Aequat. Math. 85 (2013), 449–463

0001-9054/13/030449-15
published online August 21, 2012
DOI 10.1007/s00010-012-0153-y

Aequationes Mathematicae

Volume of a doubly truncated hyperbolic tetrahedron

Alexander Kolpakov and Jun Murakami

Abstract. The present paper regards the volume function of a doubly truncated hyperbolic tetrahedron. Starting from the earlier results of J. Murakami, U. Yano and A. Ushijima, we have developed a unified approach to express the volume in different geometric cases by dilogarithm functions and to treat properly the many analytic strata of the latter. Finally, several numeric examples are given.

Mathematics Subject Classification. Primary 51M25; Secondary 51M09.

Keywords. Hyperbolic tetrahedron, Gram matrix, volume formula.

1. Introduction

The real vector space $\mathbb{R}^{1,n}$ of dimension $n+1$ with the Lorentzian inner product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n$, where $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$, is called an $(n+1)$-dimensional Lorentzian space $\mathbb{E}^{1,n}$.

Consider the twofold hyperboloid $\mathscr{H} = \{x \in \mathbb{E}^{1,n} | \langle x, x\rangle = -1\}$ and its upper sheet $\mathscr{H}^+ = \{x \in \mathbb{E}^{1,n} | \langle x, x\rangle = -1, x_0 > 0\}$. The restriction of the quadratic form induced by the Lorentzian inner product $\langle \circ, \circ\rangle$ to the tangent space to $\mathscr{H}^+$ is positive definite, and so it gives a Riemannian metric on $\mathscr{H}^+$. The space $\mathscr{H}^+$ equipped with this metric is called *the hyperboloid model* of the n-dimensional hyperbolic space and denoted by $\mathbb{H}^n$. The hyperbolic distance $d(x, y)$ between two points x and y with respect to this metric is given by the formula $\cosh d = -\langle x, y\rangle$.

Consider the cone $\mathscr{K} = \{x \in \mathbb{E}^{1,n} | \langle x, x\rangle = 0\}$ and its upper half $\mathscr{K}^+ = \{x \in \mathbb{E}^{1,n} | \langle x, x\rangle = 0, x_0 > 0\}$. A ray in $\mathscr{K}^+$ emanating from the origin corresponds to a point on the ideal boundary of $\mathbb{H}^n$. The set of such rays forms a sphere at infinity $\mathbb{S}^{n-1}_\infty \cong \partial\mathbb{H}^n$. Thus, each ray in $\mathscr{K}^+$ becomes an ideal point of $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$.

A. Kolpakov was supported by the Schweizerischer National fonds SNF no. 200021-131967/1.
J. Murakami was supported by Grant-in-Aid for Scientific Research no. 22540236.

Let p denote the radial projection of $\mathbb{E}^{1,n} \setminus \{x \in \mathbb{E}^{1,n} | x_0 = 0\}$ onto the affine hyperplane $\mathbb{P}^n_1 = \{x \in \mathbb{E}^{1,n} | x_0 = 1\}$ along a ray emanating from the origin $\mathbf{o}$. The projection p is a diffeomorphism of $\mathbb{H}^n$ onto the open n-dimensional unit ball $\mathbb{B}^n$ in $\mathbb{P}^n_1$ centred at $(1, 0, 0, \ldots, 0)$ which defines a projective model of $\mathbb{H}^n$. The affine hyperplane $\mathbb{P}^n_1$ includes not only $\mathbb{B}^n$ and its set-theoretic boundary $\partial\mathbb{B}^n$ in $\mathbb{P}^n_1$, which is canonically identified with $\mathbb{S}^{n-1}_\infty$, but also the exterior of *the compactified projective model* $\overline{\mathbb{B}^n} = \mathbb{B}^n \cup \partial\mathbb{B}^n \cong \mathbb{H}^n \cup \mathbb{S}^{n-1}_\infty$. Let $\mathrm{Ext}\,\mathbb{B}^n$ denote the exterior of $\overline{\mathbb{B}^n}$ in $\mathbb{P}^n$. Thus p could be naturally extended to a map from $\mathbb{E}^{1,n} \setminus \mathbf{o}$ onto an n-dimensional real projective space $\mathbb{P}^n = \mathbb{P}^n_1 \cup \mathbb{P}^n_\infty$, where $\mathbb{P}^n_\infty$ is the set of straight lines in the affine hyperplane $\{x \in \mathbb{E}^{1,n} | x_0 = 0\}$ passing through the origin.

Consider the one-fold hyperboloid $\mathscr{H}_\star = \{x \in \mathbb{E}^{1,n} | \langle x, x\rangle = 1\}$. Given some point u in $\mathscr{H}_\star$ define in $\mathbb{E}^{1,n}$ the half-space $\mathrm{R}_u = \{x \in \mathbb{E}^{1,n} | \langle x, u\rangle \leq 0\}$ and the hyperplane $\mathrm{P}_u = \{x \in \mathbb{E}^{1,n} | \langle x, u\rangle = 0\} = \partial\mathrm{R}_u$. Denote by Γ_u (respectively, Π_u) the intersection of R_u (respectively, P_u) with $\mathbb{B}^n$. Then Π_u is a geodesic hyperplane in $\mathbb{H}^n$, and the correspondence between the points in $\mathcal{H}_\star$ and the half-space Γ_u in $\mathbb{H}^n$ is bijective. Call the vector u normal to the hyperplane P_u (or Π_u).

Let v be a point in $\mathrm{Ext}\,\mathbb{B}^n$. Then $p^{-1}(v) \cap \mathscr{H}_\star$ consists of two points. Let $\tilde{v}$ denote one of them, so we may define *the polar hyperplane* $\Pi_{\tilde{v}}$ to v, independent of the choice of $\tilde{v} \in p^{-1}(v) \cap \mathscr{H}_\star$.

Now we descend to dimension $n = 3$. Let T be a tetrahedron in $\mathbb{P}^3_1$, that is a convex hull of four points $\{v_i\}_{i=1}^4 \subset \mathbb{P}^3_1$. We say a vertex $v \in \{v_i\}_{i=1}^4$ to be *proper* if $v \in \mathbb{B}^3$, to be *ideal* if $v \in \partial\mathbb{B}^3$ and to be *ultra-ideal* if $v \in \mathrm{Ext}\,\mathbb{B}^3$.

Let v be an ultra-ideal vertex of T. We call *a truncation of* v the operation of removing the pyramid with apex v and base $\Pi_{\tilde{v}} \cap T$. A *generalised* hyperbolic tetrahedron T is a polyhedron, possibly non-compact, of finite volume in the hyperbolic space obtained from a certain tetrahedron by polar truncation of its ultra-ideal vertices. In the case when only two vertices are truncated, we call such a generalised tetrahedron *doubly truncated*.

Depending on the dihedral angles, the polar hyperplanes may or may not intersect. In Fig. 1 a tetrahedron T with two truncated vertices is depicted. The corresponding polar planes do not intersect. We shall call this kind of generalised tetrahedron *mildly truncated*. In Fig. 2 the case when the polar planes intersect is shown. Here the tetrahedron T is truncated down to a prism, hence we shall call it *prism truncated*.

In this paper we study the volume function of a doubly truncated hyperbolic tetrahedron. The question arises first in the paper by Kellerhals [4], where a doubly truncated hyperbolic orthoscheme is considered. The whole evolution of an orthoscheme, starting from a mildly truncated one down to a Lambert cube has been investigated.

The case of a general hyperbolic tetrahedron was considered in numerous papers [1,3,8]. The case of a mildly truncated tetrahedron is due to Murakami

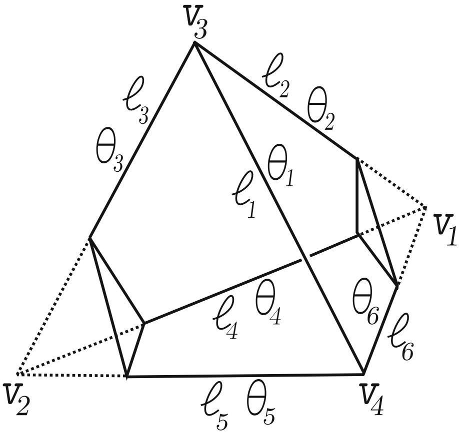

FIGURE 1. Mildly (doubly) truncated tetrahedron

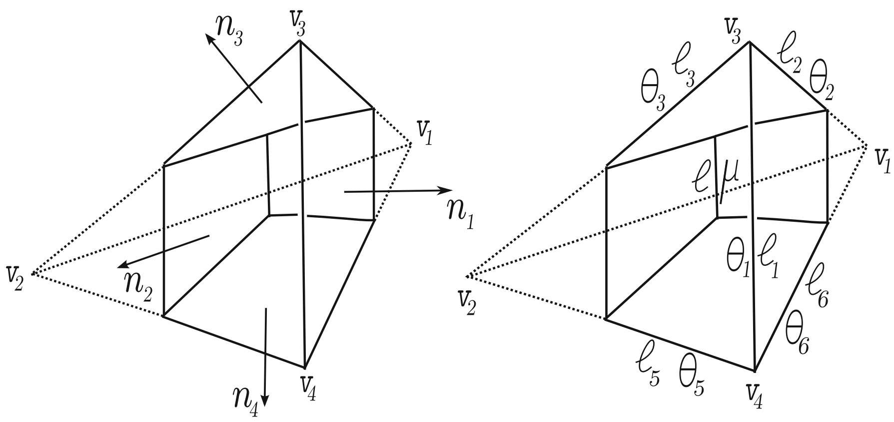

FIGURE 2. Prism (intensely) truncated tetrahedron T and its geometric parameters

and Ushijima [9,13]. The paper [13] is the starting point where the question about intense truncations of a hyperbolic tetrahedron was posed. Thus, the case of a prism truncated tetrahedron remains unattended. As we shall see later, it brings some essential difficulties. First, the structure of the volume formula should change, as first observed in [4]. Second, the branching properties of the volume function come into sight. This phenomenon was first observed for tetrahedra in the spherical space and is usually related to the use of the dilogarithm function or its analogues, see [5,9,10].

For the rest of the paper, a mildly (doubly) truncated tetrahedron is given in Fig. 1. Its dihedral angles are θ_k and its corresponding edge lengths are $\ell_k, k = \overline{1,6}$.[1]

A prism truncated tetrahedron is given in Fig. 2. The dashed edge connects the ultra-ideal vertices and corresponds to edge ℓ_4 in the previous case. The dihedral angles remain the same, except that the altitude ℓ of the prism replaces the dihedral angle θ_4. The altitude carries the dihedral angle μ and is orthogonal to the bases because of the truncation. Right dihedral angles in Figs. 1 and 2 are not indicated by symbols. The other dihedral angles and the corresponding edge length are called *essential*.

2. Preliminaries

The following propositions reveal a relationship between the Lorentzian inner product of two vectors and the mutual position of the respective hyperplanes.

Proposition 1. *Let u and v be two non-collinear points in $\mathscr{H}_\star$. Then the following holds:*

(i) *The hyperplanes Π_u and Π_v intersect if and only if $|\langle u, v\rangle| < 1$. The dihedral angle θ between them measured in $\Gamma_u \cap \Gamma_v$ is given by the formula $\cos\theta = -\langle u, v\rangle$.*

(ii) *The hyperplanes Π_u and Π_v do not intersect in $\overline{\mathbb{B}^n}$ if and only if $|\langle u, v\rangle| > 1$. They intersect in $\operatorname{Ext}\mathbb{B}^n$ and admit a common perpendicular inside $\mathbb{B}^n$ of length d given by the formula $\cosh d = \pm\langle u, v\rangle$.*[2]

(iii) *The hyperplanes Π_u and Π_v intersect on the ideal boundary $\partial\mathbb{B}^n$ only, if and only if $|\langle u, v\rangle| = 1$.*

In case (ii) *we say the hyperplanes Π_u and Π_v are ultra-parallel and in case* (iii) *they are parallel.*

Proposition 2. *Let u be a point in $\mathbb{B}^n$ and let Π_v be a geodesic hyperplane whose normal vector $v \in \mathscr{H}_\star$ is such that $\langle u, v\rangle < 0$. Then the length d of the perpendicular dropped from the point u onto the hyperplane Π_v is given by the formula $\sinh d = -\langle u, v\rangle$.*

Let T be a generalised tetrahedron in $\mathbb{H}^3$ with outward Lorentzian normals $n_i, i = \overline{1,4}$, to its faces and vertex vectors $v_i, i = \overline{1,4}$, as depicted in Fig. 2. Let G denote the Gram matrix for the normals $G = \langle n_i, n_j\rangle_{i,j=1}^4$.

The conditions under which G describes a generalised mildly truncated tetrahedron are given by [6] and [13]. For a prism truncated tetrahedron, its existence and geometry are determined by the vectors $\tilde{n}_i, i = \overline{1,6}$, where $\tilde{n}_i = n_i$

[1] For the given integers $n \geq 1, m \geq 0$, the notation $k = \overline{n, n+m}$ means that $k \in \{n, n+1, \ldots, n+m\}$.

[2] We chose the minus sign if $\Gamma_u \cap \Gamma_v = \emptyset$, otherwise we choose the plus sign.

for $i = \overline{1,4}$ and $\tilde{n}_5 = v_1, \tilde{n}_6 = v_2$. Hence the above matrix G does not necessarily have a Lorentzian signature. However, it will suffice for our purpose.

In case T is prism truncated, as Fig. 2 shows, we obtain

$$G = \begin{pmatrix} 1 & -\cos\theta_1 & -\cos\theta_2 & -\cos\theta_6 \\ -\cos\theta_1 & 1 & -\cos\theta_3 & -\cos\theta_5 \\ -\cos\theta_2 & -\cos\theta_3 & 1 & -\cosh\ell \\ -\cos\theta_6 & -\cos\theta_5 & -\cosh\ell & 1 \end{pmatrix}, \tag{1}$$

in accordance with Proposition 1.

We will *define* the edge length matrix $G^\star$ of T by

$$G^\star = \begin{pmatrix} -1 & -\cos\mu & i\sinh\ell_2 & i\sinh\ell_6 \\ -\cos\mu & -1 & i\sinh\ell_3 & i\sinh\ell_5 \\ -i\sinh\ell_2 & -i\sinh\ell_6 & -1 & -\cosh\ell_1 \\ -i\sinh\ell_6 & -i\sinh\ell_5 & -\cosh\ell_1 & -1 \end{pmatrix} \tag{2}$$

in order to obtain a Hermitian analogue of the usual edge length matrix for a mildly truncated tetrahedron (for the latter, see [13]).

By [6] or [13], we have that

$$-g^\star_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}}\sqrt{c_{jj}}}, \tag{3}$$

where c_{ij} are the corresponding (i, j) cofactors of the matrix G. The complex conjugation $g^\star_{ij} = \overline{g^\star_{ji}}, i, j = \overline{1,4}$, corresponds to a choice of the analytic strata for the square root function $\sqrt{\circ}$.

Proposition 3 ([6,13]). *The vertex $v_i, i = \overline{1,4}$, of T is proper, ideal, or ultra-ideal provided that $c_{ii} > 0, c_{ii} = 0$, or $c_{ii} < 0$, respectively.*

Hence, Propositions 1–2 imply that the matrices G and $G^\star$ agree concerning the relationship of the geometric parameters of T. The matrix $G^\star$ has complex entries since the minors $c_{ii}, i = \overline{1,4}$, in formula (3) may have different signs by Proposition 3.

To perform our computations later on in a more efficient way, we shall introduce the parameters $a_k, k = \overline{1,6}$, associated with the edges of the tetrahedron T. If T is a prism truncated tetrahedron, then we set $a_k := e^{i\theta_k}, k \in \{1, 2, 3, 5, 6\}, a_4 := e^{\ell}$ and then

$$G = \begin{pmatrix} 1 & -\frac{a_1+1/a_1}{2} & -\frac{a_2+1/a_2}{2} & -\frac{a_6+1/a_6}{2} \\ -\frac{a_1+1/a_1}{2} & 1 & -\frac{a_3+1/a_3}{2} & -\frac{a_5+1/a_5}{2} \\ -\frac{a_2+1/a_2}{2} & -\frac{a_3+1/a_3}{2} & 1 & -\frac{a_4+1/a_4}{2} \\ -\frac{a_6+1/a_6}{2} & -\frac{a_5+1/a_5}{2} & -\frac{a_4+1/a_4}{2} & 1 \end{pmatrix}. \tag{4}$$

The meaning of the parameters becomes clear if one observes the picture of a mildly truncated tetrahedron T (see Fig. 2). If two vertices v_1 and v_2 of T

become ultra-ideal and the corresponding polar hyperplanes intersect (see [13, Sections 2–3] for more basic details), the edge v_1v_2 becomes dual in a sense to the altitude ℓ of the resulting prism. Since in the case of a mildly truncated tetrahedron we have $a_k = e^{i\theta_k}$ according to [13], the altitude ℓ for now still corresponds to the parameter a_4, in order to keep consistent notation.

3. Volume formula

Let $\mathscr{U} = \mathscr{U}(a_1, a_2, a_3, a_4, a_5, a_6, z)$ denote the function

$$\begin{aligned}\mathscr{U} &= \mathrm{Li}_2(z) + \mathrm{Li}_2(a_1a_2a_4a_5z) + \mathrm{Li}_2(a_1a_3a_4a_6z) + \mathrm{Li}_2(a_2a_3a_5a_6z)\\ &\quad -\mathrm{Li}_2(-a_1a_2a_3z) - \mathrm{Li}_2(-a_1a_5a_6z) - \mathrm{Li}_2(-a_2a_4a_6z) - \mathrm{Li}_2(-a_3a_4a_5z) \end{aligned} \tag{5}$$

depending on seven complex variables $a_k, k = \overline{1,6}$ and z, where $\mathrm{Li}_2(\circ)$ is the dilogarithm function. Let z_- and z_+ be two solutions to the equation $e^{z\frac{\partial \mathscr{U}}{\partial z}} = 1$ in the variable z. According to [8], these are

$$z_- = \frac{-q_1 - \sqrt{q_1^2 - 4q_0q_2}}{2q_2} \quad \text{and} \quad z_+ = \frac{-q_1 + \sqrt{q_1^2 - 4q_0q_2}}{2q_2}, \tag{6}$$

where

$$\begin{aligned} q_0 &= 1 + a_1a_2a_3 + a_1a_5a_6 + a_2a_4a_6 + a_3a_4a_5 + a_1a_2a_4a_5\\ &\quad + a_1a_3a_4a_6 + a_2a_3a_5a_6,\\ q_1 &= -a_1a_2a_3a_4a_5a_6\left(\left(a_1 - \frac{1}{a_1}\right)\left(a_4 - \frac{1}{a_4}\right) + \left(a_2 - \frac{1}{a_2}\right)\left(a_5 - \frac{1}{a_5}\right)\right.\\ &\quad \left. + \left(a_3 - \frac{1}{a_3}\right)\left(a_6 - \frac{1}{a_6}\right)\right),\\ q_2 &= a_1a_2a_3a_4a_5a_6(a_1a_4 + a_2a_5 + a_3a_6 + a_1a_2a_6 + a_1a_3a_5 + a_2a_3a_4\\ &\quad + a_4a_5a_6 + a_1a_2a_3a_4a_5a_6). \end{aligned} \tag{7}$$

Given a function $f(x, y, \ldots, z)$, let $f(x, y, \ldots, z)\,|_{z=z_+}^{z=z_-}$ denote the difference $f(x, y, \ldots, z_-) - f(x, y, \ldots, z_+)$. Now we define the following function $\mathscr{V} = \mathscr{V}(a_1, a_2, a_3, a_4, a_5, a_6, z)$ by means of the equality

$$\mathscr{V} = \frac{i}{4}\left(\mathscr{U}(a_1, a_2, a_3, a_4, a_5, a_6, z) - z\,\frac{\partial \mathscr{U}}{\partial z}\,\log z\right)\Bigg|_{z=z_+}^{z=z_-}. \tag{8}$$

Let $\mathscr{W} = \mathscr{W}(a_1, a_2, a_3, a_4, a_5, a_6, z)$ denote the function below, which will correct the possible branching of $\mathscr{V}$ resulting from the use of (di-)logarithms:

$$\mathscr{W} = \sum_{k=1}^{6}\left(a_k\,\frac{\partial \mathscr{V}}{\partial a_k} - \frac{i}{4}\,\log e^{-4i\,a_k\,\frac{\partial \mathscr{V}}{\partial a_k}}\right)\log a_k. \tag{9}$$

Given a generalised hyperbolic tetrahedron as in Fig. 2, truncated down to a quadrilateral prism with essential dihedral angles $\theta_k, k \in \{1, 2, 3, 5, 6\}$, altitude

ℓ and dihedral angle μ along it, we set $a_k = e^{i\theta_k}, k \in \{1, 2, 3, 5, 6\}, a_4 = e^{\ell}$, as above. Then the following theorem holds.

Theorem 1. *Let T be a generalised hyperbolic tetrahedron as given in* Fig. 2*. Its volume equals*

$$\mathrm{Vol}\, T = \Re\left(-\mathscr{V} + \mathscr{W} - \frac{\mu\ell}{2}\right).$$

Note. In the statement above, the altitude length is $\ell = \Re \log a_4$ and the corresponding dihedral angle equals $\mu = -2\,\Re(a_4 \frac{\partial \mathscr{V}}{\partial a_4}) \bmod \pi$.

3.1. Preceding lemmas

Before giving a proof to Theorem 1, we need several auxiliary statements concerning the branching of the volume function.

Lemma 1. *The function $\mathscr{W}$ has a.e. vanishing derivatives $\frac{\partial \mathscr{W}}{\partial a_k}, k = \overline{1,6}$.*

Proof. Compute the derivative of (9) with respect to each $a_k, k = \overline{1,6}$, outside of its branching points and use the identities $\frac{\mathrm{d}}{\mathrm{d}z} \log z = 1/z, e^z e^w = e^{z+w}$ for all $z, w \in \mathbb{C}$. Since the branching points of a finite amount of $\log(\circ)$ and $\mathrm{Li}_2(\circ)$ functions form a discrete set in $\mathbb{C}$, the lemma follows. □

Lemma 2. *The function $\Re\,(-\mathscr{V} + \mathscr{W})$ does not branch with respect to the variables $a_k, k = \overline{1,6}$, and z.*

Proof. Let us consider a possible branching of the function defined by formula (5). Let $\mathscr{U}$ comprise only of principal strata of the dilogarithm and let $\mathscr{U}^\star$ correspond to other ones. Then we have

$$\begin{aligned}\mathscr{U}\,|_{z=z_\pm} = \mathscr{U}^\star\,|_{z=z_\pm} &+2\pi i k_0^\pm \log(z_\pm) + 2\pi i k_1^\pm \log(a_1a_2a_4a_5z_\pm)\\ &+2\pi i k_2^\pm \log(a_1a_3a_4a_6z_\pm) + 2\pi i k_3^\pm \log(a_2a_3a_5a_6z_\pm)\\ &+2\pi i k_4^\pm \log(-a_1a_2a_3z_\pm) + 2\pi i k_5^\pm \log(-a_1a_5a_6z_\pm)\\ &+2\pi i k_6^\pm \log(-a_2a_4a_6z_\pm) + 2\pi i k_7^\pm \log(-a_3a_4a_5z_\pm)\\ &+4\pi^2 k_8^\pm,\end{aligned} \tag{10}$$

with some $k_j \in \mathbb{Z}, j = \overline{0,8}$. From the above formula, it follows that

$$z_\pm \frac{\partial \mathscr{U}}{\partial z}\bigg|_{z=z_\pm} \log z_\pm = z_\pm \frac{\partial \mathscr{U}^\star}{\partial z}\bigg|_{z=z_\pm} \log z_\pm + 2\pi i \sum_{j=0}^{7} k_j^\pm \log z_\pm. \tag{11}$$

Then, according to formulas (8), (10), (11), the following expression holds for the corresponding analytic strata of the function $\mathscr{V}$:

$$\mathscr{V} = \mathscr{V}^\star - \frac{\pi}{2}\sum_{j=1}^{6} m_j \log a_j + \frac{i\pi^2}{2} m_7, \tag{12}$$

where $m_j \in \mathbb{Z}, j = \overline{1,7}$ and we have used the formula $\log(uv) = \log u + \log v + 2\pi i k, k \in \mathbb{Z}$. Hence, according to (12), we compute

$$a_j \frac{\partial \mathscr{V}}{\partial a_j} = a_j \frac{\partial \mathscr{V}^\star}{\partial a_j} - \frac{\pi m_j}{2},$$

for each $j = \overline{1,6}$. The latter implies that

$$a_j \frac{\partial \mathscr{V}}{\partial a_j} - \frac{i}{4} \log\, e^{-4i\, a_j\, \frac{\partial \mathscr{V}}{\partial a_j}} = a_j \frac{\partial \mathscr{V}^\star}{\partial a_j} - \frac{i}{4} \log\, e^{-4i\, a_j\, \frac{\partial \mathscr{V}^\star}{\partial a_j}} - \frac{\pi m_j}{2},$$

for $j = \overline{1,6}$, since we choose the principal stratum of the logarithm function $\log(\circ)$. Thus, by formula (9),

$$\begin{aligned} -\mathscr{V} + \mathscr{W} &= -\mathscr{V}^\star + \mathscr{W}^\star + \frac{\pi}{2} \sum_{j=1}^{6} m_j \log a_j - \frac{i\pi^2}{2} m_7 - \frac{\pi}{2} \sum_{j=1}^{6} m_j \log a_j \\ &= -\mathscr{V}^\star + \mathscr{W}^\star - \frac{i\pi^2}{2} m_7, \end{aligned}$$

with $m_j \in \mathbb{Z}, j = \overline{1,7}$. The proof is completed. $\square$

3.2. Proof of Theorem 1

The scheme of our proof is the following: first we show that $\frac{\partial}{\partial \theta_k} \mathrm{Vol}\, T = -\frac{\ell_k}{2}, k \in \{1,2,3,5,6\}, \frac{\partial}{\partial \mu} \mathrm{Vol}\, T = -\frac{\ell}{2}$ and second we apply the generalised Schläfli formula [7, Equation 1] to show that the volume function and the one from Theorem 1 coincide up to a constant. Finally, the remaining constant is determined.

Now, let us prove the three statements below:

(i) $\frac{\partial}{\partial \theta_1} \mathrm{Vol}\, T = -\frac{\ell_1}{2}$,

(ii) $\frac{\partial}{\partial \theta_k} \mathrm{Vol}\, T = -\frac{\ell_k}{2}$ for $k \in \{2,3,5,6\}$,

(iii) $\frac{\partial}{\partial \mu} \mathrm{Vol}\, T = -\frac{\ell}{2}$.

Note that in case (ii) it suffices to show $\frac{\partial}{\partial \theta_2} \mathrm{Vol}\, T = -\frac{\ell_2}{2}$. The statement for another $k \in \{3,5,6\}$ is completely analogous.

Let us show that the equality in case (i) holds. First, we compute

$$\begin{aligned} \frac{\partial}{\partial \theta_1} \left(\mathscr{U} - z \frac{\partial \mathscr{U}}{\partial z} \log z \right) &= i\, a_1 \frac{\partial}{\partial a_1} \left(\mathscr{U} - z \frac{\partial \mathscr{U}}{\partial z} \log z \right) \\ &= i a_1 \left(\frac{\partial \mathscr{U}}{\partial a_1} - \log z \frac{\partial}{\partial a_1} \left(z \frac{\partial \mathscr{U}}{\partial z} \right) \right) \end{aligned}$$

Upon the substitution $z := z_\pm$, we see that

$$\frac{\partial}{\partial a_1} \left(z_\pm \frac{\partial \mathscr{U}}{\partial z} (a_1, a_2, a_3, a_4, a_5, a_6, z_\pm) \right) = 0,$$

by taking the respective derivative on both sides of the identity

$$e^{z_\pm \frac{\partial \mathscr{U}}{\partial z}(a_1,a_2,a_3,a_4,a_5,a_6,z_\pm)} = 1,$$

c.f. the definition of $z_\pm$ and formula (6).

Finally, we get

$$\frac{\partial \mathscr{V}}{\partial \theta_1} = -\frac{a_1}{4}\frac{\partial \mathscr{U}}{\partial a_1}\bigg|_{z=z_+}^{z=z_-} = -\frac{1}{4}\log\left|\frac{\phi(z_-)\psi(z_+)}{\phi(z_+)\psi(z_-)}\right| + \frac{i\pi}{2}k,$$

for a certain $k \in \mathbb{Z}$, where the functions $\phi(\circ)$ and $\psi(\circ)$ are

$$\phi(z) = (1 + a_1a_2a_3z)(1 + a_1a_5a_6z),$$
$$\psi(z) = (1 - a_1a_2a_4a_5z)(1 - a_1a_3a_4a_6z).$$

The real part of the above expression is

$$\Re\frac{\partial \mathscr{V}}{\partial \theta_1} = -\frac{1}{4}\log\left|\frac{\phi(z_-)\psi(z_+)}{\phi(z_+)\psi(z_-)}\right|. \tag{13}$$

Let us set $\Delta = \det G$ and $\delta = \sqrt{\det G}$. We shall show that the expression

$$\mathscr{E} = \phi(z_-)\psi(z_+)\left(c_{34} - \delta\frac{a_1 - 1/a_1}{2}\right) - \phi(z_+)\psi(z_-)\left(c_{34} + \delta\frac{a_1 - 1/a_1}{2}\right) \tag{14}$$

is identically zero for all $a_k \in \mathbb{C}, k = \overline{1,6}$, which it actually depends on.

In order to perform the computation, the following formulas are used:

$$z_- = \frac{-\hat{q}_1 - 4\delta}{2\hat{q}_2}, \quad z_+ = \frac{-\hat{q}_1 + 4\delta}{2\hat{q}_2},$$

where $\hat{q}_l = q_l/\prod_{k=1}^{6} a_k$ for $l = 1, 2$ (cf. formulas (6)–(7)).

Note that one may consider the expression $\mathscr{E}$ as a rational function of independent variables $a_k, k = \overline{1,6}, \Delta$ and δ, making the computation easier to perform by a software routine [15]. First, we have

$$\frac{1}{\delta}\frac{\partial \mathscr{E}}{\partial a_1} = \frac{4a_1\mathscr{Y}}{\hat{q}_2^3}\frac{\partial \Delta}{\partial a_1},$$

where $\mathscr{Y} = \mathscr{Y}(a_1, a_2, a_3, a_4, a_5, a_6)$ is a certain technical term explained in Appendix. Since $\frac{\partial \Delta}{\partial a_1} = 2\delta\frac{\partial \delta}{\partial a_1}$, the above expression gives us

$$\frac{\partial \mathscr{E}}{\partial a_1} = \frac{8a_1\Delta}{\hat{q}_2^3}\frac{\partial \delta}{\partial a_1}\mathscr{Y}. \tag{15}$$

Second, we obtain

$$\frac{\partial \mathscr{E}}{\partial \delta} = -\frac{8a_1\Delta}{\hat{q}_2^3}\mathscr{Y}. \tag{16}$$

Finally, we recall that δ is a function of $a_k, k=\overline{1,6}$, and the total derivative of $\mathscr{E}$ with respect to a_1 is

$$\frac{\partial}{\partial a_1}\left(\mathscr{E}|_{\delta:=\delta(a_1,a_2,a_3,a_4,a_5,a_6)}\right)=\frac{\partial \mathscr{E}}{\partial a_1}+\frac{\partial \mathscr{E}}{\partial \delta}\frac{\partial \delta}{\partial a_1}=0,$$

according to equalities (15)–(16).

An analogous computation shows that $\frac{\partial}{\partial a_k}(\mathscr{E}|_{\delta:=\delta(a_1,a_2,a_3,a_4,a_5,a_6)})=0$ for all $k=\overline{1,6}$. Then by setting $a_k=1, k=\overline{1,6}$, we get $\Delta=-16$ and so $\delta=8i$. In this case $\mathscr{E}=0$.

Thus the equality $\mathscr{E}\equiv 0$ holds for all $a_k\in\mathbb{C}, k=\overline{1,6}$. Together with (13) it gives

$$\Re\frac{\partial \mathscr{V}}{\partial \theta_1}=-\frac{1}{4}\log\left|\frac{c_{34}+\delta\frac{a_1-a_1^{-1}}{2}}{c_{34}-\delta\frac{a_1-a_1^{-1}}{2}}\right|. \tag{17}$$

On the other hand, by formula (3), we have

$$g^\star_{34}=-\cosh\ell_1=\frac{-c_{34}}{\sqrt{c_{33}}\sqrt{c_{44}}}.$$

Here, both c_{33} and c_{44} are positive by Proposition 3, since the vertices v_3 and v_4 are proper. The formula above leads to the following equation

$$e^{2\ell_1}+2\,g^\star_{34}\,e^{\ell_1}+1=0,$$

the solution to which is determined by

$$e^{2\ell_1}=\frac{c_{34}+\delta\frac{a_1-a_1^{-1}}{2}}{c_{34}-\delta\frac{a_1-a_1^{-1}}{2}}, \tag{18}$$

in analogy to [13, Equation 5.3]. Thus, equalities (17)–(18) imply

$$\Re\frac{\partial \mathscr{V}}{\partial \theta_1}=-\frac{1}{4}\log e^{-2\ell_1}=\frac{\ell_1}{2}.$$

Together with Lemma 1, this implies that claim (i) is satisfied.

As already mentioned, in case (ii) it suffices to prove $\frac{\partial}{\partial\theta_2}\mathrm{Vol}\,T=-\frac{\ell_2}{2}$. The statement for another $k\in\{3,5,6\}$ is analogous.

By formula (3) we have that

$$g^\star_{24}=i\sinh\ell_2=\frac{-c_{24}}{\sqrt{c_{22}}\sqrt{c_{44}}}.$$

Since the vertex v_2 is ultra-ideal and the vertex v_4 is proper, by Proposition 3, $c_{22}<0$ and $c_{44}>0$. Thus,

$$\sinh\ell_2=\frac{c_{24}}{\sqrt{-c_{22}c_{44}}}.$$

The formula above implies

$$e^{2\ell_2} = -\frac{c_{24}^2 + 2c_{24}\sqrt{c_{24}^2 - c_{22}c_{44}} + c_{24}^2 - c_{22}c_{44}}{c_{22}c_{44}} \tag{19}$$

By applying Jacobi's theorem [11, Théorème 2.5.2] to the Gram matrix G, we have

$$c_{24}^2 - c_{22}c_{44} = \Delta\left(\frac{a_2 - a_2^{-1}}{2}\right)^2. \tag{20}$$

Combining (19)–(20) together, it follows that

$$-e^{2\ell_2} = \frac{c_{24} + \delta\frac{a_2 - a_2^{-1}}{2}}{c_{24} - \delta\frac{a_2 - a_2^{-1}}{2}}. \tag{21}$$

By analogy with (13), we get the formula

$$\Re\frac{\partial\mathscr{V}}{\partial\theta_2} = -\frac{1}{4}\log\left|\frac{\phi(z_-)\psi(z_+)}{\phi(z_+)\psi(z_-)}\right|, \tag{22}$$

where

$$\phi(z) = (1 + a_1a_2a_3z)(1 + a_2a_4a_6z),$$
$$\psi(z) = (1 - a_1a_2a_4a_5z)(1 - a_2a_3a_5a_6z).$$

Similar to case (i), the following relation holds:

$$\frac{\phi(z_-)\psi(z_+)}{\phi(z_+)\psi(z_-)} = \frac{c_{24} - \delta\frac{a_2 - a_2^{-1}}{2}}{c_{24} + \delta\frac{a_2 - a_2^{-1}}{2}}.$$

Then formulas (21)–(22) yield

$$\Re\frac{\partial\mathscr{V}}{\partial\theta_2} = -\frac{1}{4}\log\left|\frac{-1}{e^{2\ell_2}}\right| = \frac{\ell_2}{2}.$$

The first equality of case (ii) now follows. Carrying out an analogous computation for $\Re\frac{\partial\mathscr{V}}{\partial\theta_k}, k = 3, 5$, we obtain

$$\Re\frac{\partial\mathscr{V}}{\partial\theta_3} = \frac{\ell_3}{2} \quad \text{and} \quad \Re\frac{\partial\mathscr{V}}{\partial\theta_5} = \frac{\ell_5}{2}.$$

Thus, all equalities of case (ii) hold.

As before, by formula (3), we obtain that

$$-g_{12}^\star = \cos\mu = \frac{c_{12}}{\sqrt{c_{11}}\sqrt{c_{22}}}. \tag{23}$$

Since both vertices v_1 and v_2 are ultra-ideal, by Proposition 3, the cofactors c_{11} and c_{22} are negative. Then (23) implies the equation

$$e^{2i\mu} + \frac{2c_{12}e^{i\mu}}{\sqrt{c_{11}c_{22}}} + 1 = 0.$$

Without loss of generality, the solution we choose is

$$e^{i\mu} = \frac{-c_{12} + \sqrt{c_{12}^2 - c_{11}c_{22}}}{\sqrt{c_{11}c_{22}}}. \tag{24}$$

By squaring (24) and by applying Jacobi's theorem to the corresponding cofactors of the matrix G, the formula below follows:

$$e^{2i\mu} = \frac{c_{12} - \delta\frac{a_4 - a_4^{-1}}{2}}{c_{12} + \delta\frac{a_4 - a_4^{-1}}{2}}. \tag{25}$$

On the other hand, a direct computation shows that

$$\Re\frac{\partial \mathscr{V}}{\partial \ell} = \frac{i}{4}\,\log\left(\frac{\phi(z_-)\psi(z_+)}{\phi(z_+)\psi(z_-)}\right), \tag{26}$$

where

$$\phi(z) = (1 + a_3a_4a_5z)(1 + a_2a_4a_6z),$$
$$\psi(z) = (1 - a_1a_2a_4a_5z)(1 - a_1a_3a_4a_6z).$$

Similarly to cases (i) and (ii), we have the relation

$$\frac{\phi(z_-)\psi(z_+)}{\phi(z_+)\psi(z_-)} = \frac{c_{12} - \delta\frac{a_4 - a_4^{-1}}{2}}{c_{12} + \delta\frac{a_4 - a_4^{-1}}{2}},$$

which together with (25)–(26) yields

$$\frac{\partial \mathscr{V}}{\partial \ell} = \frac{i}{4}\log e^{2i\mu} = -\frac{\mu}{2} - \frac{\pi}{2}\,k, \quad k \in \mathbb{Z}.$$

Since, $0 \le \mu \le \pi$, we choose $k = 0$ and so $\frac{\partial \mathscr{V}}{\partial \ell} = -\frac{\mu}{2}$. The latter formula implies the equality of case (iii).

Now, by the generalised Schläfli formula [7],

$$\mathrm{dVol}\,T = -\frac{1}{2}\sum_{k\in\{1,2,3,5,6\}} \ell_k \mathrm{d}\theta_k - \frac{\ell}{2}\mathrm{d}\mu.$$

Together with the equalities of cases (i)–(iii) it yields

$$\mathrm{Vol}\,T = \Re\left(-\mathscr{V} + \mathscr{W} - \frac{\mu\ell}{2}\right) + \mathscr{C}, \tag{27}$$

where $\mathscr{C} \in \mathbb{R}$ is a constant.

Finally, we prove that $\mathscr{C} = 0$, and the theorem follows. Passing to the limit $\theta_k \to \frac{\pi}{2}, k = \overline{1,6}$, the generalised hyperbolic tetrahedron T shrinks to a point, since geometrically it tends to a Euclidean prism. Thus, we have $\mu \to \frac{\pi}{2}$ and $\ell \to 0$. By setting the limiting values above, we obtain that $a_k = i, k \in \{1,2,3,5,6\}, a_4 = 1$. Then $z_- = z_+ = 1$ by equality (6), and hence $\mathscr{V} = 0$. Since the dilogarithm function does not branch at $\pm 1, \pm i$, we have

Table 1. Some numerically computed volume values

$(\theta_1,\theta_2,\theta_3,\theta_5,\theta_6)$	(ℓ,μ)	Volume	Reference
$(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{2},\frac{\pi}{2})$	(0, 0)	0.5019205	[2]
$(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{2},\frac{\pi}{2})$	$(0.3164870, \frac{\pi}{4})$	0.4438311	[2]
$(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{2})$	(0, 0)	0.6477716	[2]
$(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{2})$	$(0.3664289, \frac{\pi}{4})$	0.5805842	[2]
$(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{5},\frac{\pi}{2},\frac{\pi}{2})$	(0, 0)	0.7466394	[2]
$(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{5},\frac{\pi}{2},\frac{\pi}{2})$	$(0.3835985, \frac{\pi}{4})$	0.6764612	[2]
$(\frac{\pi}{2},0,\frac{\pi}{2},0,\frac{\pi}{2})$	(0, 0)	0.9159659	[12]
$(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{2})$	$(0.5435350, \frac{\pi}{3})$	0.3244234	[12]
$(\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{2})$	$(0.5306375, \frac{\pi}{4})$	$0.5382759^{\flat}$	[12]
$(\frac{\pi}{2},\frac{\pi}{5},\frac{\pi}{2},\frac{\pi}{5},\frac{\pi}{2})$	$(0.4812118, \frac{\pi}{5})$	0.6580815	[12]
$(\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{2})$	$(0.4312773, \frac{\pi}{6})$	0.7299264	[12]
$(\frac{\pi}{2},\frac{\pi}{10},\frac{\pi}{2},\frac{\pi}{10},\frac{\pi}{2})$	$(0.2910082, \frac{\pi}{10})$	0.8447678	[12]

$^{\flat}$ This value is misprinted in [12, Equation 4.11]

$\mathscr{W}=0$. Since T shrinks to a point, $\operatorname{Vol} T \to 0$, which implies $\mathscr{C}=0$ by means of (27) and the proof is completed. □

4. Numeric computations

In Table 1 we have collected several volumes of prism truncated tetrahedra computed using Theorem 1 and before in [2,12]. The altitude of a prism truncated tetrahedron is computed from its dihedral angles, as the remark after Theorem 1 states. All numeric computations are carried out using the software routine "Mathematica" [15].

Acknowledgements

The first named author is grateful to Waseda University, Tokyo for hospitality during his stay in December, 2011. Part of this work was performed at the Institut Henri Poincaré, Paris in February–March, 2012.

Appendix: The term $\mathscr{Y}$ from Sect. 3.2

Let us first recall the expressions $q_k, k=0,1,2$, that are polynomials in the variables $a_k, k=\overline{1,6}$ defined by formula (7) and the expressions $\hat{q}_l, l=1,2$, defined in the proof of Theorem 1 by $\hat{q}_l = q_l/\prod_{k=1}^{6} a_k, l=1,2$. Then, the following lemma holds concerning the technical term $\mathscr{Y}$ mentioned above, which actually equals

$$
\begin{aligned}
\mathscr{Y} = {} & a_1a_2^2a_3a_4^2a_5 + a_1^2a_2a_3^2a_4^2a_5 + a_2^3a_3^2a_4^2a_5 + a_1a_2^2a_3^3a_4^2a_5 + a_1a_2^2a_4a_5^2 \\
& +a_1^2a_2a_3a_4a_5^2 + a_2^3a_3a_4a_5^2 + a_1a_2^2a_3^2a_4a_5^2 + a_1^2a_2^2a_3a_4^2a_6 + a_1a_2a_3^2a_4^2a_6 \\
& +a_1a_2^3a_3^2a_4^2a_6 + a_2^2a_3^3a_4^2a_6 + a_1^2a_2^2a_4a_5a_6 + a_1a_2a_3a_4a_5a_6 \\
& +a_1^3a_2a_3a_4a_5a_6 + a_1a_2^3a_3a_4a_5a_6 + a_1^2a_3^2a_4a_5a_6 + 2a_2^2a_3^2a_4a_5a_6 \\
& -a_1^2a_2^2a_3^2a_4a_5a_6 + a_1^4a_2^2a_3^2a_4a_5a_6 + a_1a_2a_3^3a_4a_5a_6 + a_1a_2a_3a_4^3a_5a_6 \\
& -a_1^3a_2a_3a_4^3a_5a_6+a_2^2a_3^2a_4^3a_5a_6-a_1^2a_2^2a_3^2a_4^3a_5a_6 + a_2^2a_3a_5^2a_6 - a_1^2a_2^2a_3a_5^2a_6 \\
& +a_1a_2a_3^2a_5^2a_6 - a_1^3a_2a_3^2a_5^2a_6 + a_1a_2a_4^2a_5^2a_6 + a_1^2a_3a_4^2a_5^2a_6 \\
& +2a_2^2a_3a_4^2a_5^2a_6 - a_1^2a_2^2a_3a_4^2a_5^2a_6 + a_1^4a_2^2a_3a_4^2a_5^2a_6 + a_1a_2a_3^2a_4^2a_5^2a_6 \\
& +a_1^3a_2a_3^2a_4^2a_5^2a_6 + a_1a_2^3a_3^2a_4^2a_5^2a_6 + a_1^2a_2^2a_3^3a_4^2a_5^2a_6 + a_2^2a_4a_5^3a_6 \\
& +a_1a_2a_3a_4a_5^3a_6 + a_1a_2^3a_3a_4a_5^3a_6 + a_1^2a_2^2a_3^2a_4a_5^3a_6 + a_1^2a_2a_3a_4a_6^2 \\
& +a_1a_3^2a_4a_6^2 + a_1a_2^2a_3^2a_4a_6^2 + a_2a_3^3a_4a_6^2 + a_1a_2^2a_3a_5a_6^2 - a_1^3a_2^2a_3a_5a_6^2 \\
& +a_2a_3^2a_5a_6^2 - a_1^2a_2a_3^2a_5a_6^2 + a_1^2a_2a_4^2a_5a_6^2 + a_1a_3a_4^2a_5a_6^2 + a_1a_2^2a_3a_4^2a_5a_6^2 \\
& +a_1^3a_2^2a_3a_4^2a_5a_6^2 + 2a_2a_3^2a_4^2a_5a_6^2 - a_1^2a_2a_3^2a_4^2a_5a_6^2 + a_1^4a_2a_3^2a_4^2a_5a_6^2 \\
& +a_1^2a_2^3a_3^2a_4^2a_5a_6^2 + a_1a_2^2a_3^3a_4^2a_5a_6^2 + a_1a_2^2a_4a_5^2a_6^2 + 2a_2a_3a_4a_5^2a_6^2 \\
& -a_1^2a_2a_3a_4a_5^2a_6^2 + a_1^4a_2a_3a_4a_5^2a_6^2 + a_1^2a_2^3a_3a_4a_5^2a_6^2 + a_1a_3^2a_4a_5^2a_6^2 \\
& +a_1a_2^2a_3^2a_4a_5^2a_6^2 + a_1^3a_2^2a_3^2a_4a_5^2a_6^2 + a_1^2a_2a_3^3a_4a_5^2a_6^2 + a_2a_3a_4^3a_5^2a_6^2 \\
& -a_1^2a_2a_3a_4^3a_5^2a_6^2 + a_1a_2^2a_3^2a_4^3a_5^2a_6^2 - a_1^3a_2^2a_3^2a_4^3a_5^2a_6^2 + a_2a_4^2a_5^3a_6^2 \\
& +a_1a_3a_4^2a_5^3a_6^2 + a_1a_2^2a_3a_4^2a_5^3a_6^2 + a_1^2a_2a_3^2a_4^2a_5^3a_6^2 + a_1a_2a_3a_4a_5a_6^3 \\
& +a_3^2a_4a_5a_6^3 + a_1^2a_2^2a_3^2a_4a_5a_6^3 + a_1a_2a_3^3a_4a_5a_6^3 + a_1a_2a_4^2a_5^2a_6^3 + a_3a_4^2a_5^2a_6^3 \\
& +a_1^2a_2^2a_3a_4^2a_5^2a_6^3 + a_1a_2a_3^2a_4^2a_5^2a_6^3.
\end{aligned}
$$

Lemma 3. *The above expression $\mathscr{Y}$ is not a polynomial neither in the variables q_0, q_1, q_2, nor in the variables $q_0, q_1, \hat{q}_2$.*

Proof. By setting $a_1 = a_6 := 0$, we get $q_0 = 1 + a_3a_4a_5, q_1 = q_2 = 0$ and

$$\mathscr{Y}|_{a_1=a_6:=0} = a_2^3a_3a_4a_5(a_3a_4 + a_5) = a_2^3\left(q_0|_{a_1=a_6:=0} - 1\right)(a_3a_4 + a_5).$$

As well, we have $\hat{q}_2 = a_2(a_3a_4 + a_5)$ and

$$\mathscr{Y}|_{a_1=a_6:=0} = a_2^2\left(q_0|_{a_1=a_6:=0} - 1\right)\hat{q}_2|_{a_1=a_6:=0}.$$

The former equality proves that $\mathscr{Y}$ is not a polynomial in the variables q_0, q_1, q_2. The latter shows that $\mathscr{Y}$ is not a polynomial in $q_0, q_1, \hat{q}_2$ either. $\square$

References

[1] Cho, Y., Kim, H.: On the volume formula for hyperbolic tetrahedra. Discrete Comput. Geom. **22**(3), 347–366 (1999)

[2] Derevnin, D.A., Kim, A.C.: The Coxeter Prisms in $\mathbb{H}^3$ in Recent Advances in Group Theory and Low-Dimensional Topology. pp. 35–49. Heldermann, Lemgo (2003)

[3] Derevnin, D.A., Mednykh, A.D.: A formula for the volume of a hyperbolic tetrahedron. Russ. Math. Surv. **60**(2), 346–348 (2005)
[4] Kellerhals, R.: On the volume of hyperbolic polyhedra. Math. Ann. **285**(4), 541–569 (1989)
[5] Kolpakov, A., Mednykh, A., Pashkevich, M.: Volume formula for a $\mathbb{Z}_2$-symmetric spherical tetrahedron through its edge lengths. Arkiv för Matematik. arXiv:1007.3948 (2011)
[6] Mednykh, A.D., Pashkevich, M.G.: Elementary formulas for a hyperbolic tetrahedron. Sib. Math. J. **47**(4), 687–695 (2006)
[7] Milnor, J.: The Schläfli Differential Equality in Collected Papers. I. Geometry. pp. 281–295. Publish or Perish, Houston (1994)
[8] Murakami, J., Yano, M.: On the volume of hyperbolic and spherical tetrahedron. Commun. Anal. Geom. **13**(2), 379–400 (2005)
[9] Murakami, J., Ushijima, A.: A volume formula for hyperbolic tetrahedra in terms of edge lengths. J. Geom. **83**(1–2), 153–163. arXiv:math/0402087 (2005)
[10] Murakami, J.: The volume formulas for a spherical tetrahedron. Proc. Amer. Math. Soc. **140**(9), 3289–3295. arXiv:1011.2584 (2012)
[11] Prasolov, V.: Problèmes et théorèmes d'algèbre linéaire, Enseignement des mathématiques. pp. 1–289. Cassini, Paris (2008)
[12] Szirmai, J.: Horoball packings for the Lambert-cube tilings in the hyperbolic 3-space. Beitr. Algebra Geom. **44**(1), 43–60 (2005)
[13] Ushijima, A.: A Volume Formula for Generalised Hyperbolic Tetrahedra in Mathematics and Its Applications, vol. 581. pp. 265 Springer, Berlin (2006)
[14] Vinberg, E.B.: Geometry. II: spaces of constant curvature. Encycl. Math. Sci. **29**, 139–248 (1993)
[15] Wolfram Research: Mathematica 8, a computational software routine. http://www.wolfram.com/mathematica/. Accessed 24 Oct 2011

Alexander Kolpakov
Department of Mathematics
University of Fribourg
chemin du Musée 23
1700 Fribourg, Switzerland
e-mail: kolpakov.alexander@gmail.com

Jun Murakami
Department of Mathematics
Faculty of Science and Engineering
Waseda University
3-4-1 Okubo Shinjuku-ku
Tokyo 169-8555, Japan
e-mail: murakami@waseda.jp

Received: March 21, 2012
Revised: June 30, 2012