PAID-INCURRED CHAIN RESERVING METHOD WITH DEPENDENCE MODELING

BY

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ABSTRACT

The paid-incurred chain (PIC) reserving method is a claims reserving method that allows to combine claims payments and incurred losses information in a mathematical consistent way. The main criticism on the original Bayesian log-normal PIC model presented in Merz–Wüthrich [5] is that it does not respect dependence properties within the observed data. In the present paper, we extend the original Bayesian log-normal PIC model so that dependence is modeled in an appropriate way.

KEYWORDS

Claims reserving, outstanding loss liabilities, ultimate loss prediction, claims payments, claims incurred, incurred losses, prediction uncertainty, paid-incurred chain model, PIC reserving method.

1. PAID-INCURRED CHAIN MODEL WITH DEPENDENCE MODELING

1.1. Introduction

The process of assessing the outstanding loss liabilities is an important part of the management of a non-life insurance company. Reserving actuaries need to predict the outstanding loss liabilities based on all available information. In many cases, this available information comprises claims payments observations, incurred losses data and, may be, prior loss ratio information. State-of-the-art stochastic claims reserving methods are not able to combine these three sources of information in a mathematically consistent way. Often in practice, actuaries apply different claims reserving methods to these different sources of information. The results of these different methods are then merged in an ad hoc way to obtain one prediction of the outstanding loss liabilities. Of course, such ad hoc considerations do not allow one to assess prediction uncertainty because, typically, the merging is done subjectively by pure expert opinion.

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One of the first claims reserving techniques in the actuarial literature to unify outstanding loss liability prediction based on claims payments and incurred losses information is the Munich chain ladder (MCL) method. The MCL method was introduced by Quarg–Mack [8] and its aim is to reduce the gap between the two chain ladder (CL) predictions that are based on claims payments and incurred losses data, respectively. The idea is to adjust the CL factors with paid-incurred ratios to reduce the gap between the two predictions. The difficulty with the MCL method is that it involves several parameter estimates whose precisions are difficult to quantify within a stochastic model framework.

In this paper, we revisit the Bayesian log-normal paid-incurred chain (PIC) model of Merz–Wüthrich [5]. This PIC reserving model was motivated by Posthuma *et al.* [7] who have designed a similar model bringing paid-incurred information in-line. The PIC reserving method combines claims payments and incurred losses information in such a way that Bayesian inference can be made on the missing (future) part of the claims development triangles based on both sources of information, see also Figure 1. The main criticism on the models of Posthuma *et al.* [7] and Merz–Wüthrich [5] is that they do not respect dependence properties in an appropriate way. In the present paper, we expand the initial Bayesian log-normal PIC model such that actual dependence structure found in the data is respected.

1.2. Notation and model assumptions

For the PIC model, we consider three channels of information: (i) claims payments, which refer to the payments done for reported claims; (ii) incurred losses, which correspond to the reported claim amounts; (iii) prior expert opinion. The crucial observation is that the claims payments and incurred losses time series must reach the same ultimate value, because these two time series both converge to the total ultimate claim. By choosing appropriate model assumptions, we force this property to hold true in our model.

In the following, we denote accident years by $i \in \{0, \dots, J\}$ and development years by $j \in \{0, \dots, J\}$. We assume that all claims are settled after the Jth development year. Cumulative claims payments in accident year i after j development periods are denoted by $P_{i,j}$ and the corresponding incurred losses by $I_{i,j}$. Moreover, for the ultimate claim, we assume (force) $P_{i,J} = I_{i,J}$ with probability 1, which means that ultimately (at time J) they reach the same ultimate value. For an illustration we refer to Figure 1.

We introduce the following notation, for $i \in \{0, \dots, J\}$,

$$\Xi_i = (\zeta_{i,0}; \zeta_{i,1}, \xi_{i,1}, \zeta_{i,2}, \xi_{i,2}, \dots, \zeta_{i,J}, \xi_{i,J})' \in \mathbb{R}^{2J+1},$$

with, for $j \in \{0, \dots, J\}$ and $l \in \{1, \dots, J\}$,

$$\zeta_{i,j} = \log \frac{I_{i,j}}{I_{i,j-1}} \quad \text{and} \quad \xi_{i,l} = \log \frac{P_{i,l}}{P_{i,l-1}},$$

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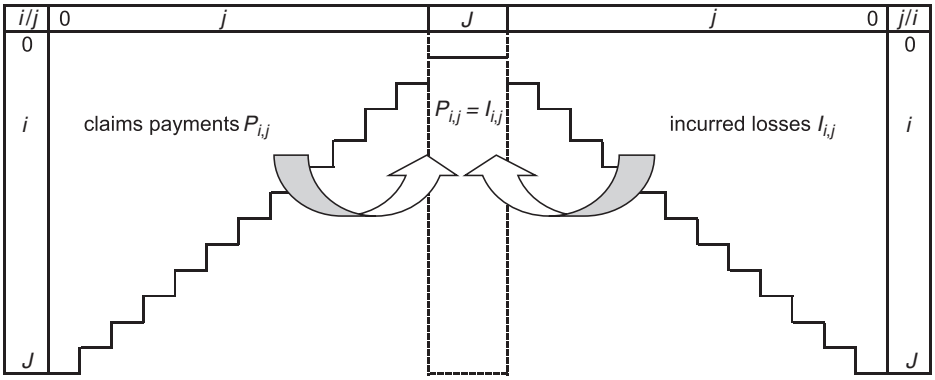

FIGURE 1: Left-hand side: cumulative claims payments $P_{i,j}$ development triangle; right-hand side: incurred losses $I_{i,j}$ development triangle; both leading to the same ultimate claim $P_{i,J} = I_{i,J}$.

where we have set $I_{i,-1} \equiv 1$. The Bayesian log-normal PIC model with dependence within the random vectors Ξ_i is then defined as follows.

Model Assumptions 1.1 (Bayesian log-normal PIC model with dependence).

- Conditionally, given $\Theta = (\Psi_0; \Psi_1, \Phi_1, \Psi_2, \Phi_2, \ldots, \Psi_J, \Phi_J)'$ and $V \in \mathbb{R}^{(2J+1)\times(2J+1)}$,
  - the random vectors $\Xi_0, \ldots, \Xi_J$ are i.i.d. with multivariate Gaussian distribution
  $$\Xi_i \sim \mathcal{N}(\Theta, V) \quad \text{for} \quad i = 0, \ldots, J;$$
  - we assume $P_{i,J} = I_{i,J}$ for all $i = 0, \ldots, J$, $\mathbb{P}$-a.s.
- V is a positive definite covariance matrix and the components of Θ are independent with prior distributions, for $j \in \{0, \ldots, J\}$ and $l \in \{1, \ldots, J\}$,
  $$\Psi_j \sim \mathcal{N}\left(\psi_j, t_j^2\right) \quad \text{and} \quad \Phi_l \sim \mathcal{N}\left(\phi_l, s_l^2\right),$$
  with prior parameters $\psi_j, \phi_l \in \mathbb{R}$ and $t_j > 0$, $s_l > 0$.

□

Remarks.

- For $V = \text{diag}(\tau_0^2; \tau_1^2, \sigma_1^2, \ldots, \tau_J^2, \sigma_J^2)$ the model above exactly gives the PIC reserving model from Merz–Wüthrich [5]. In the present paper, we now allow for general covariance matrices V (as long as they are positive definite). In (1.1) below, we give an explicit choice that will be applied to a motor third-party liability portfolio.
- The PIC model combines both cumulative payments and incurred losses data to get a unified predictor for the total ultimate claim that is based on both sources of information. Thereby, the model assumption $P_{i,J} = I_{i,J}$ guarantees that the total ultimate claim coincides for claims payments and incurred

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losses data. In particular, we obtain the identities

$$I_{i,j} = I_{i,j-1} \exp\{\zeta_{i,j}\}, \qquad \text{with initial value } I_{i,0} = \exp\{\zeta_{i,0}\},$$

and by backward recursion

$$P_{i,j-1} = P_{i,j} \exp\{-\xi_{i,j}\}, \qquad \text{with initial value } P_{i,J} = I_{i,J}.$$

Note that in comparison to Merz–Wüthrich [5], we have exchanged the role of $I_{i,j}$ and $P_{i,l}$. In the original model of Merz–Wüthrich [5], the resulting claims reserves are completely symmetric in the exchange of $I_{i,j}$ and $P_{i,l}$. If we consider the model with dependence, as in Model Assumptions 1.1 above, it is more natural to use incurred losses $I_{i,J}$ as prior for claims payments $P_{i,l}$. This means that Hertig's log-normal model [4] for $I_{i,j}$ plays the role of the prior for Gogol's claims reserving model [3] for $P_{i,l}$; see also Merz–Wüthrich [5].

- If we have prior (expert) knowledge (as a third information channel) this can be used to design the prior distribution of $\boldsymbol{\Theta}$. If there is no prior knowledge we choose non-informative priors for $\boldsymbol{\Theta}$, that is we let $t_j^2 \to \infty$ and $s_l^2 \to \infty$ for $j \in \{0, \ldots, J\}$ and $l \in \{1, \ldots, J\}$.
- The assumption $P_{i,J} = I_{i,J}$ means that all claims are settled after J development years and there is no so-called tail development factor. If there is a claims development beyond development year J, then one can extend the PIC model for the estimation of a tail development factor; see Merz–Wüthrich [6] for more details.
- Under Model Assumptions 1.1, the distribution of the ultimate claims $I_{i,J}$ is a priori equal across accident years. However, given the observed data, we observe different posterior distributions for claims of different accident years. Therefore, the PIC reserving method allows for accident year variation (see Theorem 3.2). However, if knowledge about prior differences is available, it should be incorporated in the prior means. This relaxation of the model assumption will still lead to closed-form solutions. A similar effect can be achieved by considering (volume-) adjusted observations.
- Conditional i.i.d. guarantees that we obtain a model of chain ladder type where chain ladder factors do not depend on accident year i. Of course, this model assumption requires that the data considered need to be sufficiently regular. If this is not the case, one can introduce prior differences between accident years (see also last bullet point). These more general assumptions still lead to a closed-form solution. The drawback is that the model might become over-parameterized and/or it requires to extended expert knowledge.
- The covariance matrix V allows for modeling dependence within Ξ_i. In particular, we will choose this covariance matrix such that the correlation between $\zeta_{i,j}$ and $\xi_{i,j}$ is positive because $P_{i,j}$ is contained in $I_{i,j}$ (and hence they are dependent).

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This last bullet point is motivated by the following argument: a positive change from $I_{i,j-1}$ to $I_{i,j}$ means that the claims adjusters increase their expectation in future claims payments. One part of this increased expectation is immediately paid in development period j (and hence contained in both $I_{i,j}$ and $P_{i,j}$) and the remaining increased expectation is paid with some settlement delay, which means that we also have higher expectations for $P_{i,l}$, $l > j$. This argument leads to a following possible explicit choice of the correlation matrix of the random vector Ξ_i:

$$
\begin{array}{c||cc|cc|cc|cc|c|cc}
 & \zeta_{i,0} & & \zeta_{i,1} & \xi_{i,1} & \zeta_{i,2} & \xi_{i,2} & \zeta_{i,3} & \xi_{i,3} & \zeta_{i,4} & \xi_{i,4} & \cdots & \zeta_{i,J} & \xi_{i,J} \\
\hline\hline
\zeta_{i,0} & 1 & & 0 & \rho_1 & 0 & \rho_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
\zeta_{i,1} & 0 & & 1 & \rho_0 & 0 & \rho_1 & 0 & \rho_2 & 0 & 0 & \cdots & 0 & 0 \\
\xi_{i,1} & \rho_1 & & \rho_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
\zeta_{i,2} & 0 & & 0 & 0 & 1 & \rho_0 & 0 & \rho_1 & 0 & \rho_2 & \cdots & 0 & 0 \\
\xi_{i,2} & \rho_2 & & \rho_1 & 0 & \rho_0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
\zeta_{i,3} & 0 & & 0 & 0 & 0 & 0 & 1 & \rho_0 & 0 & \rho_1 & \cdots & 0 & 0 \\
\xi_{i,3} & 0 & & \rho_2 & 0 & \rho_1 & 0 & \rho_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\hline
\zeta_{i,4} & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \rho_0 & \cdots & 0 & 0 \\
\xi_{i,4} & 0 & & 0 & 0 & \rho_2 & 0 & \rho_1 & 0 & \rho_0 & 1 & \cdots & 0 & 0 \\
\hline
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hline
\zeta_{i,J} & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & \rho_0 \\
\xi_{i,J} & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \rho_0 & 1
\end{array}
\tag{1.1}
$$

The rational behind this correlation matrix is that the incurred losses increments $\zeta_{i,j}$ are (positively) correlated to the claims payments increments $\xi_{i,j}$, $\xi_{i,j+1}$ and $\xi_{i,j+2}$ with positive correlations ρ_0, ρ_1 and ρ_2, respectively. Note that $\zeta_{i,0}$ plays the special role of the initial value for incurred losses $I_{i,0}$ (on the log scale), whereas the initial value for claims payments $P_{i,0}$ (on the log scale) can be defined by $\xi_{i,0} = \sum_{j=0}^{J} \zeta_{i,j} - \sum_{l=1}^{J} \xi_{i,l}$.

Organization: In Section 2, we derive the ultimate claim predictor for known parameters $\boldsymbol{\Theta}$ (see Corollary 2.2). The model parameter $\boldsymbol{\Theta}$ is estimated in Section 3, leading to the ultimate claim predictor in Theorem 3.2. In Section 4, we calculate the prediction uncertainty in terms of the conditional mean square error of prediction (MSEP; see Theorem 4.1). We apply the PIC method with dependence with three explicit choices of a covariance matrix of the form (1.1) to a motor third-party liability example in Section 5. In Section 6, we present the conclusions. Furthermore, all proofs are provided in the Appendix.

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2. Ultimate claim prediction for known parameters $\boldsymbol{\Theta}$

We can work either with the random vector $\Xi_i \in \mathbb{R}^{2J+1}$ (see Model Assumptions 1.1) or with the logarithmized observations given by the random vector

$$\mathbf{X}_i = (\log I_{i,0}, \log P_{i,0}, \log I_{i,1}, \log P_{i,1}, \ldots, \log I_{i,J-1}, \log P_{i,J-1}; \log I_{i,J})' \in \mathbb{R}^{2J+1}.$$

The consideration of Ξ_i was easier for the model definition and for the interpretation of the dependence structure; but often it is more straightforward if we directly work with $\mathbf{X}_i$ (under the explicit logarithmized cumulative observations). Note that there is a linear one-to-one correspondence between Ξ_i and $\mathbf{X}_i$, and we denote the (invertible) transformation matrix by B. The explicit form of B is given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & -1 & \ldots & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & -1 & \ldots & -1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ & & \vdots & & & \ddots & & \vdots & \\ 1 & 1 & 0 & 1 & 0 & \ldots & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

We then obtain the following conditional multivariate Gaussian distribution for $\mathbf{X}_i$:

$$\mathbf{X}_i|_{\{\boldsymbol{\Theta}\}} = B\, \Xi_i|_{\{\boldsymbol{\Theta}\}} \sim \mathcal{N}(\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\Theta}) = B\boldsymbol{\Theta}, \Sigma = BVB'). \tag{2.1}$$

Conditionally given the parameters $\boldsymbol{\Theta}$ (or $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\Theta})$, respectively), the random vector $\mathbf{X}_i$ is within the multivariate Gaussian distribution family. Our first aim is to study the conditional distribution of the ultimate claim $P_{i,J} = I_{i,J}$, conditionally given the parameters $\boldsymbol{\Theta}$ and the observations

$$\mathcal{D}_J = \{I_{i,j},\ P_{i,j} :\ i + j \le J,\ 0 \le i \le J,\ 0 \le j \le J\}$$

in the upper paid and incurred triangles (see Figure 1).

For accident years $i \in \{1, \ldots, J\}$, define $n = 2J + 1$ and $q = q(i) = 2(J - i + 1) \in \{2, \ldots, 2J\}$. At time J, we have for accident year i observations (given in the upper triangles $\mathcal{D}_J$)

$$\mathbf{X}_i^{(1)} = (\log I_{i,0}, \log P_{i,0}, \log I_{i,1}, \log P_{i,1}, \ldots, \log I_{i,J-i}, \log P_{i,J-i})' \in \mathbb{R}^q,$$

and we would like to predict the lower triangles given by

$$\mathbf{X}_i^{(2)} = (\log I_{i,J-i+1}, \log P_{i,J-i+1}, \ldots, \log I_{i,J-1}, \log P_{i,J-1}; \log I_{i,J})' \in \mathbb{R}^{n-q}.$$

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This provides, for $i \in \{1, \ldots, J\}$, the following decomposition $\boldsymbol{\mu} = (\boldsymbol{\mu}_i^{(1)}, \boldsymbol{\mu}_i^{(2)}) = B\boldsymbol{\Theta} \in \mathbb{R}^n$ of the conditional mean:

$$\mathbb{E}\big[\mathbf{X}_i^{(1)}\big|\boldsymbol{\Theta}\big] = \boldsymbol{\mu}_i^{(1)} = B_i^{(1)}\boldsymbol{\Theta} \in \mathbb{R}^q \quad \text{and} \quad \mathbb{E}\big[\mathbf{X}_i^{(2)}\big|\boldsymbol{\Theta}\big] = \boldsymbol{\mu}_i^{(2)} = B_i^{(2)}\boldsymbol{\Theta} \in \mathbb{R}^{n-q},$$

with partition of $B \in \mathbb{R}^{n\times n}$ given by, for $i \in \{1, \ldots, J\}$,

$$B = \begin{pmatrix} B_i^{(1)} \\ B_i^{(2)} \end{pmatrix} \quad \text{with} \quad B_i^{(1)} \in \mathbb{R}^{q\times n} \text{ and } B_i^{(2)} \in \mathbb{R}^{(n-q)\times n}.$$

In the same way, we also decompose the covariance matrix Σ which provides

$$\Sigma = \begin{pmatrix} \Sigma_i^{(11)} & \Sigma_i^{(12)} \\ \Sigma_i^{(21)} & \Sigma_i^{(22)} \end{pmatrix} \quad \text{with} \quad \Sigma_i^{(11)} \in \mathbb{R}^{q\times q}.$$

For $i = 0$, we set $q(0) = n$, $\mathbf{X}_0^{(1)} = \mathbf{X}_0 \in \mathbb{R}^n$, $\Sigma_0^{(11)} = \Sigma$ and $B_0^{(1)} = B$.

Having this notation, we provide the prediction of $\mathbf{X}_i^{(2)}$, conditionally given $\{\boldsymbol{\Theta}, \mathcal{D}_J\}$:

Lemma 2.1. *Choose an accident year $i \in \{1, \ldots, J\}$. Under Model Assumptions 1.1 we have*

$$\mathbf{X}_i^{(2)}\Big|_{\{\boldsymbol{\Theta},\mathcal{D}_J\}} = \mathbf{X}_i^{(2)}\Big|_{\{\boldsymbol{\Theta},\mathbf{X}_i^{(1)}\}} \sim \mathcal{N}\left(\widetilde{\boldsymbol{\mu}}_i^{(2)} = \boldsymbol{\mu}_i^{(2)} + \Sigma_i^{(21)}\big(\Sigma_i^{(11)}\big)^{-1}\left(\mathbf{X}_i^{(1)} - \boldsymbol{\mu}_i^{(1)}\right), \widetilde{\Sigma}_i^{(22)}\right),$$

with the conditional covariance matrix $\widetilde{\Sigma}_i^{(22)} = \Sigma_i^{(22)} - \Sigma_i^{(21)}(\Sigma_i^{(11)})^{-1}\Sigma_i^{(12)}$.

An immediate consequence of Lemma 2.1 is the following corollary.

Corollary 2.2. *Under the assumptions and notations of Lemma 2.1, we obtain for the ultimate claim $I_{i,J} = P_{i,J}$, for $i \in \{1, \ldots, J\}$,*

$$\log I_{i,J}\Big|_{\{\boldsymbol{\Theta},\mathcal{D}_J\}} \sim \mathcal{N}\big(\mathbf{e}_i'\,\widetilde{\boldsymbol{\mu}}_i^{(2)}\,,\ \mathbf{e}_i'\,\widetilde{\Sigma}_i^{(22)}\,\mathbf{e}_i\big),$$

with $\mathbf{e}_i = (0, \ldots, 0, 1)' \in \mathbb{R}^{n-q}$.

This corollary implies that, conditionally given the parameters $\boldsymbol{\Theta}$ and the observations $\mathcal{D}_J$, we get the ultimate claim predictor, for $i \in \{1, \ldots, J\}$,

$$\mathbb{E}[I_{i,J}|\boldsymbol{\Theta}, \mathcal{D}_J] = \exp\big\{\mathbf{e}_i'\,\widetilde{\boldsymbol{\mu}}_i^{(2)} + \mathbf{e}_i'\,\widetilde{\Sigma}_i^{(22)}\,\mathbf{e}_i/2\big\}. \tag{2.2}$$

In the special case of a diagonal correlation matrix (1.1), i.e. $\rho_0 = \rho_1 = \rho_2 = 0$, this is exactly the predictor derived in Corollary 2.5 of Merz–Wüthrich [5].

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3. Estimation of parameter $\boldsymbol{\Theta}$

The ultimate claim predictor (2.2) is still based on the unknown parameters $\boldsymbol{\Theta}$, namely

$$\begin{aligned}\mathbf{e}_i'\,\widetilde{\boldsymbol{\mu}}_i^{(2)} &= \mathbf{e}_i'\big(\boldsymbol{\mu}_i^{(2)} + \Sigma_i^{(21)}(\Sigma_i^{(11)})^{-1}\big(\mathbf{X}_i^{(1)} - \boldsymbol{\mu}_i^{(1)}\big)\big)\\ &= \mathbf{e}_i'\big(B_i^{(2)}\,\boldsymbol{\Theta} + \Sigma_i^{(21)}(\Sigma_i^{(11)})^{-1}\big(\mathbf{X}_i^{(1)} - B_i^{(1)}\,\boldsymbol{\Theta}\big)\big)\\ &= \Gamma_i\,\boldsymbol{\Theta} + \mathbf{e}_i'\,\Sigma_i^{(21)}(\Sigma_i^{(11)})^{-1}\,\mathbf{X}_i^{(1)},\end{aligned} \tag{3.1}$$

where we have defined

$$\Gamma_i = \mathbf{e}_i'\big(B_i^{(2)} - \Sigma_i^{(21)}\big(\Sigma_i^{(11)}\big)^{-1}B_i^{(1)}\big).$$

In particular, we see that $\mathbf{e}_i'\,\widetilde{\boldsymbol{\mu}}_i^{(2)}$ is a linear function in $\boldsymbol{\Theta}$. We aim to calculate the posterior distribution of $\boldsymbol{\Theta}$, conditionally given the observations $\mathcal{D}_J$. The σ-field generated by $\mathcal{D}_J$ is the same as the one generated by the observations $\widetilde{\mathcal{D}}_J = \{\mathbf{X}_0^{(1)}, \ldots, \mathbf{X}_J^{(1)}\}$. Therefore, by a slight abuse of notation, we identify the observations $\widetilde{\mathcal{D}}_J$ with $\mathcal{D}_J$. The likelihood of the logarithmized observations, conditionally given $\boldsymbol{\Theta}$, is then written as, see also (2.1),

$$l_{\mathcal{D}_J}(\Theta) \propto \prod_{i=0}^{J} \exp\left\{-\frac{1}{2}\big(\mathbf{X}_i^{(1)} - B_i^{(1)}\,\boldsymbol{\Theta}\big)'\big(\Sigma_i^{(11)}\big)^{-1}\big(\mathbf{X}_i^{(1)} - B_i^{(1)}\,\boldsymbol{\Theta}\big)\right\}.$$

Under Model Assumptions 1.1, the posterior distribution $u\,(\Theta|\,\mathcal{D}_J)$ of Θ, given $\mathcal{D}_J$, is given by

$$u\,(\Theta|\,\mathcal{D}_J) \;\propto\; l_{\mathcal{D}_J}(\Theta)\;\exp\left\{-\frac{1}{2}\,(\boldsymbol{\Theta} - \boldsymbol{\vartheta})'\,T^{-1}\,(\boldsymbol{\Theta} - \boldsymbol{\vartheta})\right\}, \tag{3.2}$$

where the last term is the prior density of $\boldsymbol{\Theta}$ with prior mean given by

$$\boldsymbol{\vartheta} = (\psi_0;\, \psi_1, \phi_1, \psi_2, \phi_2, \ldots, \psi_J, \phi_J)',$$

and the (diagonal) covariance matrix defined by

$$T = \operatorname{diag}\big(t_0^2;\, t_1^2, s_1^2, t_2^2, s_2^2, \ldots, t_J^2, s_J^2\big).$$

This immediately implies the following lemma.

Lemma 3.1. *Under Model Assumptions 1.1, the posterior distribution of Θ, given $\mathcal{D}_J$, is a multivariate Gaussian distribution with posterior covariance matrix*

$$T(\mathcal{D}_J) = \left(T^{-1} + \sum_{i=0}^{J}\big(B_i^{(1)}\big)'\big(\Sigma_i^{(11)}\big)^{-1}B_i^{(1)}\right)^{-1},$$

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and the posterior mean is given by

$$\boldsymbol{\vartheta}(\mathcal{D}_J) = T(\mathcal{D}_J)\left[T^{-1}\,\boldsymbol{\vartheta} + \sum_{i=0}^{J}(B_i^{(1)})'(\Sigma_i^{(11)})^{-1}\mathbf{X}_i^{(1)}\right].$$

Remarks on credibility theory.
We define the matrix $S = \left(\sum_{i=0}^{J}\left(B_i^{(1)}\right)'\left(\Sigma_i^{(11)}\right)^{-1}B_i^{(1)}\right)^{-1}$, and the credibility weights (I is the identity matrix)

$$A = (T^{-1} + S^{-1})^{-1}S^{-1} \quad \text{and} \quad I - A = (T^{-1} + S^{-1})^{-1}T^{-1},$$

see also formula (7.11) in Bühlmann–Gisler [2]. This then implies for the posterior covariance matrix $T(\mathcal{D}_J) = AS = (I - A)T$ and we obtain the linear credibility formula for the posterior mean

$$\boldsymbol{\vartheta}(\mathcal{D}_J) = (I - A)\,\boldsymbol{\vartheta} + A\,\mathbf{Y},$$

with compressed data $\mathbf{Y} = S\left[\sum_{i=0}^{J}\left(B_i^{(1)}\right)'\left(\Sigma_i^{(11)}\right)^{-1}\mathbf{X}_i^{(1)}\right]$, see Chapters 7 and 8 in Bühlmann–Gisler [2]. That is, the posterior mean $\boldsymbol{\vartheta}(\mathcal{D}_J)$ is a credibility weighted average between the prior mean $\boldsymbol{\vartheta}$ and the observations $\mathbf{Y}$ with credibility weight A.

Theorem 3.2 (PIC predictor). *Under Model Assumptions 1.1, we obtain the ultimate claim predictor for $I_{i,J} = P_{i,J}$, given observations $\mathcal{D}_J$,*

$$\widehat{I}_{i,J} = \mathbb{E}[I_{i,J}|\mathcal{D}_J] = \exp\Big\{\Gamma_i\,\boldsymbol{\vartheta}(\mathcal{D}_J) + \mathbf{e}_i'\,\Sigma_i^{(21)}\left(\Sigma_i^{(11)}\right)^{-1}\mathbf{X}_i^{(1)} + \Gamma_i\,T(\mathcal{D}_J)\,\Gamma_i'/2 + \mathbf{e}_i'\,\widetilde{\Sigma}_i^{(22)}\,\mathbf{e}_i/2\Big\}.$$

Theorem 3.2 gives the ultimate claim predictor that is now based on claims payments, incurred losses and prior expert information. In contrast to Merz–Wüthrich [5] we can now easily choose any meaningful covariance matrix V for Ξ_i.

4. Prediction uncertainty

In order to analyze the prediction uncertainty, we can now study the posterior predictive distribution of $\mathbf{I}_J = (I_{1,J}, \ldots, I_{J,J})$, which exactly corresponds to the column of unknown ultimate claims, given the observations $\mathcal{D}_J$. If $g(\cdot)$ is a sufficiently nice function, we obtain

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$$\mathbb{E}[g(\mathbf{I}_J)|\,\mathcal{D}_J] = \int_{\mathbf{x}\in\mathbb{R}^J} g(\mathbf{x})\; f\,(\mathbf{x}|\,\mathcal{D}_J)$$
$$d\mathbf{x} = \int_{\mathbf{x}\in\mathbb{R}^J,\boldsymbol{\Theta}} g(\mathbf{x})\; f\,(\mathbf{x}|\,\boldsymbol{\Theta},\mathcal{D}_J)\; u\;(\Theta|\,\mathcal{D}_J)\; d\mathbf{x}\, d\boldsymbol{\Theta},$$

where $u\;(\Theta|\,\mathcal{D}_J)$ denotes the posterior density of Θ, given $\mathcal{D}_J$. Because the densities $f\,(\mathbf{x}|\,\boldsymbol{\Theta},\mathcal{D}_J)$ and $u\;(\Theta|\,\mathcal{D}_J)$ are explicitly given by Corollary 2.2, Lemma 3.1 and the conditional independence of accident years, given $\boldsymbol{\Theta}$, we can calculate these predictive values $\mathbb{E}[g(\mathbf{I}_J)|\,\mathcal{D}_J]$ numerically, for example using Monte Carlo simulations. This allows for the analysis of any uncertainty and risk measure.

The probably most popular uncertainty measure in actuarial practice that has a closed-form solution is the so-called conditional MSEP. If we consider the total ultimate claim $\sum_{i=1}^J I_{i,J}$ and the corresponding predictor $\sum_{i=1}^J \widehat{I}_{i,J}$, the conditional MSEP is given by (see Wüthrich–Merz [9], Section 3.1)

$$\begin{aligned}&\mathrm{msep}_{\sum_{i=1}^J I_{i,J}\left|\mathcal{D}_J\right.}\left(\sum_{i=1}^J \widehat{I}_{i,J}\right)\\ &\quad= \mathbb{E}\left[\left(\sum_{i=1}^J I_{i,J} - \sum_{i=1}^J \widehat{I}_{i,J}\right)^2\middle|\,\mathcal{D}_J\right] = \mathrm{Var}\left(\sum_{i=1}^J I_{i,J}\middle|\,\mathcal{D}_J\right),\end{aligned}$$

where we have used $\widehat{I}_{i,J} = \mathbb{E}\left[\,I_{i,J}\middle|\,\mathcal{D}_J\right]$ in our Bayesian model. Henceforth, we need to calculate this last conditional variance in order to obtain the conditional MSEP.

Theorem 4.1. *Under Model Assumptions 1.1, the conditional MSEP is given by*

$$\begin{aligned}&\mathrm{msep}_{\sum_{i=1}^J I_{i,J}\left|\mathcal{D}_J\right.}\left(\sum_{i=1}^J \widehat{I}_{i,J}\right)\\ &\quad= \sum_{i,k=1}^J \widehat{I}_{i,J}\;\widehat{I}_{k,J}\;\left(\exp\left\{\Gamma_i\;T(\mathcal{D}_J)\;\Gamma_k' + \mathbf{e}_i'\;\widetilde{\Sigma}_i^{(22)}\;\mathbf{e}_i\;1_{\{i=k\}}\right\} - 1\right),\end{aligned}$$

with $\widehat{I}_{i,J} = \mathbb{E}\left[\,I_{i,J}\middle|\,\mathcal{D}_J\right]$ *given by Theorem 3.2.*

Thus, we obtain a closed-form solution for both, the ultimate claim predictors $\widehat{I}_{i,J}$ and the corresponding prediction errors, measured by the conditional MSEP.

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5. Motor third-party liability example

We apply the PIC model with dependence to the motor third-party liability data given in Tables 4 and 5. In Model Assumptions 1.1, we work with logarithmized paid claims ratios $\zeta_{i,k}$ and logarithmized incurred losses ratios $\xi_{i,k}$, respectively. That means that we have to transform the data in Tables 4 and 5 and we will use in our following calculations logarithmized ratios $\zeta_{i,k}$ for paid claimants and $\xi_{i,k}$ for incurred losses as assumed in Model Assumptions 1.1. Due to the fact that there is no expert knowledge for the specific choice of the means in the prior distributions, we choose in Model Assumptions 1.1 non-informative priors for Ψ_l and Φ_j, i.e. we let $t_j^2 \to \infty$ and $s_l^2 \to \infty$. This implies asymptotically for the credibility weight $A = I$ in Lemma 3.1 and no prior knowledge of the prior means ϕ_j and ψ_l of the prior distributions Φ_j and Ψ_l is used in our calculations.

For Model Assumptions 1.1, it remains to choose a suitable covariance matrix V. Here we present three different choices of correlation matrices of the type (1.1), which are motivated by an ad hoc estimate for the correlation matrix given below. This estimator for the correlation matrix of the type (1.1) should be seen as an intuitive proposal of a covariance structure and not as an estimator being optimal in some sense. In Section 5.1, we provide a correlation analysis of the motor third-party liability example motivating suitable correlation choices.

5.1. Correlation matrix choice

The choice of the correlation matrix type in formula (1.1) reduces the number of parameters to be estimated in comparison to the estimation of a full correlation matrix. Note that we decided for structure (1.1) by pure expert choice. In formula (1.1), we then have to choose ρ_l for $l \in \{0, 1, 2\}$ as

$$\rho_l = \mathrm{Cor}(\zeta_{i,k}, \xi_{i,k+l}) \quad \text{for} \quad i = 1, \ldots, J \quad \text{and} \quad k = 1, \ldots, J - l,$$

and $\rho_l = \mathrm{Cor}(\zeta_{i,0}, \xi_{i,l})$ for $l \in \{1, 2\}$. We propose the following ad hoc estimators for ρ_l for $l \in \{0, 1, 2\}$:

1. For the unknown means $\Psi_k = \mathbb{E}(\zeta_{i,k})$, $\Phi_k = \mathbb{E}(\xi_{i,k})$ and variances $\sigma^2_{\zeta_k} = \mathrm{Var}(\zeta_{i,k})$ and $\sigma^2_{\xi_k} = \mathrm{Var}(\xi_{i,k})$, we use sample estimates

$$\hat{\Psi}_k = \frac{1}{J-k+1} \sum_{i=0}^{J-k} \zeta_{i,k}, \quad k = 0, \ldots, J,$$

$$\hat{\sigma}^2_{\zeta_k} = \frac{1}{J-k} \sum_{i=0}^{J-k} (\zeta_{i,k} - \hat{\Psi}_k)^2, \quad k = 0, \ldots, J-1,$$

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and

$$\hat{\Phi}_k = \frac{1}{J-k+1} \sum_{i=0}^{J-k} \xi_{i,k}, \quad k = 1, \ldots, J,$$

$$\hat{\sigma}^2_{\xi_k} = \frac{1}{J-k} \sum_{i=0}^{J-k} (\xi_{i,k} - \hat{\Phi}_k)^2, \quad k = 1, \ldots, J-1.$$

Since for the estimation of the last variance parameters $\hat{\sigma}^2_{\zeta_J}$ and $\hat{\sigma}^2_{\xi_J}$ there is only one observation in the observed triangle, we use the extrapolation formula

$$\hat{\sigma}^2_{\zeta_J} = \min\left\{\hat{\sigma}^2_{\zeta_{J-2}}, \hat{\sigma}^2_{\zeta_{J-1}}, \hat{\sigma}^4_{\zeta_{J-2}}/\hat{\sigma}^2_{\zeta_{J-1}}\right\} \text{ and } \hat{\sigma}^2_{\xi_J} = \min\left\{\hat{\sigma}^2_{\xi_{J-2}}, \hat{\sigma}^2_{\xi_{J-1}}, \hat{\sigma}^4_{\xi_{J-2}}/\hat{\sigma}^2_{\xi_{J-1}}\right\}.$$

2. We consider for accident year $i \in \{0, \ldots, J\}$ the standardized observations

$$\tilde{\zeta}_{i,k} = \frac{\zeta_{i,k} - \hat{\Psi}_k}{\hat{\sigma}_{\zeta_k}} \text{ for } k = 0, \ldots J-i, \quad \tilde{\xi}_{i,k} = \frac{\xi_{i,k} - \hat{\Phi}_k}{\hat{\sigma}_{\xi_k}} \text{ for } k = 1, \ldots J-i.$$

3. For $l \in \{0, 1, 2\}$, we use the standard correlation estimator for ρ_l, based on the dataset

$$\left\{(\tilde{\zeta}_{i,k}, \tilde{\xi}_{i,k+l}) \middle| k \in \{1, \ldots, J-l\}, i \in \{1, \ldots, J-k-l\}\right\}. \tag{5.1}$$

According to our correlation estimators above, we obtain for $\hat{\rho}_l$ with $l = 0, 1, 2, 3$ ($\hat{\rho}_3$ is defined analogous to the other cases) as a function of the number of observations out of the set (5.1) (the observations are ordered in the way that we first pick up a development period k and then all accident years i up to $J-k-l$) being used for estimating ρ_l Figure 2. We see in Figure 2 that the assumption of positive correlatedness between $\zeta_{i,k}$ and $\xi_{i,k+l}$ for $l \in \{0, 1, 2\}$ is evident. For $l = 3$ or higher time lags, the correlation estimation is comparably small (about 5%) and will therefore be neglected in our following considerations. For the sample estimators, stated above and based on the full data set in (5.1) we obtain:

$\hat{\rho}_0$	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\rho}_3$
23%	27%	28%	5%

(5.2)

In order to also get correlation sensitivities, we make three explicit correlation choices (cases 1–3 in Table 1), with the correlation estimation results in (5.2) being a point of orientation for our correlation choice and compare it with the uncorrelated case (case 0) of Merz–Wüthrich [5] (see Table 1). For these correlation choices, we have to check whether the resulting covariance matrix V is positive definite (see Model Assumptions 1.1). To calculate V, we use the identity

$$V = \mathrm{Var}^{1/2}\mathrm{Cor}_l\mathrm{Var}^{1/2},$$

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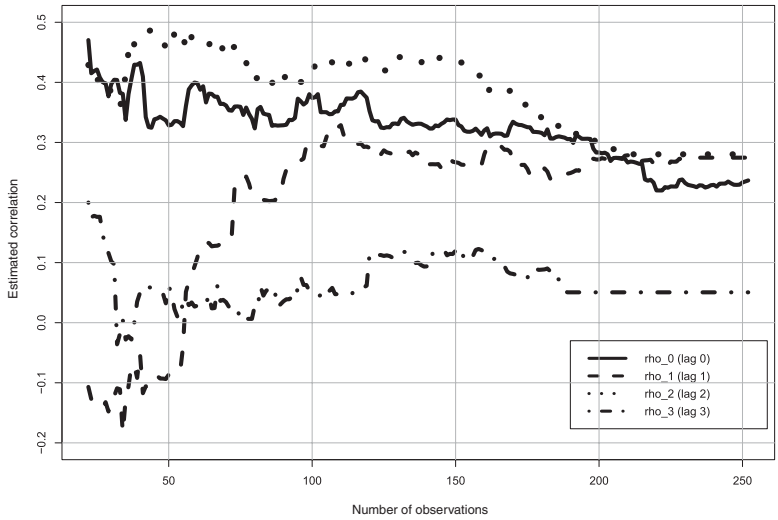

FIGURE 2: Correlation estimators $\hat{\rho}_l$ for ρ_l for $l \in \{0, 1, 2, 3\}$ as a function of the number of observations used for the estimation.

TABLE 1

INDEPENDENT CASE AND THREE EXPLICIT CHOICES OF CORRELATIONS.

Case 0	Case 1	Case 2	Case 3
$\rho_0 = \rho_1 = \rho_2 = 0\%$	$\rho_0 = 30\%$, $\rho_1 = 25\%$, $\rho_2 = 40\%$	$\rho_0 = 30\%$, $\rho_1 = 25\%$, $\rho_2 = 30\%$	$\rho_0 = 25\%$, $\rho_1 = 25\%$, $\rho_2 = 30\%$

where Cor_l denotes the correlation matrix in (1.1) for the cases $l \in \{1, 2, 3\}$ and Var denotes the diagonal matrix with the standard variance estimates of $(\zeta_{i,0}; \zeta_{i,1}, \xi_{i,1}, \zeta_{i,2}, \xi_{i,2}, \ldots, \zeta_{i,J}, \xi_{i,J})' \in \mathbb{R}^{2J+1}$ on its diagonal, i.e. $\mathrm{Var} = \mathrm{diag}\left(\hat{\sigma}^2_{\zeta_0}; \hat{\sigma}^2_{\zeta_1}, \hat{\sigma}^2_{\xi_1}, \ldots, \hat{\sigma}^2_{\zeta_J}, \hat{\sigma}^2_{\xi_J}\right)$. Since $\mathrm{Var}^{1/2}$ is diagonal, we only have to check whether the matrix Cor_l is positive definite. The eigenvalues of the correlation matrix Cor_l are for our choices of ρ_0, ρ_1 and ρ_2 in the cases $l = 1, 2, 3$ strictly positive, the smallest being $9,2 \times 10^{-12}$ and the largest being $3,8 \times 10^{-03}$ and hence the corresponding covariance matrix V fulfills Model Assumptions 1.1.

Based on these choices for the covariance matrix V, we calculate the ultimate claim reserves and the conditional MSEP.

Remarks.

- We choose by means of explorative data analysis explicit covariance matrices V. This was partly done by intuitive expert knowledge.
- In our Bayesian model the derivation of an optimal estimator of the correlation matrix having good statistical properties is not trivial and should be

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subject to more statistical research. Therefore, we present an ad hoc estimator for the correlation matrix and use the resulting estimates as an orientation for our three explicit choices of correlation (cases 1–3).

- Model Assumptions 1.1 allow for arbitrary covariance matrices as long as they are positive definite. If sufficient data for a robust estimation of the $(n(n+1)/2)$ entries of a general covariance matrix are available, there is no need to reduce to correlations up to lag 2. However, we believe that (due to overparametrization) arbitrary correlation structures are no feasible alternative and expert opinion always needs to specify additional structure.
- Positive definiteness of V should always be checked because most intuitive choices do not always provide a positive definite correlation matrix.

5.2. Claims reserves and prediction uncertainty

5.2.1. *Claims reserves at time J.* We consider the expected outstanding loss liabilities (claims reserves)

$$\hat{R}(\mathcal{D}_J) = E[I_{i,J}|\mathcal{D}_J] - P_{i,J-i}.$$

The percental difference between claims reserves with and without dependence is denoted by $D(\hat{R}(\mathcal{D}_J))$. We observe in Table 2 that in the first case the claims reserves are about 6% lower than the claims reserves without dependence. In the other two cases, the difference is still about 3%. This shows that the specific choice of correlation has a crucial impact on the claims reserves. The deeper reason for the higher claims reserves in case 0 is that there has been a slight change in the claims adjuster's philosophy that influences $I_{i,J}$. In cases 1–3, this change in philosophy is smoothed more over time which should result in more reliable claims reserves.

5.2.2. *Prediction uncertainty at time J.* In Table 3, we provide the $\text{msep}^{1/2}$ for our explicit correlation choices above and for the classical PIC model without dependence (see Table 1). We observe that the prediction uncertainty measured in terms of the conditional MSEP is higher than in the classical PIC model. The reason is that positive correlations in Model Assumptions 1.1 increase the correlations in the ultimate outcomes, and hence the uncertainties. This implies that not considering dependence in the PIC model clearly underestimates the overall uncertainties.

6. Conclusions

In the original PIC model of Merz–Wüthrich [5], a unified predictor of the ultimate claim based on incurred losses and claims payments and the corresponding prediction uncertainty in terms of the conditional MSEP can be derived analytically. The main criticism is that the original PIC model does not allow for modeling dependencies between claims payments and incurred losses as it is observed in the data.

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TABLE 2

CLAIMS RESERVES IN THE CLASSICAL PIC MODEL AND PIC MODEL WITH DEPENDENCE.

Acc. year	$\hat{R}(\mathcal{D}_J)$ Case 0	$\hat{R}(\mathcal{D}_J)$ Case 1	$D(\hat{R}(\mathcal{D}_J))$ Case 1	$\hat{R}(\mathcal{D}_J)$ Case 2	$D(\hat{R}(\mathcal{D}_J))$ Case 2	$\hat{R}(\mathcal{D}_J)$ Case 3	$D(\hat{R}(\mathcal{D}_J))$ Case 3
1	7.726	7.729	0,0%	7.729	0,0%	7.728	0,0%
2	12.084	12.090	0,0%	12.089	0,0%	12.087	0,0%
3	15.196	15.537	2,2%	15.423	1,5%	15.397	1,3%
4	9.916	8.291	16,4%	8.664	12,6%	8.718	12,1%
5	20.746	21.310	2,7%	21.169	2,0%	21.096	1,7%
6	23.675	24.111	1,8%	24.102	1,8%	24.047	1,6%
7	33.328	33.410	0,2%	33.749	1,3%	33.683	1,1%
8	35.740	37.369	4,6%	37.327	4,4%	37.146	3,9%
9	40.144	38.695	3,6%	39.669	1,2%	39.767	0,9%
10	53.888	48.764	9,5%	51.602	4,2%	51.788	3,9%
11	62.825	59.284	5,6%	61.134	2,7%	61.233	2,5%
12	79.164	77.724	1,8%	78.716	0,6%	78.352	1,0%
13	89.437	81.510	8,9%	85.614	4,3%	85.572	4,3%
14	88.300	79.565	9,9%	82.942	6,1%	83.358	5,6%
15	122.534	107.575	12,2%	115.540	5,7%	116.508	4,9%
16	126.151	108.955	13,6%	117.667	6,7%	118.831	5,8%
17	126.202	119.794	5,1%	122.695	2,8%	122.682	2,8%
18	127.522	124.947	2,0%	126.287	1,0%	125.897	1,3%
19	152.078	143.847	5,4%	147.725	2,9%	148.060	2,6%
20	185.586	170.054	8,4%	175.798	5,3%	177.062	4,6%
21	251.803	246.960	1,9%	248.818	1,2%	248.554	1,3%
Total	1.664.045	1.567.522	5,8%	1.614.459	3,0%	1.617.568	2,8%

TABLE 3

PREDICTION UNCERTAINTY $\text{msep}^{1/2}$ FOR THE CLASSICAL PIC MODEL AND THE PIC MODEL WITH DEPENDENCE.

	$\text{msep}^{1/2}$ Case 0	$\text{msep}^{1/2}$ Case 1	$\text{msep}^{1/2}$ Case 2	$\text{msep}^{1/2}$ Case 3
Total	40.606	48.010	49.145	48.922
in % of claims reserves	2,44%	3,06%	3,04%	3,02%

In this paper, we generalize the original PIC model so that it allows for modeling dependence between claims payments and incurred losses data. This is motivated by the fact that, on the one hand, claims payments are contained in incurred losses data and, on the other hand, incurred losses contain additional information, which influences future claims payments data. The data in our example (see Tables 4 and 5) confirm this hypothesis of the dependence between claims payments and incurred losses data (see Figure 2).

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We have seen in the sensitivity analysis that dependence modeling in the PIC method has a crucial impact on the claims reserves and the corresponding MSEPs (see Tables 2 and 3). Note that the classical PIC model of Merz–Wüthrich [5] underestimates the prediction uncertainty (see Table 5), due to the missing dependence in its structure within accident years. Furthermore, for a better understanding of the influence of prior choices on the reserves and its uncertainty, it might be useful to provide a sensitivity analysis of the method to the choice of priors, which should be subject to an extended case study in future work.

Concluding, the benefits of the PIC method with dependence modeling are that

- two different channels of information are combined to get a unified ultimate loss predictor;
- dependence structures between paid and incurred data can be modeled appropriately;
- prior expert knowledge can be used to design the prior distributions of the parameter vector $\boldsymbol{\Theta}$, otherwise we can choose non-informative priors for $\boldsymbol{\Theta}$. Prior expert opinion should also be used for the design of appropriate correlation structures;
- we can calculate the ultimate claim and the conditional MSEP analytically;
- the full predictive distribution can be derived via Monte Carlo simulations. This allows for the calculation of any risk measure like value-at-risk or expected shortfall.

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APPENDIX

APPENDIX. PROOFS

In this appendix, we prove all the statements.

Proof of Lemma 2.1. Conditionally given the parameters $\mathbf{\Theta}$, we have that Ξ_i for different accident years i are independent and hence so are $\mathbf{X}_i$. Therefore, conditionally given the parameters $\mathbf{\Theta}$ and $\mathcal{D}_J$, only the observations with accident year i within $\mathcal{D}_J$ are relevant for the study of the distribution of $\mathbf{X}_i$. This proves the first equality of the claim in Lemma 2.1. The second distributional claim is then a well-known result for multivariate Gaussian distributions. For the Schur complement of $\Sigma_i^{(11)}$ in Σ, we also refer to Section A.5.5 in Boyd–Vandenberghe [1]. This proves the lemma.

□

Proof of Lemma 3.1. From (3.2) it immediately follows that the posterior distribution of $\mathbf{\Theta}$, given $\mathcal{D}_J$, is a multivariate Gaussian distribution. The first and second moments are then found by squaring out all terms.

□

Proof of Theorem 3.2. Using the tower property for conditional expectations, we obtain from formulas (2.2) and (3.2):

$$\begin{aligned}\mathbb{E}[I_{i,J}|\mathcal{D}_J] &= \mathbb{E}\left[\mathbb{E}[I_{i,J}|\mathbf{\Theta},\mathcal{D}_J]|\mathcal{D}_J\right]\\ &= \mathbb{E}\left[\exp\{\Gamma_i\,\mathbf{\Theta}\}|\,\mathcal{D}_J\right]\ \exp\left\{\mathbf{e}_i'\,\Sigma_i^{(21)}\left(\Sigma_i^{(11)}\right)^{-1}\mathbf{X}_i^{(1)}+\mathbf{e}_i'\,\widetilde{\Sigma}_i^{(22)}\,\mathbf{e}_i/2\right\}.\end{aligned}$$

But then the proof is a consequence from Lemma 3.1.

□

Proof of Theorem 4.1. With the variance decoupling formula, we obtain

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{J} I_{i,J}\,\middle|\,\mathcal{D}_J\right) &= \sum_{i,k=1}^{J}\mathrm{Cov}\left(I_{i,J},I_{k,J}\,\middle|\,\mathcal{D}_J\right)\\ &= \sum_{i,k=1}^{J}\mathrm{Cov}\left(\mathbb{E}[I_{i,J}|\mathbf{\Theta},\mathcal{D}_J],\mathbb{E}[I_{k,J}|\mathbf{\Theta},\mathcal{D}_J]\,\middle|\,\mathcal{D}_J\right)+\sum_{i=1}^{J}\mathbb{E}[\mathrm{Var}(I_{i,J}|\mathbf{\Theta},\mathcal{D}_J)|\mathcal{D}_J],\end{aligned}$$

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where for the second term we have used the conditional independence of different accident years, given $\boldsymbol{\Theta}$. Thus, we need to calculate these last two terms. Using Corollary 2.2 and (3.2),

$$\mathbb{E}\left[\, I_{i,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J \right] = \exp\left\{\Gamma_i\, \boldsymbol{\Theta} + \mathbf{e}_i'\, \Sigma_i^{(21)} \left(\Sigma_i^{(11)}\right)^{-1} \mathbf{X}_i^{(1)} + \mathbf{e}_i'\, \widetilde{\Sigma}_i^{(22)}\, \mathbf{e}_i/2\right\},$$

and

$$\mathrm{Var}\left(\, I_{i,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J\right) = \mathbb{E}\left[\, I_{i,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J\right]^2 \left(\exp\left\{\mathbf{e}_i'\, \widetilde{\Sigma}_i^{(22)}\, \mathbf{e}_i\right\} - 1\right).$$

We first treat the second term. It provides

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{Var}\left(\, I_{i,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J\right) \middle| \mathcal{D}_J\right] \\
&\quad = \mathbb{E}\left[\exp\left\{2\Gamma_i\, \boldsymbol{\Theta} + 2\mathbf{e}_i'\, \Sigma_i^{(21)} (\Sigma_i^{(11)})^{-1}\, \mathbf{X}_i^{(1)} + \mathbf{e}_i'\, \widetilde{\Sigma}_i^{(22)}\, \mathbf{e}_i\right\} \middle| \mathcal{D}_J\right] \left(\exp\left\{\mathbf{e}_i'\, \widetilde{\Sigma}_i^{(22)}\, \mathbf{e}_i\right\} - 1\right) \\
&\quad = \mathbb{E}\left[\, I_{i,J} \middle| \mathcal{D}_J\right]^2 \exp\left\{\Gamma_i\, T(\mathcal{D}_J)\, \Gamma_i'\right\} \left(\exp\left\{\mathbf{e}_i'\, \widetilde{\Sigma}_i^{(22)}\, \mathbf{e}_i\right\} - 1\right).
\end{aligned}$$

For the first term we need to consider

$$\begin{aligned}
&\mathrm{Cov}\left(\mathbb{E}\left[\, I_{i,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J\right], \mathbb{E}\left[\, I_{k,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J\right] \middle| \mathcal{D}_J\right) \\
&\quad = \exp\left\{\mathbf{e}_i'\, \Sigma_i^{(21)} \left(\Sigma_i^{(11)}\right)^{-1} \mathbf{X}_i^{(1)} + \mathbf{e}_i'\, \widetilde{\Sigma}_i^{(22)}\, \mathbf{e}_i/2 + \mathbf{e}_k'\, \Sigma_i^{(21)} (\Sigma_i^{(11)})^{-1}\, \mathbf{X}_k^{(1)} + \mathbf{e}_k'\, \widetilde{\Sigma}_i^{(22)}\, \mathbf{e}_k/2\right\} \\
&\qquad \times \mathrm{Cov}\left(\exp\left\{\Gamma_i\, \boldsymbol{\Theta}\right\}, \exp\left\{\Gamma_k\, \boldsymbol{\Theta}\right\} \middle| \mathcal{D}_J\right).
\end{aligned}$$

For this last covariance term, we have

$$\begin{aligned}
&\mathrm{Cov}\left(\exp\left\{\Gamma_i\, \boldsymbol{\Theta}\right\}, \exp\left\{\Gamma_k\, \boldsymbol{\Theta}\right\} \middle| \mathcal{D}_J\right) \\
&\qquad = \mathbb{E}\left[\exp\left\{\Gamma_i\, \boldsymbol{\Theta}\right\} \middle| \mathcal{D}_J\right]\, \mathbb{E}\left[\exp\left\{\Gamma_k\, \boldsymbol{\Theta}\right\} \middle| \mathcal{D}_J\right]\, \left(\exp\left\{\Gamma_i\, T(\mathcal{D}_J)\, \Gamma_k'\right\} - 1\right).
\end{aligned}$$

Henceforth, we obtain for the first term

$$\begin{aligned}
&\mathrm{Cov}\left(\mathbb{E}\left[\, I_{i,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J\right], \mathbb{E}\left[\, I_{k,J} \middle| \boldsymbol{\Theta}, \mathcal{D}_J\right] \middle| \mathcal{D}_J\right) \\
&\qquad = \mathbb{E}\left[\, I_{i,J} \middle| \mathcal{D}_J\right] \mathbb{E}\left[\, I_{k,J} \middle| \mathcal{D}_J\right] \left(\exp\left\{\Gamma_i\, T(\mathcal{D}_J)\, \Gamma_k'\right\} - 1\right).
\end{aligned}$$

Collecting all the terms and using $\widehat{I}_{i,J} = \mathbb{E}\left[\, I_{i,J} \middle| \mathcal{D}_J\right]$ completes the proof.

□

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TABLE 4

CUMULATIVE CLAIMS PAYMENTS $P_{i,j}$, $i + j \leq 21$, DATA FROM A MOTOR THIRD-PARTY LIABILITY EXAMPLE.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	136.367	195.757	213.788	227.545	240.136	249.647	260.260	271.207	285.847	290.814	300.035	305.988	312.930	316.524	319.988	323.268	325.798	326.231	327.722	328.616	332.724	337.137
1	143.135	212.658	233.989	252.155	267.259	280.669	295.863	310.584	322.938	332.021	341.670	349.924	356.908	361.126	367.795	370.548	373.051	376.942	378.314	379.716	381.388	
2	146.469	219.759	241.983	257.063	272.848	285.437	313.398	329.903	340.223	353.075	362.015	371.042	375.094	379.430	382.385	389.057	391.468	395.824	397.531	399.724		
3	158.518	232.128	256.752	276.593	292.807	310.757	322.837	339.751	352.613	366.707	378.735	385.394	394.505	402.618	409.044	412.422	415.624	421.409	424.117			
4	158.633	224.457	249.797	267.676	285.455	303.548	320.282	340.976	352.487	361.300	374.500	388.449	397.848	402.989	408.151	414.016	416.098	419.528				
5	153.215	225.074	249.688	267.753	285.294	307.116	324.791	341.238	353.420	369.549	382.016	390.301	395.206	403.634	406.302	407.819	411.082					
6	153.185	215.699	235.609	255.384	272.749	290.988	304.081	319.717	334.457	352.992	372.879	383.645	394.634	401.194	407.377	410.387						
7	150.974	217.545	242.400	260.473	279.436	299.797	317.991	336.679	352.929	373.339	397.542	407.145	416.136	429.445	435.980							
8	141.432	205.018	225.339	241.315	260.098	277.061	296.286	312.645	330.538	338.629	349.021	357.775	366.468	372.513								
9	141.554	207.510	230.597	250.393	272.538	294.008	321.253	346.836	366.865	381.705	391.678	404.292	411.770									
10	141.899	206.157	229.510	246.710	262.735	280.171	303.956	324.354	343.041	356.874	368.163	380.622										
11	145.037	215.127	240.970	260.457	280.524	304.118	322.331	345.629	357.081	370.673	384.000											
12	135.739	203.999	232.176	250.014	277.500	298.976	323.555	339.853	352.098	364.883												
13	135.350	209.545	236.220	256.710	276.576	293.467	305.436	320.329	336.143													
14	132.847	203.592	227.902	249.914	270.477	286.129	301.347	317.801														
15	135.951	205.450	229.862	250.624	266.371	280.202	300.874															
16	131.151	193.635	215.365	234.202	247.325	262.034																
17	130.188	190.262	213.586	226.115	242.768																	
18	118.505	174.622	192.852	206.808																		
19	118.842	177.671	199.872																			
20	121.011	185.856																				
21	132.116																					

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TABLE 5

INCURRED LOSSES $I_{i,j}$, $i + j \leq 21$, DATA FROM A MOTOR THIRD-PARTY LIABILITY EXAMPLE.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	370.661	369.491	358.572	344.401	346.665	342.370	344.807	344.529	346.520	344.774	345.346	344.592	339.747	340.214	338.824	338.687	338.119	337.285	336.834	336.271	336.647	337.137
1	419.289	416.218	393.048	384.661	383.508	376.001	382.602	383.968	384.388	383.803	387.951	389.023	386.576	389.510	388.477	388.833	389.687	390.600	389.190	388.147	388.548	
2	437.568	427.975	415.222	407.791	406.460	418.175	411.091	406.724	410.959	413.780	414.044	410.973	410.142	409.120	410.870	414.860	412.483	410.468	410.964	410.768		
3	450.610	448.418	431.294	424.361	432.825	427.773	436.947	434.592	437.282	437.315	438.579	439.037	440.049	439.942	439.561	437.789	437.218	439.781	438.979			
4	468.979	450.942	432.591	435.162	423.583	420.876	424.588	424.632	429.180	431.788	429.673	433.284	430.848	432.002	431.379	429.732	429.539	428.875				
5	443.189	434.315	433.325	425.332	424.921	422.432	421.233	422.904	424.484	426.661	431.462	431.676	434.368	433.599	431.820	432.994	433.037					
6	435.307	422.149	412.660	405.190	402.367	402.597	411.096	412.873	420.973	432.325	437.067	433.610	435.824	435.656	434.796	436.011						
7	475.948	438.817	432.413	436.710	436.904	443.156	448.347	463.171	468.412	476.070	474.593	473.324	477.058	472.283	473.524							
8	447.021	422.678	405.919	399.462	400.047	398.297	406.939	404.834	409.056	411.421	412.002	410.739	409.744	413.587								
9	457.229	444.054	436.390	436.853	442.292	453.494	456.363	460.272	459.591	456.975	455.336	454.500	453.068									
10	462.989	464.776	447.833	432.893	432.124	442.743	451.994	451.534	450.528	450.845	448.398	442.810										
11	484.915	468.800	454.958	447.601	461.106	470.358	465.346	468.879	461.537	456.753	453.919											
12	462.028	429.610	438.929	454.797	468.116	468.721	469.907	463.823	459.524	452.385												
13	450.908	456.030	476.259	483.129	476.952	464.941	453.391	445.089	434.103													
14	426.385	428.504	456.796	449.886	445.397	432.021	412.353	402.565														
15	461.078	477.458	480.960	471.869	462.978	444.670	437.203															
16	444.123	430.684	433.664	419.422	403.126	396.903																
17	433.830	407.931	393.723	371.800	361.853																	
18	418.202	374.855	338.598	324.790																		
19	426.853	373.282	351.590																			
20	410.810	394.477																				
21	405.597																					

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