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The Critical Z-Invariant Ising Model via Dimers: Locality Property

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Abstract: We study a large class of critical two-dimensional Ising models, namely *critical Z-invariant Ising models*. Fisher (J Math Phys 7:1776–1781, 1966) introduced a correspondence between the Ising model and the dimer model on a decorated graph, thus setting dimer techniques as a powerful tool for understanding the Ising model. In this paper, we give a full description of the dimer model corresponding to the critical Z-invariant Ising model, consisting of explicit expressions which only depend on the *local geometry* of the underlying isoradial graph. Our main result is an explicit local formula for the inverse Kasteleyn matrix, in the spirit of Kenyon (Invent Math 150(2):409–439, 2002), as a contour integral of the discrete exponential function of Mercat (Discrete period matrices and related topics, 2002) and Kenyon (Invent Math 150(2):409–439, 2002) multiplied by a local function. Using results of Boutillier and de Tilière (Prob Theor Rel Fields 147(3–4):379–413, 2010) and techniques of de Tilière (Prob Th Rel Fields 137(3–4):487–518, 2007) and Kenyon (Invent Math 150(2):409–439, 2002), this yields an explicit local formula for a natural Gibbs measure, and a local formula for the free energy. As a corollary, we recover Baxter's formula for the free energy of the critical Z-invariant Ising model (Baxter, in Exactly solved models in statistical mechanics, Academic Press, London, 1982), and thus a new proof of it. The latter is equal, up to a constant, to the logarithm of the normalized determinant of the Laplacian obtained in Kenyon (Invent Math 150(2):409–439, 2002).

1. Introduction

In [Fis66], Fisher introduced a correspondence between the two-dimensional Ising model defined on a graph G, and the dimer model defined on a decorated version of this graph. Since then, dimer techniques have been a powerful tool for solving pertinent questions

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about the Ising model, see for example the paper of Kasteleyn [Kas67], and the book of Mc Coy and Wu [MW73]. In this paper, we follow this approach to the Ising model.

We consider a large class of critical Ising models, known as *critical Z-invariant Ising models*, introduced in [Bax86]. More precisely, we consider Ising models defined on graphs which have the property of having an *isoradial embedding*. We suppose that the Ising coupling constants naturally depend on the geometry of the embedded graph, and are such that the model is invariant under *star-triangle* transformations of the underlying graph, *i.e.* such that the Ising model is *Z-invariant*. We suppose moreover that the coupling constants are *critical* by imposing a *generalized self-duality property*. The standard Ising model on the square, triangular and honeycomb lattice at the critical temperature are examples of critical Z-invariant Ising models. In the mathematics literature, the critical Z-invariant Ising model has been studied by Mercat [Mer01b], where the author proves equivalence between existence of Dirac spinors and criticality. In [CS09], Chelkak and Smirnov prove a fundamental contribution by establishing universality and conformal invariance in the scaling limit. Technical aspects of this paper rely on the paper [CS10] by the same authors, which provides a comprehensive toolbox for discrete complex analysis on isoradial graphs. Note that some key ideas of the first paper [CS09] were presented in [Smi06]. In [BdT08], we give a complete description of the equivalent dimer model, in the case where the underlying graph is periodic. The critical Z-invariant Ising model has also been widely studied in the physics literature, see for example [Bax86, AYP87, AP07, Mar97, Mar98, CS06].

Let $G = (V(G), E(G))$ be an infinite, locally finite, isoradial graph with critical coupling constants on the edges. Then by Fisher, the Ising model on G is in correspondence with the dimer model on a decorated graph $\mathcal{G}$, with a well chosen positive weight function ν on the edges. We refer to this model as the *critical dimer model on the Fisher graph $\mathcal{G}$ of G*. It is defined as follows. A *dimer configuration* of $\mathcal{G}$ is a subset of edges of $\mathcal{G}$ such that every vertex is incident to exactly one edge of M. Let $\mathcal{M}(\mathcal{G})$ be the set of dimer configurations of $\mathcal{G}$. Dimer configurations of $\mathcal{G}$ are chosen according to a probability measure, known as *Gibbs measure*, satisfying the following properties. If one fixes a perfect matching in an annular region of $\mathcal{G}$, then perfect matchings inside and outside of this annulus are independent. Moreover, the probability of occurrence of an interior matching is proportional to the product of its edge-weights given by the weight function ν.

The key objects used to obtain explicit expressions for relevant quantities of the dimer model are the *Kasteleyn matrix*, denoted by K, and its inverse. A Kasteleyn matrix is an oriented adjacency matrix of the graph $\mathcal{G}$, whose coefficients are weighted by the function ν, introduced by Kasteleyn [Kas61]. Our main result is Theorem 1, consisting of an explicit *local* expression for an inverse K^{-1} of the Kasteleyn matrix K. It can loosely be stated as follows: refer to Sect. 4.2 for detailed definitions and to Theorem 5 for a precise statement.

Theorem 1. *Let x, y be two vertices of $\mathcal{G}$. Then the infinite matrix K^{-1}, whose coefficient $K^{-1}_{x,y}$ is given below, is an inverse Kasteleyn matrix,*

$$K^{-1}_{x,y} = \frac{1}{(2\pi)^2} \oint_{\mathcal{C}_{x,y}} f_x(\lambda) f_y(-\lambda) \operatorname{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \log \lambda \mathrm{d}\lambda + C_{x,y},$$

where f_x is a complex-valued function depending on the vertex x only; $\operatorname{Exp}_{\mathbf{x},\mathbf{y}}$ *is the discrete exponential function introduced in [Mer01a], see also [Ken02]; $C_{x,y}$ is a constant equal to $\pm\frac{1}{4}$ when x and y are close, and 0 otherwise; $\mathcal{C}_{x,y}$ is a simple closed curve oriented counterclockwise containing all poles of the integrand, and avoiding a half-line $d_{x,y}$ starting from zero.*

Before stating implications of Theorem 1 on the critical dimer model on the Fisher graph $\mathcal{G}$, let us make a few comments.

1. Theorem 1 is in the spirit of the work of Kenyon [Ken02], where the author obtains explicit local expressions for the critical Green's function defined on isoradial graphs, and for the inverse Kasteleyn matrix of the critical dimer model defined on bipartite isoradial graphs. Surprisingly both the expression for the Green's function of [Ken02] and our expression for the inverse Kasteleyn matrix involve the discrete exponential function. This relation is pushed even further in Corollary 14, see the comment after the statement of Theorem 3. Note that the expression for the inverse Kasteleyn matrix obtained in [Ken02] does not hold in our setting, since the dimer model we consider is defined on the Fisher graph $\mathcal{G}$ of G, which is *not* bipartite and not *isoradial*.
2. Theorem 1 should also be compared with the explicit expression for the inverse Kasteleyn matrix obtained in [BdT08], in the case where the graph $\mathcal{G}$ is periodic. Then, by the uniqueness statement of Proposition 5 of [BdT08], the two expressions coincide. The proofs are nevertheless totally different in spirit, and we have not yet been able to understand the identity by an explicit computation, except in the case where $G = \mathbb{Z}^2$.
3. The most interesting features of Theorem 1 are the following. First, there is no periodicity assumption on the graph $\mathcal{G}$. Secondly, the expression for $K^{-1}_{x,y}$ is *local*, meaning that it only depends on the local geometry of the underlying isoradial graph G. More precisely, it only depends on an edge-path of G between vertices $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ of G, naturally constructed from x and y. This implies that changing the isoradial graph G away from this path does not change the expression for $K^{-1}_{x,y}$. Thirdly, explicit computations of $K^{-1}_{x,y}$ become tractable, whereas, even in the periodic case, they remain very difficult with the explicit expression for $K^{-1}_{x,y}$ given in [BdT08].
4. The structure of the proof of Theorem 1 is taken from [Ken02]. The idea is to find complex-valued functions that are in the kernel of the Kasteleyn matrix K, then to define K^{-1} as a contour integral of these functions, and to define the contours of integration in such a way that $KK^{-1} = \mathrm{Id}$. The great difficulty lies in actually finding the functions that are in the kernel of K, since there is no general method to construct them; and in defining the contours of integration. Indeed, the Fisher graph $\mathcal{G}$ is obtained from an isoradial graph G, but has a more complicated structure, so that the geometric argument of [Ken02] does not work. The proof of Theorem 1 being long, it is postponed until the last section of this paper.

Using Theorem 1, an argument similar to [dT07b], and the results obtained in [BdT08], yields Theorem 2 giving an explicit expression for a Gibbs measure on $\mathcal{M}(\mathcal{G})$. When the graph $\mathcal{G}$ is periodic, this measure coincides with the Gibbs measure obtained in [BdT08] as the weak limit of Boltzmann measures on a natural toroidal exhaustion of the graph $\mathcal{G}$. A precise statement is given in Theorem 9 of Sect. 5.1.

Theorem 2. *There is a unique Gibbs measure $\mathcal{P}$ defined on $\mathcal{M}(\mathcal{G})$, such that the probability of occurrence of a subset of edges $\{e_1 = x_1y_1, \dots, x_ky_k\}$ in a dimer configuration of $\mathcal{G}$, chosen with respect to the Gibbs measure $\mathcal{P}$ is:*

$$\mathcal{P}(e_1, \dots, e_k) = \left(\prod_{i=1}^{k} K_{x_i,y_i}\right) \mathrm{Pf}\left((K^{-1})^T_{\{x_1,y_1,\dots,x_k,y_k\}}\right),$$

where K^{-1} is given by Theorem 1, and $(K^{-1})_{\{x_1,y_1,\dots,x_k,y_k\}}$ is the sub-matrix of K^{-1}, whose lines and columns are indexed by vertices $\{x_1, y_1, \dots, x_k, y_k\}$.

Theorem 2 is a result about Gibbs measures with no periodicity assumption on the underlying graph. There are only very few examples of such instances in statistical mechanics. Moreover, the Gibbs measure $\mathcal{P}$ inherits the properties of the inverse Kasteleyn matrix K^{-1} of Theorem 1: it is local, and allows for explicit computations, examples of which are given in Appendix A.

Let us now assume that the graph G is $\mathbb{Z}^2$-periodic. Using Theorem 1, and the techniques of [Ken02], we obtain an explicit expression for the free energy of the critical dimer model on the graph $\mathcal{G}$, see Theorem 12, depending only on the angles of the fundamental domain. Using Fisher's correspondence between the Ising and dimer models, this yields a new proof of Baxter's formula for the free energy of the critical Z-invariant Ising model, denoted by f_I, see Sect. 5.2 for definitions.

Theorem 3 ([Bax89]).

$$f_I = -|V(G_1)|\frac{\log 2}{2} - \sum_{e\in E(G_1)} \left[\frac{\theta_e}{\pi}\log\tan\theta_e + \frac{1}{\pi}\left(L(\theta_e) + L\left(\frac{\pi}{2} - \theta_e\right)\right)\right].$$

Note that the free energy f_I of the critical Z-invariant Ising model is, up to a multiplicative constant $-\frac{1}{2}$ and an additive constant, the logarithm of the normalized determinant of the Laplacian obtained by Kenyon [Ken02]. See also Corollary 14, where we explicitly determine the constant of proportionality relating the characteristic polynomials of the critical Laplacian and of the critical dimer model on the Fisher graph $\mathcal{G}$, whose existence had been established in [BdT08].

Outline of the paper

Section 2: Definition of the critical Z-invariant Ising model.
Section 3: Fisher's correspondence between the Ising and dimer models.
Section 4: Definition of the Kasteleyn matrix K. Statement of Theorem 5, giving an explicit local expression for the coefficient $K^{-1}_{x,y}$ of the inverse Kasteleyn matrix. Asymptotic expansion of $K^{-1}_{x,y}$, as $|x-y|\to\infty$, using techniques of [Ken02].
Section 5: Implications of Theorem 5 on the critical dimer model on the Fisher graph $\mathcal{G}$: Theorem 9 gives an explicit local expression for a natural Gibbs measure, and Theorem 12 gives an explicit local expression for the free energy. Corollary: Baxter's formula for the free energy of the critical Z-invariant Ising model.
Section 6: Proof of Theorem 5.

2. The Critical Z-Invariant Ising Model

Consider an unoriented finite graph $G = (V(G), E(G))$, together with a collection of positive real numbers $J = (J_e)_{e\in E(G)}$ indexed by the edges of G. The *Ising model on G with coupling constants J* is defined as follows. A *spin configuration* σ of G is a function of the vertices of G with values in $\{-1, +1\}$. The probability of occurrence of a spin configuration σ is given by the *Ising Boltzmann measure*, denoted P^J:

$$P^J(\sigma) = \frac{1}{Z^J}\exp\left(\sum_{e=uv\in E(G)} J_e\sigma_u\sigma_v\right),$$

where $Z^J = \sum_{\sigma\in\{-1,1\}^{V(G)}}\exp\left(\sum_{e=uv\in E(G)} J_e\sigma_u\sigma_v\right)$, is the *Ising partition function*.

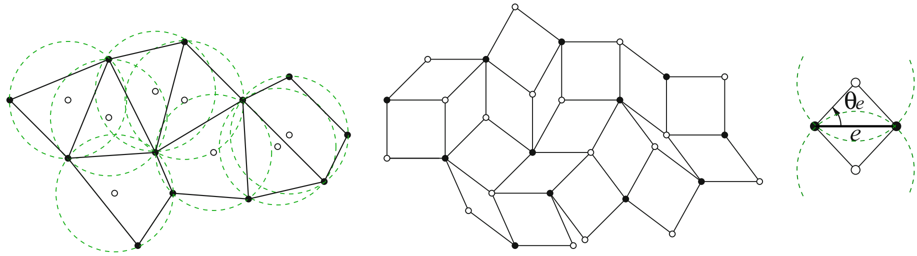


Fig. 1. *Left*: example of isoradial graph. *Center*: corresponding diamond graph. *Right*: rhombus half-angle associated to an edge e of the graph

We consider Ising models defined on a class of embedded graphs which have an additional property called *isoradiality*. A graph G is said to be *isoradial* [Ken02], if it has an embedding in the plane such that every face is inscribed in a circle of radius 1. We ask moreover that all circumcenters of the faces are in the closure of the faces. From now on, when we speak of the graph G, we mean the graph together with a particular isoradial embedding in the plane. Examples of isoradial graphs are the square and the honeycomb lattice, see Fig. 1 (left) for a more general example of an isoradial graph.

To such a graph is naturally associated the *diamond graph*, denoted by $G^\diamond$, defined as follows. Vertices of $G^\diamond$ consist in the vertices of G, and the circumcenters of the faces of G. The circumcenter of each face is then joined to all vertices which are on the boundary of this face, see Fig. 1 (center). Since G is isoradial, all faces of $G^\diamond$ are side-length-1 rhombi. Moreover, each edge e of G is the diagonal of exactly one rhombus of $G^\diamond$; we let θ_e be the half-angle of the rhombus at the vertex it has in common with e, see Fig. 1 (right). The additional condition imposed on circumcenters ensures that we can glue rhombic faces of $G^\diamond$ along edges to get the whole plane, without having some "upside-down" rhombi.

The same construction can be done for infinite and toroidal isoradial graphs, in which case the embedding is on a torus. When the isoradial graph is infinite and non-periodic, in order to ensure that the embedding is locally finite, we assume that there exists an $\varepsilon > 0$ such that the half-angle of every rhombus of $G^\diamond$ lies between ε and $\frac{\pi}{2} - \varepsilon$. This implies in particular that vertices of G have bounded degree.

It is then natural to choose the coupling constants J of the Ising model defined on an isoradial graph G, to depend on the geometry of the embedded graph: let us assume that J_e is a function of θ_e, the rhombus half-angle assigned to the edge e.

We impose two more conditions on the coupling constants. First, we ask that the Ising model on G with coupling constants J as above is Z*-invariant*, that is, invariant under *star-triangle* transformations of the underlying graph. Next, we impose that the Ising model satisfies a generalized form of *self-duality*. These conditions completely determine the coupling constants J, known as *critical coupling constants*: for every edge e of G,

$$J(\theta_e) = \frac{1}{2} \log\left(\frac{1 + \sin\theta_e}{\cos\theta_e}\right). \tag{1}$$

The Z-invariant Ising model on an isoradial graph with this particular choice of coupling constants is referred to as the *critical Z-invariant Ising model*. This model was introduced by Baxter in [Bax86]. A more detailed definition is given in [BdT08].

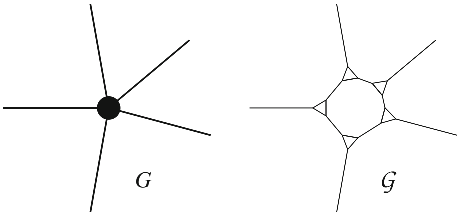


Fig. 2. *Left*: a vertex of G with its incoming edges. *Right*: corresponding decoration in $\mathcal{G}$

3. Fisher's Correspondence Between the Ising and Dimer Models

Fisher [Fis66] exhibits a correspondence between the Ising model on any graph G drawn on a surface without boundary, and the dimer model on a "decorated" version of G. Before explaining this correspondence, let us first recall the definition of the dimer model.

3.1. Dimer model. Consider a finite graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, and suppose that edges of $\mathcal{G}$ are assigned a positive weight function $\nu = (\nu_e)_{e \in E(\mathcal{G})}$. The *dimer model on $\mathcal{G}$ with weight function ν* is defined as follows.

A *dimer configuration M* of $\mathcal{G}$, also called *perfect matching*, is a subset of edges of $\mathcal{G}$ such that every vertex is incident to exactly one edge of M. Let $\mathcal{M}(\mathcal{G})$ be the set of dimer configurations of the graph $\mathcal{G}$. The probability of occurrence of a dimer configuration M is given by the *dimer Boltzmann measure*, denoted $\mathcal{P}^\nu$:

$$\mathcal{P}^\nu(M) = \frac{\prod_{e \in M} \nu_e}{\mathcal{Z}^\nu},$$

where $\mathcal{Z}^\nu = \sum_{M \in \mathcal{M}(\mathcal{G})} \prod_{e \in M} \nu_e$ is the *dimer partition function*.

3.2. Fisher's correspondence. Consider an Ising model on a finite graph G embedded on a surface without boundary, with coupling constants J. We use the following slight variation of Fisher's correspondence [Fis66].

The decorated graph, on which the dimer configurations live, is constructed from G as follows. Every vertex of degree k of G is replaced by a *decoration* consisting of $3k$ vertices: a triangle is attached to every edge incident to this vertex, and these triangles are linked by edges in a circular way, see Fig. 2. This new graph, denoted by $\mathcal{G}$, is also embedded on the surface without boundary and has vertices of degree 3. It is referred to as the *Fisher graph* of G.

Fisher's correspondence uses the high temperature expansion[1] of the Ising partition function, see for example [Bax89]:

$$Z^J = \left(\prod_{e \in E(G)} \cosh(J_e) \right) 2^{|V(G)|} \sum_{\mathsf{C} \in \mathsf{P}} \prod_{e \in \mathsf{C}} \tanh(J_e),$$

[1] It is also possible to use the *low temperature expansion*, if the spins of the Ising model do not sit on the vertices but on the faces of G.

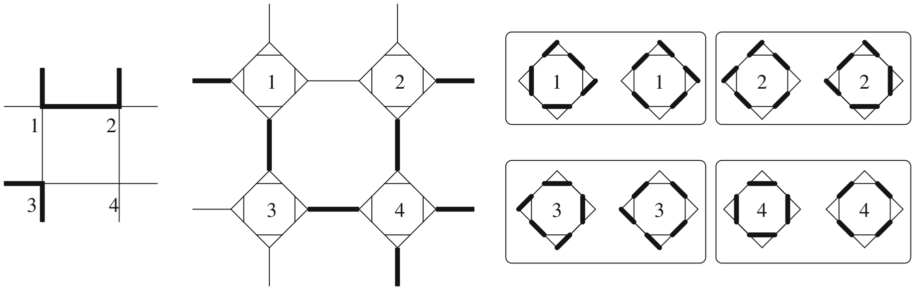


Fig. 3. Polygonal contour of $\mathbb{Z}^2$, and corresponding dimer configurations of the associated Fisher graph

where P is the family of all polygonal contours drawn on G, for which every edge of G is used at most once. This expansion defines a measure on the set of polygonal contours P of G: the probability of occurrence of a polygonal contour C is proportional to the product of the weights of the edges it contains, where the weight of an edge e is $\tanh(J_e)$.

Here comes the correspondence: to any contour configuration C coming from the high-temperature expansion of the Ising model on G, we associate $2^{|V(G)|}$ dimer configurations on $\mathcal{G}$: edges present (resp. absent) in C are absent (resp. present) in the corresponding dimer configuration of $\mathcal{G}$. Once the state of these edges is fixed, there is, for every decorated vertex, exactly two ways to complete the configuration into a dimer configuration. Figure 3 gives an example in the case where G is the square lattice $\mathbb{Z}^2$.

Let us assign, to an edge e of $\mathcal{G}$, weight $\nu_e = 1$, if it belongs to a decoration; and weight $\nu_e = \coth J_e$, if it corresponds to an edge of G. Then the correspondence is measure-preserving: every contour configuration C has the same number ($2^{|V(G)|}$) of images by this correspondence, and the product of the weights of the edges in C, $\prod_{e\in\mathsf{C}} \tanh(J_e)$ is proportional to the weight $\prod_{e\notin\mathsf{C}} \coth(J_e)$ of any of its corresponding dimer configurations for a proportionality factor, $\prod_{e\in E(G)} \tanh(J_e)$, which is independent of C.

As a consequence of Fisher's correspondence, we have the following relation between the Ising and dimer partition functions:

$$Z^J = \left(\prod_{e\in E(G)} \sinh(J_e)\right) \mathcal{Z}^\nu. \qquad (2)$$

Fisher's correspondence between Ising contour configurations and dimer configurations naturally extends to the case where G is an infinite planar graph.

3.3. Critical dimer model on Fisher graphs. Consider a critical Z-invariant Ising model on an isoradial graph G on the torus, or on the whole plane. Then, the dimer weights of the corresponding dimer model on the Fisher graph $\mathcal{G}$ are:

$$\nu_e = \begin{cases} 1 & \text{if } e \text{ belongs to a decoration,} \\ \nu(\theta_e) = \cot\left(\frac{\theta_e}{2}\right) & \text{if } e \text{ comes from an edge of } G. \end{cases}$$

We refer to these weights as *critical dimer weights*, and to the corresponding dimer model as *critical dimer model on the Fisher graph* $\mathcal{G}$.

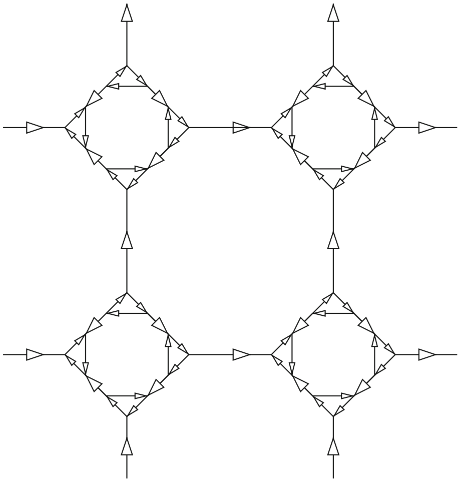


Fig. 4. An example of Kasteleyn orientation of the Fisher graph of $\mathbb{Z}^2$, in which every triangle of every decoration is oriented clockwise

4. Kasteleyn Matrix on Critical Infinite Fisher Graphs

In the whole of this section, we let G be an infinite isoradial graph, and $\mathcal{G}$ be the corresponding Fisher graph. We suppose that edges of $\mathcal{G}$ are assigned the dimer critical weight function denoted by ν. Recall that $G^\diamond$ denotes the diamond graph associated to G.

4.1. Kasteleyn and inverse Kasteleyn matrix. The key object used to obtain explicit expressions for the dimer model on the Fisher graph $\mathcal{G}$ is the *Kasteleyn matrix* introduced by Kasteleyn in [Kas61]. It is a weighted, oriented adjacency matrix of the graph $\mathcal{G}$ defined as follows.

A *Kasteleyn orientation* of $\mathcal{G}$ is an orientation of the edges of $\mathcal{G}$ such that all elementary cycles are *clockwise odd*, i.e. when traveling clockwise around the edges of any elementary cycle of $\mathcal{G}$, the number of co-oriented edges is odd. When the graph is planar, such an orientation always exists [Kas67]. For later purposes, we need to keep track of the orientation of the edges of $\mathcal{G}$. We thus build a specific Kasteleyn orientation of the graph $\mathcal{G}$ in the following way: choose a Kasteleyn orientation of the planar graph obtained from $\mathcal{G}$ by contracting each decoration triangle to a point, which exists by Kasteleyn's theorem [Kas67], and extend it to $\mathcal{G}$ by imposing that every triangle of every decoration is oriented clockwise. Refer to Fig. 4 for an example of such an orientation in the case where $G = \mathbb{Z}^2$.

The *Kasteleyn matrix* corresponding to such an orientation is an infinite matrix, whose rows and columns are indexed by vertices of $\mathcal{G}$, defined by:

$$K_{x,y} = \varepsilon_{x,y} \nu_{xy},$$

where

$$\varepsilon_{x,y} = \begin{cases} 1 & \text{if } x \sim y, \text{ and } x \to y \\ -1 & \text{if } x \sim y, \text{ and } x \leftarrow y \\ 0 & \text{otherwise.} \end{cases}$$

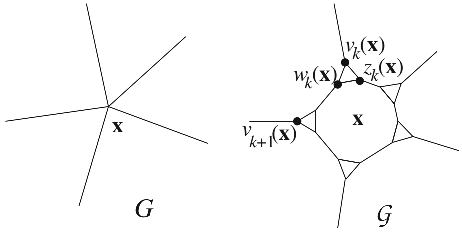


Fig. 5. Notations for vertices of $\mathcal{G}$

Note that K can be interpreted as an operator acting on $\mathbb{C}^{V(\mathcal{G})}$:

$$\forall f \in \mathbb{C}^{V(\mathcal{G})}, \quad (Kf)_x = \sum_{y \in V(\mathcal{G})} K_{x,y} f_y.$$

An *inverse of the Kasteleyn matrix* K, denoted K^{-1}, is an infinite matrix whose rows and columns are indexed by vertices of $\mathcal{G}$, and which satisfies $KK^{-1} = \text{Id}$.

4.2. Local formula for an inverse Kasteleyn matrix. In this section, we state Theorem 5 proving an explicit *local* expression for the coefficients of an inverse K^{-1} of the Kasteleyn matrix K. This inverse is the key object for the critical dimer model on the Fisher graph $\mathcal{G}$. Indeed, it yields an explicit local expression for a Gibbs measure on dimer configurations of $\mathcal{G}$, see Sect. 5.1. It also allows for a simple derivation of Baxter's formula for the free energy of the critical Z-invariant Ising model, see Sect. 5.2.

This section is organized as follows. The explicit expression for the coefficients of K^{-1} given by Theorem 5 below, see also Theorem 1, is a contour integral of an integrand involving two quantities:

- a complex-valued function depending on a complex parameter, and on the vertices of the graph $\mathcal{G}$ only, defined in Sect. 4.2.2.
- the *discrete exponential function* which first appeared in [Mer01a], see also [Ken02]. It is a complex valued function depending on a complex parameter, and on an edge-path between pairs of vertices of the graph G, defined in Sect. 4.2.3.

Theorem 5 is then stated in Sect. 4.2.4. Since the proof is long, it is postponed until Sect. 6. In Sect. 4.2.5, using the same technique as [Ken02], we give the asymptotic expansion of the coefficient $K^{-1}_{x,y}$ of the inverse Kasteleyn matrix, as $|x - y| \to \infty$.

4.2.1. Preliminary notations. From now on, vertices of $\mathcal{G}$ are written in normal symbol, and vertices of G in boldface. Let x be a vertex of $\mathcal{G}$, then x belongs to the decoration corresponding to a unique vertex of G, denoted by $\mathbf{x}$. Conversely, vertices of $\mathcal{G}$ of the decoration corresponding to a vertex $\mathbf{x}$ of G are labeled as follows, refer to Fig. 5 for an example. Let $d(\mathbf{x})$ be the degree of the vertex $\mathbf{x}$ in G, then the corresponding decoration of $\mathcal{G}$ consists of $d(\mathbf{x})$ triangles, labeled from 1 to $d(\mathbf{x})$ in counterclockwise order. For the k^{th} triangle, let $v_k(\mathbf{x})$ be the vertex incident to an edge of G, and let $w_k(\mathbf{x})$, $z_k(\mathbf{x})$ be the two other vertices, in counterclockwise order, starting from $v_k(\mathbf{x})$.

Later on, when no confusion occurs, we will drop the argument $\mathbf{x}$ in the above labeling. Define a vertex x of $\mathcal{G}$ to be of *type* 'v', if $x = v_k(\mathbf{x})$ for some $k \in \{1, \ldots, d(\mathbf{x})\}$, and similarly for '$w$' and '$z$'.

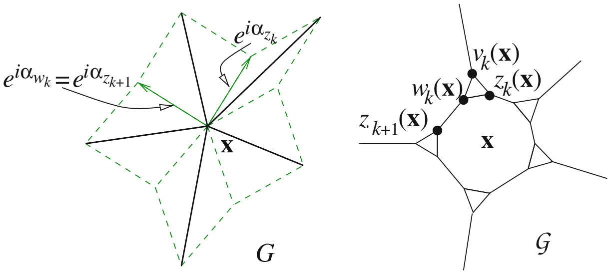


Fig. 6. Rhombus vectors of the diamond graph $G^\diamond$ assigned to vertices of $\mathcal{G}$

The isoradial embedding of the graph G fixes an embedding of the corresponding diamond graph $G^\diamond$. There is a natural way of assigning rhombus unit-vectors of $G^\diamond$ to vertices of $\mathcal{G}$: for every vertex $\mathbf{x}$ of G, and every $k \in \{1, \dots, d(\mathbf{x})\}$, let us associate the rhombus unit-vector $e^{i\alpha_{w_k(\mathbf{x})}}$ to $w_k(\mathbf{x})$, $e^{i\alpha_{z_k(\mathbf{x})}}$ to $z_k(\mathbf{x})$, and the two rhombus-unit vectors $e^{i\alpha_{w_k(\mathbf{x})}}$, $e^{i\alpha_{z_k(\mathbf{x})}}$ to $v_k(\mathbf{x})$, as in Fig. 6. Note that $e^{i\alpha_{w_k(\mathbf{x})}} = e^{i\alpha_{z_{k+1}(\mathbf{x})}}$.

4.2.2. Complex-valued function on the vertices of $\mathcal{G}$. Let us introduce the complex-valued function defined on vertices of $\mathcal{G}$ and depending on a complex parameter, involved in the integrand of the contour integral defining K^{-1} given by Theorem 5. Define $f : V(\mathcal{G}) \times \mathbb{C} \to \mathbb{C}$, by:

$$\begin{aligned} f(w_k(\mathbf{x}), \lambda) &:= f_{w_k(\mathbf{x})}(\lambda) = \frac{e^{i\frac{\alpha_{w_k(\mathbf{x})}}{2}}}{e^{i\alpha_{w_k(\mathbf{x})}} - \lambda}, \\ f(z_k(\mathbf{x}), \lambda) &:= f_{z_k(\mathbf{x})}(\lambda) = -\frac{e^{i\frac{\alpha_{z_k(\mathbf{x})}}{2}}}{e^{i\alpha_{z_k(\mathbf{x})}} - \lambda}, \\ f(v_k(\mathbf{x}), \lambda) &:= f_{v_k(\mathbf{x})}(\lambda) = f_{w_k(\mathbf{x})}(\lambda) + f_{z_k(\mathbf{x})}(\lambda), \end{aligned} \tag{3}$$

for every $\mathbf{x} \in G$, and every $k \in \{1, \dots, d(\mathbf{x})\}$. In order for the function f to be well defined, the angles $\alpha_{w_k(\mathbf{x})}$, $\alpha_{z_k(\mathbf{x})}$ need to be well defined mod 4π, indeed half-angles need to be well defined mod 2π. Let us define them inductively as follows, see also Fig. 7. The definition strongly depends on the Kasteleyn orientation introduced in Sect. 4.1. Fix a vertex $\mathbf{x}_0$ of G, and set $\alpha_{z_1(\mathbf{x}_0)} = 0$. Then, for vertices of $\mathcal{G}$ in the decoration of a vertex $\mathbf{x} \in G$, define:

$$\begin{aligned} &\alpha_{w_k(\mathbf{x})} = \alpha_{z_k(\mathbf{x})} + 2\theta_k(\mathbf{x}), \text{ where } \theta_k(\mathbf{x}) > 0 \text{ is the rhombus half-angle of Fig. 7,} \\ &\alpha_{z_{k+1}(\mathbf{x})} = \begin{cases} \alpha_{w_k(\mathbf{x})} & \text{if the edge } w_k(\mathbf{x})z_{k+1}(\mathbf{x}) \text{ is oriented from } w_k(\mathbf{x}) \text{ to } z_{k+1}(\mathbf{x}) \\ \alpha_{w_k(\mathbf{x})} + 2\pi & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

Here is the rule defining angles in the neighboring decoration, corresponding to a vertex $\mathbf{y}$ of G. Let k and ℓ be indices such that $v_k(\mathbf{x})$ is adjacent to $v_\ell(\mathbf{y})$ in $\mathcal{G}$. Then, define:

$$\alpha_{w_\ell(\mathbf{y})} = \begin{cases} \alpha_{w_k(\mathbf{x})} - \pi & \text{if the edge } v_k(\mathbf{x})v_\ell(\mathbf{y}) \text{ is oriented from } v_k(\mathbf{x}) \text{ to } v_\ell(\mathbf{y}) \\ \alpha_{w_k(\mathbf{x})} + \pi & \text{otherwise.} \end{cases} \tag{5}$$

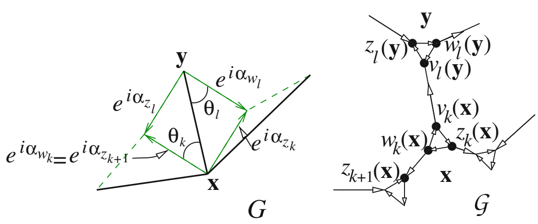


Fig. 7. Notations for the definition of the angles in $\mathbb{R}/4\pi\mathbb{Z}$

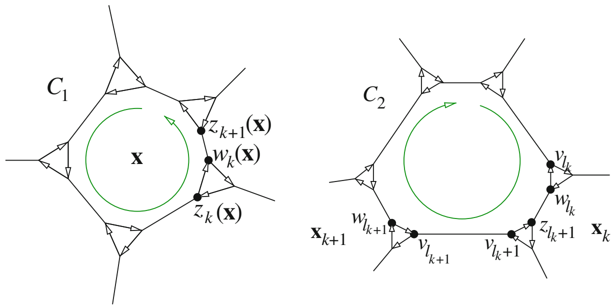


Fig. 8. The two types of cycles on which the definition of the angles in $\mathbb{R}/4\pi\mathbb{Z}$ needs to be checked. *Left*: C_1, the inner cycle of a decoration, oriented counterclockwise. *Right*: C_2, a cycle coming from the boundary of a face of G, oriented clockwise

Lemma 4. *For every vertex* $\mathbf{x}$ *of* G*, and every* $k \in \{1, \dots, d(\mathbf{x})\}$*, the angles* $\alpha_{w_k(\mathbf{x})}$*,* $\alpha_{z_k(\mathbf{x})}$*, are well defined in* $\mathbb{R}/4\pi\mathbb{Z}$*.*

Proof. It suffices to show that when doing the inductive procedure around a cycle of $\mathcal{G}$, we obtain the same angle modulo 4π. There are two types of cycles to consider: inner cycles of decorations, and cycles of $\mathcal{G}$ coming from the boundary of a face of G. These cycles are represented in Fig. 8.

Let C_1 be a cycle of the first type, that is C_1 is the inner cycle of a decoration corresponding to a vertex $\mathbf{x}$ of G, oriented counterclockwise. Let $d = d(\mathbf{x})$ be the degree of the vertex $\mathbf{x}$ in G, then:

$$C_1 = (z_1(\mathbf{x}), w_1(\mathbf{x}), \dots, z_d(\mathbf{x}), w_d(\mathbf{x}), z_1(\mathbf{x})) = (z_1, w_1, \dots, z_d, w_d, z_1).$$

According to our choice of Kasteleyn orientation, all edges of this cycle are oriented counterclockwise with respect to the face surrounded by C_1, except an odd number n of edges of the form $w_k z_{k+1}$. By definition of the angles (4), when jumping from z_k to z_{k+1}, the change in angle is $2\theta_k$ if $w_k z_{k+1}$ is oriented counterclockwise, and $2\theta_k + 2\pi$ otherwise. The total change along the cycle is thus,

$$\sum_{k=1}^{d} 2\theta_k + 2\pi n = 2\pi(n+1) \equiv 0 \mod 4\pi.$$

Let C_2 be a cycle of the second type, around a face touching m decorations corresponding to vertices $\mathbf{x}_1, \dots, \mathbf{x}_m$ of G. Suppose that C_2 is oriented clockwise,

$$C_2 = \big(w_{\ell_1}(\mathbf{x}_1), z_{\ell_1+1}(\mathbf{x}_1), v_{\ell_1+1}(\mathbf{x}_1), \dots,$$
$$\dots, v_{\ell_m}(\mathbf{x}_m), w_{\ell_m}(\mathbf{x}_m), z_{\ell_m+1}(\mathbf{x}_m), v_{\ell_m+1}(\mathbf{x}_m), v_{\ell_1}(\mathbf{x}_1), w_{\ell_1}(\mathbf{x}_1)\big).$$

The cycle C_2 contains $4m$ edges, and the number n of co-oriented edges along C_2 is odd by definition of a Kasteleyn orientation. By our choice of Kasteleyn orientation, the only edges that may be co-oriented along the cycle, are either of the form $w_{\ell_k}(\mathbf{x}_k)z_{\ell_k+1}(\mathbf{x}_k)$, or of the form $v_{\ell_k+1}(\mathbf{x}_k)v_{\ell_{k+1}}(\mathbf{x}_{k+1})$.

For every $k \in \{1, \ldots, m\}$, there are two different contributions to the total change in angle from $w_{\ell_k}(\mathbf{x}_k)$ to $w_{\ell_{k+1}}(\mathbf{x}_{k+1})$. First, from $w_{\ell_k}(\mathbf{x}_k)$ to $z_{\ell_k+1}(\mathbf{x}_k)$, by (4), there is a contribution of 2π if the corresponding edge is co-oriented. Then, from $z_{\ell_k+1}(\mathbf{x}_k)$ to $w_{\ell_{k+1}}(\mathbf{x}_{k+1})$, by (5), the angle is changed by $2\theta_{\ell_k+1}(\mathbf{x}_k) - \pi = 2\theta_{\ell_{k+1}}(\mathbf{x}_{k+1}) - \pi$, plus an extra contribution of 2π if the edge $v_{\ell_k+1}(\mathbf{x})v_{\ell_{k+1}}(\mathbf{x}_{k+1})$ is co-oriented. Thus the total change in angle is:

$$\sum_{k=1}^{m}(2\theta_{\ell_k}(\mathbf{x}_k) - \pi) + 2\pi n.$$

But the terms $\pi - 2\theta_{\ell_k}(\mathbf{x}_k)$ are the angles of the rhombi at the center of the face surrounded by C_2, which sum to 2π. Therefore, the total change is equal to $2\pi(n-1)$ which is congruent to 0 mod 4π. □

These angles at vertices are related to the notion of spin structure on surface graphs (see [CR07,CR08,Kup98]). A spin structure on a surface is equivalent to the data of a vector field with even index singularities. Kuperberg explained in [Kup98] how to construct from a Kasteleyn orientation a vector field with odd index singularities at vertices of G. One can then obtain the spin structure by merging the singularities into pairs using a reference dimer configuration.

The angles $\alpha_{w_k(\mathbf{x})}$ and $\alpha_{z_k(\mathbf{x})}$ defined here are directly related to the direction of the vector field with even index singularities obtained from Kuperberg's construction applied to our choice of Kasteleyn orientation, and from the pairing of singularities corresponding to the following reference dimer configuration:

- $w_k(\mathbf{x}) \leftrightarrow z_k(\mathbf{x})$,
- $v_k(\mathbf{x}) \leftrightarrow v_\ell(\mathbf{y})$ if they are neighbors.

4.2.3. Discrete exponential functions. Let us define the *discrete exponential function*, denoted Exp, involved in the integrand of the contour integral defining K^{-1} given by Theorem 5. This function first appeared in [Mer01a], see also [Ken02]. In order to simplify notations, we use a different labeling of the rhombus vectors of $G^\diamond$. Let $\mathbf{x}, \mathbf{y}$ be two vertices of G, and let $\mathbf{y} = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1} = \mathbf{x}$ be an edge-path of G from $\mathbf{y}$ to $\mathbf{x}$. The complex vector $\mathbf{x}_{j+1} - \mathbf{x}_j$ is the sum of two unit complex numbers $e^{i\beta_j} + e^{i\gamma_j}$ representing edges of the rhombus in $G^\diamond$ associated to the edge $\mathbf{x}_j\mathbf{x}_{j+1}$.

Then, $\mathrm{Exp} : V(G) \times V(G) \times \mathbb{C} \to \mathbb{C}$ is defined by:

$$\mathrm{Exp}(\mathbf{x}, \mathbf{y}, \lambda) := \mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) = \prod_{j=1}^{n}\left(\frac{e^{i\beta_j} + \lambda}{e^{i\beta_j} - \lambda}\right)\left(\frac{e^{i\gamma_j} + \lambda}{e^{i\gamma_j} - \lambda}\right). \tag{6}$$

The function is well defined (independent of the choice of edge-path of G from $\mathbf{y}$ to $\mathbf{x}$) since the product of the multipliers around a rhombus is 1.

4.2.4. Inverse Kasteleyn matrix. We now state Theorem 5 proving an explicit local formula for the coefficients of an inverse K^{-1} of the Kasteleyn matrix K, constructed from the functions f defined in (3) and the discrete exponentials defined in (6). The vertices x and y of $\mathcal{G}$ in the statement should be thought of as being one of $w_k(\mathbf{x})$, $z_k(\mathbf{x})$, $v_k(\mathbf{x})$ for some $\mathbf{x} \in G$ and some $k \in \{1, \dots, d(\mathbf{x})\}$, and similarly for y. The proof of Theorem 5 is postponed until Sect. 6.

Theorem 5. *Let x, y be any two vertices of $\mathcal{G}$. Then the infinite matrix K^{-1}, whose coefficient $K^{-1}_{x,y}$ is given by* (7) *below, is an inverse Kasteleyn matrix,*

$$K^{-1}_{x,y} = \frac{1}{(2\pi)^2} \oint_{\mathcal{C}_{x,y}} f_x(\lambda) f_y(-\lambda)\, \mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \log \lambda \mathrm{d}\lambda + C_{x,y}. \tag{7}$$

The contour of integration $\mathcal{C}_{x,y}$ is a simple closed curve oriented counterclockwise containing all poles of the integrand, and avoiding the half-line $d_{x,y}$ starting from zero.[2] *The constant $C_{x,y}$ is given by*

$$C_{x,y} = \begin{cases} \frac{1}{4} & \text{if } x = y = w_k(\mathbf{x}) \\ -\frac{1}{4} & \text{if } x = y = z_k(\mathbf{x}) \\ \frac{(-1)^{n(x,y)}}{4} & \text{if } \mathbf{x} = \mathbf{y},\ x \neq y, \text{ and } x \text{ and } y \text{ are of type `}w\text{' or `}z\text{'} \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where $n(x, y)$ is the number of edges oriented clockwise in the clockwise arc from x to y of the inner cycle of the decoration corresponding to $\mathbf{x}$.

Remark 6. The explicit expression for $K^{-1}_{x,y}$ given in (7) has the very interesting feature of being *local*, *i.e.* it only depends on the geometry of the embedding of the isoradial graph G on a path between $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are vertices of $G^\diamond$ at distance at most 2 from $\mathbf{x}$ and $\mathbf{y}$, constructed in Sect. 6.3. A nice consequence of this property is the following. For $i = 1, 2$, let G_i be an isoradial graph with corresponding Fisher graph $\mathcal{G}_i$. Let K_i be the Kasteleyn matrix of the graph $\mathcal{G}_i$, whose edges are assigned the dimer critical weight function. Let $(K_i)^{-1}$ be the inverse of K_i given by Theorem 5. If G_1 and G_2 coincide on a ball B, and if the Kasteleyn orientations on $\mathcal{G}_1$ and $\mathcal{G}_2$ are chosen to be the same in the ball B, then for every couple of vertices (x, y) in B at distance at least 2 from the boundary, we have

$$(K_1)^{-1}_{x,y} = (K_2)^{-1}_{x,y}.$$

4.2.5. Asymptotic expansion of the inverse Kasteleyn matrix. As a corollary to Theorem 5, and using computations analogous to those of Theorem 4.3 of [Ken02], we obtain the asymptotic expansion for the coefficient $K^{-1}_{x,y}$ of the inverse Kasteleyn matrix, as $|\mathbf{x} - \mathbf{y}| \to \infty$. In order to give a concise statement, let us introduce the following simplified notations. When x is of type 'w' or 'z', let $e^{i\alpha}$ denote the corresponding rhombus unit-vector of $G^\diamond$; and when x is of type 'v', let $e^{i\alpha_1}$, $e^{i\alpha_2}$ be the two corresponding rhombus unit-vectors of $G^\diamond$, defined in Sect. 4.2.1. Moreover, set:

$$\epsilon_x = \begin{cases} 1 & \text{if } x \text{ is of type `}w\text{' or `}v\text{'} \\ -1 & \text{if } x \text{ is of type `}z\text{'}. \end{cases}$$

[2] In most cases, the half-line $d_{x,y}$ is oriented from $\hat{\mathbf{x}}$ to $\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ are vertices of $G^\diamond$ at distance at most two from $\mathbf{x}$, $\mathbf{y}$, constructed in Sect. 6.3. Refer to this section and Remark 19 for more details.

A superscript "prime" is added to these notations for the vertex y, and ϵ_y is defined in a similar way.

Corollary 7. *The asymptotic expansion, as* $|\mathbf{x}-\mathbf{y}| \to \infty$, *of the coefficient* $K^{-1}_{x,y}$ *of the inverse Kasteleyn matrix of Theorem 5 is*:

$$K^{-1}_{x,y} = \frac{\epsilon_x \epsilon_y}{2\pi} \begin{cases} \Im\left(\dfrac{e^{i\frac{\alpha+\alpha'}{2}}}{\mathbf{x}-\mathbf{y}}\right) + o\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) & \text{if } x \text{ and } y \text{ are of type `}w\text{' or `}z\text{'},\\ \Im\left(\dfrac{e^{i\frac{\alpha}{2}}(e^{i\frac{\alpha'_1}{2}} - e^{i\frac{\alpha'_2}{2}})}{\mathbf{x}-\mathbf{y}}\right) + o\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) & \text{if } x \text{ is of type `}w\text{' or `}z\text{', } y \text{ is of type `}v\text{'},\\ \Im\left(\dfrac{(e^{i\frac{\alpha_1}{2}} - e^{i\frac{\alpha_2}{2}})(e^{i\frac{\alpha'_1}{2}} - e^{i\frac{\alpha'_2}{2}})}{\mathbf{x}-\mathbf{y}}\right) + o\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) & \text{if } x \text{ and } y \text{ are of type `}v\text{'}. \end{cases}$$

Proof. We follow the proof of Theorem 4.3 of [Ken02]. By Theorem 5, $K^{-1}_{x,y}$ is given by the integral:

$$K^{-1}_{x,y} = \frac{1}{(2\pi)^2} \int_{\mathcal{C}_{x,y}} f_x(\lambda) f_y(-\lambda) \operatorname{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \log \lambda \mathrm{d}\lambda + C_{x,y}, \tag{9}$$

where $\mathcal{C}_{x,y}$ is a simple closed curve oriented counterclockwise containing all poles of the integrand, and avoiding the half-line $d_{x,y}$ starting from 0 in the direction from $\hat{\mathbf{x}}$ to $\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ are vertices of $G^\diamond$, at distance at most 2 from $\mathbf{x}$ and $\mathbf{y}$, constructed in Sect. 6.3. We choose $d_{x,y}$ to be the origin for the angles, *i.e.* the positive real axis $\mathbb{R}_+$. The coefficient $C_{x,y}$ equals 0 whenever x and y are not in the same decoration. It will be a fortiori the case when they are far away from each other.

It suffices to handle the case where both x and y are of type 'w' or 'z'. The other cases are then derived using the relation between the functions f:

$$f_{v_k}(\lambda) = f_{w_k}(\lambda) + f_{z_k}(\lambda).$$

Define,

$$\xi = \mathbf{x} - \mathbf{y} = \sum_{j=1}^{n} e^{i\beta_j} + e^{i\gamma_j},$$

where $e^{i\beta_1}, e^{i\gamma_1}, \ldots, e^{i\beta_n}, e^{i\gamma_n}$ are the steps of a path in $G^\diamond$ from $\mathbf{y}$ to $\mathbf{x}$. When $\mathbf{x}$ and $\mathbf{y}$ are far apart, since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are at distance at most 2 from $\mathbf{x}$ and $\mathbf{y}$, the direction of $\overrightarrow{\mathbf{xy}}$ is not very different from that of $\overrightarrow{\hat{\mathbf{x}}\hat{\mathbf{y}}}$, ensuring that $\Re(\xi) < 0$.

The contour $\mathcal{C}_{x,y}$ can be deformed to a curve running counterclockwise around the ball of radius R (R large) around the origin from the angle 0 to 2π, then along the positive real axis, from R to r (r small), then clockwise around the ball of radius r from the angle 2π to 0, and then back along the real axis from r to R. The rational fraction $f_x(\lambda) f_y(-\lambda) \operatorname{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$ behaves like $O(1)$ when λ is small, and like $O\left(\frac{1}{\lambda^2}\right)$, when λ is large. As a consequence, the contribution to the integral around the balls of radius r

and R converges to 0, as we let $r \to 0$ and $R \to \infty$. The logarithm differs by $2i\pi$ on the two sides of the ray $\mathbb{R}_+$, the integral (9) is therefore equal to

$$K^{-1}_{x,y} = \frac{1}{2i\pi}\int_0^\infty f_x(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\mathrm{d}\lambda.$$

As in [Ken02], when $|\mathbf{x}-\mathbf{y}|$ is large, the main contribution to this integral comes from a neighborhood of the origin and a neighborhood of infinity. When λ is small, we have:

$$\frac{e^{i\frac{\alpha}{2}}}{e^{i\alpha}-\lambda} = e^{-i\frac{\alpha}{2}+O(\lambda)}, \quad \frac{e^{i\frac{\alpha'}{2}}}{e^{i\alpha'}+\lambda} = e^{-i\frac{\alpha'}{2}+O(\lambda)}, \quad \frac{e^{i\beta}+\lambda}{e^{i\beta}-\lambda} = \exp\left(2e^{-i\beta}\lambda + O(\lambda^3)\right).$$

Thus, for small values of λ:

$$\begin{aligned} f_x(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) &= \epsilon_x\epsilon_y \frac{e^{i\frac{\alpha}{2}}}{e^{i\alpha}-\lambda}\frac{e^{i\frac{\alpha'}{2}}}{e^{i\alpha'}+\lambda}\prod_{j=1}^n \frac{e^{i\beta_j}+\lambda}{e^{i\beta_j}-\lambda}\frac{e^{i\gamma_j}+\lambda}{e^{i\gamma_j}-\lambda} \\ &= \epsilon_x\epsilon_y e^{-i\frac{\alpha+\alpha'}{2}}\exp\left(2\bar{\xi}\lambda + O(\lambda) + O(n\lambda^3)\right). \end{aligned}$$

Integrating this estimate from 0 to $\frac{1}{\sqrt{n}}$ gives

$$\begin{aligned} \int_0^{\frac{1}{\sqrt{n}}} f_x(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\mathrm{d}\lambda &= \epsilon_x\epsilon_y\int_0^{\frac{1}{\sqrt{n}}} e^{-i\frac{\alpha+\alpha'}{2}}\exp\left(2\bar{\xi}\lambda + O(n^{-\frac{1}{2}})\right)\mathrm{d}\lambda \\ &= -\epsilon_x\epsilon_y\frac{e^{-i\frac{\alpha+\alpha'}{2}}}{2\bar{\xi}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right). \end{aligned}$$

Similarly, when λ is large, we have:

$$\begin{aligned} &\frac{e^{i\frac{\alpha}{2}}}{e^{i\alpha}-\lambda} = -\frac{e^{i\frac{\alpha}{2}+O(\lambda^{-1})}}{\lambda}, \quad \frac{e^{i\frac{\alpha'}{2}}}{e^{i\alpha'}+\lambda} = \frac{e^{i\frac{\alpha'}{2}+O(\lambda^{-1})}}{\lambda}, \\ &\frac{e^{i\beta}+\lambda}{e^{i\beta}-\lambda} = -\exp\left(2e^{i\beta}\lambda^{-1} + O(\lambda^{-3})\right). \end{aligned}$$

Thus for large values of λ,

$$f_x(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) = -\epsilon_x\epsilon_y\frac{e^{i\frac{\alpha+\alpha'}{2}}}{\lambda^2}\exp\left(2\xi\lambda^{-1} + O(\lambda^{-1}) + O(n\lambda^{-3})\right).$$

Computing the integral of this estimate for $\lambda \in (\sqrt{n}, \infty)$ gives

$$\int_{\sqrt{n}}^\infty f_x(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\mathrm{d}\lambda = \epsilon_x\epsilon_y\frac{e^{i\frac{\alpha+\alpha'}{2}}}{2\xi}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right).$$

The rest of the integral between $n^{-\frac{1}{2}}$ and $n^{\frac{1}{2}}$ is negligible (see [Ken02]). As a consequence,

$$K^{-1}_{x,y} = \frac{\epsilon_x\epsilon_y}{2i\pi}\left(\frac{e^{i\frac{\alpha+\alpha'}{2}}}{2\xi}(1+o(1)) - \frac{e^{-i\frac{\alpha+\alpha'}{2}}}{2\bar{\xi}}(1+o(1))\right)$$
$$= \frac{\epsilon_x\epsilon_y}{2\pi}\Im\left(\frac{e^{i\frac{\alpha+\alpha'}{2}}}{\mathbf{x}-\mathbf{y}}\right) + o\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right).$$

□

Remark 8. Using the same method with an expansion of the integrand to a higher order would lead to a more precise asymptotic expansion of $K^{-1}_{x,y}$.

5. Critical Dimer Model on Infinite Fisher Graphs

Let $\mathcal{G}$ be an infinite Fisher graph obtained from an infinite isoradial graph G. Assume that edges of $\mathcal{G}$ are assigned the dimer critical weight function ν. In this section, we give a full description of the critical dimer model on the Fisher graph $\mathcal{G}$, consisting of explicit expressions which only depend on the local geometry of the underlying isoradial graph G. More precisely, in Sect. 5.1, using the method of [dT07b], we give an explicit local formula for a Gibbs measure on dimer configurations of $\mathcal{G}$, involving the inverse Kasteleyn matrix of Theorem 5. When the graph is periodic, this measure coincides with the Gibbs measure of [BdT08], obtained as the weak limit of Boltzmann measures on a natural toroidal exhaustion. Then, in Sect. 5.2, we assume that the graph $\mathcal{G}$ is periodic and, using the method of [Ken02], we give an explicit local formula for the free energy of the critical dimer model on $\mathcal{G}$. As a corollary, we obtain Baxter's celebrated formula for the free energy of the critical Z-invariant Ising model.

5.1. Local formula for the critical dimer Gibbs measure. A *Gibbs measure* on the set of dimer configurations $\mathcal{M}(\mathcal{G})$ of $\mathcal{G}$, is a probability measure on $\mathcal{M}(\mathcal{G})$, which satisfies the following. If one fixes a perfect matching in an annular region of $\mathcal{G}$, then perfect matchings inside and outside of this annulus are independent. Moreover, the probability of occurrence of an interior matching is proportional to the product of the edge weights.

In order to state Theorem 9, we need the following definition. Let $\mathcal{E}$ be a finite subset of edges of $\mathcal{G}$. The *cylinder set* $A_{\mathcal{E}}$ is defined to be the set of dimer configurations of $\mathcal{G}$ containing the subset of edges $\mathcal{E}$; it is often convenient to identify $\mathcal{E}$ with $A_{\mathcal{E}}$. Let $\mathcal{F}$ be the σ-algebra of $\mathcal{M}(\mathcal{G})$ generated by the cylinder sets $(A_{\mathcal{E}})_{\mathcal{E}\text{ finite}}$. Recall that K denotes the infinite Kasteleyn matrix of the graph $\mathcal{G}$, and let K^{-1} be the matrix inverse given by Theorem 5.

Theorem 9. *There is a unique probability measure $\mathcal{P}$ on $(\mathcal{M}(\mathcal{G}),\mathcal{F})$, such that for every finite collection of edges $\mathcal{E}=\{e_1=x_1y_1,\dots,e_k=x_ky_k\}\subset E(\mathcal{G})$, the probability of the corresponding cylinder set is*

$$\mathcal{P}(A_{\mathcal{E}}) = \mathcal{P}(e_1,\dots,e_k) = \left(\prod_{i=1}^{k} K_{x_i,y_i}\right)\mathrm{Pf}\left((K^{-1})^T_{\{x_1,y_1,\dots,x_k,y_k\}}\right), \qquad (10)$$

where K^{-1} is given by Theorem 5, and $(K^{-1})_{\{x_1,y_1,\dots,x_k,y_k\}}$ is the sub-matrix of K^{-1} whose rows and columns are indexed by vertices $\{x_1,y_1,\dots,x_k,y_k\}$.

Moreover, $\mathcal{P}$ is a Gibbs measure. When $\mathcal{G}$ is $\mathbb{Z}^2$-periodic, $\mathcal{P}$ is the Gibbs measure obtained as the weak limit of the Boltzmann measures $\mathcal{P}_n$ on the toroidal exhaustion $\{\mathcal{G}_n = \mathcal{G}/(n\mathbb{Z}^2)\}_{n\geq 1}$ of $\mathcal{G}$.

Proof. The idea is to use Kolmogorov's extension theorem. The structure of the proof is taken from [dT07b]. Let $\mathcal{E} = \{e_1 = x_1y_1, \dots e_n = x_ny_n\}$ be a finite subset of edges of $\mathcal{G}$. Denote by $\mathcal{F}_{\mathcal{E}}$ the σ-algebra generated by the cylinders $(A_{\mathcal{E}'})_{\mathcal{E}'\subset\mathcal{E}}$. We define an additive function $\mathcal{P}_{\mathcal{E}}$ on $\mathcal{F}_{\mathcal{E}}$, by giving its value on every cylinder set $A_{\mathcal{E}'}$, with $\mathcal{E}' = \{e_{i_1}, \dots, e_{i_k}\} \subset \mathcal{E}$:

$$\mathcal{P}_{\mathcal{E}}(A_{\mathcal{E}'}) = \left(\prod_{j=1}^{k} K_{x_{i_j},y_{i_j}}\right) \mathrm{Pf}\left((K^{-1})^T_{\{x_{i_1},y_{i_1},\dots,x_{i_k},y_{i_k}\}}\right).$$

It is a priori not obvious that $\mathcal{P}_{\mathcal{E}}$ defines a probability measure on $\mathcal{F}_{\mathcal{E}}$. Let us show that this is indeed the case. Let $V_{\mathcal{E}}$ be the subset of vertices of G to which the decorations containing $x_1, y_1, \dots, x_n, y_n$ retract. Define Q to be a simply-connected subset of rhombi of $G^{\diamond}$ containing all rhombi adjacent to vertices of $V_{\mathcal{E}}$.

By Proposition 1 of [dT07b], there exists a $\mathbb{Z}^2$-periodic rhombus tiling of the plane containing Q, which we denote by $G^{\diamond p}$. Moreover, the rhombus tiling $G^{\diamond p}$ is the diamond graph of a unique isoradial graph G^p whose vertices contain the subset $V_{\mathcal{E}}$. Let $\mathcal{G}^p$ be the Fisher graph of G^p, then the graphs $\mathcal{G}$ and $\mathcal{G}^p$ coincide on a ball B containing $x_1, y_1, \dots, x_n, y_n$.

Endow edges of $\mathcal{G}^p$ with the critical weights and a periodic Kasteleyn orientation, coinciding with the orientation of the edges of $\mathcal{G}$ on the common ball B. We know by Remark 6, that the coefficients of the inverses $(K^p)^{-1}$ and K^{-1} given by Theorem 5 are equal for all pairs of vertices in B.

On the other hand, we know by Corollary 7 that the coefficient $(K^p)^{-1}_{x,y} \to 0$, as $|\mathbf{x} - \mathbf{y}| \to \infty$. By Proposition 5 of [BdT08], stating uniqueness of the inverse Kasteleyn matrix decreasing at infinity in the periodic case, we deduce that $(K^p)^{-1}$ is in fact the inverse computed in [BdT08] by Fourier transform. In Theorem 6 of [BdT08], we use the inverse $(K^p)^{-1}$ to construct the Gibbs measure $\mathcal{P}^p$ on $\mathcal{M}(\mathcal{G}^p)$ obtained as the weak limit of Boltzmann measures on the natural toroidal exhaustion of $\mathcal{G}^p$, which has the following explicit expression: let $\{e'_1 = x'_1y'_1, \dots, e'_k = x'_ky'_k\}$ be a subset of edges of $\mathcal{G}^p$, then

$$\mathcal{P}^p(e'_1, \dots e'_k) = \left(\prod_{j=1}^{k} K^p_{x'_jy'_j}\right) \mathrm{Pf}\left(((K^p)^{-1})^T_{\{x'_1,\dots,y'_k\}}\right).$$

In particular, the expression of the restriction of $\mathcal{P}^p$ to events involving edges in $\mathcal{E}$ (seen as edges of $\mathcal{G}^p$) is equal to the formula defining $\mathcal{P}_{\mathcal{E}}$. As a consequence, $\mathcal{P}_{\mathcal{E}}$ is a probability measure on $\mathcal{F}_{\mathcal{E}}$.

The fact that Kolmogorov's consistency relations are satisfied by the collection of probability measures $(\mathcal{P}_{\mathcal{E}})_{\mathcal{E}}$ is immediate once we notice that any finite number of these probability measures can be interpreted as finite-dimensional marginals of a Gibbs measure on dimer configurations of a periodic graph with a large enough fundamental domain. The measure $\mathcal{P}$ of Theorem 9 is thus the one given by Kolmogorov's extension theorem applied to the collection $(\mathcal{P}_{\mathcal{E}})_{\mathcal{E}}$. The Gibbs property follows from the Gibbs property of the probability measures $\mathcal{P}^p$. □

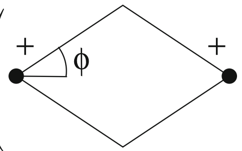
The probability of single edges are computed in details in Appendix A, using the explicit form for K^{-1} given by Theorem 5. Consider a rhombus of $G^\diamond$ with vertices $\mathbf{x}, \mathbf{t}, \mathbf{y}, \mathbf{u}$: $\mathbf{x}$ and $\mathbf{y}$ are vertices of G; $\mathbf{t}$ and $\mathbf{u}$ are vertices of G^*. Let θ be the half-angle of this rhombus, measured at $\mathbf{x}$. Denote by e the edge of $\mathcal{G}$ coming from the edge $\mathbf{xy}$ of G in the decoration process. Then,

Proposition 10. *The probability of occurrence $\mathcal{P}(e)$ of the edge e in a dimer configuration of $\mathcal{G}$ is:*

$$\mathcal{P}(e) = \frac{1}{2} + \frac{\pi - 2\theta}{2\pi \cos\theta}. \tag{11}$$

From this, we can deduce the following probability for the Ising model on the dual graph G^*. In Fisher's correspondence, the presence of the edge e in the dimer configuration corresponds to the fact that $\mathbf{xy}$ is not covered by a piece of contour. In the low temperature expansion of the Ising model, this thus means that the two spins at $\mathbf{t}$ and $\mathbf{u}$ have the same sign. Since they can be both + or −, the probability that they are both +, as a function of $\phi = \frac{\pi}{2} - \theta$, the angle of the rhombus measured at $\mathbf{u}$, is:

Corollary 11.

$\mathcal{P}$ () $= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi - 2\theta}{2\pi\cos\theta}\right) = \frac{1}{4} + \frac{\phi}{2\pi \sin\phi}.$

5.2. Free energy of the critical dimer model. In this section, we suppose that the isoradial graph G is periodic, so that the corresponding Fisher graph $\mathcal{G}$ is. A natural exhaustion of $\mathcal{G}$ by toroidal graphs is given by $\{\mathcal{G}_n\}_{n\geq 1}$, where $\mathcal{G}_n = \mathcal{G}/n\mathbb{Z}^2$, and similarly for G. The graphs $\mathcal{G}_1$ and G_1 are known as the *fundamental domains* of $\mathcal{G}$ and G respectively. In order to shorten notations in Theorem 12 below and in the proof, we write $|E_n| = |E(G_n)|$, and $|V_n| = |V(G_n)|$.

The *free energy per fundamental domain* of the critical dimer model on the Fisher graph $\mathcal{G}$ is denoted f_D, and is defined by:

$$f_D = -\lim_{n\to\infty} \frac{1}{n^2} \log \mathcal{Z}_n^\nu,$$

where $\mathcal{Z}_n^\nu$ is the dimer partition function of the graph $\mathcal{G}_n$, whose edges are assigned the critical weight function ν.

Theorem 12. *The free energy per fundamental domain of the critical dimer model on the Fisher graph $\mathcal{G}$ is given by:*

$$\begin{aligned} f_D &= -(|E_1| + |V_1|)\frac{\log 2}{2} \\ &\quad + \sum_{e\in E(G_1)} \left[\frac{\pi - 2\theta_e}{2\pi}\log\tan\theta_e - \frac{1}{2}\log\cot\frac{\theta_e}{2} - \frac{1}{\pi}\left(L(\theta_e) + L\left(\frac{\pi}{2} - \theta_e\right)\right)\right], \end{aligned}$$

where L is the Lobachevsky function, $L(x) = -\int_0^x \log 2\sin t \, dt$.

Proof. For the proof, we follow the argument given by Kenyon [Ken02] to compute the normalized determinant of the Laplacian and of the Dirac operator on isoradial graphs. See also [dT07a] for a detailed computation of the free energy of dimer models on isoradial bipartite graphs.

In [BdT08], we proved that

$$f_D = -\frac{1}{2}\iint_{\mathbb{T}^2} \log\det \hat{K}(z,w)\frac{\mathrm{d}z}{2i\pi z}\frac{\mathrm{d}w}{2i\pi w},$$

where $\hat{K}(z,w)$ is the Fourier transform of the infinite periodic Kasteleyn matrix of the critical dimer model on $\mathcal{G}$.

We need the following definition of [KS05]. Recall that $G^\diamond$ is the diamond graph associated to the isoradial graph G. A *train-track* of $G^\diamond$ is a path of edge-adjacent rhombi of $G^\diamond$, which does not turn: on entering a face, it exits along the opposite edge. As a consequence, each rhombus in a train-track has an edge parallel to a fixed unit vector. We assume that each train-track is extended as far as possible in both directions, so that it is a bi-infinite path.

The idea of the proof is to first understand how the free energy f_D is changed when the isoradial embedding of the graph G is modified by tilting a family of parallel train-tracks. If we can compute the free energy in an extreme situation, that is when all the rhombi are flat, *i.e* with half-angles equal to 0 or $\pi/2$, then we can compute the free energy of the initial graph by integrating the variation of the free energy along the deformation from the trivial flat embedding back to the initial graph, by tilting all the families of train-tracks successively until all rhombi recover their original shape. Let G^{flat} be the trivial flat isoradial graph, and let $\mathcal{G}^{\text{flat}}$ be the corresponding Fisher graph.

Consider a train-track T of $G^\diamond$. All the rhombi of the train-track have a common parallel, with an angle equal to α. Let us compute the derivative of f_D as we tilt the train-track T as well as all its copies. The function $\log\det \hat{K}(z,w)$ is integrable over the unit torus, is a differentiable function of α for all $(z,w)\in\mathbb{T}^2\backslash\{1,1\}$, and the derivative is also uniformly integrable on the torus. The dimer free energy f_D is thus differentiable with respect to α, and

$$\begin{aligned}\frac{\mathrm{d}f_D}{\mathrm{d}\alpha} &= -\frac{1}{2}\iint_{\mathbb{T}^2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\det\hat{K}(z,w)\frac{\mathrm{d}z}{2i\pi z}\frac{\mathrm{d}w}{2i\pi w}\\ &= -\frac{1}{2}\sum_{u,v\in V(\mathcal{G}_1)}\iint_{\mathbb{T}^2}\hat{K}^{-1}(z,w)_{v,u}\frac{\mathrm{d}\hat{K}(z,w)_{u,v}}{\mathrm{d}\alpha}\frac{\mathrm{d}z}{2i\pi z}\frac{\mathrm{d}w}{2i\pi w}\\ &= -\sum_{e=uv\in E(\mathcal{G}_1)}K^{-1}_{v,u}\frac{\mathrm{d}K_{u,v}}{\mathrm{d}\alpha} = -\sum_{e\in E(G_1)}\mathcal{P}(e)\frac{\mathrm{d}\log\nu_e}{\mathrm{d}\alpha}.\end{aligned}$$

In the last line, we used the uniqueness of the inverse Kasteleyn matrix whose coefficients decrease at infinity, given by Proposition 5 of [BdT08]. We also used the fact that if $e=uv$, then $\nu_e=|K_{u,v}|$ and $\mathcal{P}(e)=K_{u,v}K^{-1}_{v,u}$. Note that the sum is restricted to edges coming from $E(G_1)$, because they are the only edges of $\mathcal{G}_1$ with a weight depending on an angle.

There is in this sum a term for every rhombus in the fundamental domain G_1, and that term only depends on the half-angle of this rhombus. The variation of the free energy of the dimer model along the deformation from $\mathcal{G}^{\text{flat}}$ to $\mathcal{G}$ is thus, up to an additive constant, the sum over all rhombi in a fundamental domain of the variation of a function $\mathsf{f}(\theta)$ depending only on the geometry of the rhombus:

$$\Delta f_D = f_D(\mathcal{G}) - f_D(\mathcal{G}^{\text{flat}}) = \sum_{e \in E(G_1)} \mathsf{f}(\theta_e) - \mathsf{f}(\theta_e^{\text{flat}}),$$

with

$$\frac{\mathrm{d}\mathsf{f}(\theta)}{\mathrm{d}\theta} = -\mathcal{P}(\theta) \frac{\mathrm{d} \log \nu(\theta)}{\mathrm{d}\theta},$$

where $\mathcal{P}(\theta)$ is the probability of occurrence in a dimer configuration of the edge of $\mathcal{G}$ coming from an edge of G, whose half rhombus-angle is θ; and θ_e^{flat} is the angle, equal to 0 or $\frac{\pi}{2}$, of the degenerate rhombus in G^{flat} associated to the edge e.

Recalling that by definition of ν, and by Eq. (11):

$$\nu(\theta) = \cot \frac{\theta}{2}, \quad \mathcal{P}(\theta) = \frac{1}{2} + \frac{\pi - 2\theta}{2\pi \cos \theta},$$

and using the fact that $\frac{\mathrm{d}}{\mathrm{d}\theta} \log \cot \frac{\theta}{2} = -\frac{1}{\sin \theta}$, we deduce an explicit formula for the derivative of $\mathsf{f}(\theta)$:

$$\frac{\mathrm{d}\mathsf{f}(\theta)}{\mathrm{d}\theta} = \left(\frac{1}{2} + \frac{\pi - 2\theta}{2\pi \cos \theta} \right) \frac{1}{\sin \theta} = \frac{1}{2 \sin \theta} + \frac{\pi - 2\theta}{\pi \sin 2\theta}.$$

Using integration by parts, a primitive of $\frac{\mathrm{d}\mathsf{f}(\theta)}{\mathrm{d}\theta}$ is given by:

$$\mathsf{f}(\theta) = \frac{1}{2} \log \tan \frac{\theta}{2} + \frac{\pi - 2\theta}{2\pi} \log \tan \theta - \frac{1}{\pi} \left(L(\theta) + L\left(\frac{\pi}{2} - \theta\right) \right).$$

The problem is that $\mathsf{f}(\theta)$ goes to $-\infty$ when θ approaches 0, and thus the free energy diverges when the graph G becomes flat. One can instead study s_D, the *entropy* of the corresponding dimer model, defined by

$$s_D = -f_D - \sum_{e \in E(\mathcal{G}_1)} \mathcal{P}(e) \log \nu_e, \tag{12}$$

which behaves well when the embedding degenerates, since it is insensible to gauge transformations, and thus provides a better measure of the disorder in the model.

The sum in (12) is over all edges in the fundamental domain $\mathcal{G}_1$ of the Fisher graph $\mathcal{G}$. But since edges of the decoration have weight 1, only the edges coming from edges of G contribute. Thus, as for f_D, the variation of s_D can be written as the sum of contributions $\mathsf{s}(\theta_e)$ of all rhombi in a fundamental domain G_1,

$$\Delta s_D = s_D(\mathcal{G}) - s_D(\mathcal{G}^{\text{flat}}) = \sum_{e \in E(G_1)} \mathsf{s}(\theta_e) - \mathsf{s}(\theta_e^{\text{flat}}), \tag{13}$$

where

$$\begin{aligned}
\mathsf{s}(\theta) &= -\mathsf{f}(\theta) - \mathcal{P}(\theta) \log \nu(\theta) \\
&= -\frac{1}{2} \log \tan \frac{\theta}{2} - \frac{\pi - 2\theta}{2\pi} \log \tan \theta \\
&\quad + \frac{1}{\pi} \left(L(\theta) + L\left(\frac{\pi}{2} - \theta\right) \right) - \left(\frac{1}{2} + \frac{\pi - 2\theta}{2\pi \cos \theta} \right) \log \cot \frac{\theta}{2} \\
&= \frac{\pi - 2\theta}{2\pi} \left(\log \cot \theta - \frac{\log \cot \frac{\theta}{2}}{\cos \theta} \right) + \frac{1}{\pi} \left(L(\theta) + L\left(\frac{\pi}{2} - \theta\right) \right).
\end{aligned}$$

Since $\lim_{\theta\to 0}\left(\log\cot\theta - \frac{\log\cot\frac{\theta}{2}}{\cos\theta}\right) = -\log 2$, and $L(0) = L\left(\frac{\pi}{2}\right) = 0$, we deduce that the values of $\mathsf{s}(\theta^{\text{flat}})$ are given by the following limits:

$$\lim_{\theta\to 0}\mathsf{s}(\theta) = -\frac{1}{2}\log 2, \quad \text{and} \quad \lim_{\theta\to\frac{\pi}{2}}\mathsf{s}(\theta) = 0.$$

Let us now evaluate $s_D(\mathcal{G}^{\text{flat}})$. Since the sum of rhombus-angles around a vertex is 2π, and since rhombus half-angles are equal to 0 or $\pi/2$, there is in G^{flat}, around each vertex, exactly two rhombi with half-angle θ^{flat} equal to $\frac{\pi}{2}$. Let us analyze the dimer model on $\mathcal{G}_n^{\text{flat}}$: the "long" edges with weight equal to ∞ (corresponding to rhombus half-angles 0) are present in a random dimer configuration with probability 1. The configuration of the long edges is thus frozen. The "short" ones with weight equal to 1 (corresponding to rhombus half-angles $\pi/2$) are present in a random dimer configuration with probability $\frac{1}{2}+\frac{1}{\pi}$.

As noted above, around every vertex of G_n^{flat}, there are exactly two "short" edges. This implies that "short" edges form a collection of k disjoint cycles covering all vertices of the graph G_n^{flat}, for some positive integer k. Such a cycle cannot be trivial, because surrounding a face would require an infinite number of flat rhombi, but there is only a finite number of them in G_n^{flat}. Since these cycles are disjoint, their number is bounded by the number of edges crossing the "boundary" of G_n^{flat}. Therefore $k = O(n) = o(n^2)$.

Since all "long" edges are taken in a random dimer configuration, this implies that, for every cycle of "short" ones, edges are either all present, or all absent. The logarithm of the number of configurations for these cycles grows slower than $O(n^2)$, and therefore does not contribute to the entropy. The main contribution comes from the decorations, which have two configurations each:

$$s_D(\mathcal{G}^{\text{flat}}) = |V_1|\log 2.$$

The total number of "short" edges is $|V_n|$ (two halves per vertex of G_n), and thus the number of long ones is $|E_n| - |V_n|$. From (13), we deduce that:

$$\begin{aligned} s_D &= (|E_1| + |V_1|)\frac{\log 2}{2} \\ &\quad + \sum_{e\in E(G_1)}\left[\frac{\pi - 2\theta_e}{2\pi}\left(\log\cot\theta_e - \frac{\log\cot\frac{\theta_e}{2}}{\cos\theta_e}\right) + \frac{1}{\pi}\left(L(\theta_e) + L\left(\frac{\pi}{2} - \theta_e\right)\right)\right], \end{aligned}$$

and using (12), we deduce Theorem 12. □

The *free energy per fundamental domain* of the critical Z-invariant Ising model, denoted f_I, is defined by:

$$f_I = -\lim_{n\to\infty}\frac{1}{n^2}\log Z_n^J,$$

where Z_n^J is the partition function of the critical Z-invariant Ising model on the toroidal graph G_n. In [Bax86], Baxter gives an explicit expression for the free energy of Z-invariant Ising models (not only critical), by transforming the graph G with star-triangle transformations to make it look like large pieces of $\mathbb{Z}^2$ glued together, and making use of the celebrated computation of Onsager [Ons44] on $\mathbb{Z}^2$. As a corollary to Theorem 12, we obtain an alternative proof of this formula at the critical point.

Theorem 13 ([Bax86]).

$$f_I = -|V_1|\frac{\log 2}{2} - \sum_{e\in E(G_1)}\left[\frac{\theta_e}{\pi}\log\tan\theta_e + \frac{1}{\pi}\left(L(\theta_e) + L\left(\frac{\pi}{2}-\theta_e\right)\right)\right].$$

Proof. Using the high temperature expansion, and Fisher's correspondence, we have from Eq. (2) that:

$$Z_n^J = \left(\prod_{e\in E(G_n)}\sinh J(\theta_e)\right)\mathcal{Z}_n^\nu = \left(\prod_{e\in E(G_1)}\sinh J(\theta_e)\right)^{n^2}\mathcal{Z}_n^\nu.$$

As a consequence,

$$f_I = f_D - \sum_{e\in E(G_1)}\log\sinh J(\theta_e).$$

Moreover,

$$\sinh J(\theta) = \sinh\log\sqrt{\frac{1+\sin\theta}{\cos\theta}} = \sqrt{\frac{\tan\frac{\theta}{2}\tan\theta}{2}},$$

so that by Theorem 12:

$$\begin{aligned} f_I &= -(|E_1|+|V_1|)\frac{\log 2}{2} \\ &\quad + \sum_{e\in E(G_1)}\left[\frac{\pi-2\theta_e}{2\pi}\log\tan\theta_e - \frac{1}{2}\log\cot\frac{\theta_e}{2} - \frac{1}{\pi}\left(L(\theta_e)+L\left(\frac{\pi}{2}-\theta_e\right)\right)\right] \\ &\quad +|E_1|\frac{\log 2}{2} - \sum_{e\in E(G_1)}\frac{1}{2}\left[\log\tan\frac{\theta_e}{2} + \log\tan\theta_e\right] \\ &= -|V_1|\frac{\log 2}{2} - \sum_{e\in E(G_1)}\left[\frac{\theta_e}{\pi}\log\tan\theta_e + \frac{1}{\pi}\left(L(\theta_e)+L\left(\frac{\pi}{2}-\theta_e\right)\right)\right]. \end{aligned}$$

□

The *critical Laplacian matrix* on G is defined in [Ken02] by:

$$\Delta_{u,v} = \begin{cases} \tan\theta_{uv} & \text{if } u\sim v \\ -\sum_{u'\sim u}\tan\theta_{uu'} & \text{if } u=v \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic polynomial* of the critical Laplacian on G, denoted $P_\Delta(z,w)$, is defined by $P_\Delta(z,w) = \det\widehat{\Delta}(z,w)$, where $\widehat{\Delta}(z,w)$ is the Fourier transform of the critical Laplacian Δ.

In a similar way, the *characteristic polynomial* of the critical dimer model on the Fisher graph $\mathcal{G}$, denoted by $P(z,w)$ is defined by $P(z,w) = \det\widehat{K}(z,w)$, where $\widehat{K}(z,w)$ is the Fourier transform of the Kasteleyn matrix K of the graph $\mathcal{G}$ with critical weights.

In Theorem 8 of [BdT08], we prove that both characteristic polynomials are equal up to a non-zero multiplicative constant. Using Theorem 12, and [Ken02], we can now determine this constant explicitly.

Corollary 14. *The characteristic polynomials of the critical dimer model on the Fisher graph $\mathcal{G}$, and of the critical Laplacian on the graph G, are related by the following explicit multiplicative constant:*

$$\forall (z,w)\in\mathbb{C}^2,\qquad P(z,w)=2^{|V_1|}\prod_{e\in E(G_1)}\left(\cot^2\frac{\theta_e}{2}-1\right)P_\Delta(z,w).$$

Proof. Let $c\neq 0$, be the constant of proportionality given by Theorem 8 of [BdT08]:

$$P(z,w)=cP_\Delta(z,w).$$

Taking the logarithm on both sides, and integrating with respect to (z,w) over the unit torus yields:

$$\log c=-2f_D-\iint_{\mathbb{T}^2}\log P_\Delta(z,w)\frac{\mathrm{d}z}{2i\pi z}\frac{\mathrm{d}w}{2i\pi w}. \tag{14}$$

Moreover, by [Ken02] we have:

$$\iint_{\mathbb{T}^2}\log P_\Delta(z,w)\frac{\mathrm{d}z}{2i\pi z}\frac{\mathrm{d}w}{2i\pi w}=2\sum_{e\in E(G_1)}\left[\frac{\theta_e}{\pi}\log\tan\theta_e+\frac{1}{\pi}\left(L(\theta_e)+L\left(\frac{\pi}{2}-\theta_e\right)\right)\right].$$

Plugging this and Theorem 12 into (14), we obtain:

$$\begin{aligned}\log c&=(|E_1|+|V_1|)\log 2+\sum_{e\in E(G_1)}\left(\log\cot\frac{\theta_e}{2}+\log\cot\theta_e\right)\\&=\log\left(2^{|E_1|+|V_1|}\prod_{e\in E(G_1)}\cot\frac{\theta_e}{2}\cot\theta_e\right)\\&=\log\left(2^{|V_1|}\prod_{e\in E(G_1)}\left(\cot^2\frac{\theta_e}{2}-1\right)\right).\end{aligned}$$

□

6. Proof of Theorem 5

Let us recall the setting: G is an infinite isoradial graph, and $\mathcal{G}$ is the corresponding Fisher graph whose edges are assigned the dimer critical weight function; K denotes the infinite Kasteleyn matrix of the critical dimer model on the graph $\mathcal{G}$, and defines an operator acting on functions of the vertices of $\mathcal{G}$. We now prove Theorem 5, *i.e.* we show that $KK^{-1}=\mathrm{Id}$, where K^{-1} is given by:

$$\forall x,\ y\in V(\mathcal{G}),\quad K^{-1}_{x,y}=\frac{1}{4\pi^2}\oint_{\mathcal{C}_{x,y}}f_x(\lambda)f_y(-\lambda)\operatorname{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log\lambda\,\mathrm{d}\lambda+C_{x,y},$$

where f_x and f_y are defined by Eq. (3), $\operatorname{Exp}_{\mathbf{x},\mathbf{y}}$ by Eq. (6), and $C_{x,y}$ by Eq. 8.

This section is organized as follows. In Sect. 6.1, we prove that the function $f_x(\lambda)\operatorname{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$, seen as a function of $x\in V(\mathcal{G})$, is in the kernel of the Kasteleyn operator K. Then, in Sect. 6.2, we give the general idea of the argument, inspired from

[Ken02], used to prove $KK^{-1} = \mathrm{Id}$. This motivates the delicate part of the proof. Indeed, for this argument to run through, it is not enough to have the contour of integration $\mathcal{C}_{x,y}$ to be defined as a simple closed curve containing all poles of the integrand, and avoiding *the half-line* $d_{x,y}$; we need it to *avoid an angular sector* $s_{x,y}$*, which contains the half-line* $d_{x,y}$ *and avoids all poles of the integrand*. These angular sectors are then defined in Sect. 6.3. The delicate part of the proof, which strongly relies on the definition of the angular sectors, is given in Sect. 6.4.

6.1. Kernel of K. In this section we prove that the function $f_x(\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$, seen as a function of $x \in V(\mathcal{G})$, is in the kernel of the Kasteleyn operator K. Recall that the function f_x is defined in Eq. (3), and $\mathrm{Exp}_{\mathbf{x},\mathbf{y}}$ in Eq. (6).

Proposition 15. *Let* x, y *be two vertices of* $\mathcal{G}$*, and let* x_1, x_2, x_3 *be the three neighbors of* x *in* $\mathcal{G}$*, then for every* $\lambda \in \mathbb{C}$*:*

$$\sum_{i=1}^{3} K_{x,x_i} f_{x_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) = 0.$$

Proof. There are three cases to consider, depending on whether the vertex x is of type 'w', 'z' or 'v'.

If $x = w_k(\mathbf{x})$ *for some* $k \in \{1,\ldots,d(\mathbf{x})\}$. Then the three neighbors of x are $x_1 = z_k(\mathbf{x})$, $x_2 = z_{k+1}(\mathbf{x})$ and $x_3 = v_k(\mathbf{x})$. Since x, x_1, x_2, x_3 all belong to the same decoration, we omit the argument $\mathbf{x}$. By our choice of Kasteleyn orientation, we have:

$$K_{w_k,z_k} = -1, \quad K_{w_k,z_{k+1}} = \varepsilon_{w_k,z_{k+1}}, \quad K_{w_k,v_k} = 1.$$

By definition of the angles in $\mathbb{R}/4\pi\mathbb{Z}$ associated to vertices of $\mathcal{G}$, see (4), we have:

$$\alpha_{z_{k+1}} = \begin{cases} \alpha_{w_k} & \text{if } \varepsilon_{w_k,z_{k+1}} = 1 \\ \alpha_{w_k} + 2\pi & \text{if } \varepsilon_{w_k,z_{k+1}} = -1. \end{cases}$$

Using the definition of the function f, we deduce that:

$$K_{w_k,z_{k+1}} f_{z_{k+1}}(\lambda) = -f_{w_k}(\lambda). \tag{15}$$

As a consequence,

$$\begin{aligned} &\sum_{i=1}^{3} K_{x,x_i} f_{x_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) \\ &\quad= \left[K_{w_k,z_k} f_{z_k}(\lambda) + K_{w_k,z_{k+1}} f_{z_{k+1}}(\lambda) + K_{w_k,v_k} f_{v_k}(\lambda)\right]\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \\ &\quad= \left[-f_{z_k}(\lambda) - f_{w_k}(\lambda) + f_{v_k}(\lambda)\right]\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda). \end{aligned} \tag{16}$$

Since by definition we have $f_{v_k}(\lambda) = f_{w_k}(\lambda) + f_{z_k}(\lambda)$, we deduce that (16) is equal to 0.

If $x = z_k(\mathbf{x})$ *for some* $k \in \{1,\ldots,d(\mathbf{x})\}$. Then the three neighbors of x are $x_1 = w_{k-1}(\mathbf{x})$, $x_2 = w_k(\mathbf{x})$ and $x_3 = v_k(\mathbf{x})$. By our choice of Kasteleyn orientation, we have:

$$K_{z_k,w_{k-1}} = \varepsilon_{z_k,w_{k-1}}, \quad K_{z_k,w_k} = 1, \quad K_{z_k,v_k} = -1.$$

Similarly to the case where $x = w_k$, we have:

$$K_{z_k,w_{k-1}} f_{w_{k-1}}(\lambda) = f_{z_k}(\lambda).$$

As a consequence,

$$\begin{aligned}
&\sum_{i=1}^{3} K_{x,x_i} f_{x_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) \\
&\quad = \left[K_{z_k,w_{k-1}} f_{w_{k-1}}(\lambda) + K_{z_k,w_k} f_{w_k}(\lambda) + K_{z_k,v_k} f_{v_k}(\lambda)\right]\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \\
&\quad = \left[f_{z_k}(\lambda) + f_{w_k}(\lambda) - f_{v_k}(\lambda)\right]\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda). \qquad (17)
\end{aligned}$$

Again, we deduce that (17) is equal to 0.

If $x = v_k(\mathbf{x})$ *for some* $k \in \{1, \dots, d(\mathbf{x})\}$. Then the three neighbors of x are $x_1 = z_k(\mathbf{x})$, $x_2 = w_k(\mathbf{x})$ and $x_3 = v_\ell(\mathbf{x}')$, where ℓ and $\mathbf{x}'$ are such that $v_k(\mathbf{x}) \sim v_\ell(\mathbf{x}')$. Although all vertices do not belong to the same decoration, we omit the arguments $\mathbf{x}$ and $\mathbf{x}'$, knowing that the index k (resp. ℓ) refers to the vertex $\mathbf{x}$ (resp. $\mathbf{x}'$). By our choice of Kasteleyn orientation, we have:

$$K_{v_k,z_k} = 1, \quad K_{v_k,w_k} = -1, \quad K_{v_k,v_\ell} = \varepsilon_{v_k,v_\ell} \cot\left(\frac{\alpha_{w_k} - \alpha_{z_k}}{4}\right).$$

Using the fact that $\mathrm{Exp}_{\mathbf{x}',\mathbf{y}}(\lambda) = \mathrm{Exp}_{\mathbf{x}',\mathbf{x}}(\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$, we deduce:

$$\begin{aligned}
&\sum_{i=1}^{3} K_{x,x_i} f_{x_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) \\
&\quad = \left[K_{v_k,z_k} f_{z_k}(\lambda) + K_{v_k,w_k} f_{w_k}(\lambda) + K_{v_k,v_\ell}\,\mathrm{Exp}_{\mathbf{x}',\mathbf{x}}(\lambda) f_{v_\ell}(\lambda)\right]\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \\
&\quad = \left[f_{z_k}(\lambda) - f_{w_k}(\lambda) + \varepsilon_{v_k,v_\ell} \cot\left(\frac{\alpha_{w_k} - \alpha_{z_k}}{4}\right)\mathrm{Exp}_{\mathbf{x}',\mathbf{x}}(\lambda) f_{v_\ell}(\lambda)\right]\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda).
\end{aligned}$$

Moreover, by definition of the angles in $\mathbb{R}/4\pi\mathbb{Z}$ associated to vertices of $\mathcal{G}$, see (5), we have:

$$\alpha_{w_\ell} = \alpha_{w_k} - \varepsilon_{v_k,v_\ell}\pi, \quad \alpha_{z_\ell} = \alpha_{z_k} - \varepsilon_{v_k,v_\ell}\pi,$$

and using the definition (3) of the function f, we deduce:

$$f_{v_\ell}(\lambda) = \left(\frac{e^{i\frac{\alpha_{w_\ell}}{2}}}{e^{i\alpha_{w_\ell}} - \lambda} - \frac{e^{i\frac{\alpha_{z_\ell}}{2}}}{e^{i\alpha_{z_\ell}} - \lambda}\right) = i\varepsilon_{v_k,v_\ell}\left(\frac{e^{i\frac{\alpha_{w_k}}{2}}}{e^{i\alpha_{w_k}} + \lambda} - \frac{e^{i\frac{\alpha_{z_k}}{2}}}{e^{i\alpha_{z_k}} + \lambda}\right),$$

so that:

$$\begin{aligned}
&K_{v_k,v_\ell}\,\mathrm{Exp}_{\mathbf{x}',\mathbf{x}}(\lambda) f_{v_\ell}(\lambda) \\
&\quad = i\cot\left(\frac{\alpha_{w_k} - \alpha_{z_k}}{4}\right)\frac{(e^{i\alpha_{w_k}} + \lambda)(e^{i\alpha_{z_k}} + \lambda)}{(e^{i\alpha_{w_k}} - \lambda)(e^{i\alpha_{z_k}} - \lambda)}\left(\frac{e^{i\frac{\alpha_{w_k}}{2}}}{e^{i\alpha_{w_k}} + \lambda} - \frac{e^{i\frac{\alpha_{z_k}}{2}}}{e^{i\alpha_{z_k}} + \lambda}\right) \\
&\quad = f_{w_k}(\lambda) - f_{z_k}(\lambda). \qquad (18)
\end{aligned}$$

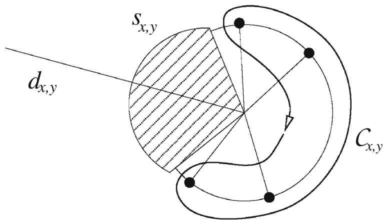


Fig. 9. Definition of the contour of integration $\mathcal{C}_{x,y}$ of the integral term of $K^{-1}_{x,y}$. The poles of the integrand are the thick points

As a consequence,

$$\sum_{i=1}^{3} K_{x,x_i} f_{x_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) = \left[f_{z_k}(\lambda) - f_{w_k}(\lambda) + f_{w_k}(\lambda) - f_{z_k}(\lambda)\right]\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) = 0. \tag{19}$$

□

6.2. General idea of the argument. The general argument used to prove Theorem 5, *i.e.* $KK^{-1} = \mathrm{Id}$, is inspired from [Ken02], where Kenyon computes a local explicit expression for the inverse of the Kasteleyn matrix of the critical dimer model on a bipartite, isoradial graph. It cannot be applied as such to our case of the critical dimer model on the Fisher graph $\mathcal{G}$ of G, which is *not* isoradial, but it is nevertheless useful to sketch the main ideas.

Let x, y be two vertices of $\mathcal{G}$, and let us assume that the contour of integration $\mathcal{C}_{x,y}$ of the integral term of $K^{-1}_{x,y}$, is defined to be a simple closed curve oriented counterclockwise, containing all poles of the integrand and avoiding an angular sector $s_{x,y}$, which contains the half-line $d_{x,y}$, see Fig. 9.

The argument runs as follows. Let x_1, x_2, x_3 be the three neighbors of x in $\mathcal{G}$. When $x \neq y$, the goal is to show that the intersection of the three sectors $\bigcap_{i=1}^{3} s_{x_i,y}$ is non empty. Then, the three contours $\mathcal{C}_{x_i,y}$ can be continuously deformed to a common contour $\mathcal{C}$ without meeting any pole, and

$$\begin{aligned}\sum_{i=1}^{3} K_{x,x_i} \oint_{\mathcal{C}_{x_i,y}} f_{x_i}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda) \log\lambda \frac{\mathrm{d}\lambda}{(2\pi)^2} \\ = \oint_{\mathcal{C}} \left(\sum_{i=1}^{3} K_{x,x_i} f_{x_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda)\right) f_y(-\lambda) \log\lambda \frac{\mathrm{d}\lambda}{(2\pi)^2} = 0,\end{aligned} \tag{20}$$

since by Proposition 15, the sum in brackets is zero. Note that in [Ken02], Kenyon has a result similar to Proposition 15, giving functions that are in the kernel of the Kasteleyn matrix of the dimer model on a bipartite, isoradial graph.

When $x = y$, the goal is to show that the intersection of the three sectors is empty, and explicitly compute:

$$\sum_{i=1}^{3} K_{x,x_i} \oint_{\mathcal{C}_{x_i,x}} f_{x_i}(\lambda) f_x(-\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{x}}(\lambda) \log\lambda \frac{\mathrm{d}\lambda}{(2\pi)^2} = 1. \tag{21}$$

In our case things turn out to be more complicated, since we cannot define angular sectors $s_{x,y}$ so that (20) and (21) hold as such. We define them in such a way that when x and y do not belong to the same triangle of a decoration $\mathbf{x}$, the intersection of the three sectors $\bigcap_{i=1}^{3} s_{x_i,y}$ is non-empty, and (20) holds. Then, in order for $(KK^{-1})_{x,y}$ to be equal to $\delta_{x,y}$ in all other cases, we need to have an additional constant $C_{x,y}$ in the expression of the inverse $K^{-1}_{x,y}$.

Remark 16. Let us mention that in [Ken02], angular sectors are not constructed explicitly neither for the inverse Kasteleyn operator nor for the Green function of the Laplacian on G, since geometric considerations suffice. For the latter, the construction has been carried out in [BMS05], by interpreting the graph G, if the train-tracks of G have only d possible directions, as the projection of a monotonic surface in $\mathbb{Z}^d$ on the plane. The sectors are then defined as the projection of octants, that are sewed together to get a branch covering of the plane.

The sectors appearing here are defined using another approach. Although the two constructions are related,[3] our geometric situation is complicated by the presence of the decorations in $\mathcal{G}$, and there is no natural notion of branched covering in this case. Moreover, we do not restrict ourselves on the number of possible directions for the train-tracks. Since our construction for the generic case also applies to [Ken02], it is thus slightly more general than [BMS05].

6.3. Definition of the angular sectors $s_{x,y}$. Let x, y be two vertices of $\mathcal{G}$. In this section, we construct the angular sector $s_{x,y}$ containing the half-line $d_{x,y}$ and avoiding all poles of the integrand of $K^{-1}_{x,y}$. In order to do this, we first recall some facts about isoradial graphs, then we encode the poles of the integrand of $K^{-1}_{x,y}$ in an edge-path $\gamma_{x,y}$ of the diamond graph $G^\diamond$. Using this path $\gamma_{x,y}$, we define the angular sector $s_{x,y}$.

6.3.1. Minimal paths. Let G be an infinite isoradial graph, and $G^\diamond$ be the associated diamond graph, and let $\mathbf{x}$, $\mathbf{y}$ be two vertices of G. We now define a *minimal path* of $G^\diamond$ from $\mathbf{x}$ to $\mathbf{y}$. This definition uses the notion of *train-track* introduced in Sect. 5.2. We say that a train-track of $G^\diamond$ *separates* $\mathbf{x}$ from $\mathbf{y}$, if when deleting it, $\mathbf{x}$ and $\mathbf{y}$ are in two distinct connected components. Now, observe that each edge of $G^\diamond$ belongs to a unique train-track whose direction is given by that edge, it is called the *train-track corresponding to the edge*. An edge-path γ of $G^\diamond$, from $\mathbf{x}$ to $\mathbf{y}$, is called *minimal*, if all train-tracks corresponding to edges of γ separate $\mathbf{x}$ from $\mathbf{y}$, and all those train-tracks are distinct. In general there is not uniqueness of the minimal path, but all minimal paths consist of the same steps, taken in a different order.

6.3.2. Encoding the poles of the integrand of $K^{-1}_{x,y}$. Let x, y be two vertices of $\mathcal{G}$, and let $\mathbf{x}$, $\mathbf{y}$ be the corresponding vertices of G. In this section, we define an edge-path $\gamma_{x,y}$ of the diamond graph $G^\diamond$ encoding the poles of the integrand $f_x(\lambda) f_y(-\lambda) \operatorname{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \log\lambda$ of $K^{-1}_{x,y}$.

By definition of the exponential function and of a minimal path, the poles of $\operatorname{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$ are encoded in the steps of a minimal path of $G^\diamond$, oriented from $\mathbf{y}$ to $\mathbf{x}$. It would be natural to see the poles of the integrand of $K^{-1}_{x,y}$, as representing the steps of a path of $G^\diamond$ passing

[3] In [BMS05], the sectors are a priori defined, then the vertices are partitioned in classes sharing the same sector. Here we construct for each vertex in $\mathcal{G}$ its own sector.

through $\mathbf{y}$ and $\mathbf{x}$, obtained by adding the steps corresponding to the poles of $f_x(\lambda)$ and $f_y(-\lambda)$. However the situation is a bit trickier due to possible cancellations of these poles with factors appearing in the numerator of $\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$. This is taken into account in the following way.

By definition, the function $f_x(\lambda)$ has either 1 or 2 poles, denoted by $\{e^{i\alpha_j}\}_j$, where $j = 1$ or $j = \{1, 2\}$. Let $T_x = \{T_x^j\}_j$ be the set of corresponding train-tracks. Similarly, define $\{-e^{i\alpha'_j}\}_j$ to be the set of poles of $f_y(-\lambda)$, and $T_y = \{T_y^j\}_j$ to be the corresponding train-tracks.

Let us start from a minimal path $\gamma_{x,y}$ from $\mathbf{y}$ to $\mathbf{x}$. For each j, we do the following procedure: if T_y^j separates $\mathbf{y}$ from $\mathbf{x}$, then the pole $-e^{i\alpha'_j}$ is cancelled by the exponential, and we leave $\gamma_{x,y}$ unchanged. If not, this pole remains, and we extend $\gamma_{x,y}$ by adding the step $-e^{i\alpha'_j}$ at the beginning of $\gamma_{x,y}$. The path obtained is still a path in $G^\diamond$. Denote by $\hat{\mathbf{y}}$ the new starting point of $\gamma_{x,y}$ at distance at most 2 of $\mathbf{y}$.

When dealing with a pole of $f_x(\lambda)$, one needs to be careful since, even when the corresponding train-track separates $\mathbf{y}$ from $\mathbf{x}$, the exponential function might not cancel the pole if it has already canceled the same pole of $f_y(-\lambda)$. This happens when T_x and T_y have a common train-track. The procedure to extend $\gamma_{x,y}$ runs as follows: for each j, if T_x^j separates $\mathbf{y}$ from $\mathbf{x}$, and $T_x^j \notin T_y$, then the pole $e^{i\alpha_j}$ is canceled by the exponential function, and we leave $\gamma_{x,y}$ unchanged. If not, this pole remains, and we extend $\gamma_{x,y}$ by attaching the step $e^{i\alpha_j}$ at the end of $\gamma_{x,y}$. The extended path $\gamma_{x,y}$ obtained in this way is still a path in $G^\diamond$, starting from $\hat{\mathbf{y}}$. Denote by $\hat{\mathbf{x}}$ its ending point, which is at distance at most 2 from $\mathbf{x}$.

Note that the way to extend $\gamma_{x,y}$ may not be unique, but the set of steps are always the same, only the order can differ.

6.3.3. Obtaining an angular sector $s_{x,y}$ from $\gamma_{x,y}$. Now that we have encoded all the poles of the integrand in a path $\gamma_{x,y}$, we want to construct the sector $s_{x,y}$ avoided by the contour $\mathcal{C}_{x,y}$ on which the integral is taken in the definition of $K^{-1}_{x,y}$.

Let us first suppose that $\gamma_{x,y}$ is a minimal path joining $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ in $G^\diamond$. Lemma 17 below describes the repartition of the steps of $\gamma_{x,y}$ on the unit circle. Since the result holds for all minimal paths in $G^\diamond$, we introduce the following more general notations. Let $\mathbf{u}, \mathbf{v}$ be two vertices of $G^\diamond$, and let $\gamma = \{\mathbf{u} = \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n = \mathbf{v}\}$ be a minimal path from $\mathbf{u}$ to $\mathbf{v}$. Denote by $e^{i\theta_j}$ the unit complex number representing the edge from $\mathbf{u}_{j-1}$ to $\mathbf{u}_j$, with $\theta_{j+1} - \theta_j \in (-\pi, \pi)$. The angles θ_j can be defined in $\mathbb{R}$ as follows. Fix some initial value for θ_0, and let $\theta_j = \theta_{j-1} + (\theta_j - \theta_{j-1})$.

Lemma 17. *Let γ be a minimal path as above. Then, there exists an angular sector of size greater than π containing none of the steps $e^{i\theta_j}$ of γ. In other words, the angles θ_j defined in $\mathbb{R}$ satisfy:*

$$\forall j, k, \quad \left|\theta_j - \theta_k\right| < \pi.$$

Proof. Let us begin with a preliminary remark. Take T_1 and T_2 two intersecting train-tracks separating $\mathbf{u}$ from $\mathbf{v}$, and orient the edges parallel to the directions of these train-tracks. The two train-tracks separate the plane into four quadrants, each containing a vertex of the rhombus R at the intersection of T_1 and T_2. Then the edges of R must have the following orientation: the two edges attached to the vertex in the same component as $\mathbf{u}$ (resp. as $\mathbf{v}$) are outgoing (resp. incoming). See Fig. 10 for an illustration.

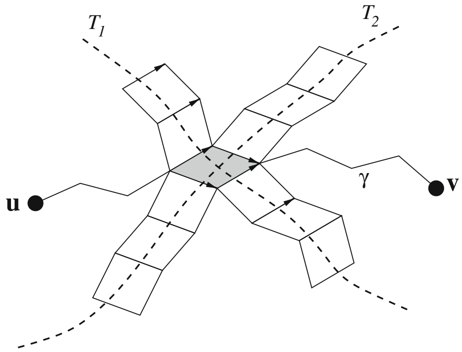


Fig. 10. Orientation of a rhombus at the intersection of two train-tracks separating **u** from **v**

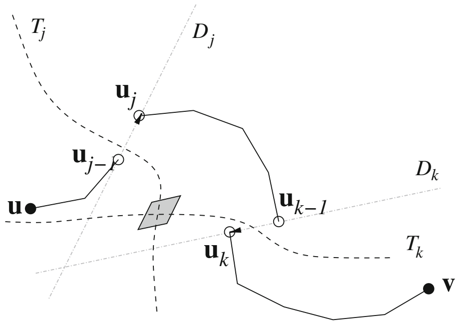


Fig. 11. Two train tracks T_j and T_k separating **u** from **v**, with $|\theta_j - \theta_k| > \pi$

Now suppose that the conclusion of the lemma is not true. Since the path is minimal, all train-tracks are distinct, so that $|\theta_j - \theta_k| \neq \pi$. Thus, there exists two edges in γ with angles $|\theta_j - \theta_k| > \pi$. Without loss of generality, we can suppose that $j < k$ and $\theta_j - \theta_k > \pi$. The situation is represented in Fig. 11.

Let D_j (resp. D_k) be the straight line defined by the vector $e^{i\theta_j}$ (resp. $e^{i\theta_k}$). Then, since the angle $\theta_j - \theta_k$ is larger than π, these two lines must cross below γ. Consider the quadrant defined by D_j and D_k, containing the part of γ from $\mathbf{u}_{j-1}$ to $\mathbf{u}_k$. Then, the part below γ contains both T_j and T_k. These two train-tracks cannot exit above γ since each of them crosses γ exactly once, by definition of minimality. So they must exit below γ. Suppose that they do not intersect, then at least one of them, say T_j, must intersect the line D_j again, but this would imply having negative angles in the rhombi defining this train-track, thus yielding a contradiction. As a consequence, T_j and T_k must intersect. Let R be the rhombus at the intersection. Now have a look at the orientation of the edges of R on Fig. 12. The two edges attached to the vertex in the same quadrant as **y** are not pointing outward, as it should be from Fig. 10. Therefore, the hypothesis we started from is false, and thus

$$\forall j, k \quad |\theta_j - \theta_k| < \pi.$$

□

In the extension procedure of $\gamma_{x,y}$, we may have added steps breaking the minimality property: indeed it is possible that a single train-track corresponds to two different steps

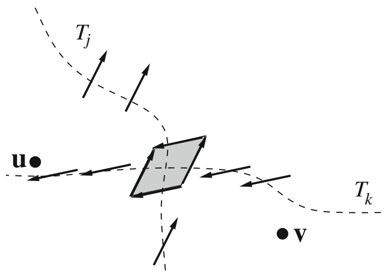


Fig. 12. Orientation of the edges of the rhombus at the intersection between T_j and T_k. Compare with the correct orientation on Fig. 10

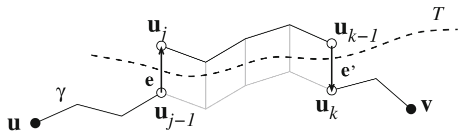


Fig. 13. A quasi-minimal path γ

of $\gamma_{x,y}$. This is the case when a train-track is common to both T_x and T_y. Note that since each of T_x, T_y contains at most two train-tracks, this might occur for at most two train-tracks. If this occurs for exactly one train-track T, we say that the path $\gamma_{x,y}$ is *quasi-minimal*.

Lemma 18 below is the analog, for quasi-minimal paths, of Lemma 17 describing the repartition of its steps on the unit circle. Let γ be a general quasi-minimal path from $\mathbf{u}$ to $\mathbf{v}$, with the notations as above. Denote by T the unique train-track corresponding to two different steps $\mathbf{e}$, $\mathbf{e}'$ of γ. Then T crosses γ twice, *i.e.* an even number of times, implying that T does not separate $\mathbf{u}$ from $\mathbf{v}$. Moreover, if $\pm e^{i\theta}$ denotes the direction of T, then one of $\mathbf{e}$ or $\mathbf{e}'$ is $\pm e^{i\theta}$, and the other is $\mp e^{i\theta}$, see Fig. 13.

Lemma 18. *Let γ be a quasi-minimal path from* $\mathbf{u}$ *to* $\mathbf{v}$*, and let T denote the only train-track corresponding to the steps* $\mathbf{e}$, $\mathbf{e}'$ *of γ. Then there exists an angular sector of size exactly π, delimited by the steps* $\mathbf{e}$, $\mathbf{e}'$ *containing none of the steps $e^{i\theta_j}$ of γ in its interior.*

Proof. Denote $\mathbf{e} = (\mathbf{u}_{j-1}, \mathbf{u}_j)$, respectively $\mathbf{e}' = (\mathbf{u}_{k-1}, \mathbf{u}_k)$, with $j < k$. Suppose, as in the proof of the previous lemma, that there are two steps $e^{i\theta_\ell}$ and $e^{i\theta_m}$ such that $|\theta_\ell - \theta_m| \geq \pi$, with $\ell < m$. Then these two steps must be $\mathbf{e}$ and $\mathbf{e}'$. The other pairs of edges are excluded, by Lemma 17 applied to the pieces of γ from $\mathbf{u}$ to $\mathbf{u}_{k-1}$ and from $\mathbf{u}_j$ to $\mathbf{v}$, that are minimal, and to the minimal path from $\mathbf{u}$ to $\mathbf{v}$ through $\mathbf{u}_{j-1}$ and $\mathbf{u}_k$, using the same steps as γ, except $\mathbf{e}$ and $\mathbf{e}'$. Therefore, $\theta_j = \theta_k + \pi$, and the other angles are all in one of the two sectors of angle π delimited by $\mathbf{e}$ and $\mathbf{e}'$. □

As a consequence of the two previous lemmas, when the path γ is minimal or quasi-minimal, the unit circle deprived from the steps of γ,

$$S^1 \backslash \{e^{i\theta_1}, \ldots, e^{i\theta_n}\},$$

has a connected component S_γ of size at least π. Let us now define the angular sector $s_{x,y}$ avoided by the contour $\mathcal{C}_{x,y}$ on which the integral is taken in the definition of $K^{-1}_{x,y}$.

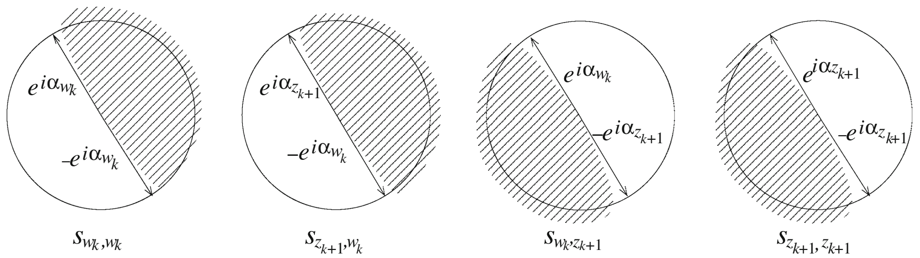


Fig. 14. Definition of the angular sector $s_{x,y}$ (*dashed line*) in Case 1: from left to right, $(x, y) = (w_k, w_k), (z_{k+1}, w_k), (w_k, z_{k+1}), (z_{k+1}, z_{k+1})$

Definition 6.1 (Generic definition of $s_{x,y}$). *Let x and y be two vertices of $\mathcal{G}$. Let $\gamma_{x,y}$ be the path of $G^\diamond$ constructed as above. When the connected component $S_{\gamma_{x,y}}$ exists and is unique, the* **angular sector** *$s_{x,y}$ is defined to be the open cone $\{re^{i\theta}\ |r > 0, \theta \in S_{\gamma_{x,y}}\}$.*

Remark 19.

1. The definition of the sector $s_{x,y}$ depends on $\gamma_{x,y}$ only through the set of steps, but is independent of the order of these steps. The sector is thus well defined given only the poles of the integrand.
2. The angular sector $s_{x,y}$ always contains the half-line $d_{x,y}$ starting from 0, in the direction from $\hat{\mathbf{x}}$ to $\hat{\mathbf{y}}$. Thus, we can define the contour of integration $\mathcal{C}_{x,y}$ to be a closed contour oriented counterclockwise and avoiding the closed half-line $d_{x,y}$.

This definition has to be adapted in two particular cases, for different reasons:

Case 1. There are two components of size π in S^1 deprived from the steps of $\gamma_{x,y}$. This occurs when the path $\gamma_{x,y}$ has two steps, and crosses one train-track back and forth. This can only happen in the following four cases. Since all vertices involved belong to the same decoration, we omit the argument $\mathbf{x}$ in the notations. Recall that $e^{i\alpha_{w_k}} = e^{i\alpha_{z_{k+1}}}$:

$$(x, y) = \begin{cases} (w_k, z_{k+1}) \text{ then, } \gamma_{x,y} \text{ is the path } -e^{i\alpha_{z_{k+1}}}, e^{i\alpha_{w_k}}, \\ (z_{k+1}, w_k) \text{ then } \gamma_{x,y} \text{ is the path } -e^{i\alpha_{w_k}}, e^{i\alpha_{z_{k+1}}}, \\ (w_k, w_k) \text{ then } \gamma_{x,y} \text{ is the path } -e^{i\alpha_{w_k}}, e^{i\alpha_{w_k}}, \\ (z_{k+1}, z_{k+1}) \text{ then } \gamma_{x,y} \text{ is the path } -e^{i\alpha_{z_{k+1}}}, e^{i\alpha_{z_{k+1}}}. \end{cases}$$

In these four cases, we set the conventions given by Fig. 14 for the angular sector $s_{x,y}$. These are natural in the light of the proof of Theorem 5, see Sect. 6.4.

Case 2. When $\text{Card}\{T_x \cap T_y\} = 2$, the path $\gamma_{x,y}$ does not fit in any of the two categories of paths above. This situation occurs when x and y are equal or neighbors in $\mathcal{G}$, and both of type 'v'. That is $(x, y) = (v_k(\mathbf{x}), v_k(\mathbf{x}))$, or $(x, y) = (v_k(\mathbf{x}), v_\ell(\mathbf{x}'))$, with $\mathbf{x} \sim \mathbf{x}'$ in G, and k and ℓ such that $v_k(\mathbf{x}) \sim v_\ell(\mathbf{x}')$.

Suppose first that $(x, y) = (v_k(\mathbf{x}), v_\ell(\mathbf{x}'))$. Then $f_{v_k(\mathbf{x})}(\lambda)$ and $f_{v_\ell(\mathbf{x}')}(-\lambda)$ have the same poles $e^{i\alpha_{w_k(\mathbf{x})}}, e^{i\alpha_{z_k(\mathbf{x})}}$. The exponential function $\text{Exp}_{\mathbf{x},\mathbf{x}'}(\lambda)$, cancels one of the pair of poles, and adds two new ones $-e^{i\alpha_{w_k(\mathbf{x})}}$, $-e^{i\alpha_{z_k(\mathbf{x})}}$.

If now $(x, y) = (v_k(\mathbf{x}), v_k(\mathbf{x}))$. Then $f_{v_k(\mathbf{x})}(\lambda)$ has poles $e^{i\alpha_{w_k(\mathbf{x})}}$, $e^{i\alpha_{z_k(\mathbf{x})}}$, and $f_{v_k(\mathbf{x})}(-\lambda)$ has opposite poles. Moreover, $\text{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) = 1$, so that it cancels no pole.

In both cases the path $\gamma_{x,y}$ follows the boundary of the rhombus associated to the edge $\mathbf{xx}'$ in G. We set the conventions given by Fig. 15 below for the angular sector $s_{x,y}$. These are natural in the light of the proof of Theorem 5, see Sect. 6.4.

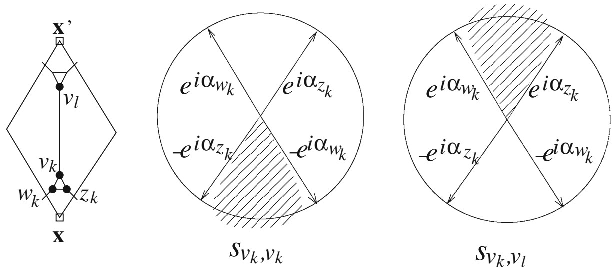


Fig. 15. *Left*: a piece of the graph $\mathcal{G}$ for the situations described in Case 2. Center: angular sector $s_{v_k(\mathbf{x}),v_k(\mathbf{x})}$. *Right*: angular sector $s_{v_k(\mathbf{x}),v_\ell(\mathbf{x}')}$

6.4. Proof of Theorem 5. In this section, we prove Theorem 5, *i.e.* $KK^{-1} = \mathrm{Id}$. Let I and C be the infinite matrices whose coefficients are the integral and constant part of K^{-1} respectively:

$$I_{x,y} = \frac{1}{(2\pi)^2}\oint_{\mathcal{C}_{x,y}} f_x(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log\lambda\,\mathrm{d}\lambda,$$
$$C_{x,y} = C_{x,y}.$$

Our goal is to show that $(KK^{-1})_{x,y} = (KI)_{x,y} + (KC)_{x,y} = \delta_{x,y}$. Let $x_1,\ x_2,\ x_3$ be the three neighbors of x in $\mathcal{G}$. Then,

$$\begin{aligned}(KI)_{x,y} &= \sum_{i=1}^{3}\oint_{\mathcal{C}_{x_i,y}} K_{x,x_i} f_{x_i}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda)\log\lambda\frac{\mathrm{d}\lambda}{(2\pi)^2},\\ (KC)_{x,y} &= \sum_{i=1}^{3} K_{x,x_i} C_{x_i,y}.\end{aligned} \tag{22}$$

In Proposition 22 below, we handle the part $(KI)_{x,y}$: we prove that as soon as x and y do not belong to the same triangle of a decoration, $\bigcap_{i=1}^{3} s_{x_i,y} \neq \emptyset$, so that by the general argument of Sect. 6.2, $(KI)_{x,y} = 0$; when x and y belong to the same triangle of a decoration, we explicitly compute $(KI)_{x,y}$. Then in Lemma 23, we handle the part $(KC)_{x,y}$: we show that the constants $C_{x,y}$ are defined so that $(KK^{-1})_{x,y} = \delta_{x,y}$.

In proving Proposition 22, we have to take into account the fact that x can be of three types, 'w','z' or 'v', *i.e.* $x = w_k(\mathbf{x}), z_k(\mathbf{x})$ or $v_k(\mathbf{x})$ for some $k \in \{1,\dots,d(\mathbf{x})\}$. The next proposition gives relations between these three cases for KI, so that although it might seem technical at first, it is in fact very useful. Indeed, it avoids lengthy repetitions in the proof of Proposition 22. Note that whenever no confusion occurs, we omit the argument $\mathbf{x}$.

Proposition 20. *For every vertex y of $\mathcal{G}$, the quantities $(KI)_{w_k,y}$, $(KI)_{z_k,y}$ and $(KI)_{v_k,y}$ satisfy the following:*

1. $$(KI)_{w_k,y} = -(KI)_{z_k,y} = \left[\left(-\oint_{\mathcal{C}_{z_k,y}} + \oint_{\mathcal{C}_{v_k,y}}\right) f_{z_k}(\lambda) + \left(-\oint_{\mathcal{C}_{w_k,y}} + \oint_{\mathcal{C}_{v_k,y}}\right) f_{w_k}(\lambda)\right] \times f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log\lambda\frac{\mathrm{d}\lambda}{(2\pi)^2}.$$

2. $$(KI)_{v_k,y} = \left[\left(\oint_{\mathcal{C}_{z_k,y}} - \oint_{\mathcal{C}_{v_k,y}}\right) f_{z_k}(\lambda) + \left(-\oint_{\mathcal{C}_{w_k,y}} + \oint_{\mathcal{C}_{v_k,y}}\right) f_{w_k}(\lambda)\right] \times f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log\lambda \frac{d\lambda}{(2\pi)^2}.$$

Proof of Proposition 20. The argument extensively uses relations between the functions $\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) f_x(\lambda)$ for neighboring vertices x. In Eqs. (16), (17), (18), (19) of the proof of Proposition 15, we explicitly computed

$$\sum_{i=1}^{3} K_{x,x_i} f_{x_i}(\lambda)\,\mathrm{Exp}_{\mathbf{x}_i,\mathbf{y}}(\lambda),$$

for $x = w_k(\mathbf{x})$, $z_k(\mathbf{x})$ and $v_k(\mathbf{x})$, respectively. Using this, Eq. (22), and the fact that $f_{v_k}(\lambda) = f_{w_k}(\lambda) + f_{z_k}(\lambda)$, we obtain the following.

If $x = w_k(\mathbf{x})$, then the three neighbors of x are $x_1 = z_k(\mathbf{x})$, $x_2 = z_{k+1}(\mathbf{x})$, $x_3 = v_k(\mathbf{x})$, and by (16) we have:

$$(KI)_{w_k,y} = \left(-\oint_{\mathcal{C}_{z_k,y}} f_{z_k}(\lambda) - \oint_{\mathcal{C}_{z_{k+1},y}} f_{w_k}(\lambda) + \oint_{\mathcal{C}_{v_k,y}} [f_{w_k}(\lambda) + f_{z_k}(\lambda)]\right) \times f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log\lambda \frac{d\lambda}{(2\pi)^2}. \tag{23}$$

If $x = z_k(\mathbf{x})$, then the three neighbors of x are $x_1 = w_{k-1}(\mathbf{x})$, $x_2 = w_k(\mathbf{x})$, $x_3 = v_k(\mathbf{x})$, and by (17) we have:

$$(KI)_{z_k,y} = \left(\oint_{\mathcal{C}_{w_{k-1},y}} f_{z_k}(\lambda) + \oint_{\mathcal{C}_{w_k,y}} f_{w_k}(\lambda) - \oint_{\mathcal{C}_{v_k,y}} [f_{w_k}(\lambda) + f_{z_k}(\lambda)]\right) \times f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log\lambda \frac{d\lambda}{(2\pi)^2}. \tag{24}$$

If $x = v_k(\mathbf{x})$, then the three neighbors of x are $x_1 = z_k(\mathbf{x})$, $x_2 = w_k(\mathbf{x})$, $x_3 = v_\ell(\mathbf{x}')$, where ℓ and $\mathbf{x}'$ are such that $v_k(\mathbf{x}) \sim v_\ell(\mathbf{x}')$ in $\mathcal{G}$. Recalling that the index ℓ refers to the decoration $\mathbf{x}'$, we also omit the arguments $\mathbf{x}$ and $\mathbf{x}'$. Using (18) and (19), we have:

$$(KI)_{v_k,y} = \left(\oint_{\mathcal{C}_{z_k,y}} f_{z_k}(\lambda) - \oint_{\mathcal{C}_{w_k,y}} f_{w_k}(\lambda) + \oint_{\mathcal{C}_{v_\ell,y}} [f_{w_k}(\lambda) - f_{z_k}(\lambda)]\right) \times f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log\lambda \frac{d\lambda}{(2\pi)^2}. \tag{25}$$

As a consequence of (23), (24), (25), Proposition 20 is proved, if we show that, for every vertex y of $\mathcal{G}$, $\mathcal{C}_{z_{k+1},y} = \mathcal{C}_{w_k,y}$, and $\mathcal{C}_{v_k,y} = \mathcal{C}_{v_\ell,y}$, which is equivalent to proving:

$$s_{z_{k+1},y} = s_{w_k,y}, \tag{26}$$
$$s_{v_k,y} = s_{v_\ell,y}. \tag{27}$$

Recall that the construction of the angular sector $s_{x,y}$ given in Sect. 6.3, relies on the path $\gamma_{x,y}$ encoding the poles of the integrand $f_x(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\log(\lambda)$ of $K^{-1}_{x,y}$.

Recall also that the generic definition of the sector (Definition 6.1) has to be adapted in two specific situations (Case 1, Case 2). These cases need to be treated separately here as well.

Generic case. By Remark 19, in the generic case, the definition of the sector $s_{x,y}$ only depends on the set of poles, hence it suffices to show that the left- and right-hand sides of (26), (27) have the same set of poles.

The vertices z_{k+1} and w_k belong to the same decoration $\mathbf{x}$, so that the exponential function has the same poles in both cases. Moreover, the functions $f_{z_{k+1}}(\lambda)$ and $f_{w_k}(\lambda)$ have the same pole $e^{i\alpha_{z_{k+1}}} = e^{i\alpha_{w_k}}$, thus proving (26). Let us now prove (27). We have:

$$
\begin{aligned}
f_{v_\ell}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x}',\mathbf{y}}(\lambda) &= f_{v_\ell}(\lambda)\,\mathrm{Exp}_{\mathbf{x}',\mathbf{x}}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)\\
&= \frac{1}{K_{v_k,v_\ell}}(f_{w_k}(\lambda) - f_{z_k}(\lambda)) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \text{ (by Eq. (18))}.
\end{aligned}
$$

Moreover, by definition of $f_{v_k}(\lambda)$:

$$
f_{v_k}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) = (f_{w_k}(\lambda) + f_{z_k}(\lambda)) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda),
$$

so that the left and right-hand sides of (27) have the same set of poles.

Case 1. This case only occurs when $y = w_k$ or $y = z_{k+1}$ in Eq. (26). Referring to Fig. 14, we see that by definition:

$$
s_{z_{k+1},z_{k+1}} = s_{w_k,z_{k+1}}, \quad \text{and} \quad s_{z_{k+1},w_k} = s_{w_k,w_k}.
$$

Case 2. This case only occurs when $y = v_k$ or $y = v_\ell$ in Eq. (27). Referring to Fig. 15, we see that by definition:

$$
s_{v_k,v_k} = s_{v_\ell,v_k}, \quad \text{and} \quad s_{v_k,v_\ell} = s_{v_\ell,v_\ell}.
$$

□

A direct consequence of Proposition 20 and the general argument of Sect. 6.2 is the following corollary, which greatly reduces the number of computations for $(KI)_{x,y}$.

Corollary 21. 1. *If* $s_{w_k,y} \cap s_{v_k,y} \neq \emptyset$, *then* $(KI)_{w_k,y} = -(KI)_{z_k,y} = -(KI)_{v_k,y}$

$$
= \left[\left(-\oint_{\mathcal{C}_{z_k,y}} + \oint_{\mathcal{C}_{v_k,y}}\right) f_{z_k}(\lambda)\right] f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \log\lambda \frac{\mathrm{d}\lambda}{(2\pi)^2}.
$$

2. *If* $s_{z_k,y} \cap s_{v_k,y} \neq \emptyset$, *then* $(KI)_{w_k,y} = -(KI)_{z_k,y} = (KI)_{v_k,y}$

$$
= \left[\left(-\oint_{\mathcal{C}_{w_k,y}} + \oint_{\mathcal{C}_{v_k,y}}\right) f_{w_k}(\lambda)\right] f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda) \log\lambda \frac{\mathrm{d}\lambda}{(2\pi)^2}.
$$

3. *If* $s_{z_k,y} \cap s_{w_k,y} \cap s_{v_k,y} \neq \emptyset$, *then* $(KI)_{w_k,y} = (KI)_{z_k,y} = (KI)_{v_k,y} = 0$.

We now state Proposition 22 computing the matrix product KI of Eq. (22). Whenever no confusion occurs, we drop the argument $\mathbf{x}$ in $w_k(\mathbf{x})$, $z_k(\mathbf{x})$, $v_k(\mathbf{x})$.

Proposition 22. *For all vertices x and y of $\mathcal{G}$, we have:*

$$(KI)_{x,y} = \begin{cases} \frac{1}{2} & \text{if } (x,y) = (w_k, w_k),\ (z_k, z_k),\ (v_k, w_k),\ (v_k, z_k) \\ -\frac{1}{2} & \text{if } (x,y) = (z_k, w_k),\ (w_k, z_k) \\ 1 & \text{if } (x,y) = (v_k, v_k) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let x and y be two vertices of $\mathcal{G}$. Then $x = w_k(\mathbf{x})$, $z_k(\mathbf{x})$ or $v_k(\mathbf{x})$ for some $k \in \{1, \ldots, d(\mathbf{x})\}$. In order to use Corollary 21, we need to understand the intersection properties of the three angular sectors $s_{w_k,y}$, $s_{z_k,y}$, $s_{v_k,y}$. The proof is divided in three cases, depending on the definition of these sectors: the generic one (Definition 6.1), and the two specific situtations (Case 1, Case 2).

Generic case. Suppose that all three sectors are constructed according to the generic definition. This excludes the case where $y \in \{w_k, z_{k+1}, z_k, w_{k-1}, v_k, v_\ell\}$.

Since $f_{v_k}(\lambda) = f_{w_k}(\lambda) + f_{z_k}(\lambda)$, the function $f_{v_k}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$ contains all poles of $f_{w_k}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$ and $f_{z_k}(\lambda) f_y(-\lambda)\,\mathrm{Exp}_{\mathbf{x},\mathbf{y}}(\lambda)$. Using Definition 6.1, and Remark 19, this implies that $s_{v_k,y} \subset s_{w_k,y}$, and $s_{v_k,y} \subset s_{z_k,y}$. As a consequence,

$$s_{w_k,y} \cap s_{z_k,y} \cap s_{v_k,y} = s_{v_k,y}.$$

Moreover $s_{v_k,y}$ is an angular sector of size at least π, so that using Point 3 of Corollary 21, we deduce that $(KI)_{x,y} = 0$.

Case 1. This case only occurs when $y = w_k, z_{k+1}, z_k$, or w_{k-1}. In each of these cases, one of the pairs $\{y, z_k\}$, $\{y, w_k\}$ requires the use of the convention depicted in Fig. 14 to construct the corresponding sector. For these four cases, the angular sectors $s_{z_k,y}$, $s_{w_k,y}$, $s_{v_k,y}$ are drawn on Fig. 16.

From *c*) and *d*) we see that when $y = z_{k+1}$ or $y = w_{k-1}$, the intersection $s_{z_k,y} \cap s_{w_k,y} \cap s_{v_k,y} \neq \emptyset$. Thus, using Point 3 of Corollary 21, we deduce that $(KI)_{x,y} = 0$. In the remaining two cases we do explicit computations.

- *Computations for $y = w_k$.* In this case, see Fig. 16 *a*), we have $s_{z_k,w_k} \cap s_{v_k,w_k} \neq \emptyset$. Hence, by Point 2 of Corollary 21, we know that:

$$\begin{aligned}(KI)_{w_k,w_k} &= -(KI)_{z_k,w_k} = (KI)_{v_k,w_k} \\ &= \left[\left(-\oint_{\mathcal{C}_{w_k,w_k}} + \oint_{\mathcal{C}_{v_k,w_k}}\right) f_{w_k}(\lambda)\right] f_{w_k}(-\lambda) \log \lambda \frac{\mathrm{d}\lambda}{(2\pi)^2}.\end{aligned}$$

Using the definition of $f_{w_k}(\lambda)$, and denoting by $\mathcal{C}$ a generic simple closed curve oriented counterclockwise, containing all poles of the integrand, and avoiding a half-line starting from 0, yields:

$$\begin{aligned}\frac{1}{(2\pi)^2}\oint_{\mathcal{C}} f_{w_k}(\lambda) f_{w_k}(-\lambda) \log \lambda \mathrm{d}\lambda &= -\frac{1}{(2\pi)^2}\oint_{\mathcal{C}} \frac{e^{i\alpha_{w_k}}}{(\lambda - e^{i\alpha_{w_k}})(\lambda + e^{i\alpha_{w_k}})} \log \lambda \mathrm{d}\lambda \\ &= -\frac{i}{4\pi}\left[\log_{\mathcal{C}}(e^{i\alpha_{w_k}}) - \log_{\mathcal{C}}(-e^{i\alpha_{w_k}})\right]. \qquad (28)\end{aligned}$$

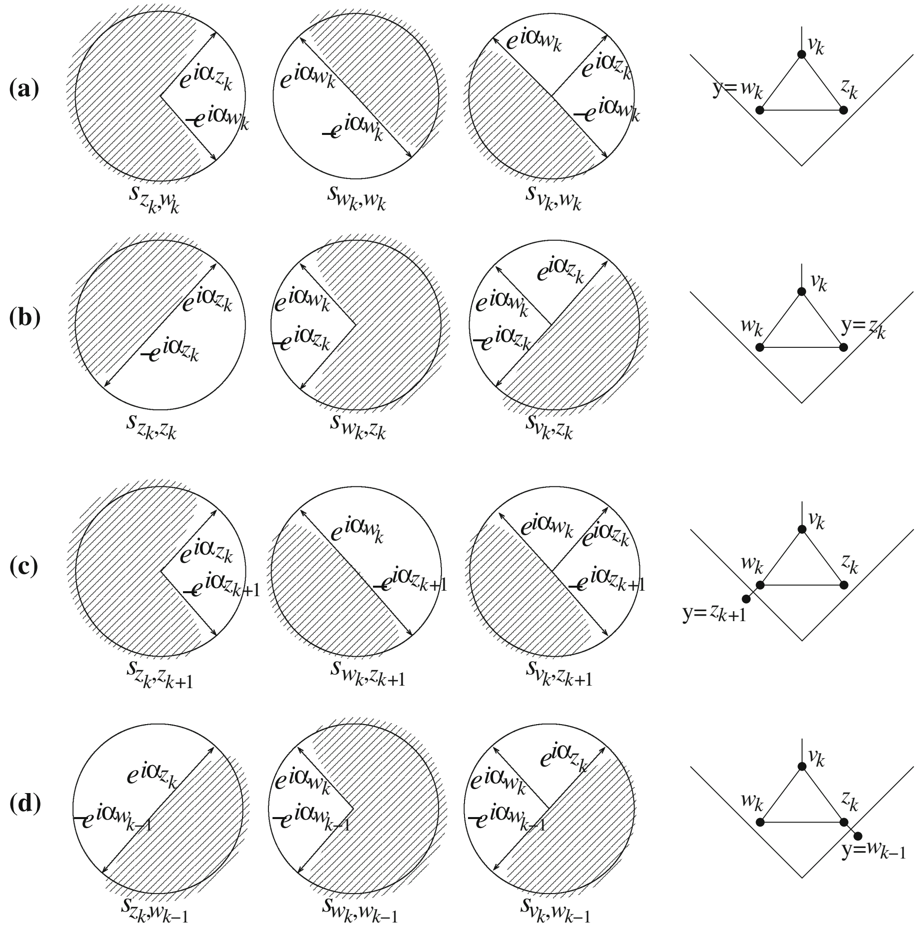


Fig. 16. Angular sectors $s_{z_k,y}$, $s_{w_k,y}$, $s_{v_k,y}$ in Case 1

As a consequence, see Fig. 16 *a*)

$$\begin{aligned}(KI)_{w_k,w_k} &= -(KI)_{z_k,w_k} = (KI)_{v_k,w_k}\\ &= -\frac{i}{4\pi}\Big[-\log_{\mathcal{C}_{w_k,w_k}}(e^{i\alpha_{w_k}}) + \log_{\mathcal{C}_{w_k,w_k}}(-e^{i\alpha_{w_k}})\\ &\qquad + \log_{\mathcal{C}_{v_k,w_k}}(e^{i\alpha_{w_k}}) - \log_{\mathcal{C}_{v_k,w_k}}(-e^{i\alpha_{w_k}})\Big]\\ &= -\frac{i}{4\pi}[-i\alpha_{w_k} + i(\alpha_{w_k}+\pi) + i\alpha_{w_k} - i(\alpha_{w_k}-\pi)] = \frac{1}{2}.\end{aligned}$$

- *Computations for* $y = z_k$. In this case, see Fig. 16 *b*), we have $s_{w_k,z_k} \cap s_{v_k,z_k} \neq \emptyset$. Hence, by Point 1 of Corollary 21, we know that:

$$\begin{aligned}(KI)_{w_k,z_k} &= -(KI)_{z_k,z_k} = -(KI)_{v_k,z_k}\\ &= \left[\left(-\oint_{\mathcal{C}_{z_k,z_k}} + \oint_{\mathcal{C}_{v_k,z_k}}\right) f_{z_k}(\lambda)\right] f_{z_k}(-\lambda)\log\lambda \frac{\mathrm{d}\lambda}{(2\pi)^2}.\end{aligned}$$

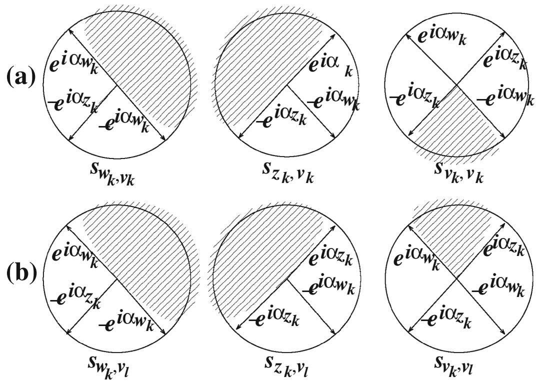


Fig. 17. Angular sectors $s_{w_k,y}$, $s_{z_k,y}$, $s_{v_k,y}$ in Case 2

Using the definition of $f_{z_k}(\lambda)$, yields $f_{z_k}(\lambda)f_{z_k}(-\lambda) = -\frac{e^{i\alpha_{z_k}}}{(\lambda - e^{i\alpha_{z_k}})(\lambda + e^{i\alpha_{z_k}})}$, so that using Fig. 16 *b*), we obtain:

$$\begin{aligned}(KI)_{w_k,z_k} &= -(KI)_{z_k,z_k} = -(KI)_{v_k,z_k} \\ &= -\frac{i}{4\pi}\Big[-\log_{\mathcal{C}_{z_k,z_k}}(e^{i\alpha_{z_k}}) + \log_{\mathcal{C}_{z_k,z_k}}(-e^{i\alpha_{z_k}}) \\ &\qquad + \log_{\mathcal{C}_{v_k,z_k}}(e^{i\alpha_{z_k}}) - \log_{\mathcal{C}_{v_k,z_k}}(-e^{i\alpha_{z_k}})\Big] \\ &= -\frac{i}{4\pi}[-i\alpha_{z_k} + i(\alpha_{z_k} - \pi) + i\alpha_{z_k} - i(\alpha_{z_k} + \pi)] = -\frac{1}{2}.\end{aligned}$$

Case 2. This case can only occur when $y = v_k(\mathbf{x})$ or $y = v_\ell(\mathbf{x}')$, with $v_k(\mathbf{x}) \sim v_\ell(\mathbf{x}')$. In these two cases, the angular sectors $s_{w_k,y}$, $s_{z_k,y}$, $s_{v_k,y}$ are drawn on Fig. 17.

From *b*) we see that $(KI)_{w_k,v_\ell} = (KI)_{z_k,v_\ell} = (KI)_{v_k,v_\ell} = 0$. For $y = v_k(\mathbf{x})$, we do explicit computations.

- *Computations for $y = v_k$*. We use 1 and 2 of Proposition 20. Let us first compute:

$$\left(-\oint_{\mathcal{C}_{z_k,v_k}} + \oint_{\mathcal{C}_{v_k,v_k}}\right) f_{z_k}(\lambda) f_{v_k}(-\lambda)\log\lambda \frac{\mathrm{d}\lambda}{(2\pi)^2}.$$

Using the definition of $f_{z_k}(\lambda)$, $f_{v_k}(\lambda)$, and denoting by $\mathcal{C}$ a generic contour, we have:

$$\begin{aligned}&\frac{1}{(2\pi)^2}\oint_{\mathcal{C}} f_{z_k}(\lambda) f_{v_k}(-\lambda)\log\lambda \mathrm{d}\lambda \\ &= -\frac{1}{4\pi^2}\oint_{\mathcal{C}}\left(\frac{e^{i\alpha_{z_k}}}{(\lambda - e^{i\alpha_{z_k}})(\lambda + e^{i\alpha_{z_k}})} - \frac{e^{i\frac{\alpha_{z_k}+\alpha_{w_k}}{2}}}{(\lambda - e^{i\alpha_{z_k}})(\lambda + e^{i\alpha_{w_k}})}\right)\log\lambda\,\mathrm{d}\lambda \\ &= -\frac{i}{4\pi}\left[\log_{\mathcal{C}}(e^{i\alpha_{z_k}}) - \log_{\mathcal{C}}(-e^{i\alpha_{z_k}}) - \frac{\log_{\mathcal{C}}(e^{i\alpha_{z_k}}) - \log_{\mathcal{C}}(-e^{i\alpha_{w_k}})}{\cos\left(\frac{\alpha_{w_k}-\alpha_{z_k}}{2}\right)}\right]. \qquad (29)\end{aligned}$$

As a consequence:

$$\left(-\oint_{\mathcal{C}_{z_k,v_k}}+\oint_{\mathcal{C}_{v_k,v_k}}\right) f_{z_k}(\lambda) f_{v_k}(-\lambda)\log\lambda\frac{\mathrm{d}\lambda}{(2\pi)^2}$$
$$=\frac{1}{4\pi}\left[-\pi+\frac{\pi-(\alpha_{w_k}-\alpha_{z_k})}{\cos\left(\frac{\alpha_{w_k}-\alpha_{z_k}}{2}\right)}-\pi-\frac{\pi-(\alpha_{w_k}-\alpha_{z_k})}{\cos\left(\frac{\alpha_{w_k}-\alpha_{z_k}}{2}\right)}\right]$$
$$=-\frac{1}{2}.$$

Let us now compute,

$$\left(-\oint_{C_{w_k,v_k}}+\oint_{C_{v_k,v_k}}\right) f_{w_k}(\lambda) f_{v_k}(-\lambda)\log\lambda\frac{\mathrm{d}\lambda}{(2\pi)^2}.$$

Exchanging z_k with w_k in (29) yields:

$$\frac{1}{(2\pi)^2}\oint_{\mathcal{C}} f_{w_k}(\lambda) f_{v_k}(-\lambda)\log\lambda\mathrm{d}\lambda$$
$$=-\frac{i}{4\pi}\left[\log_{\mathcal{C}}(e^{i\alpha_{w_k}})-\log_{\mathcal{C}}(-e^{i\alpha_{w_k}})-\frac{\log_{\mathcal{C}}(e^{i\alpha_{w_k}})-\log_{\mathcal{C}}(-e^{i\alpha_{z_k}})}{\cos\left(\frac{\alpha_{w_k}-\alpha_{z_k}}{2}\right)}\right]. \quad (30)$$

As a consequence:

$$\left(-\oint_{\mathcal{C}_{w_k,v_k}}+\oint_{\mathcal{C}_{v_k,v_k}}\right) f_{w_k}(\lambda) f_{v_k}(-\lambda)\log\lambda\frac{\mathrm{d}\lambda}{(2\pi)^2}$$
$$=\frac{1}{4\pi}\left[\pi-\frac{\pi-(\alpha_{w_k}-\alpha_{z_k})}{\cos\left(\frac{\alpha_{w_k}-\alpha_{z_k}}{2}\right)}+\pi+\frac{\pi-(\alpha_{w_k}-\alpha_{z_k})}{\cos\left(\frac{\alpha_{w_k}-\alpha_{z_k}}{2}\right)}\right]$$
$$=\frac{1}{2}.$$

Thus, by Proposition 20, we deduce:

$$(KI)_{w_k,v_k}=0,\quad (KI)_{z_k,v_k}=0,\quad (KI)_{v_k,v_k}=1.$$

□

By Proposition 22, proving that $KK^{-1}=\mathrm{Id}$, amounts to proving the following lemma for KC.

Lemma 23. *For all vertices x and y of $\mathcal{G}$, we have:*

$$(KC)_{x,y}=\begin{cases}\frac{1}{2} & \text{if } (x,y)=(w_k,w_k),\ (z_k,z_k),\ (w_k,z_k),\ (z_k,w_k)\\ -\frac{1}{2} & \text{if } (x,y)=(v_k,w_k),\ (v_k,z_k)\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By definition of C, we have $C_{x,y}=0$, as soon as x or y is of type 'v'. As a consequence, using Eq. (22) and our choice of Kasteleyn orientation, we deduce:

$$\begin{aligned}(KC)_{w_k,y}&=-C_{z_k,y}+\varepsilon_{w_k,z_{k+1}}C_{z_{k+1},y},\\(KC)_{z_k,y}&=\varepsilon_{z_k,w_{k-1}}C_{w_{k-1},y}+C_{w_k,y},\\(KC)_{v_k,y}&=C_{z_k,y}-C_{w_k,y}.\end{aligned}\tag{31}$$

Let us first prove that for every vertex y of $\mathcal{G}$, we have $\varepsilon_{w_k,z_{k+1}}C_{z_{k+1},y}=C_{w_k,y}$, or equivalently:

$$C_{z_{k+1},y}=\varepsilon_{w_k,z_{k+1}}C_{w_k,y}.\tag{32}$$

When y belongs to a different decoration than x, then both sides of (32) are equal to 0. Let us thus suppose that y is in the same decoration as x. If $y\neq z_{k+1}$, then by definition:

$$C_{z_{k+1},y}=\frac{(-1)^{n(z_{k+1},y)}}{4}=\frac{1}{4}(-1)^{n(z_{k+1},w_k)}(-1)^{n(w_k,y)}=\varepsilon_{w_k,z_{k+1}}C_{w_k,y},$$

so that (32) holds. If $y=z_{k+1}$, then by definition, $C_{z_{k+1},z_{k+1}}=-\frac{1}{4}$. Moreover,

$$\varepsilon_{w_k,z_{k+1}}C_{w_k,z_{k+1}}=\varepsilon_{w_k,z_{k+1}}\frac{(-1)^{n(w_k,z_{k+1})}}{4}=\varepsilon_{w_k,z_{k+1}}\frac{\varepsilon_{z_{k+1},w_k}}{4}=-\frac{1}{4},$$

so that (32) also holds in this case. As a consequence, Eq. (31) becomes:

$$(KC)_{w_k,y}=(KC)_{z_k,y}=-(KC)_{v_k,y}=-C_{z_k,y}+C_{w_k,y}.$$

Let us now end the proof of Lemma 23. If y does not belong to the same decoration as x, or if y belongs to the same decoration as x and is of type 'v', then $C_{w_k,y}=C_{z_k,y}=0$, so that:

$$(KC)_{w_k,y}=(KC)_{z_k,y}=(KC)_{v_k,y}=0.$$

If y belongs to the same decoration as x, but not to the same triangle of the decoration, then

$$C_{w_k,y}=\frac{(-1)^{n(w_k,y)}}{4}=\frac{1}{4}(-1)^{n(w_k,z_k)}(-1)^{n(z_k,y)}=\varepsilon_{z_k,w_k}C_{z_k,y}=C_{z_k,y},$$

since by our choice of Kasteleyn orientation, $\varepsilon_{z_k,w_k}=1$. Thus,

$$(KC)_{w_k,y}=(KC)_{z_k,y}=(KC)_{v_k,y}=0.$$

If $y=w_k$, then by definition $C_{w_k,w_k}=\frac{1}{4}$, and $C_{z_k,w_k}=-C_{w_k,z_k}=-\frac{1}{4}$, thus:

$$(KC)_{w_k,w_k}=(KC)_{z_k,w_k}=-(KC)_{v_k,w_k}=\frac{1}{2}.$$

Finally, if $y=z_k$, then by definition $C_{w_k,z_k}=\frac{1}{4}$, and $C_{z_k,z_k}=-\frac{1}{4}$, so that:

$$(KC)_{w_k,z_k}=(KC)_{z_k,z_k}=-(KC)_{v_k,w_k}=\frac{1}{2}.$$

□

Acknowledgements. We would like to thank Richard Kenyon for asking the questions solved in this paper.

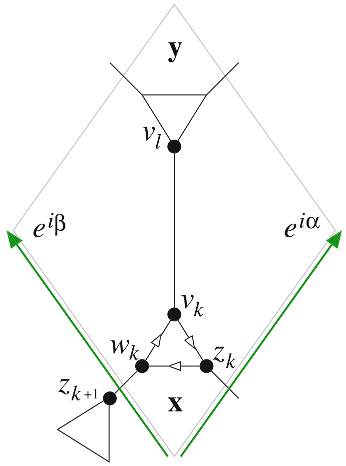


Fig. 18. A piece of the Fisher graph $\mathcal{G}$ near a rhombus with half-angle θ, adjacent to the decorations of **x** and **y**. The sides of the rhombus are represented by the unit vectors $e^{i\alpha}$ and $e^{i\beta}$, with $\beta - \alpha = 2\theta$

A. Probability of Occurrence of Single Edges

Let us compute the probability of occurrence of single edges in dimer configurations of the Fisher graph $\mathcal{G}$ chosen with respect to the Gibbs measure $\mathcal{P}$ of Theorem 9. Every edge of $\mathcal{G}$ is of the form $w_k z_k$, $w_k z_{k+1}$, $w_k v_k$ or $v_k v_\ell$ as represented on Fig. 18. The vertices z_k, w_k, z_{k+1} and v_k belong to the decoration of **x**, and v_ℓ belongs to that of **y**. The edge of G joining **x** and **y** is the diagonal of a rhombus with half-angle θ. In order to simplify notations, let us write

$$\alpha_{z_k} = \alpha, \qquad \alpha_{w_k} = \beta = \alpha + 2\theta.$$

By Theorem 9, we know that the probability of an edge $e = uv$ of $\mathcal{G}$ is given by

$$\mathcal{P}(e) = K_{u,v} \operatorname{Pf} \begin{bmatrix} K^{-1}_{u,u} & K^{-1}_{v,u} \\ K^{-1}_{u,v} & K^{-1}_{v,v} \end{bmatrix} = K_{u,v} K^{-1}_{v,u},$$

where the coefficient $K^{-1}_{u,v}$ of the inverse Kasteleyn matrix is given by Theorem 5.

Probability of the edge $w_k z_k$. The vertices w_k and z_k belong to the same decoration **x**, so that there is no contribution from the exponential function, since $\operatorname{Exp}_{\mathbf{x},\mathbf{x}} = 1$. By our choice of Kasteleyn orientation, we have $K_{z_k,w_k} = 1$. Using Formula (7) and the definition of the function f, yields:

$$\begin{aligned} \mathcal{P}(w_k z_k) = K_{z_k,w_k} K^{-1}_{w_k,z_k} &= \frac{1}{4\pi^2} \oint_{\mathcal{C}_1} f_{w_k}(\lambda) f_{z_k}(-\lambda) \log(\lambda) \mathrm{d}\lambda + C_{w_k,z_k} \\ &= \frac{1}{4\pi^2} \oint_{\mathcal{C}_1} - \frac{e^{\frac{i\beta}{2}}}{e^{i\beta} - \lambda} \frac{e^{\frac{i\alpha}{2}}}{e^{i\alpha} + \lambda} \log(\lambda) \mathrm{d}\lambda + \frac{1}{4}, \end{aligned}$$

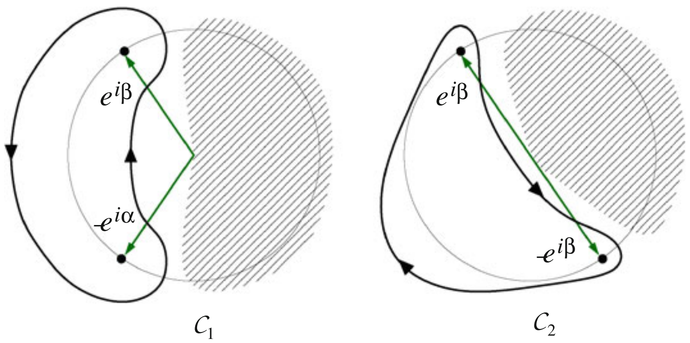


Fig. 19. *Left*: the contour $\mathcal{C}_1 = \mathcal{C}_{w_k,z_k}$ involved in the integral term of $K^{-1}_{w_k,z_k}$. *Right*: the contour $\mathcal{C}_2 = \mathcal{C}_{z_{k+1},w_k}$ involved in the integral term of $K^{-1}_{z_{k+1},w_k}$. In both cases, the arrows represent the poles of the integrand, and the shaded zone is the angular sector avoided by the contour

where $\mathcal{C}_1 = \mathcal{C}_{w_k,z_k}$ is the contour defined according to Case 1 of Sect. 6.3.3, represented on Fig. 19. The integral is evaluated by Cauchy's theorem:

$$\frac{1}{4\pi^2}\oint_{\mathcal{C}_1} \frac{e^{i\frac{\alpha+\beta}{2}}\log(\lambda)}{(\lambda - e^{i\beta})(\lambda + e^{i\alpha})}\mathrm{d}\lambda = \frac{i}{2\pi}e^{i\frac{\alpha+\beta}{2}}\left(\frac{\log_{\mathcal{C}_1}(e^{i\beta}) - \log_{\mathcal{C}_1}(-e^{i\alpha})}{e^{i\alpha}+e^{i\beta}}\right)$$
$$= \frac{i}{4\pi\cos(\frac{\beta-\alpha}{2})}(i\beta - i(\alpha+\pi))$$
$$= \frac{\pi - 2\theta}{4\pi\cos\theta}.$$

Therefore, $\mathcal{P}(w_k z_k) = \frac{1}{4} + \frac{\pi - 2\theta}{4\pi\cos\theta}$.

Probability of the edge $w_k z_{k+1}$. Since the vertices w_k and z_{k+1} belong to the same decoration, there is no contribution from the exponential function. Moreover, by Eq. (15), we know that $K_{w_k,z_{k+1}} f_{z_{k+1}}(\lambda) = -f_{w_k}(\lambda)$, and by Eq. (32), we have $K_{w_k,z_{k+1}} C_{z_{k+1},w_k} = C_{w_k,w_k} = \frac{1}{4}$. Therefore,

$$\mathcal{P}(w_k z_{k+1}) = K_{w_k,z_{k+1}} K^{-1}_{z_{k+1},w_k}$$
$$= \frac{1}{4\pi^2}\oint_{\mathcal{C}_2} K_{w_k,z_{k+1}} f_{z_{k+1}}(\lambda) f_{w_k}(-\lambda)\log(\lambda)\mathrm{d}\lambda + K_{w_k,z_{k+1}} C_{z_{k+1},w_k}$$
$$= -\frac{1}{4\pi^2}\oint_{\mathcal{C}_2} f_{w_k}(\lambda) f_{w_k}(-\lambda)\log(\lambda)\mathrm{d}\lambda + \frac{1}{4}.$$

where $\mathcal{C}_2 = \mathcal{C}_{z_{k+1},w_k}$ is the contour defined according to Case 2 of Sect. 6.3.3, represented on Fig. 19. Using Eq. (28) where we computed this integral for a generic contour $\mathcal{C}$, we obtain:

$$-\frac{1}{4\pi^2}\oint_{\mathcal{C}_2} f_{w_k}(\lambda) f_{w_k}(-\lambda)\log\lambda\mathrm{d}\lambda = \frac{i}{4\pi}[\log_{\mathcal{C}_2}(e^{i\beta}) - \log_{\mathcal{C}_2}(-e^{i\beta})]$$
$$= \frac{i}{4\pi}(i\beta - i(\beta+\pi)) = \frac{1}{4}.$$

We conclude that $\mathcal{P}(w_k z_{k+1}) = \frac{1}{2}$.

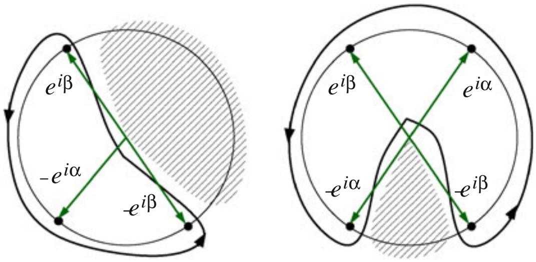


Fig. 20. *Left*: the contour $\mathcal{C}_3 = \mathcal{C}_{w_k,v_k}$ involved in the integral term of $K^{-1}_{w_k,v_k}$. *Right*: the contour $\mathcal{C}_4 = \mathcal{C}_{v_\ell,v_k}$ involved in the integral term of $K^{-1}_{v_\ell,v_k}$

Probability of the edges $w_k v_k$ an $z_k v_k$. Again, there is no contribution from the exponential function. By our choice of Kasteleyn orientation, we have $K_{v_k,w_k} = -1$, and by definition we have $C_{v_k,w_k} = 0$. Using Formula (7), this yields:

$$\mathcal{P}(w_k v_k) = K_{v_k,w_k} K^{-1}_{w_k,v_k} = -\frac{1}{4\pi^2} \oint_{\mathcal{C}_3} f_{w_k}(\lambda) f_{v_k}(-\lambda) \log(\lambda) \mathrm{d}\lambda,$$

where $\mathcal{C}_3 = \mathcal{C}_{w_k,v_k}$ is the contour defined according to Case 2 of Sect. 6.3.3, represented on Fig. 20. Using Eq. (30) where we computed this integral for a generic contour $\mathcal{C}$, we obtain:

$$\begin{aligned}\mathcal{P}(w_k v_k) &= \frac{i}{4\pi}\left(\log_{\mathcal{C}_3}(e^{i\beta}) - \log_{\mathcal{C}_3}(-e^{i\beta}) - \frac{\log_{\mathcal{C}_3}(e^{i\beta}) - \log_{\mathcal{C}_3}(-e^{i\alpha})}{\cos\left(\frac{\beta-\alpha}{2}\right)}\right)\\ &= \frac{i}{4\pi}\left(i\beta - i(\beta+\pi) - \frac{i\beta - i(\alpha+\pi)}{\cos\theta}\right)\\ &= \frac{1}{4} - \frac{\pi - 2\theta}{4\pi\cos\theta}.\end{aligned}$$

By symmetry, this is also the probability of occurrence of the edge $z_k v_k$.

Probability of the edge $v_k v_\ell$. By definition, we have $C_{v_\ell,v_k} = 0$, and by Eq. (18), we know that:

$$K_{v_k,v_\ell} f_{v_\ell}(\lambda)\,\mathrm{Exp}_{\mathbf{y},\mathbf{x}}(\lambda) = f_{w_k}(\lambda) - f_{z_k}(\lambda).$$

Using Formula (7), this yields:

$$\begin{aligned}\mathcal{P}(v_k v_\ell) &= K_{v_k,v_\ell} K^{-1}_{v_\ell,v_k}\\ &= \frac{1}{4\pi^2}\oint_{\mathcal{C}_4} K_{v_k,v_\ell} f_{v_\ell}(\lambda) f_{v_k}(-\lambda)\,\mathrm{Exp}_{\mathbf{y},\mathbf{x}}(\lambda)\log\lambda\,\mathrm{d}\lambda\\ &= \frac{1}{4\pi^2}\oint_{\mathcal{C}_4}\left[f_{w_k}(\lambda) - f_{z_k}(\lambda)\right] f_{v_k}(-\lambda)\log(\lambda)\mathrm{d}\lambda\\ &= \frac{1}{4\pi^2}\oint_{\mathcal{C}_4} f_{w_k}(\lambda) f_{v_k}(-\lambda)\log(\lambda)\mathrm{d}\lambda - \frac{1}{4\pi^2}\oint_{\mathcal{C}_4} f_{z_k}(\lambda) f_{v_k}(-\lambda)\log(\lambda)\mathrm{d}\lambda,\end{aligned}$$

where $\mathcal{C}_4 = \mathcal{C}_{v_\ell, v_k}$ is the contour defined according to Case 3 of Sect. 6.3.3, represented on Fig. 20. Using Eq. (30), the first term is equal to:

$$\begin{aligned}
&\frac{1}{4\pi^2}\oint_{\mathcal{C}_4} f_{w_k}(\lambda) f_{v_k}(-\lambda)\log(\lambda)\mathrm{d}\lambda \\
&= -\frac{i}{4\pi}\left(\log_{\mathcal{C}_4}(e^{i\beta}) - \log_{\mathcal{C}_4}(-e^{i\beta}) - \frac{\log_{\mathcal{C}_4}(e^{i\beta}) - \log_{\mathcal{C}_4}(-e^{i\alpha})}{\cos\left(\frac{\beta-\alpha}{2}\right)}\right) \\
&= -\frac{i}{4\pi}\left(i\beta - i(\beta-\pi) - \frac{i\beta - i(\alpha+\pi)}{\cos\theta}\right) \\
&= \frac{1}{4} + \frac{\pi - 2\theta}{4\pi\cos\theta}.
\end{aligned}$$

By Eq. (29) and Fig. 20, the second term is equal to:

$$\begin{aligned}
&-\frac{1}{4\pi^2}\oint_{\mathcal{C}_4} f_{z_k}(\lambda) f_{v_k}(-\lambda)\log(\lambda)\mathrm{d}\lambda = \\
&= \frac{i}{4\pi}\left(\log_{\mathcal{C}_4}(e^{i\alpha}) - \log_{\mathcal{C}_4}(-e^{i\alpha}) - \frac{\log_{\mathcal{C}_4}(e^{i\alpha}) - \log_{\mathcal{C}_4}(-e^{i\beta})}{\cos\left(\frac{\beta-\alpha}{2}\right)}\right) \\
&= \frac{i}{4\pi}\left(i\alpha - i(\alpha+\pi) - \frac{i\alpha - i(\beta-\pi)}{\cos\theta}\right) \\
&= \frac{1}{4} + \frac{\pi - 2\theta}{4\pi\cos\theta}.
\end{aligned}$$

As a consequence $\mathcal{P}(v_k v_\ell) = \frac{1}{2} + \frac{\pi-2\theta}{2\pi\cos\theta}$.

Let us make a few simple comments about the values of these probabilities.

1. The value $\frac{1}{2}$ for the probability of the edge $w_k z_{k+1}$ can be explained as follows. By Fisher's correspondence, once the configuration of the edges coming from edges of G attached to the decoration of $\mathbf{x}$ is fixed, there are two possibilities for the dimer covering inside the decoration, which both have the same weight. There is always one of the two possibilities containing the edge $w_k z_{k+1}$. Therefore, this edge appears in a random dimer configuration half of the time.
2. Notice that $\mathcal{P}(w_k z_k) = \frac{1}{2}\mathcal{P}(v_k v_\ell)$. This is explained by the fact that the edge $w_k z_k$ appears only if $v_k v_\ell$ is not present in the dimer configuration, and it appears only in one of the two allowed configurations, once the state of the edges coming from edges of G is fixed.
3. Using the two previous points, and the fact that the probability of the edges incident to a given vertex must sum to 1, one can deduce the probability of all the edges from the probability of the edge $v_k v_\ell$.

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