Level Sets of Multiparameter Stable Processes

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We establish the correct Hausdorff measure function for the level sets of additive strictly stable processes derived from strictly stable processes satisfying Taylor's condition (A). This leads naturally to a characterization of local time in terms of the corresponding Hausdorff measure function of the level set.

1. INTRODUCTION AND RESULTS

In this article, we wish to extend the approach of Ref. 7 to additive (or multi-parameter) α-stable processes $(0 < \alpha < 2)\underline{X}\colon IR^N \to IR^d$ given by

$$\underline{X}(t_1, \ldots, t_N) = \sum_{j=1}^{N} X_j(t_j),$$

where $(X_j(t)\colon -\infty < t < \infty)_{i=1}^{N}$ are independant two sided—strictly α-stable processes on IR^d starting at 0; we assume that the processes are genuinely d-dimensional, that is, for each j, the distribution of $X_j(1)$ is not supported by a strict subspace of IR^d. We will suppose for simplicity that the processes $X_j(.)$, $j = 1, 2, \ldots, N$ are i.i.d. but, as will be seen, this is not necessary. This implies for $\alpha \neq 1$ that for $t > 0$,

$$E[e^{i<u,X_j(t)>}] = \exp\left(-|t|\int_{S_d} |<\theta,u>|^{\alpha}\left(1 - i\,\mathrm{sign}\left(<\theta,u>\right)\tan\left(\frac{\pi\alpha}{2}\right)\right)\Gamma\left(d\theta\right)\right),$$

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while for $\alpha = 1$,

$$E[e^{i<u,X_j(t)>}] = \exp\left(-|t| \int_{S_d} |<\theta, u>|^\alpha (1 + i\frac{2}{\pi}\text{sign}(<\theta, u>)Ln(|<\theta, u>|))\Gamma(d\theta)\right),$$

where for each "direction" $1 \leqslant k \leqslant d$, $\int_{S_d} \theta_k \Gamma(d\theta) = 0$ (see, e.g. Ref. 11). We suppose also that the processes satisfy Taylor's condition (A) (see the next section for a statement). This is automatic for $\alpha \geqslant 1$. We discuss loosening of these conditions at the end of the article. The reason for these assumptions is to profit from the work of Ref. 12 where these assumptions are in force.

For the moment we will restrict attention to one sided, one-dimensional time processes $(X(t) : t \geqslant 0)$ (as with the $X_j(.)$, we neglect the vector notation even though $X(t)$ may well be a multi-dimensional quantity in order to distinguish the one-dimensional time process from the multi-dimensional time process).

For these processes, for $t > 0$ the sigma field $\mathcal{F}_t$ will represent the completion of the sigma field generated by $\{X(s) : s \leqslant t\}$, in the multi-time context we will use $\mathcal{F}_{\underline{t}}$ for $\underline{t} = (t_1, t_2, \ldots, t_N)$ with $t_i \geqslant 0 \; \forall \; i$, the sigma field generated by $\{X_i(s) : s \leqslant t_i : i = 1, 2, \ldots, N\}$. We hope that context will be sufficiently strong to escape any confusion. The property that we will most often use is that for $0 \leqslant s < t$ the density of $X_j(t) - X_j(s)$ is bounded by $\frac{K}{(t-s)^{d/\alpha}}$ for some constant, K, depending on the specific process but not on the particular s and t. This bound follows from the boundedness of the density at time one and the scaling relations

$$\forall c > 0 \quad (c^{-1/\alpha}X(ct) : -\infty < t < \infty) \overset{D}{=} (X(t) : -\infty < t < \infty)$$

(see, e.g. Ref. 11).

A key element is the local time functional L defined (at least) on intervals (or rectangles) $I = I_1 \times I_2 \times \cdots I_N (I_i$ intervals in $I\!R$.), by

$$L(I) = \lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^d} \lambda_N(\underline{t} \in I : |\underline{X}(\underline{t})|_\infty < \epsilon).$$

This local time will exist a.s. if and only if $N \geqslant d/\alpha$ (see, e.g. Ref. 4 or Theorem 1.1 of Ref. 5), indeed if $N < d/\alpha$, then a.s. $\underline{X}(\underline{t})$ will be nonzero for all nonzero $\underline{t}$ (see, e.g. Ref. 8 or Ref. 9).

For a set $E \subset I\!R^N$, and a continuous increasing function ϕ on the reals with $\phi(0) = 0$, the measure $m_\phi\{E\}$ is defined as the (increasing) limit of $m_\phi^\delta\{E\}$ where the latter quantity is the infimum over all coverings of

E by cubes $\{J_k\}_{k \geqslant 1}$ such that for each k, $|J_k|$, the diameter of J_k, is less than δ, of

$$\sum_k \phi(|J_k|)$$

(see, e.g. Ref. 2 or Ref. 10). We show

Theorem 1.1. For $\underline{X}$ as above with $N > d/\alpha$, there exists a finite, strictly positive constant c so that a.s. $\forall$ finite interval $I = I_1 \times I_2 \cdots \times I_N \subseteq IR^N$,

$$m_\phi\{\underline{t} \in I : \underline{X}(\underline{t}) = 0\} = cL(I),$$

where $\phi(v) = v^{N-d/\alpha}(\log\log(\frac{1}{v}))^{d/\alpha}$ and L is the local time.

The result for the Brownian case has already been shown in Ref. 7. As before this question is "half" solved by Xiao[12] and Khoshenevisan et al.;[4] it was noted by Khoshenevisan et al.[4] that the arguments of the former paper allied with its own results led to the conclusion that

$$m_\phi\{\underline{t} \in I : \underline{X}(\underline{t}) = 0\} > 0 \quad \text{a.s. if } L(I) > 0.$$

Essentially our task reduces to proving a converse inequality

Proposition 1.2. $\exists$ nonrandom $K < \infty$ so that for any closed bounded interval I, not intersecting the axes, a.s.

$$m_\phi\{t \in I : \underline{X}(t) = 0\} < K\,L(I). \qquad (*)$$

Given this result the arguments of Section 5 of Ref. 7 lead easily to Theorem 1.1 given the similarity of the approach we merely provide a sketch. One first notes that, given the continuity properties of the local times and Proposition 1.2

$$m_\phi\{\underline{t} \in I : \underline{X}(\underline{t}) = 0\} = cL(I)$$

will hold a.s. for all intervals I if it holds for each interval a.s. We consider for concreteness the interval $I = [0, 1]^N$. The function $f: IR^d \to IR_+$ is defined as

$$E^{\underline{x}}[m_\phi\{\underline{t} \in [0, 1]^N : \underline{X}(\underline{t}) = 0\}] = E[m_\phi\{\underline{t} \in [0, 1]^N : \underline{X}(\underline{t}) = 0\}|\underline{X}(\underline{0}) = \underline{x}].$$

This function is easily seen (via Proposition 1.2 and continuity properties of local times) to be continuous in $\underline{x}$ and integrable. Furthermore by scaling for any interval $\prod_j [\frac{i_j}{2^n}, \frac{i_j+1}{2^n}]$,

$$E\left[m_\phi\left\{\underline{t} \in \prod_j \left[\frac{i_j}{2^n}, \frac{i_j+1}{2^n}\right] : \underline{X}(\underline{t})=0\right\} | \underline{X}\left(\frac{i_1}{2^n} \frac{i_2}{2^n} \cdots \frac{i_N}{2^n}\right)=\underline{x}\right] = f(\underline{x}2^{n/\alpha})2^{-(N-d/\alpha)n}.$$

Proof of Theorem 1.1 is concluded by showing that

$$\sum_{0 \leqslant i_1, i_2, \ldots, i_N \leqslant 2^N-1} f\left(\underline{X}\left(\left(\frac{i_1}{2^n} \frac{i_2}{2^n} \cdots \frac{i_N}{2^n}\right)\right)2^{n/\alpha}\right)2^{-(N-d/\alpha)n} \rightarrow m_\phi\{\underline{t} \in I : \underline{X}(\underline{t})=0\}$$

in probability and that

$$\sum_{0 \leqslant i_1, i_2, \ldots, i_N \leqslant 2^N-1} f\left(\underline{X}\left(\left(\frac{i_1}{2^n} \frac{i_2}{2^n} \cdots \frac{i_N}{2^n}\right)\right)2^{n/\alpha}\right)2^{-(N-d/\alpha)n} \rightarrow \int f(\underline{x})d\underline{x}\,L([0,1]^N)$$

(again in probability).

The approach will be a modification of Ref. 7 to find a useful lower bound on the probability of large values of $L([0,t]^N)$ for real t small starting from an initial value

$$\|\underline{X}(0)\|_\infty < \frac{t^{1/\alpha}}{\left(\log\log(\frac{1}{t})\right)^{1/\alpha}}.$$

The key step will be, essentially, to condition on an event $A = A(N, t)$ so that $(\underline{X}(\underline{s}) : \underline{s} \in [0, t]^N)$ conditioned on event A is a new process

$$\left(\underline{Y}^A(\underline{s}) : \underline{s} \in [0, t]^N\right)$$

and that for this process the local time $L^Y([0, t]^N)$ is, with "reasonable" probability, h, as $t \rightarrow 0$ at least $C_A t^N$, where C_A is of the order $\frac{1}{t^{d/\alpha}}(\log\log(\frac{1}{t}))^{d/\alpha}$. Hence for any t small,

$$P\left(L([0,t]^N) \geqslant C_A t^N\right) \geqslant h\,P(A(N, t)).$$

We have many available possibilites for the process $\underline{Y}^A(.)$. There is a trade off between naturalness of process $(\underline{Y}^A(\underline{s}) : \underline{s} \in [0, t]^N)$ and the ease of calculations; we have opted for a process where calculations are reasonable, though it is not Markov. Nonmarkovness notwithstanding, the resulting process will be "positive recurrent" in a simple way.

The paper is organized as follows. In Section 2, we consider non-Markov but "positive recurrent" processes derived from our original (one-

dimensional time) stable process via suppressing certain types of jump and augmenting (by a bounded factor) the rates of others. In the succeeding section, we exploit this process to gain lower bounds on large deviation probabilities for the multi-dimensional time stable processes. In the last section, we establish Proposition 1.2, closely following the approach of Ref. 7 and then discuss extensions of the approach.

We will make use of various vectorial conventions: as previously mentioned, in a multi-dimensional context, an interval will refer to a subset that is a Cartesian product of one dimensional intervals. We will write for vectors $\underline{t}$ and $\underline{s}$ $\in IR^N$ that $\underline{t} \leqslant \underline{s}$ if for each $i \in \{1, 2, \ldots, N\}, t_i \leqslant s_i$, equally $\underline{t} < \underline{s}$ is to be understood componentwise. For a vector $\underline{u} = (u_1, u_2, \ldots, u_N)$ and a constant c, $\underline{u} + c$ is simply the vector $(u_1 + c, u_2 + c, \ldots, u_N + c)$. For a set $A \subset IR^r$ for r a positive integer and a real c, cA will denote the set $\{c\underline{x} : \underline{x} \in A\}$. For $\underline{y}$, $\underline{x}$ $\in IR^r$, $< \underline{x}, \underline{y} >$ and $\underline{x}.\underline{y}$ will both denote the usual scaler product. For $\underline{x}$ in Euclidean space, $B(\underline{x}, v)$ will signify the Euclidean ball of radius v centred at $\underline{x}$. The indicator functions will be written as I_A where A may be either a random event or a logical condition. Equally if we consider points on a sphere $\underline{\theta}$, the ball on the sphere $B(\underline{\theta}, \epsilon)$ will simply denote those points on the sphere within Euclidean distance ϵ of $\underline{\theta}$. The constant C_d will denote the volume of the unit ball in IR^d.

2. THE POSITIVE RECURRENT PROCESS

In this section, we consider various processes derived from one-dimensional time indexed strictly stable processes conditioned to not make any jumps exceeding a certain threshold. The object is, ultimately, to construct a "positive recurrent" process $(Z^v(t) : t \geqslant 0)$ "close" (in a sense that will be made clear) to the original stable process which will have spatial scale v and temporal scale v^α in the sense that

    (i)   for all times $t \geqslant v^\alpha$, $Z^v(t)$ will be of "order" v;
    (ii) $Z^v(t+s) - Z^v(t)$ will be of order $s^{1/\alpha} \wedge v$.

This process will not be (quite) Markov.

We first consider $X^v(.)$ the process with jumps of magnitude greater than v suppressed. Then by "adding in" (or even reinforcing) some suppressed jumps of order v, we obtain a centred Levy process $Y^v(.)$. This process is then modified to become the target process $Z^v(.)$.

For an α stable process on IR^d, the Lévy measure $\rho(d\underline{x})$, $\underline{x} \in IR^d$ is of the form (see Ref. 11)

$$\rho(d\underline{x}) = \frac{dR}{R^{\alpha+1}} \rho'(d\underline{\theta})$$

for ρ' a measure on the sphere of radius 1 and

$$R, \underline{\theta} \text{ is } \underline{x} \text{ in polar co-ordinates.}$$

We suppose that the α-stable process in IR^d, $(X(t):t \geqslant 0)$ satisfies Taylors condition (A): $p_t(x,y) > 0$ for all strictly positive t where $p_t(.,.)$ is the density kernal for $X_t(\cdot)$.

We will at the end of the paper discuss how to extend our results to processes with drift.

Let $(X^v(t):t \geqslant 0)$ be the Lévy process with the same drift terms as $(X(t):t \geqslant 0)$ but with Lévy measure equal to

$$\rho(d\underline{x})I_{|\underline{x}|\leqslant v}.$$

Equally one can describe $(X^v(t):t \geqslant 0)$ as the process obtained from $(X(t):t \geqslant 0)$ by suppressing all jumps of magnitude greater than v. We note here that $(X^v(s):0 \leqslant s \leqslant t)$ can be simply seen as $(X(s):0 \leqslant s \leqslant t)$ conditioned on an event of probability e^{-kt/v^α} where $k = \rho\{x:|x| \geqslant 1\}$ and so on time scales of the order v^α, $X^v(.)$ will exhibit much the same behaviour as $X(.)$.

While the expectation for $X(t)$ need not exist, that of $X^v(t)$ will certainly be defined and because of the stationary increments property and the stability it must be equal to $\underline{c}tv^{1-\alpha}$ where $\underline{c}$ is a fixed vector in IR^d. We now show that we can find a convenient Lévy process absolutely continuous with respect to $X(.)$ of zero mean and with the size of all jumps bounded by $2v$.

Proposition 2.1.   There exists $M < \infty$ and $A \subset B(\underline{0}, 2) \backslash B(\underline{0}, 1)$ so that $\forall\ v > 0$, the Lévy process $Y^v(.)$ with Lévy measure

$$\rho^Y(dx) = \rho(dx)I_{|x|\leqslant v} + M\rho(dx)I_{vA}(x),$$

satisfies

$$E[Y^v(t)] = 0\ \forall t\ > 0.$$

It is only necessary to treat $v = 1$ by scaling. The case $\alpha > 1$ is easiest and we tackle it first. In this case $E[X(1)] = 0$. Suppose that $E[X^1(1)] \neq 0$ (otherwise we may simply take A to be the empty set). Then we choose A to be $B(\underline{0}, 2) \backslash B(\underline{0}, 1)$ and M to be $\int_{B(\underline{0},1)^c} |x|\rho(dx)$ divided by $\int_{B(\underline{0},2)\backslash B(\underline{0},1)} |x|\rho(dx)$.

We now suppose that $\alpha \leqslant 1$. In this case condition (A) and strict stability implies condition

(A'): the measure ρ is not concentrated on any half space in IR^d.

In this case the proposition is a consequence of two lemmas.

Lemma 2.2. Under condition (A') there exists an $\epsilon > 0$ so that for every direction $\underline{\theta}$ on the unit sphere S^d, $\rho'(\gamma : \langle\underline{\theta}, \gamma\rangle \ < \ -2\epsilon) \geqslant 2\epsilon$ and consequently $\exists \epsilon > 0, \delta > 0$ so that $\forall \ \underline{\theta} \in S^d$, $\exists \gamma : \langle \gamma, \underline{\theta}\rangle \ \leqslant -\frac{3}{2}\epsilon$ so that $\rho'(B(\gamma, \epsilon)) > \delta$.

Proof. The first part is simply a compactness argument: suppose not then $\forall n \ \exists \ \underline{\theta}_n \in S_d$ so that the condition fails with $\epsilon = 1/n$. By compactness of S_d, $\exists$ a subsequence $\underline{\theta}_{n_k} \to \underline{\theta}_0$. Then the condition (A') is violated by the half space $\{x : \langle x, \underline{\theta}_0\rangle \ \geqslant 0\}$. Given such an $\epsilon > 0$ and any $\underline{\theta} \in S_d$, we can cover the set $\{\gamma : \langle \gamma, \underline{\theta}\rangle \leqslant -2\epsilon\}$ with a finite number, $M = M_\theta$, of ϵ-balls, $B(\gamma_1, \epsilon), \ldots, B(\gamma_i, \epsilon), \ldots, B(\gamma_M, \epsilon)$ where for each i, $\gamma_i \in \{\gamma :\langle \gamma, \underline{\theta}\rangle \leqslant -2\epsilon\}$. This cover must also cover $\{\gamma : \langle \gamma, \theta'\rangle \ \leqslant \ -\frac{3}{2}\epsilon\}$ for all $\underline{\theta}'$ in a neighbourhood, C_θ, of $\underline{\theta}$ and we may also take C_θ so that for each i, $\gamma_i \in \{\gamma :\langle \gamma, \underline{\theta}'\rangle \leqslant -\frac{3}{2}\epsilon\}$ for all $\theta' \in C_\theta$. Again we take a finite collection $C_{\theta_1}, C_{\theta_2}, \ldots, C_{\theta_n}$ covering the sphere and let $M = \max_{1\leqslant i\leqslant n}\{M_{\theta_i}\}$, then for any θ we have $\gamma \in \{\gamma' :\langle \gamma', \theta\rangle \leqslant -\frac{3}{2}\epsilon\}$ so that

$$\rho'(B(\gamma, \epsilon)) \geqslant \epsilon/M.$$

The result now follows by reducing ϵ. $\square$

Lemma 2.3. Under condition (A'), for some $\epsilon > 0$, there exist disjoint $B(\theta_1, \epsilon), B(\underline{\theta}_2, \epsilon), \ldots, \ B(\underline{\theta}_d, \epsilon)$ so that $\underline{x}_i = \int_{B(\underline{\theta}_i, \epsilon)} \rho'(d\underline{\theta})\underline{\theta}$ are linearily independent.

Proof. Let ϵ, δ be fixed as in Lemma 2.2 and such that the previous lemma holds for $K\epsilon, K^{d+2}\delta$ where K is large but fixed. We choose $\underline{\theta}_1, \ldots, \underline{\theta}_d$ in succession (and so $\underline{x}_1, \ldots, \underline{x}_d$ where $\underline{x}_i = \int_{B(\underline{\theta}_i, \epsilon)} \underline{\theta}\rho'(d\underline{\theta})$).

First $\underline{\theta}_1$ is chosen so that $\rho'(B(\underline{\theta}_1, \epsilon)) > \delta$ (which is possible if K is large).

Now, having chosen the first k points $\underline{\theta}_1, \ldots, \underline{\theta}_k$, let A be the subspace generated by $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k$ and V a halfspace whose boundary contains A. By Lemma 2.2 there is $\underline{\theta}_{k+1}$ so that

$$\rho'(B(\underline{\theta}_{k+1}, \epsilon)) > \delta,$$
$$B(\underline{\theta}_{k+1}, \epsilon) \cap B(\underline{\theta}_i, \epsilon) \neq \emptyset \quad i = 1, 2, \ldots, k$$

and so that there exists $\underline{\Theta} :< \underline{\Theta}, \underline{x}_i > = 0$ $i = 1, 2, \ldots, k$ and $< \underline{\Theta}, \underline{\theta}' > \leqslant -\frac{K}{2}\epsilon \; \forall \; \underline{\theta}' \in B(\theta_{k+1}, \epsilon)$. This implies that

$$\underline{x}_{k+1} = \int_{B(\theta_{k+1}, \epsilon)} \rho'(d\theta)\underline{\theta}$$

is not a linear combination of $\underline{x}, \ldots, \underline{x}_k$.

Continuing we obtain $\underline{\theta}_1, \underline{\theta}_2, \ldots \underline{\theta}_d, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_d$. $\qquad \square$

We now return to Proposition 2.1 for the case $\alpha \leqslant 1$, which is now a simple consequence of these two lemmas.

Proof. Suppose without loss of generality that $E[X^1(1)] = \underline{x}_0 \neq \underline{0}$. Let K be the set of points $\underline{x} \in IR^d$ of the form

$$\underline{x} = \sum_{j=1}^{R} c_j \int_{A_j} \underline{\theta} \rho'(d\underline{\theta}),$$

where the A_j are Borel subsets of S^d, the c_j are strictly positive and the vectors $\int_{A_j} \underline{\theta}\rho'(d\underline{\theta})$ span IR^d. Clearly K is an open cone in IR^d. Now note that, given the decomposition of the Levy measure ρ via ρ', if $-\underline{x}_0 \in K$, then we can find $M > 0$ and Borel subset H_{x_0} of $B(\underline{0}, 2) \backslash B(\underline{0}, 1)$ so that

$$-\underline{x}_0 = M \int_{H_{x_0}} u\rho(du).$$

Thus the proposition will quickly follow if we can show that K is equal to IR^d, which will certainly follow if $\underline{0} \in K$. Since K is an open cone the alternative is that $\underline{0}$ is on the boundary of K. But by the separation theorem (K is convex) there would then exist $\underline{v} \in IR^d$ so that $K \subset \{\underline{u} : < \underline{u}, \underline{v} > > 0\}$. A contradiction quickly arrives as, by the condition (A'), measure ρ' charges the unit sphere intersected with $\{\underline{u} : < \underline{u}, \underline{v} > < 0\}$. $\qquad \square$

Having defined $Y^v(.)$ we take M to be as given in Proposition 2.1. The Lévy process $Y^v(.)$ is, according to dimension d and index α, transient or null recurrent: it cannot be positive recurrent. We now proceed to modify this process so that while it loses the Markov property, it is "positive recurrent". We choose $M_1 >> 1$ but not depending on v and we define process $(Z^v(t) : t \geqslant 0)$ so that over time intervals $[iv^\alpha, (i + \frac{1}{2})v^\alpha)$ it evolves as the Lévy process $Y^v(.)$, while over the interval $[(i + \frac{1}{2})v^\alpha, (i + 1)v^\alpha)$ by the rule:

(a) if $|Z^v(iv^\alpha)| \leqslant M_1 v$, then on the interval $[(i + \frac{1}{2})v^\alpha, (i + 1)v^\alpha)$, $Z^v(.)$ evolves as the process $(Y^v(t) : t \geqslant 0)$, but

(b)   if $|Z^v(iv^\alpha)| > M_1 v$, then on the interval $[(i + \frac{1}{2})v^\alpha, (i + 1)v^\alpha)$, $Z^v(.)$ evolves as the Levy process $Y^v(.)$ except that the jumps of size $\frac{1}{2}v < |y| < 2v$ for y with $< y, Z^v(iv^\alpha) > > 0$ are suppressed and jumps in $2v < |y| < 4v$ with $< y, Z^v(iv^\alpha) > < 0$ are unsuppressed.

Even though the process $Z^v(.)$ is not Markov, it evolves as a Lévy process on each interval $[iv^\alpha, (i + \frac{1}{2})v^\alpha]$ and on each interval $[(i + \frac{1}{2})v^\alpha, (i + 1)v^\alpha]$ (though which law depends on the value $Z^v(iv^\alpha)$) and so we have easily

Lemma 2.4.   Given $0 \leqslant s < t$, there exists a constant, c_1 (not depending on v) so that the conditional density of $Z^v(t)$ given $\mathcal{F}_s = \sigma\{Z^v(u) : u \leqslant s\}$ is bounded by the maximum of c_1/v^d and $c_1/(t - s)^{d/\alpha}$.

Proof.   We treat first the case $t - s \leqslant v^\alpha/2$. In this case the conditional distribution of $Z^v(t) - Z^v(s)$ given $\mathcal{F}_s$ can be written as $Z_1 + Z_2$ where the Z_i are conditionally independent and Z_1 is equal in distribution to $X^{\frac{v}{2}}(t - s)$. $X^{\frac{v}{2}}(t - s)$ is the random variable $X(t - s)$ conditioned on an event (no "large" jumps) of probability bounded away from zero. Accordingly the density of Z_1 is bounded by $c/(t - s)^{d/\alpha}$ for some c not depending on v, t or s. So the density of $Z_1 + Z_2$ is also so bounded. The general case now follows as the L_∞ bound of the density of $Z^v(t)$ given $\mathcal{F}_s = \sigma\{Z^v(u) : u \leqslant s\}$ is less than that for the density of $Z^v(t)$ given $\mathcal{F}_{t - v^\alpha/2}$. $\square$

Lemma 2.5.   For each $L > 0$, there exists a strictly positive d_L so that uniformly on $|x| \leqslant L$, starting from x at time 0, the conditional density of $Z^v(v^\alpha)$ is at least d_L/v^d at each point within d_L of the origin.

Proof.   We need only consider the case $v = 1$ by scaling. We may consider instead of the process $Z^1(.)$, the Lévy process $(X^{1/2}(s) : 0 \leqslant s \leqslant 1)$ since this process arises from the other by conditioning on an event of probability bounded away from zero, namely the non occurrence of any jumps of size greater than one half on the unit time interval.

We first consider for this process, starting at zero, the density at position y at time $c < 1$, $p'(c, y)$. By elementary Fourier analysis this is

equal to

$$\frac{1}{(2\pi)^d} \int_{I\ R^d} e^{-iu.y} E[e^{iu.X^{1/2}(c)}]du$$

$$= \frac{1}{(2\pi)^d} \int_{I\ R^d} e^{-iu.y} E[e^{iu.X(c)}]du \ + \ \frac{1}{(2\pi)^d} \int_{I\ R^d} e^{-iu.y}(E[e^{iu.X^{1/2}(c)}] - E[e^{iu.X(c)}])du$$

$$= \ p^X(c,y) + o(c^{d/\alpha}),$$

where $p^X(c,.)$ is the density of $X(c)$ conditioned on starting from the origin. Since $p^X(c,y) = c^{-d/\alpha} p^X(1, y/c^{1/\alpha})$ and $p^X(1,.)$ is a continuous function, we conclude that for c small enough $p'(c,y) > 1$ for all $|y| \ll c^{1/\alpha}$. Fix such a c.

For fixed x, it is easily seen, there is a strictly positive probability, $f(x)$ that, starting from x, $X^{1/2}(1-c)$ has magnitude less than $c^{1/\alpha}/4$. Thus by compactness of the ball of radius L there is a strictly positive f so that uniformly over $|x| \leqslant L$, $P^x(|X^{1/2}(1-c)| \leqslant c^{1/\alpha}/2$. From this and the Markov property at time $1-c$ we have that uniformly over $|x| \leqslant L$ the density at y is at least f for all y of magnitude less than $c^{1/\alpha}/2$. $\qquad\square$

It is easy but laborious to show

Proposition 2.6. There exists $M_2 > M_1$ and $\delta, c > 0$ not depending on v so that if

$$|Z^v(0)| > M_2 v,$$

then for $\sigma = \inf\{iv^\alpha : |Z^v(iv^\alpha)| < M_2 v\}$,

$$W_n = \ e^{c|Z^v(nv^\alpha \wedge \sigma)| + \delta(nv^\alpha \wedge \sigma)}$$

is a supermartingale.

Corollary 2.7. The discrete time Markov chain $(Z^v(nv^\alpha) : n \geqslant 0)$ has a unique stationary distribution.

Proof. We define the stopping times (with respect to the continuous time filtration)

$\sigma_0 = \inf\{t > 0 : |Z^v(t)| > M_2 v\}$ and for $i \geqslant 0$, $\qquad \sigma_i' = \inf\{nv^\alpha > \sigma_i\}$,

$\tau_i = \inf\{nv^\alpha \geqslant \sigma_i' : |Z^v(nv^\alpha)| < M_2 v\}$,

$\sigma_{i+1} = \inf\{t > \tau_i : |Z^v(t)| > M_2 v\}$. The τ_i, σ_i' are not stopping times with respect to the the natural filtration of $\{Z^v(nv^\alpha) : n \geqslant 0\}$ but this will not affect things.

Since the jumps for the continuous time process $\{Z^v(t) : t \geqslant 0\}$ are bounded by $4v$ in magnitude, it is immediate that (with the possible exception of σ_0) $|Z^v(\sigma_i)| \leqslant (M_2 + 4)v \ \forall \ i$. From this and the Markov property

(given $\mathcal{F}_{jv^\alpha}$) on intervals $[jv^\alpha, (j+1)v^\alpha]$, we have (again with the possible exception of σ_0) that there exists non random K so that for all i and v,

$$E[e^{\delta|Z^v(\sigma_i')|}|\mathcal{F}_{\sigma_i}] \leqslant K.$$

This and Proposition 2.6 ensure that

$$E[e^{\delta(\tau_i - \sigma_i')}|\mathcal{F}_{\sigma_i}] \leqslant K'.$$

From this we easily deduce that for all initial values $z \in IR^d$, the distributions

$$\mu_n(A) = \frac{1}{n}\sum_{i=1}^n P^z(Z^v(iv^\alpha) \in A),$$

satisfy

$$\exists K_1, \ K_2 < \infty, c > 0 \quad \text{so that for all } x \ \limsup_{n\to\infty} \mu_n(B(0, xv)^c) \leqslant K_1 e^{-K_2 x}.$$

This inequality firstly implies that the measures $\{\mu_n\}_{n\geqslant 1}$ are tight and thus (see, e.g. Ref. 1) possess a convergent subsequence in the weak topology. Any limit is necessarily a stationary distribution for the semigroup P (see, e.g. Ref. 6, Proposition 1.8) and will inherit the exponetial bound on tail probabilities. This ensures that a stationary distribution, μ^v, for the discrete time process exists and that it satisfies $\mu^v(B(0, vx)^c) \leqslant K_1 e^{-K_2 x}$ for all positive x.

To establish uniqueness of the stationary measure, note first that by Lemma 2.5, any two realisations $Z^v(.)$ and $Z^{v'}(.)$ may be coupled together so that after first meeting (at a time iv^α) they are equal thereafter and so that for $L > 0$ fixed and all positive integer n,

$$P(Z^v(nv^\alpha) = Z^{v'}(nv^\alpha)|Z^v(mv^\alpha), \ Z^{v'}(mv^\alpha) \ \ 0 \leqslant m < n) > d_L C_d (d_L)^d > 0,$$

if both $Z^v((n-1)v^\alpha)$ and $Z^{v'}((n-1)v^\alpha)$ are less than Lv in magnitude.

By our exponential bounds we have that if L is fixed sufficiently large then with probability tending to one as n tends to infinity,

$$\frac{\#i < n : |Z^v(iv^\alpha)| < L}{n} > \frac{3}{4}$$

and likewise for $Z^{v'}(.)$. Thus with probability tending to one as n tends to infinity

$$\frac{\#i < n : |Z^v(iv^\alpha)|, |Z^{v'}(iv^\alpha)| < L}{n} > \frac{1}{2}.$$

This implies that for the above coupling with probability tending to one as n tends to infinity, $Z^v(nv^\alpha) = Z^{v'}(nv^\alpha)$. By standard arguments this yields the desired unicity. $\qquad\square$

We are ultimately interested in the behaviour of the continuous time process

As noted, the process $(Z^v(t):t\geqslant 0)$ is non-Markov. However given stationary measure μ^v for the discrete time process, we call $\pi^{v,c}$ the stationary measure for the continuous time process $(Z^v(t):t\geqslant 0)$, where

$$\pi^{v,c}(A)=\frac{1}{v^\alpha}\int_0^{v^\alpha}P^{\mu^v}(Z^v(s)\in A)ds.$$

We now assemble some elementary properties of this stationary distribution. The following comes directly from the results for μ^v and Lemmas 2.4 and 2.5.

Proposition 2.8.   The "stationary" probability $\pi^{v,c}$ for the process $(Z^v(t):t\geqslant 0)$ is absolutely continuous with respect to Lebesgue measure with density bounded by c_2/v^d and greater than c_3/v^d in a neighbourhood of the origin containing $B(0,vc_4)$ for some strictly positive c_4 not depending on v.

The results and arguments given above enable one to conclude

Proposition 2.9.   There are strictly positive $H(>1)$ and c_5 so that for the continuous time process $(Z^v(t):t\geqslant 0)$:

(i)   irrespective of $t>s\geqslant 0$ and $Z^v(0)=y\in I\!R^d$, the conditional density of $Z^v(t)$ given $\{Z^v(r):r\leqslant s\}$ is bounded by $\frac{1}{c_5}(\frac{1}{v^d}\vee\frac{1}{(t-s)^{d/\alpha}})$;

(ii) for all $t>Hv^\alpha$, and all $Z^v(0)$ of magnitude less than v, the conditional density of $Z^v(t)$ in $B(0,c_5v)$ is bounded below by c_5/v^d.

Given this we can establish the key bounds for the local times at zero of the multiparameter process $\underline{X}$.

Proposition 2.10.   For process $\underline{Z}^v(t_1,\ldots,t_N)=\sum_{j=1}^N Z_j^v(t_j)$, where Z_j^v are independent Markov processes with the same semigroup as $Z^v(.)$ above and with $|Z_j^v(0)|\leqslant v\ \forall\ j$

(i)   there exists strictly positive c_6 not depending on v so that for all $t\geqslant 2Hv^\alpha$ (H the constant of Proposition 2.9), $E[L([\frac{t}{2},t]^N)]\geqslant c_6t^N/v^d$;

(ii) there exists finite positive c_7 not depending on v so that for all $t\geqslant 2Hv^\alpha$ (H the constant of Proposition 2.9), $E[(L([\frac{t}{2},t]^N)])^2]\leqslant c_7t^{2N}/v^{2d}$.

Proof. For $(s_1, s_2, \ldots, s_N) \in [\frac{t}{2}, t]^N$ with $s_i \geqslant H v^\alpha \ \forall \ i$, we have that $Z^v(s_1, \ldots, s_N)$ has density at zero at least $c_5^{(d+1)(N-1)} (\frac{(C_d)^{N-1}}{N^{(N-1)d}}) \frac{c_5}{v^d}$ where, recall, C_d is the volume of a radius one sphere in d dimensions. So, see Ref. 3,

$$E\left[L\left(\left[\frac{t}{2}, t\right]^N\right)\right] \geqslant c_6 t^N / v^d$$

with $c_6 = c_5^{(d+1)(N-1)} (\frac{(C_d)^{N-1}}{N^{(N-1)d}}) c_5 / 2^N$.

For the second bound we have (see, e.g. Ref. 3), that

$$E\left[\left(L\left(\left[\frac{t}{2}, t\right]^N\right)\right)^2\right] = \int_{[\frac{t}{2}, t]^N} \int_{[\frac{t}{2}, t]^N} d\underline{s} \, d\underline{s}' \, p(\underline{0}, \underline{0}, \underline{s}, \underline{s}'),$$

where $p(\underline{x}, \underline{y}, \underline{s}, \underline{s}')$ is the joint density of $\underline{Z}^v(\underline{s})$ and $\underline{Z}^v(\underline{s}')$ at $(\underline{x}, \underline{y})$. Now for given $\underline{s}$ and $\underline{s}'$, let index i be such that $|s_i - s_i'| = \sup_{1 \leqslant j \leqslant n} |s_j - s_j'|$. Without loss of generality suppose that $|s_i - s_i'| = s_i - s_i'$. Then applying Lemma 2.4 twice and using the fact that the density of a convolution is bounded by the smallest L^∞ bound for the densities of the individual random variables, we find that

$$p(\underline{0}, \underline{0}, \underline{s}, \underline{s}') \leqslant \frac{1}{c_5^2} \frac{1}{v^d} \left(\frac{1}{v^d} \wedge \frac{1}{|s_i - s_i'|^{d/\alpha}} \right).$$

Integrating this over $[\frac{t}{2}, t]^N \times [\frac{t}{2}, t]^N$ gives the second moment bound. □

This gives the immediate corollary

Corollary 2.11. For $\underline{Z}^v(.)$ as in Proposition 2.10, above, so that $|Z_j^v(.)| \leqslant v \ \forall \ 1 \leqslant j \leqslant N$, there exists a strictly positive constant c_g not depending on the initial values of the $Z_j^v(.)$, v or t so that for all v and all $t \geqslant 2H v^\alpha$, $P(L([\frac{t}{2}, t]^N)] \geqslant c_g t^N / v^d) \geqslant c_g$.

3. APPLICATION TO THE ORIGINAL PROCESS

We now consider using the results of the preceding section to gain information about the original process X and $\underline{X}$. As already noted the process $(Z^v(t) : t \geqslant 0)$ is simply the original process $(X(t) : t \geqslant 0)$ conditioned on not making certain jumps over given intervals and having certain jump rates enhanced.

Proposition 3.1. Suppose that $|\sum_{j=1}^N X_j(0)| \leqslant Nv$ and that $t/v^\alpha > 2H$ for H the constant of Proposition 2.9, then for $c_8 \in (0, \infty)$ not depend-

ing on t or v,

$$P\left[L\left(\left[\frac{t}{2},t\right]^N\right) > c_g t^N/v^d\right] > c_g e^{-c_8 t/v^\alpha},$$

where c_g is the constant of Proposition 2.9.

Proof. By linearity of $\underline{X}$ and the stationary independent increments property of Lévy processes we may suppose that for $j = 1, 2, \ldots, N$, $|X_j(0)| \leqslant v$.

We will take, without loss of generality, t to be an integer multiple of v^α. Let $t = nv^\alpha$. The idea is to exploit Corollary 2.11, which gives information on our "artificial" process $\underline{Z}^v(.)$, to gain suitable lower bounds on probabilities for our process $\underline{X}$. We first establish some new notation while recalling how (one- dimensional time process) $Z^v(.)$ is derived from the original (one-dimensional time process) $X(.)$ by suppressing some jumps and by increasing by a (bounded) factor the jump rates for other types of jumps (all drift term are retained):

(i)   all jumps of magnitude greater than $4v$ are suppressed.
(ii) no jump of magnitude less than $\frac{1}{2}v$ is suppressed.

Furthermore for $s \in [iv^\alpha, (i + \frac{1}{2})v^\alpha)$ the set $\{y : \frac{1}{2}v \leqslant |y| \leqslant 4v\}$ is partitioned into three subsets:

J_i^1 for which the jump rate is zero,
J_i^2 on which the jump rate corresponds to ρ the Levy measure of process $X(.)$,
J_i^3 on which the jump rate corresponds to $M\rho$.

Likewise on $s \in [iv^\alpha, (i + \frac{1}{2})v^\alpha)$ the set $\{y : \frac{1}{2}v \leqslant |y| \leqslant 4v\}$ is partitioned into three subsets:

H_i^1 for which the jump rate is zero,
H_i^2 on which the jump rate corresponds to ρ the Levy measure of process $X(.)$,
H_i^3 on which the jump rate corresponds to $M\rho$.

For clarity we will work on canonical space $(\Omega, \mathcal{F}, P)$ with Ω the space of cadlag functions $\omega: IR_+^N \to IR^d$, starting from $|\omega_i(0)| \leqslant v$ $\forall$ $1 \leqslant i \leqslant d$, with $\mathcal{F}$ the usual Borel sigmafield corresponding to Skorohod topology and P the probability according to which the process $(w(s) : s \geqslant 0)$ has the law of $\underline{X}(.)$. We let P_2 be the probability on $(\Omega, \mathcal{F})$ under which $(w(s) : s \geqslant 0)$ has the law of $Z^v(.)$ and we take P_1 to be the probability on this space corresponding to the "piecewise" Levy process which on interval

$[iv^\alpha, (i + \frac{1}{2})v^\alpha)$ evolves like evolves like $Z^v(.)$ except that for jumps in J_i^3 the rates are given by ρ and not $M\rho$ and likewise on $[(i + \frac{1}{2})v^\alpha, (i + 1)v^\alpha)$ the jump rates for H_i^3 are those of $X(.)$. In words the jump rates are derived from those of $X(.)$ by making the same suppression as are made to get $Z^v(.)$ but without augmenting any rates.

We wish to prove a lower bound for $P(A)$ where A is the set of paths $\{\omega : L(\omega, [\frac{t}{2}, t]^N) > c_g t^N / v^{d/\alpha}\}$. We first observe that

$$P(A) \geqslant P(A \cap B) \ = \ P(B) P_1(A),$$

where B is the event that for all components $1 \leqslant j \leqslant N$, there are no jumps of magnitude greater than $4v^\alpha$ and on time interval $[iv^\alpha, (i + \frac{1}{2})v^\alpha)$ no jumps in J_i^1 $0 \leqslant i \leqslant n - 1$ and on time interval $[(i + \frac{1}{2})v^\alpha, (i + 1)v^\alpha)$ no jumps in H_i^1 $0 \leqslant i \leqslant n - 1$. This easily leads to the bound

$$P(A) \geqslant e^{-c_8' n} P_1(A)$$

for some constant c_8' depending only on the original process and not on n or v. This is because the Levy measure of the "suppressed" zone is always bounded by a constant times v^α. Thus to finish the proof of the lemma it will suffice to achieve a suitable lower bound for $P_1(A)$.

Since P_1 is absolutely continuous with respect to P_2, we have

$$P_1(A) \ = \ \int_A \frac{dP_1}{dP_2}(\omega) dP_2(\omega).$$

Direct calculation gives that $\frac{dP_1}{dP_2}(\omega)$ is equal to $M^{-N(\omega)} e^{M-1} \geqslant M^{-N(\omega)}$ where

$$N(\omega) = \sum_{i=0}^{n-1} \text{number of jumps in } J_i^3 \text{ during } [iv^\alpha, (i + \frac{1}{2})v^\alpha)$$
$$+ \sum_{i=0}^{n-1} \text{number of jumps in } H_i^3 \text{ during } [(i + \frac{1}{2})v^\alpha, (i + 1)v^\alpha).$$

The random variable $N(\omega)$ is a Poisson process of random rate bounded by a constant c_9 times v^α where c_9 does not depend on n or v. Hence there exists constant c_{10} so that for all $n \geqslant 1$,

$$P_2(N(\omega) > c_{10} n) \ \leqslant \ c_g/2,$$

where c_g is the constant of Corollary 2.11. Thus we have by Corollary 2.11,

$$P_1(A) \ \geqslant \ \int_{A \cap C} \frac{dP_1}{dP_2}(\omega) dP_2(\omega) \ \geqslant \ M^{-c_{10} n} c_g/2.$$

where C is the event $\{N(\omega) \leqslant c_{10}n\}$. Thus we have

$$P(A) \geqslant e^{-c_8'n} M^{-c_{10}n} c_g/2.$$

<div style="text-align: right">□</div>

We need now to transform this into a large deviation bound on the large deviations for multiparameter stable processes. The following lemma is elementary:

Lemma 3.2.   For $(X(t) : t \geqslant 0)$, with $|X(0)| \leqslant n^{1/(2d)}2^{-n/\alpha}$, the probability that there exists a negative integer i with $2^i \in [n^2 2^{-n}, 2^{-n/2}]$ and so that $|X(2^i)| \geqslant n^{2/\alpha}2^{i/\alpha}$ is bounded by c_{11}/n where c_{11} is a finite constant not depending on n.

Let us take, for $\underline{t} > 0$, the event $A(\underline{t}, n)$ to be that there exists $j \in \{1, 2, \ldots, N\}$ and a negative integer i with $2^i \in [n^2 2^{-n}, 2^{-n/2}]$ so that $|X_j(t_j + 2^i) - X_j(t_j)| \geqslant n^{2/\alpha}2^{i/\alpha}$, then we have

Corollary 3.3.   For $\underline{t} > 0$ the event $A(\underline{t}, n)$ has probability bounded by c_{12}/n for some c_{12} depending on N, α but not on n or $\underline{t}$. Furthermore if $\max_i t_i > c > 0$ then the probability of the event $A(\underline{t}, n) \cap \{|\underline{X}(\underline{t})| \leqslant n^{1/2d}2^{-n/\alpha}\}$ is less than $c_{13}2^{-dn/\alpha}/n^{1/2}$ where $c_{13} = c_{13}(c)$ depends on c but not on n or $\underline{t}$.

We can now give a bound on aberrant LIL behaviour for the local time around a time point $\underline{t} > 0$. Define the event $C(n, \underline{t})$(for such $\underline{t}$) to be $\{|\underline{X}(\underline{t})| \leqslant n^{1/2d}2^{-n/\alpha}\} \cap \{\nexists s \in [n^2 2^{-n}, 2^{-n/2}] : L([\underline{t}, \underline{t} + s]) \geqslant hs^{N-d/\alpha} \log(n)^{d/\alpha}\} \setminus A(\underline{t}, n)$. Note we have not yet fully specfied the event $C(n, \underline{t})$ as the constant h is not fixed. We will later fix it small but strictly positive in a way not depending on n.

Proposition 3.4.   If the constant h is fixed sufficiently small then for all n sufficiently large and all $\underline{t} > 0$ with $\max_i t_i > c > 0$, $P(C(n, \underline{t})) \leqslant 2c_{13}2^{-nd/\alpha}/n^{1/2}$ for c_{13} as in Corollary 3.3.

Proof.   We need only treat n large, so we will in the course of the proof use various inequalities that, though not true in total generality, hold for all n sufficiently large.

We take the finite set of integers $I = \{i_0, i_1, \ldots, i_R\}$ to be given by first taking i_0 to be the smallest integer such that $2^{i_0} > 2.2^{-n}n^2$ and then

for $1 \leqslant k \leqslant R$, $i_k = i_{k-1} + \frac{8}{\alpha} \log(n)$ where i_R is the largest such integer less than $2^{-n/2}$. For n large R will be greater than $\alpha n / 18 \log(n)$.

We take s_k to be $(n^{\frac{4}{\alpha}} - 1) 2^{i_k}$ for $0 \leqslant k < R - 1$ and $v_k = (c_{14} s_k / \log(n))^{1/\alpha}$. c_{14} is a constant not depending on n to be chosen later. For n large we have (the fact that the choice of c_{14} is still in abeyance notwithstanding) for all $0 \leqslant k < R$ that $2^{i_k} < s_k < 2^{i_{k+1}}$.

We have from Proposition 3.1 that if $0 \leqslant k < R$ and $|\underline{X}(\underline{t} + 2^{i_k})| \leqslant v_k$, then

$$P\left(L([\underline{t} + 2^{i_k}, \underline{t} + 2^{i_k} + s_k] > c_g(s_k)^N / (v_k)^d | \mathcal{F}_{\underline{t} + 2^{i_k}}\right) > c_g e^{-c_8 s_k / (v_k)^\alpha}$$
$$= c_g e^{-(c_8/c_{14}) \log(n)}.$$

We now fix c_{14} so large that for n large $c_g e^{-(c_8/c_{14}) \log(n)} > \frac{1}{n^{1/4}}$. With this value fixed the event $\{L([\underline{t} + 2^{i_k}, \underline{t} + 2^{i_k} + s_k] > c_g(s_k)^N / (v_k)^d\}$ can be rewritten as $\{L([\underline{t} + 2^{i_k}, \underline{t} + 2^{i_k} + s_k] > (c_g / c_{14}^{d/\alpha})(s_k)^{N-d/\alpha} (\log(n))^{d/\alpha}\}$. We will now choose h to be $(c_g / c_{14}^{d/\alpha})/2$ so that for n large

$$(c_g / c_{14}^{d/\alpha})(s_k)^{N-d/\alpha} > h(s_k + 2^{i_k})^{N-d/\alpha} \quad \text{for all } 0 \leqslant k < R.$$

To conclude the proof we note that if event $A(\underline{t}, n)$ holds, then for each $0 \leqslant k < R$, $|X_j(t_j + 2^{i_k}) - X_j(t_j)| \leqslant n^{2/\alpha} 2^{k/\alpha}$ and so if $A(\underline{t}, n) \cap \{|\underline{X}(\underline{t})| \leqslant 2^{-n/\alpha} n^{1/(2d)}\}$ holds then for all $0 \leqslant k < R$ (if n is sufficiently large),

$$|\underline{X}(\underline{t} + 2^{i_k})| \leqslant 2^{-n/\alpha} n^{1/(2d)} + N n^{2/\alpha} 2^{k/\alpha} \leqslant v_k.$$

Hence we have (with the above choice of h) that

$$P(C(n, \underline{t})) \leqslant P\left(A(\underline{t}, n) \cap \{|\underline{X}(\underline{t})| \leqslant 2^{-n/\alpha} n^{1/(2d\alpha)}\}\right) + \left(1 - \frac{1}{n^{1/4}}\right)^R.$$

Given our bound for R, the proof is complete. $\qquad\square$

4. HAUSDORFF MEASURE BOUNDS

In this section, we establish Proposition 1.2. We consider for concreteness the cube $[1, 2]^N$ but it will be clear that no loss of generality is entailed.

Consider the cube in $[1, 2]^N$, $\Pi_{j=1}^N (1 + \frac{i_j}{2^n}, 1 + \frac{i_j+1}{2^n}] = H_{\underline{i}}^n$. Let the Lebesgue measure of the range of $\underline{X}$ on $H_{\underline{i}}^n$ be $R_{\underline{i}}^n$.

Lemma 4.1. For positive k, $P(R_{\underline{i}}^n \geqslant k 2^{-dn/\alpha}) \leqslant C e^{-ck^{1/N}}$ for some finite, positive C, c not depending on $n, \underline{i}$.

Proof. We consider the time components separately. Let the stopping times σ_k^j (with respect to $\mathcal{F}_s^j = \sigma(\underline{X}_j(u):0 \leqslant u \leqslant s)$, the natural one for the process $(X_j(s):s \geqslant 0)$ be defined recursively by $\sigma_0^j = 1 + i_j/2^n$ and for $k \geqslant 1$,

$$\sigma_k^j = \inf\left\{t > \sigma_{k-1}^j : d\left(X_j(t), X_j\left(\sigma_{k-1}^j\right)\right) \geqslant 2^{-n/\alpha}\right\}.$$

Then we have by the strong Markov property and natural scaling properties, that there exists c not depending on j, n, k or $\underline{i}$ such that

$$P\left[\sigma_k^j > \sigma_{k-1}^j + 2^{-n} | \mathcal{F}_{\sigma_{k-1}^j}^j\right] \geqslant c$$

($\mathcal{F}_T^j$ is the usual σ-field for a stopping time T for the filtration $\mathcal{F}_s^j$) and so

$$P\left[\sigma_k^j \leqslant 1 + \frac{i_j + 1}{2^n}\right] \leqslant (1-c)^k.$$

Let $M_j^i = \sup\{k : \sigma_k^j \leqslant 1 + \frac{i_j+1}{2^n}\}$ then the range of $\underline{X}$ on $H_{\underline{i}}^n$ is contained in

$$\bigcup_{k_1=0}^{M_1^i} \bigcup_{k_2=0}^{M_2^i} \cdots \bigcup_{k_N=0}^{M_n^i} A(k_1, k_2, \ldots, k_N),$$

where

$$A(k_1, k_2, \ldots, k_N) = \left\{\underline{x}_1 + \underline{x}_2 +, \cdots \underline{x}_N : \underline{x}_j \in B(\underline{X}_j(\sigma_{k_j}^j), 2^{-n/\alpha}), j = 1, 2, \ldots, N\right\},$$

since the Lebesgue measure of $A(k_1, k_2, \ldots, k_N)$ is bounded by $C_d N^d 2^{-dn/\alpha}$ we easily have

$$R_{\underline{i}}^n \leqslant C_d N^d 2^{-dn/\alpha}(M_1^i + 1)(M_2^i + 1)\ldots(M_N^i + 1)$$

and so, given this bound

$$P\left(R_{\underline{i}}^n > c_{d,N}\left(2^{-dn/\alpha}\right)(k+1)^N\right) \leqslant N(1-c)^k$$

from which the result follows.   □

In a similar way we can show for $D_{\underline{i}}^n =$ diameter of range of $\underline{X}$ on $H_{\underline{i}}^n$.

Lemma 4.2.   There exists finite, positive C, c so that for all $n, \underline{i}$,

$$P(R_{\underline{i}}^n \geqslant k 2^{-nd/\alpha}, D_{\underline{i}}^n \geqslant v 2^{-n/\alpha}) \leqslant C e^{-ck^{1/N}} \frac{1}{v^\alpha}.$$

Proof of Proposition 1.2. We must show that there is an "economical" covering of the 0 level set on $[1, 2]^N$.

Recall the cube $H_{\underline{i}}^n$ is $\{\underline{t} : t_j \in [1 + \frac{i_j}{2^n}, 1 + \frac{i_j + 1}{2^n})\}$. We consider various cases :

(i) $\{\underline{X}(\underline{t}) : \underline{t} \in H_{\underline{i}}^n\} \cap \{\underline{0}\} = \emptyset$ (in this case the cube $H_{\underline{i}}^n$ does not intersect the level set);

(ii) $\{\underline{0}\} \subset \{\underline{X}(\underline{t}) : \underline{t} \in H_{\underline{i}}^n\}$, $R_{\underline{i}}^n \geqslant n^2 2^{-dn/\alpha}$;

(iii) $\underline{0} \in \{\underline{X}(\underline{t}) : \underline{t} \in H_{\underline{i}}^n\}$, $R_{\underline{i}}^n \leqslant n^2 2^{-dn/\alpha}$; $D_{\underline{i}}^n \geqslant n^{\frac{1}{2d}} 2^{-n/\alpha}$;

(iv) $\underline{0} \in \{\underline{X}(\underline{t}) : \underline{t} \in H_{\underline{i}}^n\}$, $R_{\underline{i}}^n \leqslant n^2 2^{-n/\alpha}$, $D_{\underline{i}}^n < n^{\frac{1}{2d}} 2^{-n/\alpha}$ but there does not exist $n^2 2^{-n} \leqslant s \leqslant 2^{-n/2}$ so that

$$L\left(\Pi_{j=1}^N \left[1 + \frac{i_j}{2^{n\alpha}}, 1 + \frac{i_j}{2^{n\alpha}} + s\right]\right) \leqslant \frac{h}{2} \, s^{N - d/\alpha} \left(\log\log\left(\frac{1}{s}\right)\right)^{d/\alpha};$$

where h is small as in Proposition 3.4,

(v)   all other cases.

Then we have by Lemma 4.1 that the probability that for cube $H_{\underline{i}}^n$ (ii) occurs is bounded by

$$\sum_{k=0}^{\infty} P(\underline{X}(\underline{t}) = \underline{0} \text{ for some } \underline{t} \in H_{\underline{i}}^n | R_{\underline{i}}^n \in [2^k n^2 2^{-dn/\alpha}, 2^{k+1} n^2 2^{-dn/\alpha})$$
$$\times P(R_{\underline{i}}^n \in [2^k n^2 2^{-dn/\alpha}, 2^{k+1} n^2 2^{-dn/\alpha}))$$
$$\leqslant \sum_{k=0}^{\infty} P\left(R_{\underline{i}}^n \geqslant 2^k n^2 2^{-dn/\alpha}\right) P\left(\underline{X}(\underline{t}) = \underline{0} \text{ for } \underline{t} \in H_{\underline{i}}^n | R_{\underline{i}}^n \in [2^k n^2 2^{-dn/\alpha}, \right.$$
$$\left. 2^{k+1} n^2 2^{-dn/\alpha})\right)$$
$$\leqslant \sum_{k=0}^{\infty} C e^{-c 2^{k/N} n^{2/N}} 2^{k+1} n^2 2^{-dn/\alpha}$$
$$\leqslant C e^{-c n^{2/N}} n^2 2^{-dn/\alpha} \leqslant C e^{-c n^{2/N}} 2^{-nd/\alpha} \text{ for } c, \, C \text{ not depending on } n, \underline{i}.$$

Here the inequality $P(\underline{X}(\underline{t}) = \underline{0} \text{ for some } \underline{t} \in H_{\underline{i}}^n | R_{\underline{i}}^n \in [2^k n^2 2^{-dn/\alpha}, 2^{k+1} n^2 2^{-dn/\alpha}) \leqslant C 2^k n^2 2^{-dn/\alpha}$ follows from the fact that the random variable $R_{\underline{i}}^n$ is independant of $\underline{X}(1 + \frac{\underline{i}}{2^n})$ which has density bounded by a constant.

For case (iii) we have, similarly, by Lemma 4.2 that the probability (uniformly on $H_{\underline{i}}^n$) that (iii) holds is bounded by

$$\sum_{k=0}^{2\log_2(n)}\Bigg(P(\underline{X}(\underline{t})=\underline{0} \text{ for some } \underline{t}\in H_{\underline{i}}^n | R_{\underline{i}}^n \in [(2^k-1)2^{-dn/\alpha}, 2^{k+1}2^{-dn/\alpha}),$$

$$D_{\underline{i}}^n > 2^{-n/\alpha}n^{1/2d}\Bigg)$$

$$\times P(R_{\underline{i}}^n \in [(2^k-1)2^{-dn/\alpha}, 2^{k+1}2^{-dn/\alpha}), D_{\underline{i}}^n > 2^{-n/\alpha}n^{1/2d})\Bigg)$$

$$\leqslant \sum_{k=0}^{2\log_2(n)} C'2^{k+1}2^{-dn/\alpha}e^{-c2^{k/N}n^{2/N}}/n^{\alpha/2d},$$

which is less than $C2^{-nd/\alpha}\frac{1}{n^{\alpha/2d}}$ for some C not depending on n or $\underline{i}$.

We have by Proposition 3.4, that the probability that $H_{\underline{i}}^n$ belongs to case (iv) is bounded by $C2^{-nd/\alpha}/n^{1/2}$.

We now define the cubes $V_{\underline{i}}^n$ by

$= \emptyset$ if (i) holds

$= H_{\underline{i}}^n$ if (ii), (iii) or (iv) hold. This collection of $\underline{i}$ will be denoted B^n,

$= \left[1+\dfrac{i_1}{2^n}, 1+\dfrac{i_1}{2^n}+s\right]\times\left[1+\dfrac{i_2}{2^n}, 1+\dfrac{i_2}{2^n}+s\right]\times\cdots\left[1+\dfrac{i_N}{2^n}, 1+\dfrac{i_N}{2^n}+s\right]$ otherwise ,

where s is the smallest element of $[n^22^{-n}, 2^{-n/2}]$ so that

$$L\left(\Pi_{j=1}^N\left[1+\frac{i_j}{2^{n\alpha}}, 1+\frac{i_j}{2^{n\alpha}}+s\right]\right)\geqslant \frac{h}{2}\ s^{N-d/\alpha}\left(\log\log\left(\frac{1}{s}\right)\right)^{d/\alpha}.$$

Let J^n be the (random) collection of $\underline{i}$ so that for cube $H_{\underline{i}}^n$ condition (v) holds. For $i\in J^n$, the cube $E_{\underline{i}}^n$ will be the cube with the same centre as $V_{\underline{i}}^n$ but with a side length five times as large. Then by the usual Vitelli covering argument, we have that there exists $I^n\subset J^n$ so that

(i)   for distinct i, j in I^n, $V_{\underline{i}}^n\cap V_{\underline{j}}^n=\emptyset$;

(ii) $\cup_{i\in J^n}V_{\underline{i}}^n\subset\cup_{i\in I^n}E_{\underline{i}}^n$.

Now the 0 level set in $[1, 2]^N$ is contained in

$$\cup_{i\in B^n}V_{\underline{i}}^n\ \cup\ \cup_{i\in I^n}E_{\underline{i}}^n.$$

For a constant K not depending on n we have

$$E\left[\sum_{i\in B^n}\phi(|D_{\underline{i}}^n|)\right]\leqslant K2^{Nn}\frac{2^{-nd/\alpha}}{n^{1/2}\wedge n^{\alpha/2d}}2^{-n(N-d/\alpha)}\log(n)\to 0$$

as $n \to \infty$. On the other hand we have that for any strictly positive ϵ,

$$\sum_{i \in I^n} \phi(|E_{\underline{i}}^n|) \leqslant 5^{N-d/\alpha} \sum_{i \in I^n} \phi(|V_{\underline{i}}^n|)$$

$$\leqslant \frac{2.5^{N-d/\alpha}}{h} \sum_{i \in I^n} L(V_{\underline{i}}^n) \leqslant \frac{2.5^{N-d/\alpha}}{h} L\left(\left[1-\epsilon, 2+\epsilon\right]^N\right)$$

(Where the second inequality is derived from the disjointness of the $V_{\underline{i}}^n$.) By continuity of the local time on intervals this latter term, for ϵ sufficiently small will be less than or equal to $\frac{2.5^{N-d/\alpha}}{h} L\left(\left[1-\epsilon, 2+\epsilon\right]^N\right)$ if the local time is strictly positive and less than any desired psoitive constant otherwise. Thus, by Fatou's Lemma, Proposition 1.2 holds.

4.1. Extensions

If stable processes which are not necessarily strict are considered, then things become a little more complicated. The arguments of Ref. 12 will still go through if drift is added to the processes. However for our arguments to essentially still go through, we would have to consider $\alpha \geqslant 1$. In $\alpha = 1$, there is still an appropiate scaling relation. In dealing with $\alpha > 1$ we have that in arriving at Proposition 2.1 one cannot use scaling as before and in fact there is not a set A so that to achieve a truncated process $Y^v(.)$ of mean zero one "unsuppresses" the jumps in vA but in fact as the spatial scale v tends to zero the perturbation required to deal with the extra drift term has an effect tending to zero. The calculations proceed after this much as in the strict stability case. If, however, the parameter α is less than one then one can directly employ the arguments of Section 2 to arrive at a process $X^v(.)$ of zero mean: the drift term locally dominates and it is not possible simply to alter the jump rates for jumps of order v to create a process of zero mean. In this case, we not only cannot show results analogous to Theorem 1.1, we suspect that they may no longer pertain: if the (uni-dimensional time)process has a strong tendency to a given direction then the creation of a large local time will be a more complex event than simply independent recurrent behaviour for the uni-dimensional processes.

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