GROUPS ACTING ON HYPERBOLIC SPACES: HARMONIC ANALYSIS AND NUMBER THEORY

(Springer Monographs in Mathematics)

By Jürgen Elstrodt, Fritz Grunewald and Jens Mennicke: 524 pp., £57.50, isbn 3 540 62745 6 (Springer, 1998).

Among discrete subgroups of Lie groups, Kleinian groups (discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$) acting on hyperbolic space, and Fuchsian groups (discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$) acting on the hyperbolic plane, are of particular interest. Although Kleinian and Fuchsian groups are closely related, there are a number of important differences. Kleinian groups cannot be continuously deformed (by Mostow rigidity), and the associated hyperbolic 3-manifolds do not carry a complex structure; on the other hand, because of the higher dimensionality, Kleinian groups give rise to more complicated and richer geometric and analytic phenomena. In the context of harmonic analysis on symmetric spaces, Kleinian and Fuchsian groups also play a special rôle. While in the Fuchsian case there are already a number of treatments, the book under review is the first monograph on the spectral theory of Kleinian groups, including Selberg's trace formula.

The study of Kleinian groups (or of Fuchsian groups) is particularly attractive because it involves learning many diverse branches of mathematics. This may be illustrated very well by listing the contents of the book under review. The first chapter is about *hyperbolic geometry* (of hyperbolic 3-space). Different models are presented, and also the point of view of Lie groups and symmetric spaces is considered. The second chapter treats the basic theory of *Kleinian groups*, including classification of parabolic elements and finiteness questions. Also discussed are Poincaré's theorem of fundamental polyhedra and the question of volumes of hyperbolic 3-manifolds. The third chapter is about *automorphic functions*. Poincaré series and Eisenstein series are introduced, the Fourier expansion of Eisenstein series in cusps being fundamental for this theory. The method of point-pair invariants is discussed, and an explicit formula for the Selberg transform is deduced. Chapters 4–6 are the heart of the book. They treat, following the lines of the already classical approach of Maass, Roelcke and Selberg, the *spectral theory* of the Laplace operator for cofinite Kleinian groups which culminates in Selberg's trace formula. The resolvent kernel is of Hilbert–Schmidt type only for cocompact Kleinian groups. The case of cofinite non-compact Kleinian groups is a good bit more difficult. Their continuous spectrum is treated, as usual, through the meromorphic continuation of Eisenstein series; the authors follow Colin de Verdière's approach through pseudo-Laplacians. The theory of eigenpackets is used (and explained) as well. As in the cocompact case, the discrete spectrum has non-negative eigenvalues with finite multiplicities, but not much is known about the size of discrete spectrum of cofinite non-compact Kleinian groups. A number of applications of trace formula are proved, for instance the prime geodesic theorem and the fact that the Selberg zeta function is an entire function of order 3. Chapter 7 and Chapter 10 are about *arithmetic Kleinian groups*. Bianchi groups $\mathrm{PSL}(2, \mathcal{O})$ (in Chapter 7) are the natural analogues of the modular group $\mathrm{PSL}(2, \mathbb{Z})$; the authors also discuss their interesting group theoretic structure. Chapter 10 contains a general construction of arithmetic Kleinian groups. A list of all thirty-two Kleinian groups which are Coxeter groups with four generators is given, some of them being non-arithmetic. The subject treated in Chapter 8 could be called *Kleinian modular forms*.

General results of the spectral theory of Kleinian groups are applied to the groups $\mathrm{PSL}(2, \mathcal{O})$ (and their Eisenstein series). This technique has been very successful in number theory. The authors obtain, among other results, an analogue of Kronecker's limit formula, results on non-vanishing of L-functions and, with the help of zeta functions, Weyl's asymptotic law on the distribution of eigenvalues. In Chapter 9 the theory of (arithmetic) Kleinian groups is used to derive classical results on *integral binary Hermitian forms*.

The book under review is very rich, very accessible, and most carefully written. The text is enriched by many original approaches and applications, fruit of the authors' long experience with the subject. This valuable book will be exceedingly useful to all who wish to learn the theory of Kleinian groups in connection with harmonic analysis and number theory.

For the reader's convenience, I have added below some references to recent related literature.

References

1. A. Borel, *Automorphic forms on* $\mathrm{SL}_2(\mathbb{R})$ (Cambridge University Press, 1997).
2. J. Dodziuk and J. Jorgenson, 'Spectral asymptotics on degenerating hyperbolic 3-manifolds', *Mem. Amer. Math. Soc.* 643 (1998).
3. H. Iwaniec, *Introduction to the spectral theory of automorphic forms* (Revista Matemática Iberoamericana, Madrid, 1995).

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INTRODUCTION TO GEOMETRIC PROBABILITY

By Daniel A. Klain and Gian-Carlo Rota: 178 pp., £12.95 (US$19.95), isbn 0 521 59654 8 (Cambridge University Press, 1997).

Geometric probability theory begins, as does this book, with Buffon's needle problem. Few readers will need reminding that its solution says that, if a needle of length L is dropped on a grid of parallel lines in a plane at distance L apart, then the probability that it meets one of the lines is $2/\pi$. From a modern perspective, this answer depends on the existence of a measure on the family of lines in the plane which is invariant under isometries.

This idea generalizes in several directions. The *affine Grassmannian* $\mathrm{Graff}(n,k)$ consists of the family of all k-dimensional flats (affine subspaces) in n-dimensional euclidean space $\mathbf{R}^n$, endowed with its natural invariant measure λ_k^n induced by the group E_n of isometries of $\mathbf{R}^n$. If $k \geqslant 1$, then a k-flat meets a compact set K if and only if it meets the convex hull of K; thus it is natural to confine one's attention to compact convex sets. If K is such a set, then the measure of the k-flats which meet K is clearly proportional to the mean $(n-k)$-volume of the projection of K on the $(n-k)$-dimensional *linear* subspaces of $\mathbf{R}^n$, and thus (in some sense) measures the $(n-k)$-dimensional size of K. In one normalization, this is the *quermassintegral* $W_k(K)$ of K introduced by Minkowski (the definition accounts for the name). Nowadays, an alternative normalization introduced by the reviewer, giving the *intrinsic volume* $V_{n-k}(K)$, is often preferred; one advantage is that $V_r(K)$ is independent of the dimension of the ambient space of K.

The intrinsic volumes V_r (denoted in this book by μ_r) are *valuations*, in that, if K_1