Adv Comput Math (2013) 38:531–561
DOI 10.1007/s10444-011-9248-6

Refinable functions for dilation families

Philipp Grohs

Received: 1 October 2010 / Accepted: 12 October 2011 /
Published online: 29 October 2011
© Springer Science+Business Media, LLC 2011

Abstract We consider a family of $d \times d$ matrices W_e indexed by $e \in E$ where (E, μ) is a probability space and some natural conditions for the family $(W_e)_{e\in E}$ are satisfied. The aim of this paper is to develop a theory of continuous, compactly supported functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ which satisfy a refinement equation of the form

$$\varphi(x) = \int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha)\varphi\,(W_e x - \alpha)\, d\mu(e)$$

for a family of filters $a_e : \mathbb{Z}^d \to \mathbb{C}$ also indexed by $e \in E$. One of the main results is an explicit construction of such functions for any reasonable family $(W_e)_{e\in E}$. We apply these facts to construct scaling functions for a number of affine systems with composite dilation, most notably for shearlet systems.

Keywords Refinable functions · Composite dilation wavelets · Shearlets

Mathematics Subject Classifications (2010) 42C15 · 42C40 · 65T99 · 68U10

1 Introduction

The motivation for this paper is to develop a theory which, in analogy to wavelet MRA theory, allows one to build affine tight frames for $L_2(\mathbb{R}^d)$ with possibly more than one dilation matrix involved in the affine scaling. The main

Communicated by Tomas Sauer.

P. Grohs (✉)
ETH Zürich, Zurich, Switzerland
e-mail: pgrohs@math.ethz.ch

application we have in mind is the shearlet transform which forms an affine system with the affine scaling given by the composition of shear matrices and anisotropic diagonal matrices [19]. But also most of the previously studied composite dilation MRA's [10, 11] fall into our framework and therefore our results are interesting in this respect, too. To the best of our knowledge no continuous and compactly supported scaling functions for composite dilation systems have been known up to date and our results provide such functions constructively.

To motivate why it is interesting to study MRA systems with more than one dilation, we quickly recall the recently introduced shearlet transform [19].

1.1 Wavelets

In order to describe our view of the shearlet transform we first begin with a very brief description of the success story of wavelets. The reader probably knows all this, if not, we refer to the classical reference [5]. Wavelets have many ancestors, among which we would like to mention the Littlewood-Paley (LP) decomposition. It decomposes a function into separate parts each one with frequency support in an octave. In harmonic analysis this has been a very useful tool for a long time; unlike conventional Fourier methods, it conveniently allows to describe a large variety of (global) function spaces. Wavelets can be regarded as a localized version of the LP decomposition, that is, given a decomposition

$$f = \sum_{j \in \mathbb{Z}} P_j f$$

where P_j has frequency support in the j-th octave band, a wavelet decomposition further decomposes each $P_j f$ into spatially localized parts

$$P_j f = \sum_{\alpha \in \mathbb{Z}^d} P_{j,\alpha} f.$$

It turns out that one can find functions, so-called wavelets, ψ with

$$P_{j,\alpha} f = \left\langle f, 2^{j/2} \psi \left(2^j \cdot - \alpha \right) \right\rangle.$$

The first wavelet constructions have been carried out with the LP decomposition in mind. In particular, they usually had compact frequency support, a feature which is very undesirable for many applications.

Things changed dramatically when it was realized that wavelets had an equivalent counterpart in signal processing, namely subband coding. This insight led to the development of the notion of Multiresolution Analysis (MRA). The key idea is to consider a nested sequence of subspaces

$$\dots V_{-1} \subset V_0 \subset V_1 \subset \dots$$

of $L_2(\mathbb{R}^d)$ such that $V_j = \{ f(2\cdot) : \ f \in V_{j-1} \}$ and such that V_0 is spanned by the translates of a function φ. In terms of the previous discussion of the LP decomposition, the spaces V_j contain all frequency bands $\leq j$ and thus the orthogonal projection of any function f onto V_j corresponds to $\sum_{k\leq j} P_k f$. The MRA-analogue of the frequency projections P_j is the orthogonal projection onto the complement W_j of V_{j-1} in V_j. It turns out that it is possible to represent the complement W_1 of V_0 in V_1 as the closed span of the translates of a function $\psi \in V_1$ (actually one usually needs a finite number of functions but this fact is not relevant for the present discussion), and consequently to have $W_j = \mathrm{cls}_{L_2(\mathbb{R}^d)}\big(\mathrm{span}\big(\{\psi(2^j \cdot -\alpha) : \alpha \in \mathbb{Z}^d\}\big)\big)$. This function ψ can be constructed as a finite linear combination

$$\psi(\cdot) = \sum_{\alpha\in\mathbb{Z}^d} q(\alpha)\varphi(2\cdot -\alpha).$$

It turns out that by choosing the weights $q(\alpha)$ correctly (this is a purely algebraic problem) the system

$$\left\{\varphi(\cdot-\alpha) :\ \alpha \in \mathbb{Z}^d\right\} \cup \left\{2^{j/2}\psi\left(2^j\cdot -\alpha\right) :\ j \geq 0,\ \alpha \in \mathbb{Z}\right\}$$

constitutes a tight frame for $L_2(\mathbb{R}^d)$, i.e.

$$f = \sum_{\alpha\in\mathbb{Z}^d} \langle f, \varphi(\cdot-\alpha)\rangle\, \varphi(\cdot-\alpha) + \sum_{j\in\mathbb{Z}}\sum_{\alpha\in\mathbb{Z}^d} \left\langle f, 2^{j/2}\psi\left(2^j\cdot -\alpha\right)\right\rangle 2^{j/2}\psi\left(2^j\cdot-\alpha\right)$$

for all $f \in L_2(\mathbb{R}^d)$. This method of construction of ψ for a given MRA is called the *unitary extension principle* [22]. What makes this construction so useful is that

1. all the theoretical results from the LP-decomposition can be retained in this general construction,
2. this construction is extremely general, for example one can choose φ, ψ to be compactly supported, piecewise polynomial or symmetric w.r.t. some symmetry group just to name a few properties, and
3. the decomposition and reconstruction of a given function can be performed in linear time using filter operations. This is a direct consequence of the MRA construction.

All these three properties are crucial for the wavelet transform to be a useful tool in all kinds of numerical problems.

1.2 Shearlets

Wavelets also have some shortcomings. When the dimension d is greater than one, the decomposition into frequency bands is not sufficient to capture the subtle geometric phenomena arising in multivariate functions at microscopic scales like for example edges in images. To remedy this difficulty, another decomposition has become popular in harmonic analysis in the 1970's; the Second Dyadic Decomposition (SDD) which further decomposes the frequency bands into wedges obeying the parabolic scaling relation length $\sim$ width^2 [23]. In the spirit of the early wavelet constructions which were based on the

LP-decomposition, Candes and Donoho constructed tight frames of bivariate functions, called curvelets, which are based on the SDD-decomposition [2, 3]. Like the early wavelet constructions, these functions all have compact frequency support. Moreover, the structure of this system is not as simple as the structure of wavelet systems, where the tight frame is constructed from one single function ψ by dilation and translation. In contrast to this, in the framework of the shearlet transform it is possible to generate tight frames which are adapted to the SDD and which are generated from one single bivariate function [9, 17, 19]. We briefly describe the construction.

Take a wavelet ψ_1 which has frequency support in a small annulus and a low-pass function ψ_2 which is also bandlimited. Define a shearlet ψ via $\hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_2}{\xi_1})$. If $\psi_1,\ \psi_2$ satisfy some additional assumptions, then there exists a (infinitely supported) smooth function φ such that the system

$$\left\{\varphi(\cdot - \alpha) : \alpha \in \mathbb{Z}^2\right\} \cup \left\{\det(W)^{j/2}\psi\left(U_e W^j \cdot -\alpha\right) : j \in \mathbb{Z},\ -2^j < e < 2^j,\ \alpha \in \mathbb{Z}^2\right\}$$

constitutes a tight frame for the space of square-integrable functions with frequency support in a cone, where we have put

$$U_e := \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W := \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

The function

$$P_{e,j} f := \sum_{\alpha \in \mathbb{Z}^d} \left\langle f, \det(W)^{j/2}\psi\left(U_e W^j \cdot -\alpha\right)\right\rangle \det(W)^{j/2}\psi\left(U_e W^j \cdot -\alpha\right)$$

can be interpreted as a projection onto a parabolic wedge in the frequency domain. In particular, the shearlet (and also the curvelet-) transform can be seen as a localized LP-type decomposition for the SDD.

An important question is if, similar to the MRA constructions for wavelets described above, it is possible to construct shearlet tight frames more effectively and generally in an MRA setting. If we want to carry over the wavelet construction to the shearlet setting, we quickly see that the associated scaling spaces should be of the form

$$V_j^{\text{shear}} = \text{cls}_{L_2(\mathbb{R}^d)}\left(\text{span}\left(\left\{\varphi\left(U_e W^j \cdot -\alpha\right) :\ -2^j < e < 2^j,\ \alpha \in \mathbb{Z}^2\right\}\right)\right)$$

with some scaling function φ and $j \geq 0$. The first crucial property that needs to be satisfied for a shearlet MRA construction is the nestedness of the scaling spaces; we need $V_j^{\text{shear}} \subset V_{j+1}^{\text{shear}}$. This holds if and only if

$$\varphi(\cdot) = \sum_{e \in \{-1,0,1\}} \sum_{\alpha \in \mathbb{Z}^2} \frac{1}{3} a_e(\alpha) \varphi\left(U_e W \cdot -\alpha\right) \tag{1}$$

for some sequences a_e (the fact that we only need to consider $e \in \{-1, 0, 1\}$ follows from the commuting relation $WU_e = U_{2e}W$). We call such functions φ which satisfy (1) refinable.

In the present paper we are concerned with the construction of such functions φ, but in a more general framework: We develop a constructive theory of functions which are refinable with respect to a dilation family $(W_e)_{e\in E}$ as in (5) below, the shearlet dilation family $(U_e W)_{e\in\{-1,0,1\}}$ being a special case.

1.3 Previous work

There have been several previous attempts at constructing a suitable MRA structure for shearlet systems, but none of them is completely satisfactory. The work that is closest to ours is the theory of composite dilation systems [10, 11] which is very similar to our construction but there is one crucial difference: The scaling spaces defined in [11] are defined as

$$V_j^{\text{shear,cd}} = \text{cls}_{L^2(\mathbb{R}^d)}\left(\text{span}\left(\left\{\varphi\left(U_l W^j \cdot -\alpha\right) : \ l \in \mathbb{Z}, \ \alpha \in \mathbb{Z}^2 \right\}\right)\right),$$

e.g. all shear directions are already thrown into the scaling space $V_0^{\text{shear,cd}}$. It is easy to see that in this case no useful scaling function φ exists such that the scaling spaces are nested and such that the sheared translates of φ constitute a frame for the scaling space: Just observe that any useful scaling function satisfies $\hat{\varphi}(0) \neq 0$ and apply Corollary 1.6.3 (b) of [21].[1] A consequence is that, for constructing shearlet MRA's, these spaces are not suitable.

Another interesting development is [18], where adaptive subdivision schemes are introduced in order to obtain a directional decomposition and [14], where similar ideas are employed and an extension principle is proven, allowing for the construction of directional tight frames. In this connection we would also like to mention the contourlet transform [7] which builds on somewhat similar ideas and which is already a well-established tool in computer graphics.

Despite the many interesting developments and ideas in all these works, to this date no MRA construction exists which produces genuine shearlet frames.

Also for other MRA systems with more than one dilation matrix the results are rather sparse. There exist constructions of refinable functions satisfying a refinement relation analogous to (1), but all these constructions produce either non-continuous Haar-type scaling functions [16], or bandlimited functions with very bad spacial localization [1]. Below we construct continuous and compactly supported refinable functions for a large variety of composite dilation systems.

1.4 Contributions

The purpose of this paper is to develop a general theory for the construction of MRA's for systems with composite dilation which contain shearlet systems. A main result of this paper is that for any reasonable dilation family a refinable function with desirable properties exists. In particular for shearlet

[1] Even though we are not aware of any published proof of this fact, it appears that this observation is not new.

systems we give several examples of such functions. In the next section, Section 2, we introduce the basic tools and notation, in particular we define a refinement procedure which will be crucial in the construction of refinable functions. Then, in Section 3 we discuss the connection between the existence of a refinable function and the convergence of the refinement procedure. In Section 4 we show, among other things, that for any dilation family there exists a refinable function which is compactly supported. In Section 5 we discuss connections between the ideas in [18] and our work and give some further examples. Finally in the paper [8] we use the results of this paper to construct an interpolating shearlet transform.

2 Notation

We fix a dimension d and denote by l_p, L_p the Lebesgue spaces $l_p(\mathbb{Z}^d)$, $L_p(\mathbb{R}^d)$ with the norms $\|\cdot\|_{l_p} := \|\cdot\|_{l_p(\mathbb{Z}^d)}$, $\|\cdot\|_{L_p} := \|\cdot\|_{L_p(\mathbb{R}^d)}$, $1 \leq p \leq \infty$. If we are considering Lebesgue spaces with respect to another measure space $\mathcal{M}$ we specify this by writing $L_p(\mathcal{M})$ and $\|\cdot\|_{L_p(\mathcal{M})}$ for the space and the norm. For a set of functions V and a function space B, we denote by $\mathrm{span}(V)$ its linear span and by $\mathrm{cls}_B(V)$ its closure in the topology of B.

Let $x, y \in \mathbb{C}^d$. Then we denote by xy the standard Euclidean inner product $xy := x^T \cdot \bar{y}$. For a function $f \in L_2 \cap L_1$ we define its *Fourier transform*

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) \exp(-ix\omega) dx, \quad \omega \in \mathbb{R}^d$$

and we extend this notion to the space of tempered distributions. For a function $a \in l_1$ we define the Fourier transform as

$$\hat{a}(\omega) := \sum_{\alpha \in \mathbb{Z}^d} a(\alpha) \exp(-i\alpha\omega), \quad \omega \in [\pi, \pi]^d.$$

Let us consider a lattice (i.e. a discrete subgroup of $\mathbb{R}^d$) $\Gamma \subset \mathbb{R}^d$. We say that a matrix $W \in \mathbb{R}^{d\times d}$ *generates the lattice* Γ if

$$W\mathbb{Z}^d = \Gamma.$$

An invertible matrix U such that both U and U^{-1} have integer entries (i.e., $U \in \mathrm{GL}(d, \mathbb{Z})$) is called *unimodular*.

Lemma 1 *Two matrices W, W' generate the same lattice if and only if there exists a unimodular matrix U such that*

$$W' = WU. \tag{2}$$

We call a matrix $W \in \mathbb{R}^{d\times d}$ *integer-expanding* if it has integer entries and all its eigenvalues are of modulus > 1.

Definition 1 (Dilation family, filter family) Let (E, μ) be a probability space e.g. $\mu(E) = 1$. A family $\mathcal{W} = (W_e)_{e \in E}$ of $d \times d$ matrices is called a *dilation family*. A family $\mathcal{A} = (a_e)_{e \in E}$, $a_e : \mathbb{Z}^d \to \mathbb{C}$ for all $e \in E$ of filters indexed by E is called a *filter family* associated with E.

We make the following assumptions for a dilation , resp. a filter family:

(Measurability of matrices) The mapping $e \in E \mapsto W_e$ is measurable.[2]
(Measurability of filters) For all $\alpha \in \mathbb{Z}^d$ the mapping $e \in E \mapsto a_e(\alpha)$ is measurable.
(Uniform integrability)

$$\sup_{e \in E} \|a_e\|_{l_1} = \sup_{e \in E} \sum_{\alpha \in \mathbb{Z}^d} |a_e(\alpha)| < \infty.$$

(Uniform norm bound)

$$\sup_{e \in E} \|W_e^{-1}\| < \infty.$$

(Affine invariance)

$$\sum_{\beta \in \mathbb{Z}^d} a_e(\alpha - W_e \beta) = 1 \quad \text{for all } e \in E,\ \alpha \in \mathbb{Z}^d. \tag{3}$$

A filter family is called *finitely supported* if all supports of the filters a_e are contained in a fixed bounded set, $e \in E$.

Remark 1 In practice, E will be a finite set, the sigma algebra will be the powerset of E and μ a weighted counting measure. Such a choice makes all the above assumptions except the affine invariance trivially true.

We impose a general assumption concerning the generation of lattices: First we need some further notation. Denoting by E^n the direct product $E \times \cdots \times E$, by E^∞ the direct product $E^{\mathbb{N}}$ and by $E_\infty := \bigcup_{n \in \mathbb{N}} E^n$, we recursively define for $\mathbf{e} := (e_1, e_2, \ldots, e_n) \in E^n$ the matrices

$$W_{\mathbf{e}} := W_{e_n} W_{(e_1, e_2, \ldots, e_{n-1})}.$$

The matrices $W_{\mathbf{e}}$, $\mathbf{e} \in E^n$ correspond to scale n. We need these different matrices belonging to the same scale to be compatible with each other in order to obtain meaningful results. In this spirit we make the following definition.

Definition 2 (Compatibility) We say that a dilation family $(W_e)_{e \in E}$ satisfies the *Lattice Compatibility Condition (LCC)* if there exists an integer-expanding matrix W such that for any $n \in \mathbb{N}$ and $\mathbf{e} \in E^n$ the matrix $W_{\mathbf{e}}^{-1}$ generates the same lattice as W^{-n} which we call $\Gamma^{(n)}$.

[2] We equip $\mathbb{C}^{d \times d}$ with the Lebesque measure.

Remark 2 The reader might wonder if the LCC follows inductively from the simpler assumption that all the matrices $(W_e^{-1})_{e\in E}$ generate the same lattice. This is not the case and a counterexample is given by

$$E = \{0, 1\}, \quad W_0 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad W_1 = \begin{pmatrix} 4 & 0 \\ 4 & 2 \end{pmatrix}.$$

It is not difficult to see that W_0^{-1}, W_1^{-1} generate the same lattice but $W_{(0,0)}, W_{(1,1)}$ don't.

Remark 3 Note that in the above definition it is possible that none of the matrices $W_e,\ e \in E$ is integer-expanding. We are not aware of any previous results dealing with the construction of refinable functions for non-integer-expanding dilation matrices, thus our results also break new grounds in this respect.

Let us now look a bit closer at the LCC for the sequence of lattices $\Gamma^{(n)} := W^{-n}\mathbb{Z}^d, n \in \mathbb{N}$. By Lemma 1 and the LCC, for every $\mathbf{e} \in E^n,\ n \in \mathbb{N}$, there exists a unimodular matrix $U_{\mathbf{e}}$ such that

$$U_{\mathbf{e}} W^n = W_{\mathbf{e}}. \tag{4}$$

A particularly convenient subclass of dilation families is introduced in the following definition.

Definition 3 (Finite type) A dilation family $(W_e)_{e\in E}$ satisfying the LCC is called *of finite type* if

$$|\{U_{\mathbf{e}} :\ \mathbf{e} \in E_\infty\}| < \infty.$$

Otherwise, $(W_e)_{e\in E}$ is called *of infinite type*.

Now we define the notion of refinability of a function with respect to a filter family and a dilation family.

Definition 4 (Refinability) A continuous function $\varphi : \mathbb{R}^d \to \mathbb{C}$ is called *refinable with respect to a dilation family* $(W_e)_{e\in E}$ *and a filter family* $(a_e)_{e\in E}$ if

$$\varphi(x) = \int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha)\varphi\left(W_e x - \alpha\right) d\mu(e) \tag{5}$$

2.1 Refinement operators.

To any filter a and any matrix M with integer entries we can associate the *subdivision operator* $S_{a,M}$ with *mask* a and *dilation matrix* M defined on d-variate sequences via

$$S_{a,M}p(\alpha) := \sum_{\beta\in\mathbb{Z}^d} a\left(\alpha - M\beta\right) p(\beta), \quad \alpha \in \mathbb{Z}^d.$$

An interpretation of such an operator is that p is some data defined on a lattice $\mathbb{Z}^d$ and $S_{a,M}$ performs an upsampling operation on this data to the finer lattice $M^{-1}\mathbb{Z}^d$. Subdivision operators can be iterated – with the resulting operator being again a subdivision operator:

Lemma 2 *Assume that a, b are two filters, and M, N two dilation matrices. Then*

$$S_{a,M}S_{b,N} = S_{c,MN}$$

with

$$c = S_{b,N}a.$$

We have

$$\|c\|_{l_1} \leq \|a\|_{l_1}\|b\|_{l_1}. \tag{6}$$

In terms of Fourier transforms we have

$$\hat{c}(\omega) = \hat{b}(\omega)\hat{a}\left(N^T\omega\right). \tag{7}$$

$S_{a,M}$ *is* translation invariant*: With σ_γ being the translation operator $c(\cdot) \mapsto c(\cdot - \gamma)$, $\gamma \in \mathbb{Z}^d$ we have*

$$S_{a,M} \circ \sigma_\gamma = \sigma_{M\gamma} \circ S_{a,M}.$$

Proof We only show (6), the rest is an easy and standard computation. To show (6) we note that the subdivision operator $S_{b,N}$ may be interpreted as an upsampling operator U_N that maps a function $p : \mathbb{Z}^d \to \mathbb{C}$ to the function $U_N p$ which is defined via

$$U_N p(\alpha) = \begin{cases} p(\beta) & \alpha = N\beta \text{ for some } \beta \in \mathbb{Z}^d \\ 0 & \text{else} \end{cases}$$

followed by convolution with b, i.e.

$$c = b * U_N a.$$

Since $\|U_N a\|_{l_1} = \|a\|_{l_1}$, the result follows from Hölder's inequality. □

The previous lemma allows us to define for each $\mathbf{e} = (e_1, \ldots, e_n) \in E^n$ a subdivision operator $S_\mathbf{e}$ with mask $a_\mathbf{e}$ and dilation matrix $W_\mathbf{e}$ satisfying the recursion

$$S_\mathbf{e} p = S_{a_{e_n}, W_n} S_{(e_1,\ldots,e_{n-1})} p.$$

As stated above the interpretation is that this operator maps a function defined on $\mathbb{Z}^d$ to an upsampling defined on $W_\mathbf{e}^{-1}\mathbb{Z}^d$.

We now define our main tool, a refinement procedure which later will be shown to construct refinable functions satisfying (5) in the limit.

Definition 5 (Refinement process) With the conventions and notation as above we define the *refinement process* $\mathcal{S} := (S^{(n)})_{n\in\mathbb{N}}$ associated with the dilation family $\mathcal{W} = (W_e)_{e\in E}$ and the filter family $\mathcal{A} = (a_e)_{e\in E}$ via

$$S^{(n)} p(\alpha) := \int_{E^n} S_{\mathbf{e}} p\,(U_{\mathbf{e}}\alpha)\, d\mu^n(\mathbf{e}), \tag{8}$$

$p \in l_\infty(\mathbb{Z}^d)$ (μ^n denoting the n-fold tensor product measure of μ on E^n).

Remark 4 From the general assumptions in Definition 4 and Lemma 2, it is easy to see that the mapping $E^n \to \mathbb{C}$, $\mathbf{e} \mapsto S_{\mathbf{e}} p(U_{\mathbf{e}}\alpha)$ is integrable with respect to $(E^n, \mu \otimes \cdots \otimes \mu)$. In particular this will allow us to use Fubini's theorem whenever necessary.

Note that the operator $S^{(n)}$ is translation invariant in the following sense:

$$S^{(n)} \circ \sigma_\gamma = \sigma_{W^n\gamma} \circ S^{(n)}. \tag{9}$$

The use of the 'norming matrices' $U_{\mathbf{e}}$ in the equation (8) allows us to interpret the operator $S^{(n)}$ as an upsampling operator which maps a function defined on $\mathbb{Z}^d$ onto an upsampled version defined on $W^{-n}\mathbb{Z}^d$. In view of this remark the following definition is natural.

Definition 6 The refinement process $\mathcal{S} = (S^{(n)})_{n\in\mathbb{N}}$ is called *convergent* if for all nonzero initial data $p \in l_\infty(\mathbb{Z}^d)$ there exists a uniformly continuous function $\varphi_p \neq 0$ such that

$$\lim_{n\to\infty} \left\| \varphi_p(x) - S^{(n)} p\left(W^n x\right)\right\|_{L_\infty(\Gamma^{(n)})} = 0. \tag{10}$$

In the remainder of this work we will study the convergence properties of $\mathcal{S}$ and the relation between convergence of $\mathcal{S}$ and the existence of a refinable function φ satisfying (5).

A big difference between the convergence theory of systems of the form $\mathcal{S}$ and usual subdivision schemes is that the refinement family $\mathcal{S}$ is not memoryless. In the study of subdivision schemes it does not matter at which level we start subdividing while for the system $\mathcal{S}$ it does. This fact makes many standard arguments from linear subdivision considerably more complicated or even impossible.

2.2 Relation to composite dilation systems.

A rather general construction of families $(W_e)_{e\in E}$ satisfying the LCC is the following: Suppose that G is a subgroup of $\mathrm{GL}(d,\mathbb{C})$ of the form

$$G = \left\{U_l W^k : l \in H,\ k \in \mathbb{Z}\right\} \tag{11}$$

where $\{U_l : l \in H\}$ is a group of unimodular matrices and W is an integer-expanding matrix. Now let $E \subseteq H$ be an arbitrary index set. We can set

$$W_e := U_e W.$$

It is now easy to see that the family $(W_e)_{e\in E}$ satisfies the assumptions of Definition 2: Since G is closed under composition, it follows that $W_{(e_1,e_2)} = W_{e_2}W_{e_1} \in G$. This implies that there exists $l_{(e_1,e_2)} \in H$ and $k \in \mathbb{Z}$ with

$$U_{l_{(e_1,e_2)}} W^k = W_{(e_1,e_2)}.$$

Since $|\det(U_l)| = 1$ for all $l \in H$ and $\det(W) \neq 1$, we can conclude that $k = 2$ and therefore we have found a unimodular matrix $U_{(e_1,e_2)} := U_{l_{(e_1,e_2)}}$ such that

$$U_{(e_1,e_2)} W^{(2)} = W_{(e_1,e_2)}.$$

But this is just the LCC for $n = 2$. The proof for general $n > 2$ follows in the same way. Systems of the form (11) have been treated in [11] in great detail and are called *composite dilation systems* there.

We summarize the above discussion in the following lemma:

Lemma 3 *Assume that H is a group of unimodular matrices. Assume further, that W is an integer-expanding matrix such that the system*

$$G := \left\{hW^k : \ h \in H, \ k \in \mathbb{Z}\right\}$$

forms a group. Then for any $E \subseteq H$ the dilation family $(hW)_{h\in E}$ satisfies the LCC.

Example 1 (Shearlet group) Maybe the most prominent composite dilation system is the shearlet system which arises from the construction above by setting

$$W = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad H = \left\{ B_j := \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} : \ j \in \mathbb{Z} \right\}.$$

The group structure of G can easily be verified from the relation

$$WB_j = B_{2j}W, \quad j \in \mathbb{Z}.$$

It is also possible to extend this example to higher dimensions.

Example 2 (Quincux dilation, [16]) Another interesting example arises from defining

$$W = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and H to be the group of symmetries of the unit square, i.e.

$$H = \{B_i : \ i = 0, \ldots, 7\},$$

where

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ B_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $B_i = -B_{i-4}$ for $i = 4, 5, 6, 7$. Again it is straightforward to verify the group property of G. This example can also be extended to higher dimensions using Coxeter groups.

Note that the dilation family in Example 2 is of finite type, whereas the dilation family in in Example 1 is of infinite type.

2.3 Moderation

We need to impose some more natural constraints on the dilation family $(W_e)_{e\in E}$, in particular, we would like that the distortion of the lattices $W^{-n}\mathbb{Z}^d$ introduced by application of the matrices $U_{\mathbf{e}}$, $\mathbf{e} \in E^n$ can be controlled somehow. In order to formalize this we introduce the following notion of a family of balls:

Definition 7 (Balls) A *family of balls* is a one-parameter family $(B_\tau)_{\tau\in\mathbb{R}_+}$ of bounded subsets of $\mathbb{R}^d$ such that

(Monotonicity)

$$\tau \leq \tau' \Rightarrow B_\tau \subseteq B_{\tau'}.$$

(Exhaustion)

$$\bigcup_{\tau\in\mathbb{R}_+} B_\tau = \mathbb{R}^d \quad \text{and} \quad \bigcap_{\tau\in\mathbb{R}^d} B_\tau = \{0\}.$$

(Subadditivity) There exists a constant $C \geq 0$ with

$$B_{\tau_1} + B_{\tau_2} + \cdots + B_{\tau_j} \subseteq B_{C(\tau_1+\tau_2+\cdots+\tau_j)} \quad \text{for all } j \in \mathbb{N}.$$

(Controlled growth) There exist constants C_1, C_2 so that

$$C_1\tau\,[-1, 1]^d \subseteq B_\tau \subseteq C_2\tau\,[-1, 1]^d \quad \text{for all } \tau \in \mathbb{R}_+.$$

Remark 5 Actually the subadditivity condition in the definition above is superfluous; it follows from the controlled growth condition. the reason why we still include it is that in the proof of Theorem 2 only the subadditivity is required and not the controlled growth.

Definition 8 (Moderate dilation family) Let $(B_\tau)_{\tau\in\mathbb{R}_+}$ be a family of balls. A dilation family $(W_e)_{e\in E}$ is called *moderate* with respect to the family $(B_\tau)_{\tau\in\mathbb{R}_+}$ if there exists a sequence $(\lambda_j)_{j\in\mathbb{N}} = O(\lambda^j)$, for some $\lambda < 1$ such that

$$W_{\mathbf{e}}^{-1} B_\tau \subseteq B_{\lambda_j\tau} \quad \text{for all } e \in E^j,\ \tau \in \mathbb{R}_+,\ j \in \mathbb{N}. \tag{12}$$

Now we can show that many interesting dilation families are moderate.

Proposition 1 *Every dilation family of finite type is moderate. The dilation family from Example* 1 *with* $E = \{-1, 0, 1\}$ *is moderate.*

Proof Recall the matrix W in Definition 2. By assumption the matrix W is integer-expanding. Therefore there exists a norm on $\mathbb{R}^s$ such that

$$\|W^{-n}x\| \le \rho\lambda^j\|x\|,$$

where $\rho > 0$ and $\lambda < 1$. Now define the family of balls

$$D_\tau := \tau D, \quad \text{where } D := \{x : \ \|x\| \le 1\}.$$

By assumption the set $\{U_{\mathbf{e}} : \ \mathbf{e} \in \bigcup_{n\in\mathbb{N}} E^n\}$ is finite. This implies that there exist numbers $\tau_{\min},\ \tau_{\max}$ such that

$$\tau_{\max} D \supseteq B := \bigcap_{\mathbf{e}\in E_\infty} U_{\mathbf{e}} D \supseteq \tau_{\min} D.$$

Now we put $B_\tau := \tau B$ and (12) follows since for $\mathbf{e} \in E^j$

$$W_{\mathbf{e}}^{-1} B \subset W_{\mathbf{e}}^{-1} U_{\mathbf{e}} D = W^{-n} D \subseteq \lambda^j \rho D \subseteq \lambda^j \tau_{\min}^{-1} \rho B.$$

To show the subadditivity property we use the triangle inequality to compute

$$\begin{aligned} B_{\tau_1} + \cdots + B_{\tau_j} &= \tau_1 B + \cdots + \tau_j B \subseteq \tau_1 \tau_{\max} D + \cdots + \tau_j \tau_{\max} D \\ &\subseteq \left(\tau_1 + \cdots + \tau_j\right) \tau_{\max} D \subseteq \frac{\tau_{\max}}{\tau_{\min}} \left(\tau_1 + \cdots + \tau_j\right) B \\ &= B_{C(\tau_1+\cdots+\tau_j)} \end{aligned}$$

with $C := \frac{\tau_{\max}}{\tau_{\min}}$. Monotonicity and exhaustion of the family (B_τ) are obvious. The controlled growth follows from the equivalence of all norms on $\mathbb{R}^d$. To show that the family from Example 1 with $E := \{-1, 0, 1\}$ is moderate, we show that

$$W_e^{-1} [-1, 1]^2 \subseteq 1/2 [-1, 1]^2, \quad e \in \{-1, 0, 1\}.$$

This is obviously true for $e = 0$, but also the case $e = -1, 1$ follows straightforwardly since for instance

$$W_1^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}$$

and clearly $\|W_1^{-1}\|_{l_\infty(\mathbb{Z}_2)\to l_\infty(\mathbb{Z}_2)} \le \frac{1}{2}$. □

In order to get a better grasp of the moderation property of a dilation family, we give an equivalent definition in terms of a spectral quantity.

Theorem 1 *A dilation family $\mathcal{W}$ is moderate if and only if the* joint spectral radius $\rho(\mathcal{W})$ *defined by*

$$\rho(\mathcal{W}) := \limsup_{n\to\infty} \sup_{\mathbf{e}\in E^n} \|W_{\mathbf{e}}^{-1}\|^{1/n},$$

with $\|\cdot\|$ any matrix norm, satisfies

$$\rho(\mathcal{W}) < 1.$$

Proof That moderation implies that $\rho(\mathcal{W}) < 1$ follows from the controlled growth condition: Indeed, with $\|\cdot\|$ being the matrix norm induced by $\|\cdot\|_\infty$ on $\mathbb{R}^d$, the controlled growth condition implies that $\|W_{\mathbf{e}}^{-1}\|$, $\mathbf{e} \in E^n$ is dominated by $C\lambda^n$, where $\lambda < 1$ and C is independent of $n, \mathbf{e}$. This in turn implies that $\rho(\mathcal{W}) < 1$. To show that $\rho(\mathcal{W}) < 1$ implies moderation we note that the condition on the spectral radius implies that for every matrix norm $\|\cdot\|$ induced from a norm $\|\cdot\|'$ on $\mathbb{R}^d$, we have $\|W_{\mathbf{e}}^{-1}x\|' \leq C\lambda^n\|x\|'$ with C depending only on $\|\cdot\|'$ and $\lambda < 1$. It follows that $\mathcal{W}$ is moderate with respect to the family of balls given by $B_\tau := \tau B$ and B the unit ball w.r.t. $\|\cdot\|'$. □

Corollary 1 *$\mathcal{W} = (W_e)_{e\in E}$ is moderate if and only if $\mathcal{W}^T := (W_e^T)_{e\in E}$ is moderate.*

Proof Just pick a matrix norm with $\|W\| = \|W^T\|$ for all $W \in \mathbb{R}^{d\times d}$ in the definition of $\rho(\mathcal{W})$. □

3 Convergence and refinability

Here we explore the basic relations between the existence of a refinable function for a dilation family and a filter family and the convergence of the associated refinement process. In analogy to the theory of stationary subdivision [4] we first show that convergence of the refinement process implies the existence of a refinable function. We also show the converse under a stability assumption. After that we show that for any dilation family and any finitely supported filter family, there exists a tempered distribution which is compactly supported and which satisfies the associated refinement relation.

3.1 Convergence implies refinability

Theorem 2 *Assume that $\mathcal{W} = (W_e)_{E\in E}$ is a dilation family and $\mathcal{A} = (a_e)_{e\in E}$ a filter family. The associated refinement process converges if and only if it converges for initial data δ. In this case the limit function $\varphi := \varphi_\delta$ satisfies the refinement equation (5) and for all initial data $p \in l_\infty(\mathbb{Z}^d)$ the limit function φ_p is given by*

$$\varphi_p(x) = \sum_{\alpha\in\mathbb{Z}^d} p(\alpha)\varphi(x-\alpha). \tag{13}$$

If $\mathcal{A}$ is finitely supported and $\mathcal{W}$ is moderate, then φ is of compact support.

Proof The first statement is just a consequence of the linearity of the operators $S^{(n)}$. Also (13) is a direct consequence of the linearity and the translation invariance (9). We go on to prove that φ satisfies the refinement equation (5) if $\mathcal{S}$ is convergent. Since φ is continuous we only need to verify (5) for points $x = W^{-n_0}\eta$ with $\eta \in \mathbb{Z}^d$. Let $n \geq n_0$. Then

$$\begin{aligned}\int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha)\varphi\left(W_e x-\alpha\right) d\mu(e) &= \int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha)\big(\varphi\left(W_e x-\alpha\right) - S^{(n)} \\ &\quad \times \delta\left(W^n\left(W_e x-\alpha\right)\right)\big)d\mu(e) \\ &\quad + \int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha) S^{(n)}\delta\left(W^n\left(W_e x-\alpha\right)\right) d\mu(e) \\ &\leq \sup_{e\in E}\|a_e\|_1 \sup_{y\in\Gamma^{(n)}} \left|\varphi(y) - S^{(n)}\delta\left(W^n y\right)\right| \\ &\quad + \int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha) S^{(n)}\delta\left(W^n\left(W_e x-\alpha\right)\right) d\mu(e) \\ &= o(1)+\int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha) S^{(n)}\delta\left(W^n\left(W_e x-\alpha\right)\right) d\mu(e).\end{aligned}$$

It remains to show that $\int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha) S^{(n)}\delta\left(W^n\left(W_e x-\alpha\right)\right) d\mu(e) \to \varphi(x)$. We have

$$\begin{aligned}&\int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha) S^{(n)}\delta\left(W^n\left(W_e x-\alpha\right)\right) d\mu(e) \\ &= \int_E \sum_{\alpha\in\mathbb{Z}^d} a_e(\alpha) \int_{\mathbf{e}\in E^n} S_{\mathbf{e}}\delta\left(W_{\mathbf{e}}\left(W_e x-\alpha\right)\right) d\mu^n(\mathbf{e})d\mu(e) \\ &= \int_E \int_{\mathbf{e}\in E^n} \sum_{\alpha\in\mathbb{Z}^d} a_{\mathbf{e}}\left(W_{\mathbf{e}}W_e x - W_{\mathbf{e}}\alpha\right) a_e(\alpha) d\mu^n(\mathbf{e})d\mu(e) \\ &= \int_E \int_{\mathbf{e}\in E^n} S_{\mathbf{e}} a_e\left(W_{\mathbf{e}}W_e x\right) d\mu^n(\mathbf{e})d\mu(e) \\ &= \int_E \int_{\mathbf{e}\in E^n} S_{\mathbf{e}} S_e\delta\left(W_{\mathbf{e}}W_e x\right) d\mu^n(\mathbf{e})d\mu(e) \\ &= \int_E \int_{\mathbf{e}\in E^n} S_{(e,\mathbf{e})}\delta\left(W_{(e,\mathbf{e})}x\right) d\mu^n(\mathbf{e})d\mu(e) = S^{(n+1)}\left(W^{n+1}x\right) \to \varphi(x).\end{aligned}$$

To show that φ is compactly supported we note that, since $S^{(n)}\delta(\alpha) \to \varphi(W^{-n}\alpha)$, it suffices to show that the support of $W^{-n}S^{(n)}\delta$ stays bounded. This follows if the support of $W_{\mathbf{e}}^{-1}a_{\mathbf{e}}$ stays bounded independently of $\mathbf{e} \in E_\infty$. For a sequence p with $\operatorname{supp}(p) \subseteq B_M$ and a mask a with $\operatorname{supp}(a) \subseteq B_N$ it is easy to see that

$$W^{-1}\operatorname{supp}\left(S_{a,W}p\right) \subseteq W^{-1}B_N + B_M.$$

Applying this inclusion inductively and assuming that all masks a_e are supported in B_M yields for $\mathbf{e} \in E^n$

$$W_{\mathbf{e}}^{-1}\text{supp}\,(S_{\mathbf{e}}\delta) \subseteq \sum_{l=1}^{n} W_{P_l(\mathbf{e})}^{-1} B_M \subseteq \sum_{l=1}^{n} B_{\lambda_l M} \subseteq B_{\sum_{l=1}^{n}\lambda_l M} \subseteq B_{\tilde{M}}$$

with $\tilde{M} := \sum_{l=1}^{\infty} \lambda_l M$. Since this bound is independent of $\mathbf{e}$ and n, this proves the compact support of φ. □

Remark 6 A careful examination of the proof above shows that compact support of φ can still be shown if we drop the controlled growth condition in the definition of a family of balls and if we replace the condition $(\lambda_i)_{i\in\mathbb{N}} = O(\lambda^i)$ for some $\lambda < 1$ by the weaker condition $\sum_{i\in\mathbb{N}} \lambda_i < \infty$. This is actually the reason why we did not define moderation in terms of the joint spectral radius in the first place.

Example 3 Continuing the Example 1, a refinable function with respect to the shearlet dilation family satisfies

$$\varphi(\cdot) = \sum_{e\in\{-1,0,1\}} \frac{1}{3} a_e(\alpha)\varphi\left(\begin{pmatrix} 4 & 2e \\ 0 & 2 \end{pmatrix} \cdot -\alpha\right).$$

These are precisely the scaling functions that we are aiming for and which ensure the nestedness of the shearlet scaling spaces as defined in the introduction. From the previous result we know that for the construction of such functions it suffices to find a convergent refinement procedure.

We now prove a partial converse to Theorem 2 under an additional stability assumption.

Definition 9 A function φ is called $\mathcal{W}$-stable if there exists a constant $\nu > 0$ such that

$$\limsup_{n\to\infty} \left\| \int_{E^n} \sum_{\alpha\in\mathbb{Z}^d} c(\mathbf{e},\alpha)\varphi\left(W_{\mathbf{e}} \cdot -\alpha\right) d\mu^n(\mathbf{e}) \right\|_{L_\infty(\Gamma^{(n)})} \geq \nu \limsup_{n\to\infty} \left\| \int_{E^n} c\left(\mathbf{e}, U_{\mathbf{e}}\cdot\right) d\mu^n(\mathbf{e}) \right\|_{l_\infty}. \quad (14)$$

Theorem 3 *Assume that φ is compactly supported, uniformly continuous and refinable with respect to a moderate dilation family $\mathcal{W}$ and a finitely supported filter family $\mathcal{A}$. If φ is $\mathcal{W}$-stable, then the associated refinement operator $\mathcal{S}$ is convergent.*

Proof We first show that with our assumptions we have that

$$\Psi(x) := \sum_{\alpha\in\mathbb{Z}^d} \varphi(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^d} \varphi(\alpha) = \Psi(0). \quad (15)$$

Since Ψ is $\mathbb{Z}^d$-periodic, this statement is definitely true for $x \in \mathbb{Z}^d = \Gamma^{(0)}$. Using the refinement relation we deduce that

$$\Psi(x) = \int_E \sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \in \mathbb{Z}^d} a_e(\alpha - W_e\beta)\,\varphi(W_e x - \alpha) = \int_E \Psi(W_e x).$$

Using this fact, we can conclude inductively that for all $x \in \bigcup_{n\in\mathbb{N}} \Gamma^{(n)}$ we have (15) and by continuity this holds for all x. Let us normalize φ so that $\Psi(0) = 1$. We want to show that the refinement procedure $\mathcal{S}$ converges. To do this we first estimate using the refinement relation and Equation (15):

$$\begin{aligned} 0 &= \left\| \varphi(x) - \int_{E^n} \sum_{\alpha \in \mathbb{Z}^d} S_{\mathbf{e}}\delta(\alpha)\varphi(W_{\mathbf{e}}x - \alpha)d\mu^n(\mathbf{e}) \right\|_{L_\infty(\Gamma^{(n)})} \\ &= \left\| \int_{E^n} \sum_{\alpha \in \mathbb{Z}^d} (\varphi(x) - S_{\mathbf{e}}\delta(\alpha))\varphi(W_{\mathbf{e}}x - \alpha)d\mu^n(\mathbf{e}) \right\|_{L_\infty(\Gamma^{(n)})} \\ &\geq \left\| \int_{E^n} \sum_{\alpha \in \mathbb{Z}^d} \left(\varphi\left(W_{\mathbf{e}}^{-1}\alpha\right) - S_{\mathbf{e}}\delta(\alpha)\right)\varphi(W_{\mathbf{e}}x - \alpha)d\mu^n(\mathbf{e}) \right\|_{L_\infty(\Gamma^{(n)})} \\ &\quad - \left\| \int_{E^n} \sum_{\alpha \in \mathbb{Z}^d} \left(\varphi\left(W_{\mathbf{e}}^{-1}\alpha\right) - \varphi(x)\right)\varphi(W_{\mathbf{e}}x - \alpha)d\mu^n(\mathbf{e}) \right\|_{L_\infty(\Gamma^{(n)})} := A - B \end{aligned}$$

We show that $B \to 0$. Assume that $\mathrm{supp}(\varphi) \subseteq K$ with K compact. Then

$$\begin{aligned} B &= \left\| \int_{E^n} \sum_{\alpha \in W_{\mathbf{e}}x - K} \left(\varphi\left(W_{\mathbf{e}}^{-1}\alpha\right) - \varphi(x)\right)\varphi(W_{\mathbf{e}}x - \alpha)d\mu^n(\mathbf{e}) \right\|_{L_\infty(\Gamma^{(n)})} \\ &\leq |K| \sup_{\mathbf{e}\in E^n} \sup_{x-y \in W_{\mathbf{e}}^{-1}K} |\varphi(x) - \varphi(y)| \to 0 \end{aligned}$$

because of the controlled growth condition on $\mathcal{W}$ and the continuity of φ. Now, since $B \to 0$ the estimates above imply that $A \to 0$. Hence we can use the stability condition to conclude that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\| \int_{E^n} \sum_{\alpha \in \mathbb{Z}^d} \left(\varphi\left(W_{\mathbf{e}}^{-1}\alpha\right) - S_{\mathbf{e}}\delta(\alpha)\right)\varphi\left(W_{\mathbf{e}}x - \alpha\right)d\mu^n(\mathbf{e}) \right\|_{L_\infty(\Gamma^{(n)})} \\ &\geq \nu \limsup_{n\to\infty} \left\| \int_{E^n} \varphi\left(W^{-n}\alpha\right) - S_{\mathbf{e}}\left(U_{\mathbf{e}}\alpha\right)d\mu^n(\mathbf{e}) \right\|_{l_\infty(\mathbb{Z}^d)} \\ &= \nu \limsup_{n\to\infty} \left\| \varphi\left(W^{-n}\alpha\right) - \int_{E^n} S_{\mathbf{e}}\left(U_{\mathbf{e}}\alpha\right)d\mu^n(\mathbf{e}) \right\|_{l_\infty(\mathbb{Z}^d)} \\ &= \nu \limsup_{n\to\infty} \left\| \varphi(x) - S^{(n)}\left(W^n x\right) \right\|_{L_\infty(\Gamma^{(n)})} \end{aligned}$$

and this means that the refinement process converges. □

Remark 7 Any function Φ that is cardinal, i.e. $\Phi(\alpha) = \delta(\alpha), \alpha \in \mathbb{Z}^d$ is $\mathcal{W}$-stable.

3.2 The normalized solution

In [4] it is shown that for any conventional subdivision scheme there exists a tempered distribution which is refinable. We show that an analogous result holds in our setting.

Theorem 4 *For every moderate dilation family $\mathcal{W}$ and every filter family $\mathcal{A}$ there exists a distribution Φ with $\hat{\Phi}(0) = 1$ that satisfies the refinement equation* (5)*. If $\mathcal{A}$ is finitely supported, Φ has compact support.*

Proof We only prove the more difficult statement for a finite family of filters $\mathcal{A}$. The rest follows along similar lines and is considerably simpler. Taking Fourier transforms, we see that Φ must satisfy

$$\begin{aligned}\hat{\Phi}(\omega) &= \int_E \det(W_e)^{-1} \sum_{\alpha \in \mathbb{Z}^d} a_e(\alpha) \exp\left(-i\alpha W_e^{-T}\omega\right) \hat{\Phi}\left(W_e^{-T}\omega\right) d\mu(e) \\ &= \int_E \det(W_e)^{-1} \hat{a}_e\left(W_e^{-T}\omega\right) \hat{\Phi}\left(W_e^{-T}\omega\right) d\mu(e).\end{aligned}$$

The obvious candidate for such a function is defined via

$$\hat{\Phi}(\omega) := \lim_{n\to\infty} \int_{E^n} \prod_{l=1}^{n} \det\left(W_{e_l}\right)^{-1} \hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right) d\mu^n(\mathbf{e}), \tag{16}$$

where $\mathbf{e} = (e_1, \ldots, e_n)$ and $P_l(\mathbf{e}) := (e_1, \ldots, e_l) \in E^l, l \leq n$. In the following we show that $\hat{\Phi}$ defined via (16) is the Fourier transform of a compactly supported tempered distribution. Using the controlled growth property of the family of balls which moderates $\mathcal{W}^T$ by Corollary 1 and the affine invariance of $\mathcal{A}$, it is straightforward to show that the right hand side of (16) converges uniformly on compact sets for $\omega \in \mathbb{C}^s$, hence we can extend $\hat{\Phi}$ to an entire function. We show that $\hat{\Phi}$ is of exponential type: First, by the finite support of $\mathcal{A}$ and elementary (and well-known) estimates we have

$$\left|\det(W_e)^{-1} \hat{a}_e(\omega) - 1\right| \leq C \exp\left(A \|\Im\omega\|_2\right) \min\left(1, \|\omega\|_2\right), \tag{17}$$

with constants A, C independent of $e \in E$: Indeed, since

$$|\exp(-i\alpha\omega) - 1| \leq 2\exp(|\Im\alpha\omega|) \min(|\alpha\omega|, 2),$$

we have, assuming that all filters a_e are supported in the set $\{\alpha \in \mathbb{Z}^d : \|\alpha\|_2 \leq L\}$, $e \in E$, $L \in \mathbb{R}_+$,

$$
\begin{aligned}
&|\det(W_e)^{-1}\hat{a}_e(\omega) - 1| \\
&\quad = \left|\sum_{\alpha\in\mathbb{Z}^d} \det(W_e)^{-1} a_e(\alpha)\exp(-i\alpha\omega) - 1\right| \\
&\quad = \left|\sum_{\alpha\in\mathbb{Z}^d} \det(W_e)^{-1} a_e(\alpha)(\exp(-i\alpha\omega) - 1)\right| \\
&\quad = \left|\sum_{\|\alpha\|_2\leq L} \det(W_e)^{-1} a_e(\alpha)(\exp(-i\alpha\omega) - 1)\right| \\
&\quad \leq \left|\det(W_e)^{-1}\right| \sup_{e\in E}\|a_e\|_1 \sup_{\|\alpha\|_2\leq L} |\exp(-i\alpha\omega) - 1| \\
&\quad \leq \left|\det(W_e)^{-1}\right| \sup_{e\in E}\|a_e\|_1 \sup_{\|\alpha\|_2\leq L} 2\exp(|\alpha\Im\omega|)\min(|\alpha\omega|, 2) \\
&\quad \leq \left|\det(W_e)^{-1}\right| \sup_{e\in E}\|a_e\|_1 2\exp(L\|\Im\omega\|_2)\min(L\|\omega\|_2, 2) \\
&\quad \leq \left|\det(W_e)^{-1}\right| \sup_{e\in E}\|a_e\|_1 2\exp(L\|\Im\omega\|_2)\max(L, 2)\min(\|\omega\|_2, 1), \qquad (18)
\end{aligned}
$$

and this is (17) with $A = L$ and $C = |\det(W_e)^{-1}| \sup_{e\in E}\|a_e\|_1 2\max(L, 2)$.

By the moderation condition we know that we have

$$\left\|W_{\mathbf{e}}^{-T}x\right\| \leq D\lambda^n\|x\| \quad \text{for} \quad \mathbf{e} \in E^n, \tag{19}$$

$\lambda < 1, n \in \mathbb{N}$, $D > 0$ and $\|\cdot\|$ (say) the Euclidean norm in $\mathbb{C}^d$. We investigate a product

$$\prod_{l=1}^{n} \det\left(W_{e_l}\right)^{-1} \hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right), \quad \mathbf{e} \in E^n. \tag{20}$$

With $k = \lceil \log_{1/\lambda}(\|\omega\|) + \log_{1/\lambda}(D)\rceil$ we have

$$
\begin{aligned}
|(20)| &\leq \prod_{l=1}^{k}\left|\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\right| \prod_{l=k+1}^{n}\left|\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\right| \\
&\leq \prod_{l=1}^{k}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\right) \prod_{l=k+1}^{n}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\left\|W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right).
\end{aligned}
$$

Since by the definition of k we have $\|W_{P_l(\mathbf{e})}^{-T}\omega\| \leq 1$ for $l > k$, we can further estimate

$$
\begin{aligned}
|(20)| &\leq \prod_{l=1}^{k}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\prod_{l=k+1}^{n}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\left\|W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\right. \\
&\leq \prod_{l=1}^{k}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\prod_{l=k+1}^{n}\left(1+C\exp(A)\left\|W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\right. \\
&\leq \prod_{l=1}^{k}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\prod_{l=k+1}^{n}\left(1+C\exp(A)D\lambda^{l}\|\omega\|\right)\right. \\
&\leq \prod_{l=1}^{k}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\prod_{l=k+1}^{n}\exp\left(C\exp(A)D\lambda^{l}\|\omega\|\right)\right. \\
&\leq \prod_{l=1}^{k}\left(1+C\exp\left(A\left\|\Im W_{P_l(\mathbf{e})}^{-T}\omega\right\|\right)\exp\left(\frac{C\exp(A)}{1-\lambda}\right)\right.
\end{aligned}
$$

Using the contractivity property (19) again we arrive at

$$
\begin{aligned}
|(20)| &\leq (1+C)^{k}\prod_{l=1}^{k}\exp\left(A\left\|W_{P_l(\mathbf{e})}^{-T}\Im\omega\right\|\right)\exp\left(\frac{C\exp(A)}{1-\lambda}\right) \\
&\leq (1+C)^{k}\prod_{l=1}^{k}\exp\left(AD\lambda^{k}\|\Im\omega\|\right)\exp\left(\frac{C\exp(A)}{1-\lambda}\right) \\
&\leq (1+C)^{k}\exp\left(\frac{AD}{1-\lambda}\|\Im\omega\|\right)\exp\left(\frac{C\exp(A)}{1-\lambda}\right) \\
&\leq (1+C)^{\log_{1/\lambda}(\|\omega\|)}\exp\left(\frac{AD}{1-\lambda}\|\Im\omega\|\right)\exp\left(\frac{C\exp(A)}{1-\lambda}\right)(1+C)^{\lceil\log_{1/\lambda}(D)\rceil+1} \\
&\leq \|\omega\|^{\log_{1/\lambda}(1+C)}\exp\left(\frac{AD}{1-\lambda}\|\Im\omega\|\right)\exp\left(\frac{C\exp(A)}{1-\lambda}\right)(1+C)^{\lceil\log_{1/\lambda}(D)\rceil+1}.
\end{aligned}
$$

Since the above bound is independent of $\mathbf{e}$, this proves that $\hat{\Phi}$ is of exponential type and thus we can apply the Paley–Wiener–Schwartz Theorem [15] to conclude that Φ is a compactly supported tempered distribution. □

Definition 10 The distribution Φ from Theorem 4 is called the *normalized solution* of the refinement equation corresponding to $\mathcal{W}, \ \mathcal{A}$.

4 Construction of refinable functions for every moderate dilation family

In this section we prove one of our main results, namely the existence of a compactly supported and continuous refinable function for every moderate dilation family. We also discuss the convergence behavior of the associated refinement process for interpolatory filters. In our analysis we generalize a number of well-known notions and results from classical wavelet theory, such as Cohen's condition. We recommend the reader to look at [20] for a comparison between classical results on wavelets and our more general results. In particular, [20] constructs (standard) refinable functions from interpolating filters using positivity conditions on the Fourier transforms of these filters. We will show analogous results for general dilation families, resulting in explicit constructions of refinable functions. Our proofs strongly build on the ideas from [20], but, compared to the situation covered there, several additional technical issues arise in our setting.

4.1 Filters with positive Fourier transform and a generalized Cohen condition

Definition 11 A filter family $\mathcal{A} = (a_e)_{e\in E}$ is called *interpolating* if

$$a_e\left(W_e\beta\right) = \delta(\beta) \quad \text{for all } \beta \in \mathbb{Z}^d. \tag{21}$$

If $\mathcal{A}$ is an interpolating filter family, then also all filter families $\mathcal{A}_n := (a_{\mathbf{e}})_{\mathbf{e}\in E^n}$ are interpolating with respect to the dilation family $\mathcal{W}_n := (W_{\mathbf{e}})_{\mathbf{e}\in E^n}$ for all $n \in \mathbb{N}$. In particular, this implies that

$$a_{\mathbf{e}}(0) = 1 \quad \text{for all } \mathbf{e} \in E_\infty,$$

or equivalently

$$\int_{[-\pi,\pi]^d} \hat{a}_{\mathbf{e}}(\omega)d\omega = (2\pi)^d \quad \text{for all } \mathbf{e} \in E_\infty. \tag{22}$$

We will need this equation in the proof of the following result which is a direct extension of the results in [20, Section 4]

Proposition 2 *Assume that $\mathcal{W}$ is a moderate dilation family and $\mathcal{A}$ is an interpolating filter family such that $\hat{a}_e(\omega) \geq 0$ for all $\omega \in \mathbb{R}^d$ and $e \in E$. Then the normalized solution is a uniformly continuous function.*

Proof We show that $\hat{\Phi} \in L_1$, where Φ is the normalized solution according to Theorem 4. By (7), the Fourier transform of $a_{\mathbf{e}}$ is given as

$$\hat{a}_{\mathbf{e}}(\omega) = \prod_{l=1}^{n} \hat{a}_{e_l}\left(W^T_{R_l(\mathbf{e})}\omega\right),$$

where $\mathbf{e} = (e_1, \ldots, e_n) \in E^n$, $R_l(\mathbf{e}) = (e_{l+1}, \ldots, e_n), l < n$, $R_n(\mathbf{e}) = \emptyset$ and $W_\emptyset := I$. Clearly, we also have $\hat{a}_{\mathbf{e}}(\omega) \geq 0$ for all $\omega \in \mathbb{R}^d$. Now we estimate

$$\int_{C_2^{-1} B_{\lambda_n^{-1}}} |\hat{\Phi}(\omega)| d\omega = \int_{C_2^{-1} B_{\lambda_n^{-1}}} \hat{\Phi}(\omega) d\omega \leq \int_{\bigcap_{\mathbf{e}\in E^n} W_{\mathbf{e}}^T [-\pi,\pi]^d} \hat{\Phi}(\omega) d\omega$$

$$\leq \int_{E^n} \int_{W_{\mathbf{e}}^T [-\pi,\pi]^d} \prod_{l=1}^{n} \det \left(W_{e_l}\right)^{-1} \hat{a}_{e_l} \left(W_{P_l(\mathbf{e})}^{-T} \omega\right)$$

$$\times \hat{\Phi} \left(W_{\mathbf{e}}^{-T} \omega\right) d\mu^n(\mathbf{e}) d\omega$$

$$= \int_{E^n} \int_{[-\pi,\pi]^d} \hat{a}_{\mathbf{e}}(\omega) \hat{\Phi}(\omega) d\omega d\mu^n(\mathbf{e})$$

$$\leq \|\hat{\Phi}\|_{L_\infty([-\pi,\pi]^d)} \int_{E^n} \int_{[-\pi,\pi]^d} |a_{\mathbf{e}}(\omega)| d\omega d\mu^n(\mathbf{e})$$

$$= \|\hat{\Phi}\|_{L_\infty([-\pi,\pi]^d)} \int_{E^n} \int_{[-\pi,\pi]^d} a_{\mathbf{e}}(\omega) d\omega d\mu^n(\mathbf{e})$$

$$\leq (2\pi)^d \|\hat{\Phi}\|_{L_\infty([-\pi,\pi]^d)},$$

where we let (B_τ) be a family of balls moderating $\mathcal{W}^T$ and C_2 the constant from the controlled growth condition. This shows that $\hat{\Phi} \in L_1$ and the proof is complete. □

Theorem 5 *For every moderate dilation family $\mathcal{W}$ there exists a finite filter family $\mathcal{A}$ and a uniformly continuous and compactly supported function φ which is refinable with respect to $\mathcal{W}$ and $\mathcal{A}$.*

Proof In [12] it is shown that there exists a dilation matrix M generating the same lattice as W and an interpolating (with respect to the dilation matrix M) filter a with positive Fourier transform. Since W and M generate the same lattice, there exists a unimodular matrix V with $M = WV$. Then

$$\sum_{\alpha\in\mathbb{Z}^d} a(W\alpha) = \sum_{V\alpha\in\mathbb{Z}^d} a(WV\alpha) = \sum_{\alpha\in Z^d} a(M\alpha) = \delta(\alpha),$$

so a is interpolating with respect to W, too. Now we put

$$a_e(\cdot) := a\left(U_e^{-1} \cdot\right), \quad e \in E.$$

The filter a_e is interpolating with respect to W_e. Furthermore, since $\hat{a}(\omega) \geq 0$ for all $\omega \in \mathbb{R}^d$, we have $\hat{a}_e(\omega) = \hat{a}\left(U_e^T \omega\right) \geq 0$ for all $e \in E$, $\omega \in \mathbb{R}^d$. By Proposition 2, the normalized solution is a continuous function. This proves the theorem. □

We want to explore the refinable function Φ in more depth. In particular we would like to know if Φ arises as the limit of the associated refinement process which would give us a constructive means to define these functions.

In order to proceed we need to make some more definitions.

Definition 12 A measurable set $K \subset \mathbb{R}^d$ is called *congruent to* $[-\pi, \pi]^d$ *modulo* $\mathbb{Z}^d$, if

(i) K contains a neighborhood of 0,
(ii) $\bigcup_{\alpha \in \mathbb{Z}^d} (K + \alpha) = \mathbb{R}^d$,
(iii) $K \cap (K + \alpha)$ is of Lebesgue measure zero for $\alpha \neq 0$.

This definition implies that K can be cut into countably many pieces $K_\alpha := K \cap ([-\pi, \pi]^d + \alpha)$, $\alpha \in \mathbb{Z}^d$ such that $\tilde{K} := \bigcup T_{-\alpha} K_\alpha$ is a disjoint union in the measure-theoretic sense by (iii) ($T_{-\alpha}$ denoting translation by $-\alpha$), and by (ii) we have the equality $\tilde{K} = [-\pi, \pi]^d$ holding in the measure theoretic sense. In particular, this implies that the system $(\exp(-i\alpha\omega))_{\alpha \in \mathbb{Z}^d}$ forms an orthonormal basis of $L_2(K)$.

Definition 13 A filter family $\mathcal{A}$ and a dilation family $\mathcal{W}$ satisfies the *generalized Cohen condition* if there exists a set K congruent to $[-\pi, \pi]^d$ modulo $\mathbb{Z}^d$ and $\mathcal{E} \subset E^\infty$ such that for all $\mathbf{e} = (e_1, e_2, \dots) \in \mathcal{E}$, $\omega \in K$ and $l \in \mathbb{N}$ we have $\mu^l(\{P_l(\mathcal{E})\}) > 0$ and

$$\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T} \omega\right) > \rho > 0 \tag{23}$$

for some positive constant ρ.

The following result generalizes [20, Theorem 4.1] to general dilation families.

Proposition 3 *With the assumptions from Proposition* 2 *assume further that the filter family and the dilation family satisfies the generalized Cohen condition. Then the normalized solution is cardinal, i.e.*

$$\Phi(\alpha) = \delta(\alpha) \quad \textit{for all } \alpha \in \mathbb{Z}^d. \tag{24}$$

In particular the refinement process $\mathcal{S}$ associated with $\mathcal{A}$ and $\mathcal{W}$ is convergent with limit function Φ.

Proof Define the sequence of functions

$$\hat{\varphi}_n(\omega) := \int_{E^n} \prod_{l=1}^{n} \det\left(W_{e_l}\right)^{-1} \hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T} \omega\right) \chi_{W_{\mathbf{e}}^T K}(\omega) d\mu^n(\mathbf{e}).$$

Since K contains a neighborhood of 0 and the controlled growth condition of the dilation family we have pointwise convergence $\hat{\varphi}_n \to \hat{\Phi}$. The crucial thing to prove the cardinality of Φ is to show that the sequence $\hat{\varphi}_n$ is dominated by an L_1-function in order to apply the dominated convergence theorem in the argument below. This is the only place where Cohen's condition is needed. Assume now that we can apply the dominated convergence theorem and exchange integrals and the limit $n \to \infty$ below. Then for $\alpha \in \mathbb{Z}^d$

$$\begin{aligned}\Phi(\alpha) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\Phi}(\omega) \exp(-i\alpha\omega) d\omega = (2\pi)^{-d} \int_{\mathbb{R}^d} \lim_{n\to\infty} \hat{\varphi}_n(\omega) \exp(-i\alpha\omega) d\omega \\ &= \lim_{n\to\infty} (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\varphi}_n(\omega) \exp(-i\alpha\omega) d\omega \\ &= \lim_{n\to\infty} (2\pi)^{-d} \int_{E^n} \int_{W_{\mathbf{e}}^T K} \prod_{l=1}^{n} \det\left(W_{e_l}\right)^{-1} \hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right) \exp\left(-i\alpha\omega\right) d\omega d\mu^n(\mathbf{e}) \\ &= \lim_{n\to\infty} (2\pi)^{-d} \int_{E^n} \int_{K} \prod_{l=1}^{n} \hat{a}_{e_l}\left(W_{R_l(\mathbf{e})}^{T}\omega\right) \exp\left(-iW_{\mathbf{e}}^T \alpha\omega\right) d\omega d\mu^n(\mathbf{e}) \\ &= \lim_{n\to\infty} (2\pi)^{-d} \int_{E^n} \int_{K} \hat{a}_{\mathbf{e}}(\omega) \exp\left(-iW_{\mathbf{e}}\alpha\omega\right) d\omega d\mu^n(\mathbf{e}) \\ &= \lim_{n\to\infty} \int_{E^n} a_{\mathbf{e}}\left(W_{\mathbf{e}}\alpha\right) = \delta(\alpha),\end{aligned}$$

hence Φ is cardinal. We still need to justify the interchange of the limit with the integral in the second line above. To do this we show that there exists a constant $A \geq 0$ such that

$$\varphi_n(\omega) \leq A\hat{\Phi}(\omega) \quad \text{for all } \omega \in \mathbb{R}^d,\ n \in \mathbb{N}. \tag{25}$$

Since we already know that $\hat{\Phi} \in L_1$, this is sufficient to apply the interchange of limits. It is not difficult to see that (25) holds if there exists a constant $B > 0$ such that

$$\hat{\Phi}(\omega) \geq B \quad \text{for all } \omega \in K. \tag{26}$$

Indeed, if we assume (26), we have the estimate

$$\hat{\varphi}_n(\omega) \leq \int_{E^n} \prod_{l=1}^{n} \det\left(W_{e_l}\right)^{-1} \hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right) \hat{\Phi}\left(W_{\mathbf{e}}^{-T}\omega\right) \frac{1}{B} d\mu^n(\mathbf{e}) = \frac{1}{B}\hat{\Phi}(\omega),$$

and this is (25) with $A = \frac{1}{B}$. Our goal now is to prove (26). To do this we first pick n_0 large enough such that there exists a constant B_1 and $\lambda < 1$ with

$$\left|1 - \det\left(W_{e_l}\right)^{-1} \hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\right| \leq B_1\lambda^l \leq \exp(-1) \tag{27}$$

for all $\mathbf{e} \in E^\infty$ and $l \geq n_0$. Since $\mathcal{W}$ is moderate such a constant exists. We define the *cylinder set*

$$[\mathcal{E}]_{n_0} := \left\{\mathbf{e} \in E^\infty : P_{n_0}(\mathbf{e}) \in P_{n_0}(\mathcal{E})\right\}$$

and note that for all $n \geq n_0$

$$\mu^n\left(P_n\left([\mathcal{E}]_{n_0}\right)\right) = \mu^{n_0}\left(P_{n_0}(\mathcal{E})\right) =: B_2 > 0.$$

By the generalized Cohen condition, we have

$$\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right) \geq \det(W_{e_l})^{-1}\rho \quad \text{for all } \omega \in K,\ \mathbf{e} \in [\mathcal{E}]_{n_0},\ l \leq n_0. \tag{28}$$

Now pick any $\mathbf{e} \in [\mathcal{E}]_{n_0}$ and write $u = \det(W)^{-1} = \det(W_e)^{-1}$ for all $e \in E$. Since

$$1 - t \geq \exp(-\exp(1)t), \quad 0 \leq t \leq \exp(-1),$$

we can estimate for $n > n_0$ and $\omega \in K$

$$\begin{aligned}
\prod_{l=1}^{n}\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right) &= \prod_{l=1}^{n_0}\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\\
&\quad \times \prod_{l=n_0+1}^{n}\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\\
&\geq \rho^{n_0}u^{n_0}\prod_{l=n_0+1}^{n}\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\\
&\geq \rho^{n_0}u^{n_0}\prod_{l=n_0+1}^{n}\left(1-\left|1-\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\right|\right)\\
&\geq \rho^{n_0}u^{n_0}\prod_{l=n_0+1}^{n}\exp\left(-\exp(1)\left|1-\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)\right|\right)\\
&\geq \rho^{n_0u^{n_0}}\exp\left(-\exp(1)B_1\sum_{l=n_0+1}^{n}\lambda^l\right)\\
&\geq \rho^{n_0}u^{n_0}\exp\left(-\exp(1)B_1\lambda^{n_0+1}\frac{1}{1-\lambda}\right) =: B_3 > 0.
\end{aligned}$$

We can now conclude our argument by estimating for $\omega \in K$

$$\begin{aligned}
\hat{\Phi}(\omega) &= \lim_{n\to\infty}\int_{E_n}\prod_{l=1}^{n}\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)d\mu^n(\mathbf{e})\\
&\geq \lim_{n\to\infty}\int_{P_n([\mathcal{E}]_{n_0})}\prod_{l=1}^{n}\det\left(W_{e_l}\right)^{-1}\hat{a}_{e_l}\left(W_{P_l(\mathbf{e})}^{-T}\omega\right)d\mu^n(\mathbf{e})\\
&\geq \lim_{n\to\infty}\int_{P_n([\mathcal{E}]_{n_0})}B_3 d\mu^n(\mathbf{e}) \geq B_2B_3 =: B > 0.
\end{aligned}$$

We have now shown that ϕ is cardinal. The last statement that the associated refinement process converges now follows from Remark 7 and Theorem 3. This finishes the proof. □

Remark 8 The careful reader will notice that in the definition of the generalized Cohen condition we actually only need to assume (23) for $l \le n_0$, where n_0 is chosen from (27). We did not include this into the original definition since in our opinion that would make the definition too technical with only a very little bit of gain in generality.

Using the previous result we can reduce the problem of constructing cardinal refinable functions for a dilation family $\mathcal{W}$ to the simpler and well-studied problem of constructing cardinal refinable functions for one single dilation matrix $W \in \mathcal{W}$.

Theorem 6 *Given a moderate dilation family $\mathcal{W}$. Assume that there exists $e_0 \in E$, $\mu(\{e_0\}) > 0$, and a mask a such that a is interpolating with respect to W_{e_0}, $\hat{a}(\omega) \ge 0$ for all $\omega \in \mathbb{R}^d$ and a satisfies the usual Cohen condition, e.g.*

$$\hat{a}\left(\left(W_{e_0}^{-T}\right)^n \omega\right) > 0 \quad \text{for } \omega \in K,\ n \in \mathbb{N}$$

and some set K congruent to $[-\pi, \pi]^d$ modulo $\mathbb{Z}^d$. Then the refinement procedure associated with the filter family

$$\mathcal{A} = (a_e)_{e\in E} := \left(a\left(U_{e_0} U_e^{-1} \cdot\right)\right)_{e\in E}$$

converges to a continuous and cardinal function Φ. If a is finitely supported, Φ is of compact support.

Proof By putting $\mathcal{E} := \{(e_0, e_0, e_0, \dots)\}$ we see that $\mathcal{W}$ and $\mathcal{A}$ satisfies the generalized Cohen condition. Furthermore $\mathcal{A}$ is interpolating. Therefore the result follows from Proposition 3. □

Theorem 7 *If there exists $e_0 \in E$, $\mu(\{e_0\}) > 0$ such that W_{e_0} is integer-expanding. Then there exists a finitely supported filter family $\mathcal{A}$ such that the associated refinement procedure converges to a compactly supported cardinal function.*

Proof In [13, Proposition 4.1] it is shown that for any integer-expanding matrix there exists a mask a which satisfies the assumptions of Theorem 6. □

4.2 Approximation properties

Define the scaling spaces

$$V_\infty^{(n)} = \mathrm{cls}_{L_\infty}\left(\mathrm{span}\left(\left\{\Phi(W_{\mathbf{e}} \cdot -\alpha) :\ \mathbf{e} \in E^n,\ \alpha \in \mathbb{Z}^d\right\}\right)\right).$$

We say that Φ has approximation order k w.r.t. the dilation family $\mathcal{W}$ if for all bounded functions $f \in C^k$ with bounded derivatives we have

$$\inf_{\varphi \in V_\infty^{(n)}} \|f - \varphi\|_\infty = O\left(|\rho(\mathcal{W})|^{-kn}\right).$$

Corollary 2 *Assume that for some $e_0 \in E$, $\mu(\{e_0\}) > 0$ the matrix W_{e_0} is integer-expanding and there exists a diagonal matrix Λ and a unimodular matrix U with $W_{e_0} = U^{-1}\Lambda U$. Then there exist compactly supported cardinal refinable functions of arbitrary approximation order.*

Proof Without loss of generality we assume that U is the identity. For a diagonal matrix Λ it is well known how to construct interpolating filters a which satisfy the Cohen condition and which satisfy sum-rules of arbitrary order [6, 20]. This means that for every polynomial p of degree k we have $S_{a,\Lambda} p\big|_{\mathbb{Z}^d} = p\big|_{\Lambda^{-1}\mathbb{Z}^d}$. Putting $a_e = a(U_e^{-1}\cdot)$, $e \in E$, we get

$$S_{\mathbf{e}} p\big|_{\mathbb{Z}^d} = p\big|_{W_{\mathbf{e}}^{-1}\mathbb{Z}^d}$$

and therefore

$$S^{(n)} p\big|_{\mathbb{Z}^d} = p\big|_{W^{-n}\mathbb{Z}^d}$$

for all polynomials p of degree $\leq k$. Now, having the above polynomial reproduction property at hand, the desired statement follows from standard arguments exploiting the locality of the refinement process and the local approximability of smooth functions by polynomials. □

Example 4 (Shearlet Dubuc–Deslauriers) We apply the above Corollary 2 to the case of the shearlet group with $E = -1, 0, 1$. Since W_0 is a diagonal matrix, the necessary assumptions are fulfilled and we can construct a whole family of cardinal refinable functions with arbitrary approximation order. To be more explicit, we can choose the filter a as the filter corresponding to a tensor product Dubuc–Deslauriers scheme which has been introduced in [6] and also studied in [20]. This gives us an explicit construction of a whole family of cardinal refinable functions satisfying (5) for the shearlet group with arbitrary approximation order, see Fig. 1.

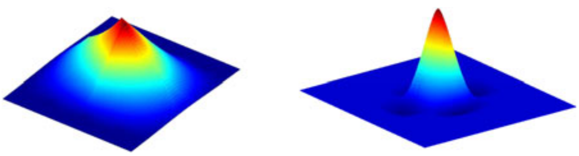

Fig. 1 Shearlet refinable function constructed from degree 1 Dubuc–Deslauriers scheme (*left*) and from degree 2 Dubuc–Deslauriers scheme (*right*)

5 Adaptive subdivision schemes

In this section we compare our refinement processes to the adaptive subdivision schemes (or actually a generalization thereof) introduced in [18]. It turns out that to each adaptive subdivision scheme we can canonically assign a refinement procedure and convergence of the adaptive subdivision scheme implies convergence of the refinement procedure. This fact gives us one more tool to construct refinable functions. As an example we construct a family of refinable functions for the shearlet dilation family based on B-splines.

Definition 14 Consider a moderate dilation family $\mathcal{W}$ and a finitely supported filter family $\mathcal{A}$. The associated *adaptive subdivision scheme* first picks a 'direction' $\mathbf{e}$ in E^∞ and considers for some initial data $c \in l_\infty$ the sequence $S_{P_n(\mathbf{e})}c$. The adaptive subdivision scheme is called convergent if for all directions $\mathbf{e} \in E^\infty$ and initial data c there exists a function $\varphi_{\mathbf{e},c}$ with

$$\lim_{n\to\infty} \left\| \varphi_{\mathbf{e},c}(x) - S_{P_n(\mathbf{e})}c\left(W_{P_n(\mathbf{e})}x\right) \right\|_{L_\infty(\Gamma^{(n)})} = 0,$$

the convergence speed being independent of $\mathbf{e}$.

Let $\eta_i, i = 1, \ldots, d$ be the canonical basis of $\mathbb{Z}^d$. Define the operator

$$\Delta c := (\Delta_1 c, \ldots, \Delta_d c)^T := (c(\cdot) - c(\cdot - \eta_1), \ldots, c(\cdot) - c(\cdot - \eta_d))^T$$

operating on $c \in l_\infty$. The following theorem has been proven in [18] for a specific dilation family $\mathcal{W}$ and the adaption to general moderate dilation families is straightforward:

Theorem 8 *The adaptive subdivision scheme associated with $\mathcal{A}$, $\mathcal{W}$ is convergent if and only if*

$$\rho_a := \limsup_{n\to\infty} \sup_{\mathbf{e}\in E^n} \|\Delta S_{\mathbf{e}}\|^{1/n} < 1.$$

The aim of this section is to prove the following theorem:

Theorem 9 *Assume that*

$$\rho_c := \limsup_{n\to\infty} \int_{E^n} \|\Delta S_{\mathbf{e}}\|^{1/n} d\mu^n(\mathbf{e}) < 1.$$

Then the refinement process associated with $\mathcal{A}$ and $\mathcal{W}$ converges. In particular, convergence of the adaptive subdivision scheme implies convergence of the refinement process.

Proof The proof of this simply adapts the arguments from [18]. Define the *semiconvolution*

$$g * c(\cdot) := \sum_{\alpha\in\mathbb{Z}^d} c(\alpha) g(\cdot - \alpha), \quad g \in L_\infty,\ c \in l_\infty.$$

Recall the matrix W associated with $\mathcal{W}$ and choose a cardinal, continuous and compactly supported refinable function g with respect to W (which exists by [18, Proposition 4.2]). We show that under our assumptions the sequence $G_n(\cdot) := \int_{E^n} \sum_{\alpha\in\mathbb{Z}^d} a_{\mathbf{e}} * g(W_{\mathbf{e}}\cdot) d\mu^n(\mathbf{e})$ converges uniformly which easily implies the convergence of $\mathcal{S}$.

Since g is refinable with respect to W, there exists a finite filter b with

$$g(\cdot) = b * g(W\cdot) = \sum_{\alpha\in\mathbb{Z}^d} g(W\cdot - \alpha).$$

Define for $e \in E$ the filter $b_e(\cdot) := b(U_e^{-1}\cdot)$. Then with $g_e(\cdot) := g(U_e\cdot)$, we have

$$g_e(\cdot) = b_e * g_e(W_e\cdot).$$

It follows that

$$G_{n+1}(\cdot) - G_n(\cdot) = \int_{E^{n+1}} \left(a_{\mathbf{e}} - S_{b_{e_{n+1}}, W_{en+1}} a_{P_n(\mathbf{e})}\right) * g(W_{\mathbf{e}}\cdot) d\mu^{n+1}(\mathbf{e}).$$

The arguments in [18, Proposition 4.19] show that $\|(a_{\mathbf{e}} - S_{b_{e_{n+1}}, W_{en+1}} a_{P_n(\mathbf{e})}) * g(W_{\mathbf{e}}\cdot)\|_{L_\infty}$ can be bounded by a constant times $\|\Delta a_{P_n(\mathbf{e})}\|_{l_\infty}$, hence

$$\|G_{n+1}(\cdot) - G_n(\cdot)\|_{L_\infty} \le \int_{E^n} \|\Delta a_{\mathbf{e}}\|_{l_\infty} d\mu(\mathbf{e}) \le C\rho_c^n$$

for some constant $C > 0$. □

Example 5 (Shearlet B-splines) Again we construct a family of refinable functions for the shearlet dilation family, this time from the well-known B-spline schemes. Consider a mask a which arises from a tensor-product B-spline scheme for the dilation matrix $W = W_0$. It is well known that

$$\|\Delta_1 S_{a,W} c\|_{l_\infty} \le 1/4\|\Delta_1 c\|_{l_\infty} \quad \text{and} \quad \|\Delta_2 S_{a,W} c\|_{l_\infty} \le 1/2\|\Delta_2 c\|_{l_\infty}.$$

Now define the filter family $a_i(\cdot) := a(U_i^{-1}\cdot)$. Then to establish that $\rho_c < 1$ it suffices to show that

$$\|\Delta S_i c\|_{l_\infty} \le \lambda\|\Delta c\|_{l_\infty}, \quad i \in E,\ \lambda < 1.$$

For $i = 0$ this clearly holds with $\lambda = 1/2$. We now investigate the case $i = 1$, the case $i = -1$ follows analogously. We have

$$S_1 c(\alpha) = S_0 c(U_1^{-1}\alpha)$$

and therefore we need to estimate

$$\begin{aligned}\Delta_1 S_1 c(\cdot) &= \Delta_1 S_0 c\left(U_1^{-1}\cdot\right) = S_0 c\left(U_1^{-1}\cdot\right) - S_0\left(U_1^{-1}(\cdot - \eta_1)\right)\\ &= S_0 c\left(U_1^{-1}\cdot\right) - S_0\left(U_1^{-1}\cdot - \eta_1\right)\end{aligned}$$

since $U_1^{-1}\eta_1 = \eta_1$, so the estimate for $\Delta_1 S_1$ is the same as for $\Delta_1 S_0$. For the other case we write

$$\begin{aligned}\Delta_2 S_1 c(\cdot) &= S_0 c\left(U_1^{-1}\cdot\right) - S_0\left(U_1^{-1}(\cdot - \eta_2)\right) = S_0 c\left(U_1^{-1}\cdot\right) - S_0\left(U_1^{-1}\cdot - \eta_2 + \eta_1\right)\\ &= S_0 c\left(U_1^{-1}\cdot\right) - S_0\left(U_1^{-1}\cdot - \eta_2\right) + S_0 c\left(U_1^{-1}\cdot - \eta_2\right) - S_0\left(U_1^{-1}\cdot - \eta_2 + \eta_1\right).\end{aligned}$$

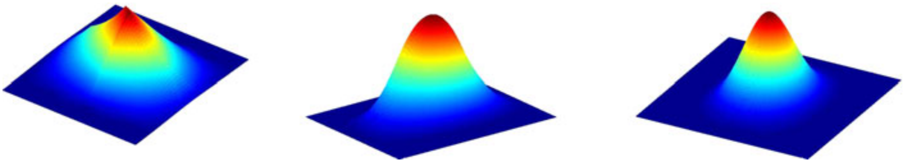

Fig. 2 Shearlet refinable function constructed from degree 1 B-spline scheme (*left*), from degree 2 B-spline scheme (*middle*), and from degree 3 B-spline scheme (*right*)

It follows that

$$\|\Delta_2 S_1 c\|_{l_\infty} \leq \|\Delta_1 S_0 c\|_{l_\infty} + \|\Delta_2 S_0 c\|_{l_\infty} \leq 1/4\|\Delta_1 c\|_{l_\infty} + 1/2\|\Delta_2 c\|_{l_\infty} \leq 3/4\|\Delta c\|_{l_\infty}$$

In summary we get that

$$\|\Delta S_1 c\|_{l_\infty} \leq 3/4\|\Delta c\|_{l_\infty}$$

which is enough to conclude that $\rho_c < 1$ and consequently that $\mathcal{S}$ is convergent. The shearlet refinable functions corresponding to the B-spline scheme of degree 1, 2, 3 are depicted in Fig. 2.

6 Future Work

In this work we provided a basis for a general construction of MRAs for systems with more than one dilation. The next natural step is to aim at deriving an extension principle that allows us to construct wavelets with composite dilation from a refinable function. Another direction of future research is the investigation of smoothness properties of the refinable functions constructed in this paper.

References

1. Blanchard, J.: Minimally supported frequency composite dilation parseval frame wavelets. J. Geom. Anal. **19**, 19–35 (2009)
2. Candes, E.J., Donoho, D.L.: Continuous curvelet transform: I. resolution of the wavefront set. Appl. Comput. Harmon. Anal. **19**, 162–197 (2003)
3. Candes, E.J., Donoho, D.L.: Continuous curvelet transform: II. discretization and frames. Appl. Comput. Harmon. Anal. **19**, 198–222 (2003)
4. Cavaretta, A.S., Dahmen, W., Micchelli, C. A.: Stationary Subdivision. American Mathematical Society (1991)
5. Daubechies, I.: Ten lectures on Wavelets. SIAM (1992)
6. Deslauriers, G., Dubuc, S.: Symmetric iterative interpolation processes. Constr. Approx. **5**, 49–68 (1989)
7. Do, M.N., Vetterli, M.: The contourlet transform: An efficient directional multiresolution image representation. IEEE Trans. Image Process. **14**, 2091–2106 (2005)
8. Grohs, P.: Interpolating composite systems. In: Proceedings of the 13th Intl. Conference on Approximation Theory. Technical Report. San Antonio (2010, to appear)
9. Guo, K. Labate, D.: Optimally sparse multidimensional representation using shearlets. SIAM J. Math. Analy. **39**, 298–318 (2007)

10. Guo, K., Lim, W.-Q., Labate, D., Weiss, G., Wilson, E.: Wavelets with composite dilations. Electron. Res. Announc. Am. Math. Soc. **10**, 78–87 (2004)
11. Guo, K., Lim, W.-Q., Labate, D., Weiss, G., Wilson, E.: Wavelets with composite dilation and their mra properties. Appl. Comput. Harmon. Anal. **20**, 202–236 (2006)
12. Han, B.: Symmetry property and construction of wavelets with a general dilation matrix. Linear Algebra Appl. **353**, 207–225 (2002)
13. Han, B.: Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix. J. Comput. Appl. Math. **155**, 43–67 (2003)
14. Han, B., Kutyniok, G., Shen, Z.: A unitary extension principle for shearlet systems. Technical Report. Universität Osnabrück (2009)
15. Hörmander, L.: The Analysis of linear Partial Differential Operators. Springer (1983)
16. Krishtal, I.A., Robinson, B.D., Weiss, G., Wilson, E.: Some simple haar-type wavelets in higher dimensions. J. Geom. Anal. **17**, 87–96 (2007)
17. Kutyniok, G., Labate, D.: Resolution of the wavefront set using continuous shearlets. Trans. Am. Math. Soc. **361**, 2719–2754 (2009)
18. Kutyniok, G., Sauer, T.: Adaptive directional subdivision schemes and shearlet multiresolution analysis. SIAM J. Math. Analy. **41**, 1436–1471 (2009)
19. Labate, D., Kutyniok, G., Lim, W.-Q., Weiss, G.: Sparse multidimensional representation using shearlets. In: SPIE Proc. 5914, Wavelets XI (San Diego, CA, 2005), pp. 254–262. SPIE, Bellingham, WA (2005)
20. Micchelli, C.A.: Interpolatory subdivision schemes and wavelets. J. Approx. Theory **86**, 41–71 (1996)
21. Ron, A. Shen, Z.: Frames and stable bases for shift-invariant subspaces of $l_2(\mathbb{R}^d)$. Can. J. Math. **47**, 1051–1094 (1995)
22. Ron, A., Shen, Z.: Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator. J. Funct. Anal. **148**(2), 408–447 (1997)
23. Stein, E.M.: Harmonic Analysis. Princeton University Press (1993)