymbol{f}, \boldsymbol{v}_a)_\Omega, \\ &\nu\,(\nabla(\boldsymbol{u}_h + \boldsymbol{u}_e), \nabla \boldsymbol{v}_b)_K - (p_h, \nabla\cdot \boldsymbol{v}_b)_K = (\boldsymbol{f}, \boldsymbol{v}_b)_K, \end{aligned} \tag{2.6}$$

for all $\boldsymbol{v}_a \in \boldsymbol{\mathcal{V}}_h^r,\ q_h \in Q_h^t,\ \boldsymbol{v}_b \in \boldsymbol{H}_0^1(K),\ K \in \mathcal{T}_h$. Since for every $K \in \mathcal{T}_h$, $\boldsymbol{v}_b|_{\partial K} = \boldsymbol{0}$, the second equation in (2.6) corresponds to the weak form of the following problem

$$\begin{aligned} -\nu\Delta \boldsymbol{u}_e + \nabla p_h &= \boldsymbol{f} + \nu\Delta \boldsymbol{u}_h \quad \text{in } K \in \mathcal{T}_h, \\ \boldsymbol{u}_e &= \boldsymbol{g}_e \quad \text{on } F \subset \partial K,\ K \in \mathcal{T}_h, \end{aligned} \tag{2.7}$$

where $\boldsymbol{g}_e$ is the solution of the following one-dimensional Poisson problem on $\mathcal{E}_h^{\text{int}}$:

$$\begin{aligned} -\nu\partial_{ss}\boldsymbol{g}_e &= \frac{1}{h_F}[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h + p_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F \quad \text{on } F \in \mathcal{E}_h^{\text{int}}, \\ \boldsymbol{g}_e &= \boldsymbol{0} \qquad \text{at the endpoints of } F. \end{aligned} \tag{2.8}$$

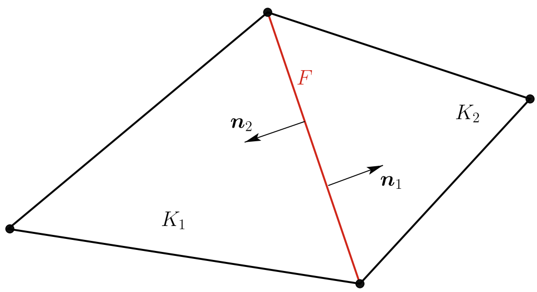


Fig. 1 Two neighboring elements $K_1, K_2 \in \mathcal{T}_h$ (with outer normals $\boldsymbol{n}_1, \boldsymbol{n}_2$) sharing the edge $F \in \mathcal{E}_h^{\text{int}}$

Here s is the curvilinear abscissa of F, $\partial_{\boldsymbol{n}}$ stands for the normal derivative operator, and $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{2\times 2}$. In addition, by $[\![\boldsymbol{w}]\!]_F$ we denote the jump of $\boldsymbol{w} \in \boldsymbol{H}^1(\mathcal{T}_h)$ across the edge F, that is

$$[\![\boldsymbol{w}]\!]_F = (\boldsymbol{w}|_{K_1})|_F \cdot \boldsymbol{n}_1 + (\boldsymbol{w}|_{K_2})|_F \cdot \boldsymbol{n}_2, \tag{2.9}$$

where $K_1, K_2 \in \mathcal{T}_h$ are such that $K_1 \cap K_2 = F$ and $\boldsymbol{n}_1, \boldsymbol{n}_2$ are the exterior normals to K_1, K_2 respectively (see Fig. 1). If F lies on $\partial\Omega$, then we take $[\![\boldsymbol{w}]\!]_F = \boldsymbol{w}\cdot\boldsymbol{n}$. Note that the conformity of the enriched space for the bubble-part of the velocity is achieved via the non-homogeneous transmission condition on $\mathcal{E}_h^{\text{int}}$ defined by (2.7)–(2.8). Now, on each $K \in \mathcal{T}_h$ set $\boldsymbol{u}_e|_K = \boldsymbol{u}_e^K + \boldsymbol{u}_e^{\partial K}$. Therefore, from (2.7) we have the auxiliary problems:

$$\begin{aligned} -\nu\Delta\boldsymbol{u}_e^K &= \boldsymbol{f} + \nu\Delta\boldsymbol{u}_h - \nabla p_h \qquad \text{in } K \in \mathcal{T}_h,\\ \boldsymbol{u}_e^K &= \boldsymbol{0} \quad \text{on } \partial K \in \mathcal{T}_h, \end{aligned}$$

and

$$\begin{aligned} -\nu\Delta\boldsymbol{u}_e^{\partial K} &= 0 \quad \text{in } K \in \mathcal{T}_h,\\ \boldsymbol{u}_e^{\partial K} &= \boldsymbol{g}_e \quad \text{on } \partial K \in \mathcal{T}_h. \end{aligned}$$

These problems are well-posed, and this implies that the second equation in (2.6) is satisfied. Then the enriched part of the solution is completely identified. A static condensation procedure (see the detailed development in [2]) allows to derive the following stabilized method: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{\mathcal{V}}_h^r \times Q_h^t$ such that

$$\begin{aligned} &\nu\,(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h)_\Omega - (p_h, \nabla\cdot\boldsymbol{v}_h)_\Omega\\ &\quad + (q_h, \nabla\cdot\boldsymbol{u}_h)_\Omega + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}(-\nu\Delta\boldsymbol{u}_h + \nabla p_h, \nu\Delta\boldsymbol{v}_h + \nabla q_h)_K \end{aligned}$$

$$+\sum_{F\in\mathcal{E}_h^{\text{int}}}\frac{h_F}{12\nu}\left([\![\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_h+p_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F,[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h+q_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F\right)_F$$

$$=(\boldsymbol{f},\boldsymbol{v}_h)_\Omega+\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}\left(\boldsymbol{f},\nu\Delta\boldsymbol{v}_h+\nabla q_h\right)_K, \tag{2.10}$$

for all $(\boldsymbol{v}_h, q_h)\in\boldsymbol{\mathcal{V}}_h^r\times Q_h^t$.

Such formulation depends on the assumption that $\boldsymbol{f}$ is piecewise constant on each element $K\in\mathcal{T}_h$. Nevertheless, as done in [2], error estimates with the same optimal order of convergence can be derived for the more general case in which $\boldsymbol{f}\in\boldsymbol{H}^1(\Omega)$. Notice that in the case in which the jump terms across $F\in\mathcal{E}_h^{\text{int}}$ are neglected, method (2.10) reduces to a Douglas–Wang stabilization method (see [15]).

The following section contains well known results that will play a key role in the construction of the error estimates.

2.3 Some technical lemmas

We will make use of two well established trace inequalities (cf. [1, Theorem 3.10])

$$\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(F)}\le C\left(h_K^{-1}\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(K)}+h_K|\boldsymbol{v}|^2_{\boldsymbol{H}^1(K)}\right)\qquad\forall\boldsymbol{v}\in\boldsymbol{H}^1(K), \tag{2.11}$$

$$\|\partial_{\boldsymbol{n}}\boldsymbol{v}\|^2_{\boldsymbol{L}^2(F)}\le C\left(h_K^{-1}|\boldsymbol{v}|^2_{\boldsymbol{H}^1(K)}+h_K|\boldsymbol{v}|^2_{\boldsymbol{H}^2(K)}\right)\qquad\forall\boldsymbol{v}\in\boldsymbol{H}^2(K), \tag{2.12}$$

for $F\subset\partial K$, where C depends also on the minimum angle of $K\in\mathcal{T}_h$.

Let $I_h:\boldsymbol{H}_0^1(\Omega)\cap C^0(\overline{\Omega})^2\to\boldsymbol{\mathcal{V}}_h^1$ be the usual Lagrange interpolation operator, $\Pi_h: L^2(\Omega)\to Q_h^t$ the L^2-projection operator, and $J_h:\boldsymbol{H}^1(\Omega)\to\boldsymbol{\mathcal{V}}_h^1$ the Clément interpolation operator (see e.g. [13,18]). These operators satisfy some well known approximation properties which we collect in the following lemma.

Lemma 2.1 (Interpolation operators) *For all* $\boldsymbol{v}\in\boldsymbol{H}^2(\Omega)$, $q\in H^1(\Omega)\cap L_0^2(\Omega)$, $K\in\mathcal{T}_h$, $F\in\mathcal{E}_h^{\text{int}}$, *the following estimates hold*

$$|\boldsymbol{v}-I_h\boldsymbol{v}|_{\boldsymbol{H}^m(K)}\le Ch_K^{2-m}|\boldsymbol{v}|_{\boldsymbol{H}^2(K)}\qquad m=0,1,2 \tag{2.13}$$

$$\|I_h\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\le C\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, \tag{2.14}$$

$$|\boldsymbol{v}-I_h\boldsymbol{v}|_{\boldsymbol{H}^m(F)}\le Ch_F^{2-m-1/2}|\boldsymbol{v}|_{\boldsymbol{H}^2(\tilde{K})}\qquad m=0,1, \tag{2.15}$$

$$|\boldsymbol{v}-J_h\boldsymbol{v}|_{\boldsymbol{H}^m(K)}\le Ch_K^{1-m}|\boldsymbol{v}|_{\boldsymbol{H}^1(\tilde{K})}\qquad m=0,1, \tag{2.16}$$

$$\|q-\Pi_h q\|_{L^2(\Omega)}\le Ch|q|_{H^1(\Omega)}, \tag{2.17}$$

$$\|\Pi_h q\|_{L^2(\Omega)}\le C\|q\|_{L^2(\Omega)}, \tag{2.18}$$

where $\tilde{K}$ *is the union of all elements* L *such that* $\bar{K}\cap\bar{L}\neq\emptyset$.

Proof For (2.13), (2.14), and (2.16)–(2.18) see e.g. [18,40]. Relation (2.15) follows from (2.13), the local mesh regularity condition (2.5) and (2.11).

Finally, owing to the continuous inf–sup condition satisfied by the couple of the spaces $H_0^1(\Omega)$ and $L_0^2(\Omega)$, it is known (cf. [28, Corollary 2.4 and Sect. 5.1]) that the following result holds.

Lemma 2.2 *For each $r_h \in Q_h \subset L_0^2(\Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ such that*

$$\nabla\cdot \boldsymbol{w} = r_h \text{ a.e. in } \Omega, \quad \text{and} \quad |\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)} \le C\, \|r_h\|_{L^2(\Omega)}.$$

3 Finite volume element approximation

In this section, starting from the FE method (2.10), and a standard finite volume mesh, we provide the main tools that stand behind our FVE formulation.

3.1 The finite volume mesh

Let $S = \{s_j, j = 1, \dots, N_h\}$ be the set of nodes of $\mathcal{T}_h$. Before defining our FVE method, let us introduce an adjoint mesh $\mathcal{T}_h^\star$ in Ω, whose elements $K_j^\star$ are closed polygons called control volumes. For constructing $\mathcal{T}_h^\star$, a general scheme for a generic triangle $K \in \mathcal{T}_h$ will be presented. If we fix an interior point b_K in every $K \in \mathcal{T}_h$ (we will choose b_K to be the barycenter of $K \in \mathcal{T}_h$), we can construct $\mathcal{T}_h^\star$ by associating to each node $s_j \in S$, a control volume $K_j^\star$, whose edges are obtained by connecting b_K with the midpoints of each edge of K, forming a so-called Donald diagram (see e.g. [22,32,40]), as shown in Fig. 2. Furthermore, if $\mathcal{T}_h$ is locally regular (see (2.5) then so is $\mathcal{T}_h^\star$, that is, there exists $C > 0$ such that $C^{-1}h^2 \le |K_j^\star| \le Ch^2$, for all $K_j^\star \in \mathcal{T}_h^\star$. In our FVE scheme, the trial function space for the velocity field associated with $\mathcal{T}_h$ is $\boldsymbol{\mathcal{V}}_h^1$, and the test function space associated with $\mathcal{T}_h^\star$ corresponds to the set of all piecewise constants. Specifically,

$$\boldsymbol{\mathcal{V}}_h^\star := \Big\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_{K_j^\star} \in \mathbb{P}_0(K_j^\star)^2 \text{ for all } K_j^\star \in \mathcal{T}_h^\star,\ \boldsymbol{v}|_{K_j^\star} = \boldsymbol{0} \text{ if } K_j^\star \text{ is a boundary volume}\Big\}.$$

It holds that $\dim(\boldsymbol{\mathcal{V}}_h^1) = \dim(\boldsymbol{\mathcal{V}}_h^\star) = N_h$. Analogously, Q_h^t, $t = 0, 1$, is the test space for the pressure field (which is associated with $\mathcal{T}_h$ and not with the adjoint mesh $\mathcal{T}_h^\star$).

The relation between the trial and test spaces is made precise by the lumping map $\mathcal{P}_h : \boldsymbol{\mathcal{V}}_h^1 \to \boldsymbol{\mathcal{V}}_h^\star$ (cf. [4]) which is defined as follows: For all $\boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h^1$,

$$\boldsymbol{v}_h(x) = \sum_{j=1}^{N_h} \boldsymbol{v}_h(s_j)\boldsymbol{\phi}_j(x) \mapsto \mathcal{P}_h\boldsymbol{v}_h(x) = \sum_{j=1}^{N_h} \boldsymbol{v}_h(s_j)\boldsymbol{\chi}_j(x) \qquad x \in \Omega,$$

where $\boldsymbol{\chi}_j$ is the characteristic function of the control volume $K_j^\star$, that is,

$$\boldsymbol{\chi}_j(x) = \begin{cases} 1 & x \in K_j^\star, \\ 0 & \text{otherwise.} \end{cases}$$

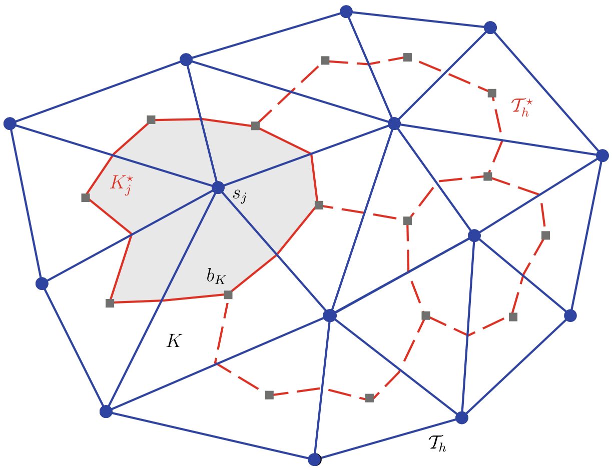


Fig. 2 Schematic representation of elements in the primal mesh $\mathcal{T}_h$ and interior node-centered control volumes of the dual mesh $\mathcal{T}_h^\star$ (in *dashed lines*)

Note that $\{\chi_j(1,0),\ \chi_j(0,1)\}_j$ provides a basis of the finite volume space $\boldsymbol{\mathcal{V}}_h^\star$. Note also that $\mathcal{P}_h$ is a one-to-one map from $\boldsymbol{\mathcal{V}}_h^1$ to $\boldsymbol{\mathcal{V}}_h^\star$. The lumping operator $\mathcal{P}_h$ allows us to recast the Petrov–Galerkin formulation as a standard Galerkin method. The following lemma (cf. [14,32]) establishes a technical result involving the previously defined transfer operator.

Lemma 3.1 *Let $K \in \mathcal{T}_h$, $F \subset \partial K$. Then, the following relations hold*

$$\int_K (\boldsymbol{v}_h - \mathcal{P}_h \boldsymbol{v}_h) = 0, \tag{3.1}$$

$$\|\boldsymbol{v}_h - \mathcal{P}_h \boldsymbol{v}_h\|_{\boldsymbol{L}^2(K)} \le C h_K |\boldsymbol{v}_h|_{\boldsymbol{H}^1(K)}, \tag{3.2}$$

for all $\boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h^1$.

Now, let $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}_h^1$ and $F \in \mathcal{E}_h^{\text{int}}$. Using the jump definition (2.9), the regularity of the mesh, and the trace inequality (2.12), we can deduce that

$$\begin{aligned}\sum_{F\in\mathcal{E}_h^{\text{int}}} h_F \,\|[\![\partial_{\boldsymbol{n}}\boldsymbol{w}]\!]_F\|^2_{L^2(F)} &\le C \sum_{F\in\mathcal{E}_h^{\text{int}}} h_F \int_F (\partial_{\boldsymbol{n}}\boldsymbol{w}|_F)^2 \\ &\le C \sum_{K\in\mathcal{T}_h} \Big(|\boldsymbol{w}|^2_{\boldsymbol{H}^1(K)} + h_K^2\, |\boldsymbol{w}|^2_{\boldsymbol{H}^2(K)}\Big).\end{aligned} \tag{3.3}$$

In the forthcoming analysis the following mesh-dependent norms will be used:

$$|\!|\!|\boldsymbol{v}|\!|\!|_h := \left(\nu|\boldsymbol{v}|^2_{\boldsymbol{H}^1(\Omega)} + \sum_{F\in\mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu}\,\|[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}]\!]_F\|^2_{L^2(F)}\right)^{1/2}, \quad \|q\|_h := \left(\sum_{K\in\mathcal{T}_h} \frac{h_k^2}{8\nu}\,|q|^2_{H^1(K)}\right)^{1/2}.$$

3.2 Construction of the stabilized FVE method

Let $(\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t$. In order to construct the underlying FVE method, we consider the discrete problem associated to the variational formulation obtained by multiplying (2.1) by $\mathcal{P}_h \boldsymbol{v}_h$ and integrating by parts over each control volume $K_j^\star \in \mathcal{T}_h^\star$, then by multiplying (2.2) by q_h and integrating by parts over each element $K \in \mathcal{T}_h$. We end up with the following finite volume element method: Find $(\boldsymbol{w}_h, r_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t$ such that

$$\tilde{a}(\boldsymbol{w}_h, \mathcal{P}_h \boldsymbol{v}_h) + \tilde{b}(r_h, \mathcal{P}_h \boldsymbol{v}_h) + (q_h, \nabla\cdot \boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \mathcal{P}_h \boldsymbol{v}_h)_\Omega \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t, \tag{3.4}$$

where the bilinear forms $\tilde{a}(\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot)$ are defined as follows:

$$\tilde{a}(\boldsymbol{w}_h, \mathcal{P}_h \boldsymbol{v}_h) = -\sum_{j=1}^{N_h} \boldsymbol{v}_h(s_j) \int_{\partial K_j^\star} \nu \partial_{\boldsymbol{n}} \boldsymbol{w}_h, \quad \tilde{b}(r_h, \mathcal{P}_h \boldsymbol{v}_h) = \sum_{j=1}^{N_h} \boldsymbol{v}_h(s_j) \int_{\partial K_j^\star} r_h \boldsymbol{n},$$

for $\boldsymbol{w}_h, \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h^1$, $q_h, r_h \in Q_h^t$. A stabilized version of (3.4) will be introduced later. Notice that since the test functions are piecewise constant, the bilinear forms do not involve area integral terms as usually happens in FE formulations of Stokes problems.

Concerning these bilinear forms, the following result will be useful to carry out the error analysis in a finite-element-fashion (see e.g. [43]).

Lemma 3.2 *For the bilinear forms $\tilde{a}(\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot)$ there holds:*

$$\tilde{a}(\boldsymbol{w}_h, \mathcal{P}_h \boldsymbol{v}_h) = \nu(\nabla \boldsymbol{w}_h, \nabla \boldsymbol{v}_h)_\Omega \qquad \forall \boldsymbol{w}_h, \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h^1, \tag{3.5}$$

$$\tilde{b}(q_h, \mathcal{P}_h \boldsymbol{v}_h) = -(q_h, \nabla\cdot \boldsymbol{v}_h)_\Omega \qquad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t. \tag{3.6}$$

Proof First, let g be a continuous function in the interior of a quadrilateral Q_j (as shown in Fig. 3) such that $\int_F g = 0$ for every edge F of Q_j. With the help of Fig. 3 it is not hard to see that the following relation holds

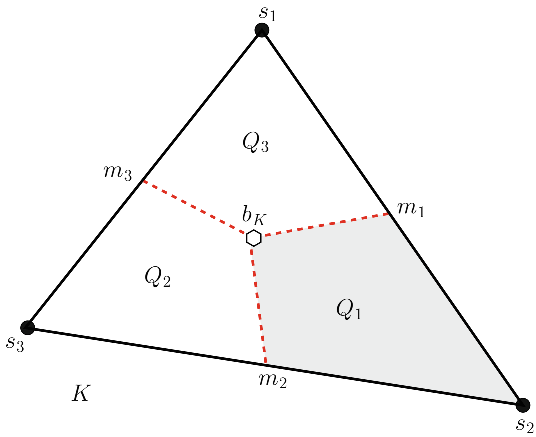
$$\sum_{j=1}^{N_h} \int_{\partial K_j^\star} g = \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{3} \int_{m_{i+1} b_K m_i} g, \tag{3.7}$$

where $m_{i+1} b_K m_i$ stands for the union of the segments $m_{i+1} b_K$ and $b_K m_i$. In the case that the index is out of bound, we take $m_{i+1} = m_i$.

Next, any $\boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h^1$ is linear on each $\overline{ab} \subset F \in \mathcal{E}_h^{\text{int}}$. Then, in particular $\int_{ab} \boldsymbol{v}_h = 1/2(a-b)(\boldsymbol{v}_h(a) + \boldsymbol{v}_h(b))$ which implies that

$$\int_{s_j s_{j+1}} \boldsymbol{v}_h = \int_{s_j m_j} \boldsymbol{v}_h(s_j) + \int_{m_j s_{j+1}} \boldsymbol{v}_h(s_{j+1}), \tag{3.8}$$

Fig. 3 A given element K of the primal mesh $\mathcal{T}_h$. The m_i's are the midpoints of the edges, b_K is the barycenter of K and the Q_i's are the quadrilaterals formed by the paths $b_K m_i s_{i+1} m_{i+1} b_K$



where s_j is a node of $\mathcal{T}_h$ and m_j is the midpoint on the edge joining s_j and s_{j+1} (if $j = 3$, then we take $s_{j+1} = s_1$).

Now, for obtaining (3.5) we use the definition of $\tilde{a}(\cdot,\cdot)$, (3.7), the fact that $\mathcal{P}_h \boldsymbol{v}_h|_{Q_i} = \boldsymbol{v}_h(s_i)$, and integration by parts twice to get

$$
\begin{aligned}
\tilde{a}(\boldsymbol{w}_h, \mathcal{P}_h \boldsymbol{v}_h) &= -\nu \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{3} \boldsymbol{v}_h(s_i) \int\limits_{m_{i+1} b_K m_i} \partial_{\boldsymbol{n}} \boldsymbol{w}_h \\
&= \nu \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{3} \boldsymbol{v}_h(s_i) \int\limits_{m_i s_{i+1} m_{i+1}} \partial_{\boldsymbol{n}} \boldsymbol{w}_h - \nu \sum_{K\in\mathcal{T}_h} \sum_{Q_i} (\Delta \boldsymbol{w}_h, \boldsymbol{v}_h(s_i))_{Q_i} \\
&= \nu \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{3} \boldsymbol{v}_h(s_i) \int\limits_{m_i s_{i+1} m_{i+1}} (\boldsymbol{v}_h(s_i) - \boldsymbol{v}_h) \cdot \partial_{\boldsymbol{n}} \boldsymbol{w}_h + \nu (\nabla \boldsymbol{w}_h, \nabla \boldsymbol{v}_h)_\Omega \\
&= \nu \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{3} \boldsymbol{v}_h(s_i) \left[\int\limits_{s_i m_i} \boldsymbol{v}_h(s_i) + \int\limits_{s_i m_{i+1}} \boldsymbol{v}_h(s_{i+1}) - \int\limits_{s_i s_{i+1}} \boldsymbol{v}_h \right] + \nu (\nabla \boldsymbol{w}_h, \nabla \boldsymbol{v}_h)_\Omega.
\end{aligned}
$$

Noticing that $\partial_{\boldsymbol{n}} \boldsymbol{v}_h$ is constant on the edges of K, and after applying (3.8), we get (3.5). For proving (3.6), we use the definition of $\mathcal{P}_h$, integration by parts and (3.1) to obtain

$$
\begin{aligned}
\tilde{b}(q_h, \mathcal{P}_h \boldsymbol{v}_h) &= \sum_{j=1}^{N_h} \boldsymbol{v}_h(s_j) \int\limits_{\partial K_j^\star} q_h \boldsymbol{n} \\
&= \sum_{j=1}^{N_h} \int\limits_{K_j^\star} \mathcal{P}_h \boldsymbol{v}_h \nabla q_h = \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{3} \int\limits_{Q_i} \mathcal{P}_h \boldsymbol{v}_h \nabla q_h
\end{aligned}
$$

$$= \sum_{K\in\mathcal{T}_h}\int_K (\mathcal{P}_h\boldsymbol{v}_h - \boldsymbol{v}_h)\nabla q_h + \sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{v}_h\nabla q_h = \sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{v}_h\nabla q_h$$
$$= -(q_h, \nabla\cdot\boldsymbol{v}_h)_\Omega.$$

Corollary 3.1 *The bilinear form $\tilde{a}(\cdot,\cdot)$ is symmetric, continuous and coercive in $\boldsymbol{\mathcal{V}}_h^1$.*

We point out that a similar analysis can be carried out if instead of considering dual meshes of Donald-type, we use the so-called Voronoi-type (see e.g. [40]) dual meshes.

With our choice for the finite dimensional spaces $\boldsymbol{\mathcal{V}}_h^1 \times Q_h^t$ (i.e., a $\mathbb{P}_1 - \mathbb{P}_1$ or a $\mathbb{P}_1 - \mathbb{P}_0$ pair) the finite volume scheme (3.4) does not satisfy the discrete inf–sup condition. Therefore we incorporate the same stabilization terms showing up in the finite element formulation (2.10). This implies that the proposed stabilized FVE method reads: Find $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t$ such that

$$\tilde{a}(\tilde{\boldsymbol{u}}_h, \mathcal{P}_h\boldsymbol{v}_h) + \tilde{b}(\tilde{p}_h, \mathcal{P}_h\boldsymbol{v}_h) + (q_h, \nabla\cdot\tilde{\boldsymbol{u}}_h)_\Omega + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}(-\nu\Delta\tilde{\boldsymbol{u}}_h + \nabla\tilde{p}_h, \nabla q_h)_K$$
$$+ \sum_{F\in\mathcal{E}_h^{\mathrm{int}}}\frac{h_F}{12\nu}([\![\nu\partial_{\boldsymbol{n}}\tilde{\boldsymbol{u}}_h + \tilde{p}_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F, [\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F)_F = (\boldsymbol{f}, \mathcal{P}_h\boldsymbol{v}_h)_\Omega$$
$$+ \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}(\boldsymbol{f}, \nabla q_h)_K,$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t$. In the light of Lemma 3.2, it can be recast as: Find $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t$ such that

$$\mathcal{C}_h\left((\tilde{\boldsymbol{u}}_h, \tilde{p}_h), (\boldsymbol{v}_h, q_h)\right) = \mathcal{F}_h(\boldsymbol{v}_h, q_h) \qquad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t, \tag{3.9}$$

where for all $(\boldsymbol{w}_h, p_h), (\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t$, the forms $\mathcal{C}_h$ and $\mathcal{F}_h$ are defined as follows

$$\mathcal{C}_h\left((\boldsymbol{w}_h, p_h), (\boldsymbol{v}_h, q_h)\right) := \nu(\nabla\boldsymbol{w}_h, \nabla\boldsymbol{v}_h)_\Omega - (p_h, \nabla\cdot\boldsymbol{v}_h)_\Omega + (q_h, \nabla\cdot\boldsymbol{w}_h)_\Omega$$
$$+ \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}(-\nu\Delta\boldsymbol{w}_h + \nabla p_h, \nabla q_h)_K$$
$$+ \sum_{F\in\mathcal{E}_h^{\mathrm{int}}}\frac{h_F}{12\nu}([\![\nu\partial_{\boldsymbol{n}}\boldsymbol{w}_h + p_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F, [\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h + q_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F)_F,$$
$$\mathcal{F}_h(\boldsymbol{v}_h, q_h) := (\boldsymbol{f}, \mathcal{P}_h\boldsymbol{v}_h)_\Omega + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}(\boldsymbol{f}, \nabla q_h)_K. \tag{3.10}$$

Obviously, one observes that since $\tilde{\boldsymbol{u}}_h, \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h^1$, the terms $\Delta\boldsymbol{v}_h$ appearing in (2.10) and also $\Delta\tilde{\boldsymbol{u}}_h$, vanish. However, the residual-related term $-\nu\Delta\tilde{\boldsymbol{u}}_h + \nabla p_h$ is used in the subsequent analysis, and hence $-\nu\Delta\tilde{\boldsymbol{u}}_h$ remains in the formulation (3.9).

4 Convergence analysis

The goal of this section is to derive the error analysis for (3.9). We will proceed by obtaining optimal error estimates in the h-norms, and in the L^2-norm.

Remark 4.1 In the whole section, we will consider that the solution $(\boldsymbol{u}, p)$ of (2.4) belongs to $[\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)] \times [H^1(\Omega) \cap L_0^2(\Omega)]$. Such regularity holds either if Ω is convex, or $\partial\Omega$ is Lipschitz-continuous, and if $\boldsymbol{f}$ fulfils certain orthogonality relations given e.g. in [6, Th. II.1].

For $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$, consider the problem: Find $(\boldsymbol{z}, s) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$
\begin{aligned}
-\nu \Delta \boldsymbol{z} + \nabla s &= \boldsymbol{\varphi} \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{z} &= 0 \qquad \text{in } \Omega, \\
\boldsymbol{z} &= \boldsymbol{0} \quad \text{on } \partial\Omega,
\end{aligned}
$$

which in its weak form reads

$$
\nu(\nabla \boldsymbol{z}, \nabla \boldsymbol{v})_\Omega + (s, \nabla \cdot \boldsymbol{v})_\Omega - (q, \nabla \cdot \boldsymbol{z})_\Omega = (\boldsymbol{v}, \boldsymbol{\varphi})_\Omega \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega), \tag{4.1}
$$

and let us recall the following regularity result (see [28]).

Lemma 4.1 *If $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$ fulfils the conditions given in* [6, Th. II.1]*, then the solution of* (4.1) *satisfies $(\boldsymbol{z}, s) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ and moreover*

$$
\|\boldsymbol{z}\|_{\boldsymbol{H}^2(\Omega)} + \|s\|_{H^1(\Omega)} \leq C \, \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.2}
$$

Lemma 4.2 (Consistency) *Let the pair $(\boldsymbol{u}, p)$ be the solution of* (2.4) *and let $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1$ be its approximation defined by the FVE method* (3.9)*. Then, if $\boldsymbol{f}$ is piecewise constant with respect to the primal triangulation $\mathcal{T}_h$, there holds that*

$$
\mathcal{C}_h \left((\boldsymbol{u} - \tilde{\boldsymbol{u}}_h, p - \tilde{p}_h), (\boldsymbol{v}_h, q_h) \right) = 0 \quad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1,
$$

that is, the FVE method (3.9) *is fully consistent.*

Proof Remark 4.1 implies that $[\![\nu \partial_{\boldsymbol{n}} \boldsymbol{u}]\!]_F$ vanishes on every internal edge F of the primal mesh. Then, using (3.10) and (3.1), the result follows.

If $\boldsymbol{f}$ is not piecewise constant, then we only obtain *asymptotic consistency* (see e.g. [18]). Moreover, the consistency error $\mathcal{C}_h \left((\boldsymbol{u} - \tilde{\boldsymbol{u}}_h, p - \tilde{p}_h), (\boldsymbol{v}_h, q_h) \right)$ induced by considering $\boldsymbol{f}$ being piecewise constant is of $\mathcal{O}(h^2)$. In fact,

$$\begin{aligned}\mathcal{C}_h\left((\boldsymbol{u}-\tilde{\boldsymbol{u}}_h, p-\tilde{p}_h),(\boldsymbol{v}_h,q_h)\right) &= \mathcal{C}_h\left((\boldsymbol{u},p),(\boldsymbol{v}_h,q_h)\right)-\mathcal{F}_h(\boldsymbol{v}_h,q_h)\\ &= (\boldsymbol{f},\boldsymbol{v}_h)_\Omega-(\boldsymbol{f},\mathcal{P}_h\boldsymbol{v}_h)_\Omega\\ &\quad+\sum_{F\in\mathcal{E}_h^{\text{int}}}\frac{h_F}{12\nu}\left([\![\nu\partial_{\boldsymbol{n}}\boldsymbol{u}]\!]_F,[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h]\!]_F\right)_F\\ &= \sum_{K\in\mathcal{T}_h}(\boldsymbol{f},\boldsymbol{v}_h-\mathcal{P}_h\boldsymbol{v}_h)_K\\ &= \sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}-\frac{1}{|K|}\int_K\boldsymbol{f},\boldsymbol{v}_h-\mathcal{P}_h\boldsymbol{v}_h\right)_K\\ &\le Ch^2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\,|\boldsymbol{v}_h|_{\boldsymbol{H}^1(\Omega)},\end{aligned}$$

for all $\boldsymbol{v}_h\in\boldsymbol{\mathcal{V}}_h^1$, by virtue of Cauchy–Schwarz inequality and Lemma 3.1.

Note that from the definition of $\mathcal{C}_h$, $\boldsymbol{\mathcal{V}}_h^1$ and that of the h-norms, the following result holds, which implies the well-posedness of (3.9).

Lemma 4.3 (Continuity and coercivity in the h-norms) *Let* $(\boldsymbol{w}_h,r_h)\in\boldsymbol{\mathcal{V}}_h^1\times Q_h^t$. *Then*

$$\begin{aligned}\mathcal{C}_h\left((\boldsymbol{w}_h,r_h),(\boldsymbol{v}_h,q_h)\right)&\le(|\!|\!|\boldsymbol{w}_h|\!|\!|_h+\|r_h\|_h)(|\!|\!|\boldsymbol{v}_h|\!|\!|_h+\|q_h\|_h),\\ \mathcal{C}_h\left((\boldsymbol{v}_h,q_h),(\boldsymbol{v}_h,q_h)\right)&=|\!|\!|\boldsymbol{v}_h|\!|\!|_h^2+\|q_h\|_h^2,\end{aligned}\tag{4.3}$$

for all $(\boldsymbol{v}_h,q_h)\in\boldsymbol{\mathcal{V}}_h^1\times Q_h^t$.

Since the Clément interpolate J_hq of $q\in H^1(\Omega)\cap L_0^2(\Omega)$, does not necessarily belong to $L_0^2(\Omega)$, we will introduce the operator L_h defined by $L_hq:=J_hq-|\Omega|^{-1}\int_\Omega J_hq$. This operator possesses the same interpolation properties (e.g. (2.16)) as J_h.

Theorem 4.1 (An optimal-order error estimate in the h-norms) *Let* $(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\in\boldsymbol{\mathcal{V}}_h^1\times Q_h^t$ *be the unique solution of* (3.9) *and* $(\boldsymbol{u},p)$ *the unique solution of* (2.4). *Then, under the assumption of* $\boldsymbol{f}$ *being piecewise constant, there exists* $C>0$ *such that*

$$|\!|\!|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h|\!|\!|_h+\|p-\tilde{p}_h\|_h\le Ch\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)}+|p|_{H^1(\Omega)}\right).$$

Proof Let $\boldsymbol{\varepsilon}=I_h\boldsymbol{u}-\boldsymbol{u}$, $\eta=L_hp-p$ be the individual errors between the exact solution and the projected solution, and let $\boldsymbol{\varepsilon}_h=I_h\boldsymbol{u}-\tilde{\boldsymbol{u}}_h$, $\eta_h=L_hp-\tilde{p}_h$ denote the error between the FVE approximation and the projection of the exact solution.

First, using (3.3) and (2.13) we have

$$\begin{aligned}\sum_{F\in\mathcal{E}_h^{\text{int}}}\frac{h_F}{12\nu}\,\|[\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{v}-I_h\boldsymbol{v})]\!]_F\|_{L^2(F)}^2&\le C\sum_{K\in\mathcal{T}_h}\left(|\boldsymbol{v}-I_h\boldsymbol{v}|_{\boldsymbol{H}^1(K)}^2+h_K^2\,|\boldsymbol{v}-I_h\boldsymbol{v}|_{\boldsymbol{H}^2(K)}^2\right)\\ &\le Ch^2\,|\boldsymbol{v}|_{\boldsymbol{H}^2(\Omega)}^2,\end{aligned}\tag{4.4}$$

and using the definition of the h-norm and Lemma 2.1, gives

$$\|\!|\boldsymbol{v} - I_h \boldsymbol{v}|\!\|_h^2 \leq \ Ch^2 \, |\boldsymbol{v}|^2_{\boldsymbol{H}^2(\Omega)}. \tag{4.5}$$

Furthermore, (2.16) and the definition of the h-norm also implies that

$$\nu^{-1} \, \|q - L_h q\|^2_{L^2(\Omega)} + \|q - L_h q\|_h^2 \leq C\nu^{-1} h^2 \, |q|^2_{H^1(\Omega)}. \tag{4.6}$$

Next, applying (4.3), Lemma 4.2 and integration by parts we get

$$\begin{aligned} \|\!|\boldsymbol{\varepsilon}_h|\!\|_h^2 + \|\eta_h\|_h^2 &= \mathcal{C}_h\left((\boldsymbol{\varepsilon}_h, \eta_h), (\boldsymbol{\varepsilon}_h, \eta_h)\right) \\ &= \mathcal{C}_h\left((\boldsymbol{u} - \tilde{\boldsymbol{u}}_h, p - \tilde{p}_h), (\boldsymbol{v}_h, q_h)\right) + \mathcal{C}_h\left((\boldsymbol{\varepsilon}, \eta), (\boldsymbol{\varepsilon}_h, \eta_h)\right) \\ &= \nu(\nabla \boldsymbol{\varepsilon}, \nabla \boldsymbol{\varepsilon}_h)_\Omega - (\eta, \nabla \cdot \boldsymbol{\varepsilon}_h)_\Omega - (\boldsymbol{\varepsilon}, \nabla \eta_h)_\Omega \\ &\quad + \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{8\nu} \left(-\nu \Delta \boldsymbol{\varepsilon} + \nabla \eta, \nabla \eta_h\right)_K \\ &\quad + \sum_{F \in \mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu} \left([\![\nu \partial_{\boldsymbol{n}} \boldsymbol{\varepsilon}]\!]_F, [\![\nu \partial_{\boldsymbol{n}} \boldsymbol{\varepsilon}_h]\!]_F\right)_F. \end{aligned} \tag{4.7}$$

Now, (4.7), Cauchy–Schwarz inequality, the definition of h-norms, a repeated application of (4.5), and (2.13), (2.15), (2.16), enable us to write

$$\begin{aligned} &\|\!|\boldsymbol{\varepsilon}_h|\!\|_h^2 + \|\eta_h\|_h^2 \\ &\leq C \left(|\boldsymbol{\varepsilon}|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{\varepsilon}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)} + \sum_{F \in \mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu} \, \|[\![\nu \partial_{\boldsymbol{n}} \boldsymbol{\varepsilon}]\!]_F\|^2_{L^2(F)} \right. \\ &\quad \left. + \sum_{K \in \mathcal{T}_h} \left[\frac{8\nu}{h_K^2} \, \|\boldsymbol{\varepsilon}\|^2_{\boldsymbol{L}^2(K)} + \frac{h_K^2}{8\nu} (\|\eta\|^2_{L^2(K)} + \|\Delta \boldsymbol{\varepsilon}\|^2_{\boldsymbol{L}^2(K)}) \right] \right)^{1/2} \\ &\quad \times \left(|\boldsymbol{\varepsilon}_h|^2_{\boldsymbol{H}^1(\Omega)} + \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{8\nu} \, \|\eta_h\|^2_{H^1(K)} + \sum_{F \in \mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu} \, \|[\![\nu \partial_{\boldsymbol{n}} \boldsymbol{\varepsilon}_h]\!]_F\|^2_{L^2(F)} \right)^{1/2} \\ &\leq C \left(\|\!|\boldsymbol{\varepsilon}|\!\|_h^2 + \|\boldsymbol{\varepsilon}\|^2_{\boldsymbol{L}^2(\Omega)} + \sum_{K \in \mathcal{T}_h} \left[\frac{8\nu}{h_K^2} \, \|\boldsymbol{\varepsilon}\|^2_{\boldsymbol{L}^2(K)} + \frac{h_K^2}{8\nu} \, \|\Delta \boldsymbol{\varepsilon}\|^2_{\boldsymbol{L}^2(K)} \right] + \|\eta\|_h^2 + \|\eta\|^2_{L^2(\Omega)} \right)^{1/2} \\ &\quad \times \left(\|\!|\boldsymbol{\varepsilon}_h|\!\|_h^2 + \|\eta_h\|_h^2 \right)^{1/2} \\ &\leq C \left(h^2 \, |\boldsymbol{u}|^2_{\boldsymbol{H}^2(\Omega)} + h^2 \, |p|^2_{H^1(\Omega)} + \sum_{K \in \mathcal{T}_h} 8\nu(1 + \nu^2) h_K^2 \, |\boldsymbol{u}|^2_{\boldsymbol{H}^1(K)} \right)^{1/2} \left(\|\!|\boldsymbol{\varepsilon}_h|\!\|_h^2 + \|\eta_h\|_h^2 \right)^{1/2}, \end{aligned}$$

which implies the following:

$$\|\!|\boldsymbol{\varepsilon}_h|\!\|_h + \|\eta_h\|_h \leq Ch \left(|\boldsymbol{u}|^2_{\boldsymbol{H}^2(\Omega)} + |p|^2_{H^1(\Omega)} \right)^{1/2}.$$

Finally, in order to get the desired result, it is sufficient to apply triangular inequality and (4.5), (4.6).

Theorem 4.2 (An optimal-order L^2-error estimate for the pressure field) *Assume that $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1$ and $(\boldsymbol{u}, p)$ are the unique solutions of (3.9) and (2.4), respectively. Then, there exists a positive constant $C > 0$ such that*

$$\|p - \tilde{p}_h\|_{L^2(\Omega)} \leq Ch\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)}\right).$$

Proof Let be $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ such that $\nabla\cdot\boldsymbol{w} = p - \tilde{p}_h$, as stated in Lemma 2.2. Further, selecting $(\boldsymbol{v}_h, q_h) = (J_h\boldsymbol{w}, 0) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1$ in Lemma 4.2 we have

$$\begin{aligned}0 &= \nu(\nabla(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h), \nabla J_h\boldsymbol{w})_\Omega - (p - \tilde{p}_h, \nabla\cdot J_h\boldsymbol{w})_\Omega\\ &\quad + \sum_{F\in\mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu}([\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h)]\!]_F, [\![\nu\partial_{\boldsymbol{n}} J_h\boldsymbol{w}]\!]_F)_F.\end{aligned}$$

Using this relation and integration by parts we obtain

$$\begin{aligned}\|p - \tilde{p}_h\|^2_{L^2(\Omega)} &= (p - \tilde{p}_h, \nabla\cdot\boldsymbol{w})_\Omega\\ &= (p - \tilde{p}_h, \nabla\cdot(\boldsymbol{w} - J_h\boldsymbol{w}))_\Omega + (p - \tilde{p}_h, \nabla\cdot J_h\boldsymbol{w})_\Omega\\ &= -\sum_{K\in\mathcal{T}_h}(\boldsymbol{w} - J_h\boldsymbol{w}, \nabla((p - \tilde{p}_h))_K + \nu(\nabla(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h), \nabla J_h\boldsymbol{w})_\Omega\\ &\quad + \sum_{F\in\mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu}([\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h)]\!]_F, [\![\nu\partial_{\boldsymbol{n}} J_h\boldsymbol{w}]\!]_F)_F.\end{aligned}$$

Then, by Cauchy–Schwarz inequality, (2.16), Lemma 2.2, (2.11), definition of h-norms, and Theorem 4.1 we can infer that

$$\begin{aligned}&\|p - \tilde{p}_h\|^2_{L^2(\Omega)}\\ &\leq \sum_{K\in\mathcal{T}_h}\|\boldsymbol{w} - J_h\boldsymbol{w}\|_{\boldsymbol{L}^2(K)}\,|p - \tilde{p}_h|_{H^1(K)} + \nu\,|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|_{\boldsymbol{H}^1(\Omega)}\,|J_h\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)}\\ &\quad + \sum_{F\in\mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu}([\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h)]\!]_F, [\![\nu\partial_{\boldsymbol{n}} J_h\boldsymbol{w}]\!]_F)_F\\ &\leq \left(\sum_{K\in\mathcal{T}_h}\frac{8\nu}{h_K^2}\,\|\boldsymbol{w} - J_h\boldsymbol{w}\|^2_{\boldsymbol{L}^2(K)} + \nu\,|J_h\boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)} + \sum_{F\in\mathcal{E}_h^{\text{int}}}\frac{h_F}{12\nu}\,\|[\![\nu\partial_{\boldsymbol{n}} J_h\boldsymbol{w}]\!]_F\|^2_{\boldsymbol{L}^2(F)}\right)^{1/2}\\ &\quad\times\left(\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}\,|p - \tilde{p}_h|^2_{\boldsymbol{H}^1(K)} + |\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|^2_{\boldsymbol{H}^1(\Omega)} + \sum_{F\in\mathcal{E}_h^{\text{int}}}\frac{h_F}{12\nu}\,\|[\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h)]\!]_F\|^2_{\boldsymbol{L}^2(F)}\right)^{1/2}\end{aligned}$$

$$\leq C\left(|\boldsymbol{w}|^2_{\boldsymbol{H}^1(\Omega)} + \nu\,|J_h\boldsymbol{w}|^2_{\boldsymbol{H}^1(\Omega)}\right)^{1/2}\left(|||\boldsymbol{u}-\tilde{\boldsymbol{u}}_h|||^2_h + \|p-\tilde{p}_h\|^2_h\right)^{1/2}$$
$$\leq Ch\,\|p-\tilde{p}_h\|_{L^2(\Omega)}\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)}\right),$$

and dividing by $\|p-\tilde{p}_h\|_{L^2(\Omega)}$, the result follows.

Theorem 4.3 (L^2-error estimate for the velocity field) *Suppose that* $(\boldsymbol{u}, p)$ *is the solution of* (2.4), *and* $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{\mathcal{V}}^1_h \times Q^1_h$ *is the approximation defined by the FVE method* (3.9). *Then the following a priori error estimate holds*

$$\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)} \leq Ch^2\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)}\right).$$

Proof First consider the dual problem (4.1) with $\boldsymbol{\varphi} = \boldsymbol{u}-\tilde{\boldsymbol{u}}_h$. Moreover, let us choose in (2.4) and (3.9), $(\boldsymbol{v}_h, q_h) = (I_h\boldsymbol{z}, \Pi_h s) \in \boldsymbol{\mathcal{V}}^1_h \times Q^1_h$, and subtract the resulting expressions. We then subtract again the result to (4.1) with the particular choice $(\boldsymbol{v}, q) = (\boldsymbol{u}-\tilde{\boldsymbol{u}}_h, p-\tilde{p}_h)$, (and again $\boldsymbol{\varphi} = \boldsymbol{u}-\tilde{\boldsymbol{u}}_h$). Next we apply Lemma 4.2 to obtain

$$\begin{aligned}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|^2_{\boldsymbol{L}^2(\Omega)} = &\sum_{F\in\mathcal{E}^{\text{int}}_h}\frac{h_F}{12\nu}([\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)]\!]_F, [\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{z}-I_h\boldsymbol{z})]\!]_F - [\![\nu\partial_{\boldsymbol{n}}\boldsymbol{z}]\!]_F)_F\\ &+\nu(\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h), \nabla(\boldsymbol{z}-I_h\boldsymbol{z}))_\Omega + (s-\Pi_h s, \nabla\cdot(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h))_\Omega\\ &-(p-\tilde{p}_h, \nabla\cdot(\boldsymbol{z}-I_h\boldsymbol{z}))_\Omega\\ &+\sum_{K\in\mathcal{T}_h}\frac{h^2_K}{8\nu}(-\nu\Delta\boldsymbol{u}+\nabla(p-\tilde{p}_h), -\nabla\Pi_h s)_K.\end{aligned}$$

We now proceed to combine Cauchy–Schwarz inequality, the definition of h-norms, (3.2), Theorems 4.1 and 4.2, (4.4), (4.5), (2.13), (2.17), (2.18), and (4.2) to deduce that

$$\begin{aligned}&\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\leq C\Bigg(\nu\,|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h|^2_{\boldsymbol{H}^1(\Omega)} + \|\nabla\cdot(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p-\tilde{p}_h\|^2_{L^2(\Omega)}\\ &\quad+\sum_{K\in\mathcal{T}_h}\frac{h^2_K}{8\nu}\|-\nu\Delta\boldsymbol{u}+\nabla(p-\tilde{p}_h)\|^2_{\boldsymbol{L}^2(K)} + \sum_{F\in\mathcal{E}^{\text{int}}_h}\frac{h_F}{12\nu}\|[\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)]\!]_F\|^2_{\boldsymbol{L}^2(F)}\Bigg)^{1/2}\\ &\quad\times\Bigg(\nu\,|\boldsymbol{z}-I_h\boldsymbol{z}|^2_{\boldsymbol{H}^1(\Omega)} + \|s-\Pi_h s\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{T}_h}\frac{h^2_K}{8\nu}\|\nabla\Pi_h s\|^2_{\boldsymbol{L}^2(K)} + \|\nabla\cdot(\boldsymbol{z}-I_h\boldsymbol{z})\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\quad+\sum_{F\in\mathcal{E}^{\text{int}}_h}\frac{h_F}{12\nu}\left[\|[\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{z}-I_h\boldsymbol{z})]\!]_F\|^2_{\boldsymbol{L}^2(F)} + \|[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{z}]\!]_F\|^2_{\boldsymbol{L}^2(F)}\right]\Bigg)^{1/2}\end{aligned}$$

$$\leq C\left(\|\!|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|\!|_h^2+\|p-\tilde{p}_h\|_h^2+\nu h^2\,|\boldsymbol{u}|^2_{\boldsymbol{H}^2(\Omega)}+\|p-\tilde{p}_h\|^2_{L^2(\Omega)}\right)^{1/2}$$
$$\times\left(\|\!|\boldsymbol{z}-I_h\boldsymbol{z}\|\!|_h^2+\|s-\Pi_hs\|^2_{L^2(\Omega)}+h^2\,|\boldsymbol{z}|^2_{\boldsymbol{H}^2(\Omega)}+h^2\,|I_h\boldsymbol{z}|^2_{\boldsymbol{H}^2(\Omega)}+\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}\,|\Pi_hs|^2_{\boldsymbol{H}^1(K)}\right)^{1/2}$$
$$\leq Ch^2\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)}+|p|_{H^1(\Omega)}\right)\left(h^2\left[|\boldsymbol{z}|^2_{\boldsymbol{H}^2(\Omega)}+|s|^2_{H^1(\Omega)}+\|s\|_h^2\right]\right)^{1/2}$$
$$\leq Ch^2\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)}+|p|_{H^1(\Omega)}\right)\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)},$$

and the proof is complete after dividing by the last term in the RHS.

It is easily seen that by using the local trace inequality (2.11) and Céa's lemma, it is possible to modify the regularity hypothesis of Theorems 4.2 and 4.3, setting $(\boldsymbol{u},p)\in[\boldsymbol{H}^{1+\delta}(\Omega)\cap\boldsymbol{H}^1_0(\Omega)]\times[H^\delta(\Omega)\cap L^2_0(\Omega)]$, $1/2<\delta\leq 1$, in order to obtain the estimate

$$\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)}+h\,\|p-\tilde{p}_h\|_{L^2(\Omega)}\leq Ch^{1+\delta}\left(|\boldsymbol{u}|_{\boldsymbol{H}^{1+\delta}(\Omega)}+|p|_{H^\delta(\Omega)}\right).$$

Remark 4.2 If we now consider $t=0$ and so the pressure field is approximated by piecewise constant functions, then the FVE method would simply read: Find $(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\in\boldsymbol{\mathcal{V}}_h^1\times Q_h^0$ such that

$$\nu(\nabla\tilde{\boldsymbol{u}}_h,\nabla\boldsymbol{v}_h)_\Omega-(\tilde{p}_h,\nabla\cdot\boldsymbol{v}_h)_\Omega+(q_h,\nabla\cdot\tilde{\boldsymbol{u}}_h)_\Omega$$
$$+\sum_{F\in\mathcal{E}_h^{\rm int}}\frac{h_F}{12\nu}\left([\![\nu\partial_{\boldsymbol{n}}\tilde{\boldsymbol{u}}_h+\tilde{p}_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F,[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_h+q_h\mathbf{I}\cdot\boldsymbol{n}]\!]_F\right)_F=(\boldsymbol{f},\mathcal{P}_h\boldsymbol{v}_h)_\Omega,$$
$$(\boldsymbol{v}_h,q_h)\in\boldsymbol{\mathcal{V}}_h^1\times Q_h^0.$$

In such case, an analysis similar to that performed in this section could be carried out. For instance, in the proof of Theorem 4.1, it will suffice to use the projection $\Pi_h:L^2(\Omega)\to Q_h^0$ instead of the normalized Clément operator L_h, and the analysis is done using the mesh-dependent norm

$$\|\!|(\boldsymbol{v},q)\|\!|_h:=\left(\nu|\boldsymbol{v}|^2_{\boldsymbol{H}^1(\Omega)}+\sum_{F\in\mathcal{E}_h^{\rm int}}\frac{h_F}{12\nu}\,\|[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{v}+q\mathbf{I}\cdot\boldsymbol{n}]\!]_F\|^2_{L^2(F)}\right)^{1/2}.$$

As in the error analysis given in this section, the obtained rates of convergence are of order h for $\|\!|(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,p-\tilde{p}_h)\|\!|_h$ and $\|p-\tilde{p}_h\|_{L^2(\Omega)}$, and of order h^2 for $\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)}$.

5 Superconvergence analysis

As briefly mentioned in Sect. 1, the present approach for establishing superconvergence estimates basically consists in projecting the FVE approximation $(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\in$

$(\boldsymbol{\mathcal{V}}_h^1 \times Q_h^1)$ into a different finite dimensional space $(\boldsymbol{\mathcal{V}}_\rho^r \times Q_\rho^t)$, $r, t \geq 0$, which corresponds to the (possibly of higher order) counterpart of $(\boldsymbol{\mathcal{V}}_h^1 \times Q_h^1)$ associated to a coarser mesh $\mathcal{T}_\rho$ of size $\rho = h^\alpha$, with $\alpha \in (0, 1)$. More precisely,

$$\boldsymbol{\mathcal{V}}_\rho^r = \{\boldsymbol{v}_\rho \in \boldsymbol{C}^0(\bar{\Omega}) : \boldsymbol{v}_\rho|_K \in \mathbb{P}_r(K)^2 \text{ for all } K \in \mathcal{T}_\rho\},$$

and

$$Q_\rho^t = \{q_\rho \in C^0(\overline{\Omega}) : q_\rho|_K \in \mathbb{P}_t(K) \text{ for all } K \in \mathcal{T}_\rho\}.$$

In addition, we will assume that the mesh $\mathcal{T}_\rho$ satisfies the so-called inverse assumption [12]: there exists $C > 0$ such that $\rho \leq C\rho_K$, for all $K \in \mathcal{T}_\rho$, and we will use the following inverse inequality (see e.g. [30])

$$\left\|v_\rho\right\|_{\boldsymbol{H}^m(K)} \leq C\rho_K^{-m} \left\|v_\rho\right\|_{\boldsymbol{L}^2(K)} \quad v_\rho \in \boldsymbol{\mathcal{V}}_\rho^r,\ K \in \mathcal{T}_\rho. \tag{5.1}$$

We continue by defining the operators $\Pi_\rho^{\mathcal{V}}$, Π_ρ^{Q} as the L^2-projections onto $\boldsymbol{\mathcal{V}}_\rho^r$ and Q_ρ^t respectively. Therefore, in particular it holds that

$$\|\boldsymbol{v} - \Pi_\rho^{\mathcal{V}}\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C\rho^s|\boldsymbol{v}|_{\boldsymbol{H}^s(\Omega)} \qquad 0 \leq s \leq r+1, \tag{5.2}$$

$$\left\|\Pi_\rho^{\mathcal{V}}\boldsymbol{v}\right\|_{\boldsymbol{L}^2(\Omega)} \leq C\left\|\boldsymbol{v}\right\|_{\boldsymbol{L}^2(\Omega)}, \tag{5.3}$$

$$\left\|q - \Pi_\rho^{Q} q\right\|_{L^2(\Omega)} \leq C\rho^s\left|q\right|_{H^s(\Omega)} \qquad 0 \leq s \leq t+1. \tag{5.4}$$

Let us also denote by I_ρ the Lagrange interpolator into $\boldsymbol{\mathcal{V}}_\rho^r$, and note that by (2.13), (5.2) and (5.1) it follows that

$$\begin{aligned}\left|\boldsymbol{v} - \Pi_\rho^{\mathcal{V}}\boldsymbol{v}\right|_{\boldsymbol{H}^1(\Omega)} &\leq \left|\boldsymbol{v} - I_\rho\boldsymbol{v}\right|_{\boldsymbol{H}^1(\Omega)} + \left|I_\rho\boldsymbol{v} - \Pi_\rho^{\mathcal{V}}\boldsymbol{v}\right|_{\boldsymbol{H}^1(\Omega)} \\ &\leq C\left(\rho^{s-1}|\boldsymbol{v}|_{\boldsymbol{H}^s(\Omega)} + \rho^{-1}\left\|I_\rho\boldsymbol{v} - \Pi_\rho^{\mathcal{V}}\boldsymbol{v}\right\|_{\boldsymbol{L}^2(\Omega)}\right) \\ &\leq C\rho^{s-1}|\boldsymbol{v}|_{\boldsymbol{H}^s(\Omega)} = Ch^{\alpha(s-1)}|\boldsymbol{v}|_{\boldsymbol{H}^s(\Omega)} \qquad 0 \leq s \leq r+1. \end{aligned} \tag{5.5}$$

When considering particularly simple domains, it is also possible to handle a different choice for $(\boldsymbol{\mathcal{V}}_\rho^r \times Q_\rho^t)$, such as B-splines or trigonometric functions as mentioned in [30]. In that case, (5.2)–(5.4) should be properly rewritten.

Theorem 5.1 (Superconvergence for the velocity) *Let* $(\boldsymbol{u}, p) \in [\boldsymbol{H}^s(\Omega) \cap \boldsymbol{H}_0^1(\Omega)] \times [H^1(\Omega) \cap L_0^2(\Omega)]$ *and* $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1$ *be the solutions of* (2.4) *and* (3.9), *respectively. Then there exists a positive constant C such that*

$$\left\|\boldsymbol{u} - \Pi_\rho^{\mathcal{V}}\tilde{\boldsymbol{u}}_h\right\|_{\boldsymbol{L}^2(\Omega)} \leq Ch^{\alpha s}|\boldsymbol{u}|_{\boldsymbol{H}^s(\Omega)} + Ch^2\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)}\right), \tag{5.6}$$

$$\left|\boldsymbol{u} - \Pi_\rho^{\mathcal{V}}\tilde{\boldsymbol{u}}_h\right|_{\boldsymbol{H}^1(\Omega)} \leq Ch^{\alpha(s-1)}|\boldsymbol{u}|_{\boldsymbol{H}^s(\Omega)} + Ch^{2-\alpha}\left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)}\right), \tag{5.7}$$

for $0 \le s \le r+1, r \ge 1$.

Proof By triangular inequality, (5.2) and definition of $\Pi_\rho^{\mathcal{V}}$ and ρ, it follows that

$$\begin{aligned}\left\|\boldsymbol{u}-\Pi_\rho^{\mathcal{V}}\tilde{\boldsymbol{u}}_h\right\|_{\boldsymbol{L}^2(\Omega)} &\le \left\|\boldsymbol{u}-\Pi_\rho^{\mathcal{V}}\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)} + \left\|\Pi_\rho^{\mathcal{V}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)\right\|_{\boldsymbol{L}^2(\Omega)}\\ &\le Ch^{\alpha s}|\boldsymbol{u}|_{\boldsymbol{H}^s(\Omega)} + \left\|\Pi_\rho^{\mathcal{V}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)\right\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{5.8}$$

The task now consists in estimating the second term in the RHS of (5.8). First, note that since $\Pi_\rho^{\mathcal{V}}$ is a L^2-projection, it satisfies

$$\left(\Pi_\rho^{\mathcal{V}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h),\boldsymbol{w}\right)_\Omega = \left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,\Pi_\rho^{\mathcal{V}}\boldsymbol{w}\right)_\Omega \qquad \forall \boldsymbol{w}\in\boldsymbol{L}^2(\Omega).$$

A combination of this relation with norm properties gives

$$\left\|\Pi_\rho^{\mathcal{V}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)\right\|_{\boldsymbol{L}^2(\Omega)} = \sup_{\boldsymbol{w}\in\boldsymbol{L}^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\left|\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,\Pi_\rho^{\mathcal{V}}\boldsymbol{w}\right)_\Omega\right|}{\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}}. \tag{5.9}$$

We now proceed to use (4.1) with the particular choices $\boldsymbol{\varphi} = \Pi_\rho^{\mathcal{V}}\boldsymbol{w}$ for some fixed $\boldsymbol{w}\in\boldsymbol{L}^2(\Omega)$, and $(\boldsymbol{v},q) = (\boldsymbol{u}-\tilde{\boldsymbol{u}}_h, p-\tilde{p}_h)$. Thus, estimates (4.2) and (5.3) imply that

$$|\boldsymbol{z}|_{\boldsymbol{H}^2(\Omega)} + |s|_{H^1(\Omega)} \le C\,\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}, \tag{5.10}$$

where $(\boldsymbol{z},s)$ is the solution of (4.1). Therefore, using Lemma 4.2 with the particular choice $(\boldsymbol{v}_h,q_h) = (I_h\boldsymbol{z},\Pi_h s)\in\boldsymbol{\mathcal{V}}_h^1\times Q_h^1$, gives

$$\begin{aligned}&\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,\Pi_\rho^{\mathcal{V}}\boldsymbol{w}\right)_\Omega\\ &\quad=\nu(\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h),\nabla(\boldsymbol{z}-I_h\boldsymbol{z}))_\Omega + (s-\Pi_h s,\nabla\cdot(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h))_\Omega - (p-\tilde{p}_h,\nabla\cdot(\boldsymbol{z}-I_h\boldsymbol{z}))_\Omega\\ &\qquad-\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}(-\nu\Delta\boldsymbol{u}+\nabla(p-\tilde{p}_h),\nabla\Pi_h s)_K - \sum_{F\in\mathcal{E}_h^{\mathrm{int}}}\frac{h_F}{12\nu}([\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)]\!]_F,[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{z}]\!]_F)_F.\end{aligned}$$

Then, from Cauchy inequality, (2.13), (2.18), (2.11), Theorems 4.1, 4.2, and estimates (5.10), (5.3) we obtain

$$\begin{aligned}&\left(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,\Pi_\rho^{\mathcal{V}}\boldsymbol{w}\right)_\Omega\\ &\quad\le \nu\,|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h|_{\boldsymbol{H}^1(\Omega)}\,|\boldsymbol{z}-I_h\boldsymbol{z}|_{\boldsymbol{H}^1(\Omega)} + \|s-\Pi_h s\|_{L^2(\Omega)}\,|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h|_{\boldsymbol{H}^1(\Omega)}\\ &\qquad+\|p-\tilde{p}_h\|_{L^2(\Omega)}\,|\boldsymbol{z}-I_h\boldsymbol{z}|_{\boldsymbol{H}^1(\Omega)} + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{8\nu}\,\|-\nu\Delta\boldsymbol{u}+\nabla(p-\tilde{p}_h)\|_{\boldsymbol{L}^2(K)}\,\|\nabla\Pi_h s\|_{\boldsymbol{L}^2(K)}\\ &\qquad+\sum_{F\in\mathcal{E}_h^{\mathrm{int}}}\frac{h_F}{12\nu}\,\|[\![\nu\partial_{\boldsymbol{n}}(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h)]\!]_F\|_{\boldsymbol{L}^2(F)}\,\|[\![\nu\partial_{\boldsymbol{n}}\boldsymbol{z}]\!]_F\|_{\boldsymbol{L}^2(F)}\end{aligned}$$

$$
\begin{aligned}
&\le \Bigg(2\nu\, |\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{\nu}\, \|p - \tilde{p}_h\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{8}\, \|\Delta \boldsymbol{u}\|^2_{\boldsymbol{L}^2(K)} \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{8\nu}\, |p - \tilde{p}_h|^2_{H^1(K)} + \sum_{F\in\mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu}\, \|[\![\nu \partial_{\boldsymbol{n}}(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h)]\!]_F\|^2_{\boldsymbol{L}^2(F)} \Bigg)^{1/2} \\
&\quad \times \Bigg(2\nu\, |\boldsymbol{z} - I_h \boldsymbol{z}|^2_{\boldsymbol{H}^1(\Omega)} + \frac{1}{\nu}\, \|s - \Pi_h s\|^2_{L^2(\Omega)} + \sum_{F\in\mathcal{E}_h^{\text{int}}} \frac{h_F}{12\nu}\, \|[\![\nu \partial_{\boldsymbol{n}} I_h \boldsymbol{z}]\!]_F\|^2_{\boldsymbol{L}^2(F)} \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{8\nu}\, \|\nabla \Pi_h s\|^2_{\boldsymbol{L}^2(K)} \Bigg)^{1/2} \\
&\le C \left(|\!|\!|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|\!|\!|^2_h + \|p - \tilde{p}_h\|^2_h + \frac{1}{\nu}\, \|p - \tilde{p}_h\|^2_{L^2(\Omega)} + \nu h^2\, |\boldsymbol{u}|^2_{\boldsymbol{H}^2(\Omega)} \right)^{1/2} \\
&\quad \times \left((\nu + 1) h^2\, |\boldsymbol{z}|^2_{\boldsymbol{H}^2(\Omega)} + h^2 \frac{(\nu+1)}{\nu}\, |s|^2_{H^1(\Omega)} \right)^{1/2} \\
&\le C h^2 \left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)} \right) \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}
$$

Finally, by applying (5.8) and (5.9) the proof of (5.6) is completed. For the second estimate, it suffices to use triangular inequality, the inverse inequality (5.1), estimate (5.5), and (5.6) to arrive at

$$
\begin{aligned}
\left| \boldsymbol{u} - \Pi_\rho^{\mathcal{V}} \tilde{\boldsymbol{u}}_h \right|_{\boldsymbol{H}^1(\Omega)} &\le \left| \boldsymbol{u} - \Pi_\rho^{\mathcal{V}} \boldsymbol{u} \right|_{\boldsymbol{H}^1(\Omega)} + \left| \Pi_\rho^{\mathcal{V}} (\boldsymbol{u} - \tilde{\boldsymbol{u}}_h) \right|_{\boldsymbol{H}^1(\Omega)} \\
&\le C h^{\alpha(s-1)} |\boldsymbol{u}|_{\boldsymbol{H}^s(\Omega)} + h^{-\alpha} \left\| \boldsymbol{u} - \Pi_\rho^{\mathcal{V}} \boldsymbol{u} \right\|_{\boldsymbol{L}^2(\Omega)} \\
&\quad + h^{-\alpha} \left\| \boldsymbol{u} - \Pi_\rho^{\mathcal{V}} \tilde{\boldsymbol{u}}_h \right\|_{\boldsymbol{L}^2(\Omega)} \\
&\le C h^{\alpha(s-1)} |\boldsymbol{u}|_{\boldsymbol{H}^s(\Omega)} + C h^{2-\alpha} \left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)} \right).
\end{aligned}
$$

Combining (2.11), (5.1), (5.5) and the definition of the h-norm, we can also conclude that the next result holds.

Corollary 5.1 *Under the hypotheses of Theorem 5.1, we have the following estimate*

$$
\begin{aligned}
\left|\!\left|\!\left| \boldsymbol{u} - \Pi_\rho^{\mathcal{V}} \tilde{\boldsymbol{u}}_h \right|\!\right|\!\right|^2_h &\le C h^{\alpha(s-1)} (h + h^\alpha + 1) |\boldsymbol{u}|_{\boldsymbol{H}^s(\Omega)} + C h^2 (h^{1-\alpha} + h^{-\alpha} + 1) \\
&\quad \times \left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)} \right),
\end{aligned}
$$

for $0 \le s \le r + 1$.

Remark 5.1 We stress that Theorem 5.1 does not provide an improvement of the convergence rate for the velocity field in the L^2-norm in the studied case of $\mathbb{P}_1$ elements. This holds even if in the postprocessing stage we use a different space for the velocity such as $\mathbb{P}_r,\ r \geq 2$. However the *superconvergence* is achieved in the $\boldsymbol{H}^1$-seminorm for $\mathbb{P}_r,\ r \geq 2$ and for $\alpha > 1/2$. The following result (which may be proved in much the same way as Theorem 5.1, by using a duality argument) yields superconvergence for the pressure field as well, even in the case $t = 1$.

Theorem 5.2 (Superconvergence for the pressure) *Assume that* $(\boldsymbol{u}, p) \in [\boldsymbol{H}^s(\Omega) \cap \boldsymbol{H}_0^1(\Omega)] \times [H^1(\Omega) \cap L_0^2(\Omega)]$ *and* $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^1$ *are the solutions of* (2.4) *and* (3.9) *respectively. Then there exists a positive constant C such that*

$$\left\| p - \Pi_\rho^Q \tilde{p}_h \right\|_{L^2(\Omega)} \leq C h^{\alpha s} |p|_{H^s(\Omega)} + C h^{2-\alpha} \left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)} \right),$$
$$\alpha > 1/2,\ 0 \leq s \leq t+1,\ t \geq 1.$$

Remark 5.2 As briefly mentioned at the end of Sect. 2.2, the derivation of (2.10) requires $\boldsymbol{f}$ to be piecewise constant. Notice however, that if $\boldsymbol{f} \in \boldsymbol{H}^1(\Omega)$, the relevant term appearing in the deduction of the error analysis for (3.9) (take for instance the proof of Theorem 4.3, and recall that we have taken $\boldsymbol{\varphi} = \boldsymbol{u} - \tilde{\boldsymbol{u}}_h$) is readily estimated as

$$\begin{aligned}
(\boldsymbol{f}, I_h\boldsymbol{z} - \mathcal{P}_h I_h \boldsymbol{z})_\Omega &= (\boldsymbol{f} - \Pi_h \boldsymbol{f}, I_h\boldsymbol{z} - \mathcal{P}_h I_h \boldsymbol{z})_\Omega \\
&\leq \|\boldsymbol{f} - \Pi_h \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|I_h\boldsymbol{z} - \mathcal{P}_h I_h \boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq C h^2 \|\boldsymbol{f}\|_{\boldsymbol{H}^1(\Omega)} |I_h \boldsymbol{z}|_{\boldsymbol{H}^1(\Omega)} \\
&\leq C h^2 \|\boldsymbol{f}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)} \\
&= C h^2 \|\boldsymbol{f}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)},
\end{aligned}$$

where we have applied Lemma 3.1, properties of I_h, Π_h, and (4.2). Then, performing an analogous analysis to that presented in [2, Appendix B], it is possible to recast the estimate of Theorem 4.3 as

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C h^2 \left(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{H}^1(\Omega)} \right).$$

Analogously, it is not difficult to extend all our convergence and superconvergence results to cover the general case $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. In such case, the estimates are of the same order than those presented in the paper.

6 A numerical test

We present an example illustrating the performance of the proposed FVE scheme on a set of triangulations of the domain $\Omega = (0,1)^2$ (see Fig. 4). In the following, by $e(\boldsymbol{u}) := \|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)}$, $e(p) := \|p - \tilde{p}_h\|_{L^2(\Omega)}$ and $E(\boldsymbol{u}, p) := \|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_h + \|p - \tilde{p}_h\|_h$ we will denote errors, and $r(\boldsymbol{u})$, $r(p)$ and $R(\boldsymbol{u}, p)$ will denote the experimental rates of convergence given by

$$r(\boldsymbol{u}) = \frac{\log(e(\boldsymbol{u})/\hat{e}(\boldsymbol{u}))}{\log(h/\hat{h})}, \quad r(p) = \frac{\log(e(p)/\hat{e}(p))}{\log(h/\hat{h})}, \quad R(\boldsymbol{u}, p) = \frac{\log(E(\boldsymbol{u}, p)/\hat{E}(\boldsymbol{u}, p))}{\log(h/\hat{h})},$$

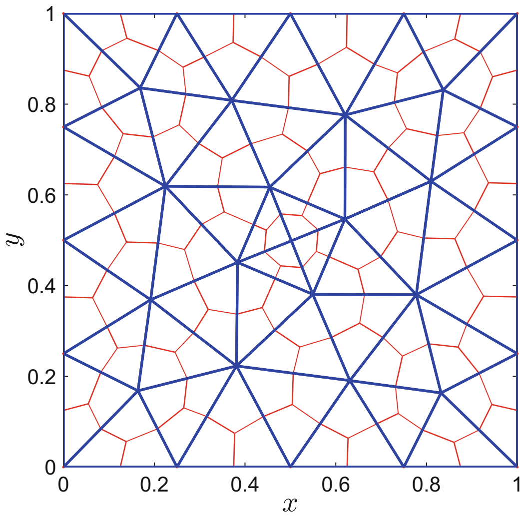


Fig. 4 Example of primal and dual meshes $\mathcal{T}_h, \mathcal{T}_h^\star$ on $\Omega = (0, 1)^2$ (17 interior nodes)

Table 1 Degrees of freedom N_h, computed errors and observed convergence rates for methods (3.9) and (6.1)

N_h	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(p)$	$r(p)$	$E(\boldsymbol{u}, p)$	$R(\boldsymbol{u}, p)$
Finite volume element method (3.9)						
121	1.2694×10^{-3}	–	3.9136×10^{-3}	–	5.3820×10^{-2}	–
449	3.4187×10^{-4}	1.8926	2.0103×10^{-3}	0.9964	2.7126×10^{-3}	0.9899
1,729	8.7094×10^{-5}	1.9728	9.8203×10^{-4}	0.9983	1.4032×10^{-3}	1.0027
6,785	2.1872×10^{-5}	1.9935	4.8360×10^{-4}	1.0219	6.6752×10^{-4}	1.0201
26,881	5.5209×10^{-6}	1.9861	2.4118×10^{-4}	1.0036	3.2989×10^{-4}	1.0168
107,009	1.3667×10^{-6}	2.0142	1.1963×10^{-4}	1.0116	1.6196×10^{-4}	1.0175
Finite volume element method (6.1)						
121	1.9446×10^{-3}	–	5.7385×10^{-3}	–	7.0973×10^{-3}	–
449	4.8838×10^{-4}	1.9920	2.6121×10^{-3}	1.1051	3.5083×10^{-3}	1.0208
1729	1.2265×10^{-4}	1.9981	1.3041×10^{-3}	1.0740	1.6941×10^{-3}	1.0457
6785	3.0518×10^{-5}	2.0015	6.1270×10^{-4}	1.0409	8.4207×10^{-4}	1.0114
26,881	7.5632×10^{-6}	2.0109	2.9861×10^{-4}	1.0372	4.1615×10^{-4}	1.0221
107,009	1.8847×10^{-6}	2.0117	1.4684×10^{-4}	1.0212	2.1062×10^{-4}	1.0093

where e and $\hat{e}$ (E and $\hat{E}$ respectively) stand for the corresponding errors computed for two consecutive meshes of sizes h and $\hat{h}$. In the implementation we have used a standard Uzawa algorithm (see e.g. [24]) in which the stopping criterion is $\left\| \tilde{p}_h^r - \tilde{p}_h^{r+1} \right\|_{L^2(\Omega)} \le 10^{-6}$.

For comparative purposes, we formulate another FVE method obtained by discarding the jump terms (that is, the FVE-counterpart of a Douglas–Wang method):

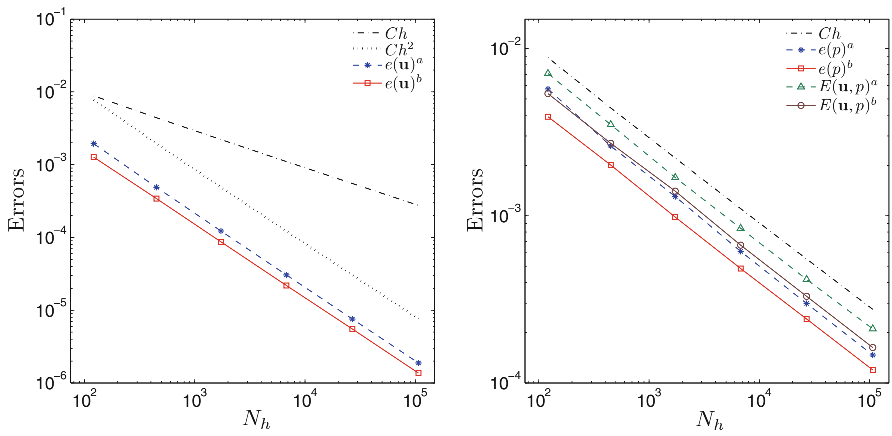


Fig. 5 Convergence histories for the FVE methods (6.1) and (3.9). The displayed quantities correspond to $e(\boldsymbol{u}) := \|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)}$, $e(p) := \|p - \tilde{p}_h\|_{L^2(\Omega)}$ and $E(\boldsymbol{u}, p) := \|\!|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|\!|_h + \|p - \tilde{p}_h\|_h$. The superscripts a and b in the legends correspond to the method (6.1) and (3.9), respectively

$$\nu(\nabla \boldsymbol{w}_h, \nabla \boldsymbol{v}_h)_\Omega - (p_h, \nabla\cdot \boldsymbol{v}_h)_\Omega + (q_h, \nabla\cdot \boldsymbol{w}_h)_\Omega + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{8\nu}(-\nu\Delta \boldsymbol{w}_h + \nabla p_h, \nabla q_h)_K$$
$$= \mathcal{F}_h(\boldsymbol{v}_h, q_h) \quad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{\mathcal{V}}_h^1 \times Q_h^t, \tag{6.1}$$

where $\mathcal{F}_h$ is defined in (3.10).

We set $\nu = 1$ and the forcing term $\boldsymbol{f}$ chosen in such a way that the exact solution of (2.1)–(2.3) is $\boldsymbol{u} = ((x_1^4 - 2x_1^3 + x_1^2)(4x_2^3 - 6x_2^2 + 2x_2), -(4x_1^3 - 6x_1^2 + 2x_1)(x_2^4 - 2x_2^3 + x_2^2))^T$, $p(x) = x_1^5 + x_2^5 - 1/3$. Notice that p satisfies $\int_\Omega p = 0$ and $(\boldsymbol{u}, p)$ has a regular behaviour in the whole domain Ω (and then the regularity assumptions of Sect. 4 are satisfied). A comparison was performed between the numerical results obtained using the methods (3.9) and (6.1). In Table 1 and Fig. 5 we depict the convergence history of this example for the approximations given by both FVE methods. In both cases, the dominant error is $E(\boldsymbol{u}, p)$. More precisely, in $E(\boldsymbol{u}, p)$ the term $\|p - \tilde{p}_h\|_h$ is dominating, followed by $|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|_{\boldsymbol{H}^1(\Omega)}$. It is clearly seen that the rates of convergence $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$ anticipated by Theorem 4.1, Theorem 4.2 and Theorem 4.3 for the formulation (3.9) are confirmed by the numerical results using (6.1) as well. In addition, we see that (3.9) shows a quantitatively better accuracy than (6.1), while the qualitative behavior given by the error slopes, is essentially the same for both formulations.

Finally, we apply a postprocessing technique by considering a coarser mesh of size $\rho = h^{2/3}$. Table 2 and Fig. 6 show the superconvergence behavior of the approximate solution when a postprocessing algorithm with Taylor-Hood ($\mathbb{P}_2 - \mathbb{P}_1$) elements is applied. It is observed that as h decreases, the convergence rate for the velocity approaches asymptotically $h^{4/3}$. This is in well accordance with the theoretical

Table 2 Degrees of freedom N_h, computed errors and observed superconvergence rates for methods (3.9) and (6.1)

N	$\left\| p - \Pi_\rho^Q \tilde{p}_h \right\|_{L^2(\Omega)}$	Rate	$\left\| \boldsymbol{u} - \Pi_\rho^{\mathcal{V}} \tilde{\boldsymbol{u}}_h \right\|_{\boldsymbol{H}^1(\Omega)}$	Rate
Finite volume element method (3.9)				
121	2.5271×10^{-3}	–	6.4310×10^{-3}	–
449	1.3038×10^{-3}	0.9548	3.6215×10^{-3}	0.8285
1,729	5.9180×10^{-4}	1.1395	1.9142×10^{-3}	0.9230
6,785	2.7442×10^{-4}	1.1870	9.0612×10^{-4}	1.0796
26,881	1.0329×10^{-4}	1.5097	3.5732×10^{-4}	1.3436
107,009	2.9706×10^{-5}	1.4928	1.4604×10^{-4}	1.2916
Finite volume element method (6.1)				
121	3.1154×10^{-3}	–	6.2179×10^{-3}	–
449	1.9147×10^{-3}	0.9011	3.1254×10^{-3}	1.0168
1,729	7.8238×10^{-4}	1.0815	1.6411×10^{-3}	0.9705
6,785	3.2394×10^{-4}	1.2755	7.4130×10^{-4}	1.0788
26,881	1.1455×10^{-4}	1.5124	2.9608×10^{-4}	1.3261
107,009	3.9812×10^{-5}	1.5076	1.1837×10^{-4}	1.3279

Postprocessing with Taylor-Hood elements and with the choice $\rho = h^{2/3}$

estimate predicted by Theorem 5.1 with the setting $r = 2, t = 1$ and $\alpha = 2/3$. The observed superconvergence rate for the pressure approaches $h^{3/2}$, which is slightly higher than the rate predicted by Theorem 5.2. A similar behavior has been also noticed in e.g. [37].

7 Conclusion

In this paper we have developed a FVE method for the Stokes problem. The discretization scheme is associated to a FE method in which an enhancement of the approximation space for the velocity field is applied following [25]. We have exploited some of the potential advantages of FVE discretizations with respect to classical finite volume methods, such as the flexibility in handling unstructured triangulations of complex geometries and that the discretization is constructed on the basis of the variational background of FE methods, therefore being more suitable to perform L^2-error analysis. This error analysis was performed for the case of piecewise linear continuous interpolation spaces only, nevertheless the same idea could be extended to a more general framework. A superconvergence analysis based on L^2-projections was also proposed, and the numerical experiments provided in this paper confirmed our theoretical findings. Finally we mention that extensions of this approach to other relevant problems, such as the generalized and transient Stokes problems, high-order approximation methods, and a posteriori error analysis are part of current and future work.

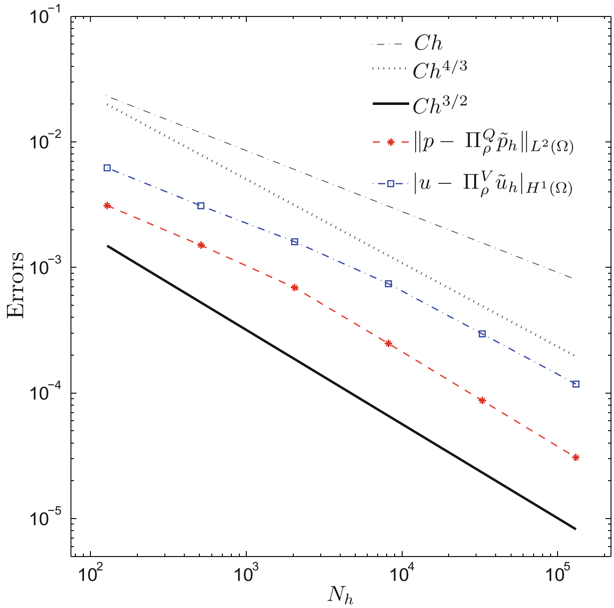


Fig. 6 Superconvergence rates obtained by a higher order postprocessing procedure. The displayed quantities correspond to $\left|\boldsymbol{u} - \Pi_\rho^{\mathcal{V}} \tilde{\boldsymbol{u}}_h\right|_{\boldsymbol{H}^1(\Omega)}$ and $\left\| p - \Pi_\rho^Q \tilde{p}_h \right\|_{\boldsymbol{L}^2(\Omega)}$

Acknowledgments The authors wish to express their gratitude to Gabriel Barrenechea for his valuable suggestions and comments concerning the stabilized method. In addition, the authors would like to acknowledge financial support by the European Research Council through the Advanced Grant *Mathcard, Mathematical Modelling and Simulation of the Cardiovascular System*, Project ERC-2008-AdG 227058. Finally, the authors thank the anonymous referees whose sharp remarks led to an important improvement in the content of the paper.

References

1. Agmon, S.: Lectures on Elliptic Boundary Value Problems. AMS, Providence (2010)
2. Araya, R., Barrenechea, G., Valentin, F.: Stabilized finite element methods based on multiscale enrichment for the Stokes problem. SIAM J. Numer. Anal. **44**, 322–348 (2006)
3. Arminjon, P., Madrane, A.: A mixed finite volume/finite element method for 2-dimensional compressible Navier–Stokes equations on unstructured grids. In: Fey, M., Jeltsch, R. (eds.) Hyperbolic Problems Theory, Numerics, Applications. International Series of Numerical Mathematics, vol 129, pp. 11–20. Birkhäuser, Basel (1998)
4. Bank, R.E., Rose, D.J.: Some error estimates for the box method. SIAM J. Numer. Anal. **24**, 777–787 (1987)
5. Bejček, M., Feistauer, M., Gallouët, T., Hájek, J., Herbin, R.: Combined triangular FV-triangular FE method for nonlinear convection-diffusion problems. ZAMM Z. Angew. Math. Mech. **87**, 499–517 (2007)
6. Bernardi, C., Raugel, G.: Méthodes d'éléments finis mixtes pour les équations de Stokes et de Navier–Stokes dans un polygone non convexe. Calcolo **18**, 255–291 (1981)
7. Bi, Ch., Ginting, V.: A residual-type a posteriori error estimate of finite volume element method for a quasi-linear elliptic problem. Numer. Math. **114**, 107–132 (2009)

8. Bürger, R., Ruiz-Baier, R., Torres, H.: A finite volume element method for a coupled transport-flow system modeling sedimentation. (2011, submitted)
9. Cai, Z.: On the finite volume element method. Numer. Math. **58**, 713–735 (1991)
10. Chou, S.C.: Analysis and convergence of a covolume method for the generalized Stokes problem. Math. Comput. **66**, 85–104 (1997)
11. Chou, S.C., Kwak, D.Y.: Multigrid algorithms for a vertex-centered covolume method for elliptic problems. Numer. Math. **90**, 441–458 (2002)
12. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
13. Clément, Ph.: Approximation by finite element functions using local regularization. RAIRO Anal. Numer. **9**, 77–84 (1975)
14. Cui, M., Ye, X.: Superconvergence of finite volume methods for the Stokes equations. Numer. Methods PDEs **25**, 1212–1230 (2009)
15. Douglas, J., Wang, J.: An absolutely stabilized finite element method for the Stokes problem. Math. Comput. **52**, 485–508 (1989)
16. El Alaoui, L.: An adaptive finite volume box scheme for solving a class of nonlinear parabolic equations. Appl. Math. Lett. **22**, 291–296 (2009)
17. El Alaoui, L., Ern, A.: Residual and hierarchical a posteriori error estimates for nonconforming mixed finite element methods. ESAIM: Math. Model. Numer. Anal. **38**, 903–929 (2004)
18. Ern, A., Guermond, J.-L.: Theory and Practice of Finite Elements. Springer-Verlag, New York (2004)
19. Eymard, R., Herbin, R., Latché, J.C.: On a stabilized colocated finite volume scheme for the Stokes problem. ESAIM: Math. Model. Numer. Anal. **40**, 501–527 (2006)
20. Eymard, R., Gallouët, T., Herbin, R., Latché, J.-C.: Convergence of the MAC scheme for the compressible Stokes equations. SIAM J. Numer. Anal. **48**, 2218–2246 (2010)
21. Ewing, R.E., Lin, T., Lin, Y.: On the accuracy of the finite volume element method based on piecewise linear polynomials. SIAM J. Numer. Anal. **39**, 1865–1888 (2002)
22. Feistauer, M., Felcman, J., Lukáčová-Medvid'ová, M.: Combined finite element-finite volume solution of compressible flow. J. Comput. Appl. Math. **63**, 179–199 (1995)
23. Feistauer, M., Felcman, J., Lukáčová-Medvid'ová, M., Warnecke, G.: Error estimates for a combined finite volume-finite element method for nonlinear convection-diffusion problems. SIAM J. Numer. Anal. **36**, 1528–1548 (1999)
24. Fortin, M., Glowinsky, R.: Méthodes de lagrangien augmenté. Dunod, Paris (1982)
25. Franca, L., Madureira, A., Valentin, F.: Towards multiscale functions: enriching finite element spaces with local but not bubble-like functions. Comput. Methods Appl. Mech. Eng. **194**, 2077–2094 (2005)
26. Gallouët, T., Herbin, R., Latché, J.C.: A convergent finite element-finite volume scheme for the compressible Stokes problem. Part I: the isothermal case. Math. Comput. **78**, 1333–1352 (2009)
27. Girault, V.: A combined finite element and marker and cell method for solving Navier–Stokes equations. Numer. Math. **26**, 39–59 (1976)
28. Girault, V., Raviart, P.A.: Finite Element Methods for Navier–Stokes Equations. Springer-Verlag, Berlin (1986)
29. Han, H., Wu, X.: A new mixed finite element formulation and the MAC method for the Stokes equations. SIAM J. Numer. Anal. **35**, 560–571 (1998)
30. Heimsund, B.-O., Tai, X.-C., Wang, J.: Superconvergence for the gradient of finite element approximations by L^2 projections. SIAM J. Numer. Anal. **40**, 1263–1280 (2002)
31. Hilhorst, D., Vohralík, M.: A posteriori error estimates for combined finite volume-finite element discretizations of reactive transport equations on nonmatching grids. Comput. Methods Appl. Mech. Eng. **200**, 597–613 (2011)
32. Ikeda, T.: Maximum principle in finite element models for convection–diffusion phenomena. In: Mathematical Studies 76. Lecture Notes in Numerical and Applied Analysis, vol. 4. North-Holland, Amsterdam (1983)
33. Lamichhane, B.P.: Inf-sup stable finite-element pairs based on dual meshes and bases for nearly incompressible elasticity. IMA J. Numer. Anal. **29**, 404–420 (2009)
34. Li, J., Wang, J., Ye, X.: Superconvergence by L^2-projections for stabilized finite element methods for the Stokes equations. Int. J. Numer. Anal. Model. **6**, 711–723 (2009)
35. Li, J., Chen, Z.: A new stabilized finite volume method for the stationary Stokes equations. Adv. Comput. Math. **30**, 141–152 (2009)
36. Li, J., Chen, Z., Wu, W.: Generalized Difference Methods for Differential Equations: Numerical Analysis of Finite Volume Methods. Marcel Dekker, New York (2000)

37. Matthies, G., Skrzypacs, P., Tobiska, L.: Superconvergence of a 3D finite element method for stationary Stokes and Navier–Stokes problems. Numer. Methods PDEs **21**, 701–725 (2005)
38. Nicaise, S., Djadel, K.: Convergence analysis of a finite volume method for the Stokes system using non-conforming arguments. IMA J. Numer. Anal. **25**, 523–548 (2005)
39. Nicolaides, R.: The covolume approach to computing incompressible flows. In: Gunzburger, M., Nicolaides, R. (eds.) Incompressible Computational Fluid Dynamics, pp. 295–333. Cambridge University Press, Cambridge (1993)
40. Quarteroni, A.: Numerical Models for Diffferential Problems. MS&A Series, vol. 2. Springer-Verlag, Milan (2009)
41. Quarteroni, A., Valli, A.: Numerical Approximation of Partial Differential Equations. Springer-Verlag, Berlin (1997)
42. Wang, J., Ye, X.: Superconvergence Analysis for the Stokes Problem by Least Squares Surface Fitting. SIAM J. Numer. Anal. **39**, 1001–1013 (2001)
43. Ye, X.: On the relationship between finite volume and finite element methods applied to the Stokes equations. Numer. Methods PDEs **5**, 440–453 (2001)
44. Ye, X.: A discontinuous finite volume method for the Stokes problems. SIAM J. Numer. Anal. **44**, 183–198 (2006)