ON THE PROBABILITY AND SEVERITY OF RUIN

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ABSTRACT

In the usual model of the collective risk theory, we are interested in the severity of ruin, as well as its probability. As a quantitative measure, we propose $G(u, y)$, the probability that for given initial surplus u ruin will occur and that the deficit at the time of ruin will be less than y, and the corresponding density $g(u, y)$. First a general answer in terms of the transform is obtained. Then, assuming that the claim amount distribution is a combination of exponential distributions, we determine g; here the roots of the equation that defines the adjustment coefficient play a central role. An explicit answer is also given in the case in which all claims are of constant size.

KEYWORDS

Ruin; severity of ruin.

1. INTRODUCTION

In the following we shall use the model and the notation of Bowers *et al.* (1987, chapter 12). Thus we consider a company with initial surplus $U(0) = u$, whose surplus at time t is given by the expression

$$U(t) = u + ct - S(t), \qquad t > 0. \tag{1}$$

Here c is the constant rate at which the premiums are received, and $S(t)$, the aggregate claims up to time t, is a compound Poisson process given by the parameter λ (the expected number of claims per unit time) and the claim amount distribution $P(x)$. It is assumed that c contains a loading. Let T denote the time of ruin (with the convention that $T = \infty$, if ruin does not occur), and let $\psi(u)$ denote the probability of ruin considered as a function of the initial surplus.

It has been argued that the probability of ruin is a very crude stability criterion. We are not just interested in the *probability* of ruin, but we also want to know

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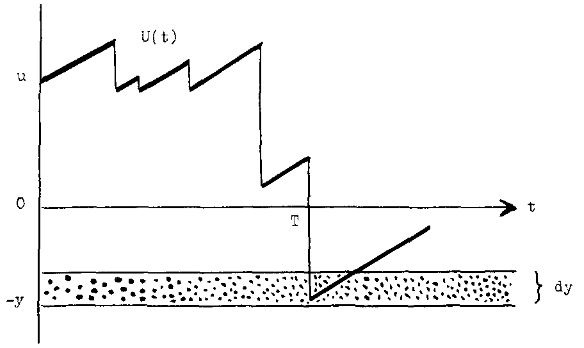

FIGURE 1. The interpretation of $g(u, y)\ dy$.

how serious the situation is when ruin occurs. To obtain a quantitative answer we introduce the function

$$G(u, y) = \Pr(T < \infty, -y < U(T) < 0), \tag{2}$$

which is a function of the variables $u \geqslant 0$ and $y \geqslant 0$ and is the probability that ruin occurs and that the deficit at the time of ruin is less than y. We shall also consider the corresponding density

$$g(u, y) = \frac{d}{dy} G(u, y) \tag{3}$$

whose existence will be shown in Section 6. Thus $g(u, y)\ dy$ is the probability that ruin occurs and that $U(T)$ will be between $-y$ and $-y + dy$ (see Figure 1). Theorem 12.2 of BOWERS *et al.* (1987) tells us that

$$g(0, y) = \frac{\lambda}{c} [1 - P(y)]. \tag{4}$$

Our main goal is to explore $g(u, y)$ in the more interesting case when u is positive.

2. A FUNCTIONAL EQUATION

According to theorem 12.2 of BOWERS *et al.* (1987), the probability that the surplus will ever fall below the initial level u and will be between $u - x$ and $u - x + dx$ when it happens for the first time is

$$\frac{\lambda}{c} [1 - P(x)]\ dx.$$

We use this and the law of total probability to see that

$$G(u, y) = \frac{\lambda}{c} \int_0^u G(u - x, y)[1 - P(x)]\ dx + \frac{\lambda}{c} \int_u^{u+y} [1 - P(x)]\ dx. \tag{5}$$

Note that $\psi(u) = G(u, \infty)$. Thus the equation for $\psi(u)$ of exercise 11 of BOWERS *et al.* (1987, chapter 12) is a special case of (5).

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Differentiating (5) with respect to y, we obtain a functional equation for g:

$$g(u, y) = \frac{\lambda}{c} \int_0^u g(u-x, y)[1 - P(x)]\, dx + \frac{\lambda}{c} [1 - P(u+y)]. \tag{6}$$

In the terminology of FELLER (1966), equations (5) and (6) are renewal equations of the defective type.

Instead of determining g directly, we shall first find its transform $\gamma(r, y)$, which is defined as

$$\gamma(r, y) = \int_0^\infty e^{ru} g(u, y)\, du. \tag{7}$$

We multiply (6) by e^{ru} and integrate over u from 0 to ∞. On the left-hand side we get $\gamma(r, y)$. If we replace the variable u by the new integration variable $z = u - x$ we can simplify the resulting double integral on the right-hand side as follows:

$$\frac{\lambda}{c} \int_0^\infty \int_0^u e^{r(u-x)} g(u-x, y)\, e^{rx} [1 - P(x)]\, dx\, du$$

$$= \frac{\lambda}{c} \int_0^\infty \int_0^\infty e^{rz} g(z, y)\, e^{rx} [1 - P(x)]\, dz\, dx$$

$$= \frac{\lambda}{c} \gamma(r, y) \int_0^\infty e^{rx} [1 - P(x)]\, dx.$$

The second term on the right-hand side can be written as

$$\frac{\lambda}{c} \int_0^\infty e^{ru} [1 - P(u+y)]\, du = \frac{\lambda}{c} e^{-ry} \int_y^\infty e^{rx} [1 - P(x)]\, dx.$$

This way we obtain from (6) a linear equation for $\gamma(r, y)$:

$$\gamma(r, y) = \frac{\lambda}{c} \gamma(r, y) \int_0^\infty e^{rx} [1 - P(x)]\, dx + \frac{\lambda}{c} e^{-ry} \int_y^\infty e^{rx} [1 - P(x)]\, dx.$$

Its solution is

$$\gamma(r, y) = (\lambda/c)\, e^{-ry} \int_y^\infty e^{rx} [1 - P(x)]\, dx \Big/ \left[1 - (\lambda/c) \int_0^\infty e^{rx} [1 - P(x)]\, dx\right]. \tag{8}$$

The remaining task is to invert this transform to obtain $g(u, y)$. In the following we shall look at a family of claim amount distributions in which this can be done in a transparent way.

3. COMBINATIONS OF EXPONENTIAL DISTRIBUTIONS

Let us assume that the claim amount distribution is a combination of exponential

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distributions, i.e. that it has a probability density function of the form

$$p(x) = \sum_{j=1}^{n} A_j \beta_j e^{-\beta_j x}, \qquad x > 0, \tag{9}$$

with β_j positive and

$$A_1 + A_2 + \cdots + A_n = 1. \tag{10}$$

In the special case when all A_j are positive, we speak of a mixture of exponential distributions. From (9) it follows that

$$1 - P(x) = \sum_{j=1}^{n} A_j \, \mathrm{e}^{-\beta_j x}, \qquad x > 0. \tag{11}$$

We substitute this into (8) and obtain

$$\gamma(r, y) = (\lambda/c) \sum_{j=1}^{n} e^{-\beta_j y} A_j/(\beta_j - r) \Bigg/ \left[1 - (\lambda/c) \sum_{j=1}^{n} A_j/(\beta_j - r)\right]. \tag{12}$$

Applying the method of partial fractions, we can write this expression in the following form:

$$\gamma(r, y) = \sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} e^{-\beta_j y}/(r_k - r). \tag{13}$$

Here $r_1, r_2, \ldots, r_n$ are the zeros of the denominator, i.e. the solutions of the equation

$$(\lambda/c) \sum_{j=1}^{n} A_j/(\beta_j - r) = 1. \tag{14}$$

It is assumed that the n roots are distinct; of course some may be complex. Condition (14) is the same as the condition that defines the adjustment coefficient, see exercise 8 of BOWERS *et al.* (1987, chapter 12). Thus the adjustment coefficient is one of the roots; without loss of generality we may set $r_1 = R$.

The coefficients C_{jk} can be calculated as follows. We multiply $\gamma(r, y)$ by $(r_m - r)$ and let $r \to r_m$. If we do this in (13), the coefficient of $\exp(-\beta_j y)$ is C_{jm}. In (12) we divide the denominator by $(r_m - r)$ and let $r \to r_m$. Since the denominator vanishes for $r = r_m$, this operation gives minus the derivative of the denominator at r_m. Thus the coefficient of $\exp(-\beta_j y)$ is

$$C_{jm} = A_j/(\beta_j - r_m) \Bigg/ \sum_{l=1}^{n} A_l/(\beta_l - r_m)^2. \tag{15}$$

Once the roots have been determined, the coefficients can be calculated easily from this formula.

The inversion of (13) is simple: one verifies that

$$g(u, y) = \sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} e^{-\beta_j y} e^{-r_k u} \tag{16}$$

satisfies (7); therefore expression (16) is the desired solution.

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These results can also be used to determine the probability of ruin. Since

$$\psi(u) = \int_0^\infty g(u, y)\, dy, \tag{17}$$

if follows from (16) that

$$\psi(u) = \sum_{k=1}^{n} C_k\, e^{-r_k u}, \tag{18}$$

where

$$C_k = \sum_{j=1}^{n} C_{jk}/\beta_j. \tag{19}$$

Formula (18) can be found in CRAMÉR (1955) and more recently in BOWERS *et al.* (1987); in both cases the discussion is limited to mixtures of exponential distributions. A related discussion can be found in DICKSON and GRAY (1984).

The class of mixtures of exponential distributions is somewhat limited; for example, the mode of such a distribution is necessarily at 0. On the contrary, the family of combinations of exponential distributions is rather rich, though not every choice of A_j, β_j gives a probability density function. A subset of this family consists of the sums of n independent exponential (β_j) distributed random variables with unequal parameters (see FELLER (1966, problem 12 of chapter I.13)). An elegant proof can be obtained by looking at the moment-generating functions and applying the method of partial fractions. Taking the limit $\beta_j \to \beta$, one obtains the $\Gamma(n, \beta)$ density. One may also show that the Gamma distribution with arbitrary values of the non-scale parameter, say $n - \delta$ with $0 < \delta < 1$, is in the closure of this class. It is sufficient to show that such a Gamma distribution is a mixture of Gamma distributions with non-scale parameter n. The definition of the Gamma function implies

$$x^{-\delta} = \int_0^\infty \frac{e^{-tx} t^{\delta - 1}}{\Gamma(\delta)}\, dt.$$

Using this we may write the Gamma $(n - \delta, 1)$ density as:

$$\frac{x^{n-\delta-1} e^{-x}}{\Gamma(n-\delta)} = \int_0^\infty \frac{(t+1)^n x^{n-1} e^{-(t+1)x}}{\Gamma(n)}\, \frac{t^{\delta-1}(t+1)^{-n}\Gamma(n)}{\Gamma(n-\delta)\,\Gamma(\delta)}\, dt.$$

We shall show in the following section that a mixture or combination of Gamma distributions can be handled in quite the same way as a combination of exponential distributions.

4. COMBINATIONS OF GAMMA DISTRIBUTIONS

The advantage of considering combinations of exponentials rather than just mixtures lies in the fact that this class also contains distributions with mode not equal to zero. Another way to include such distributions is to consider mixtures, or combinations, of Gamma distributions with integer-valued non-scale parameter.

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To avoid unnecessarily complicated formulae, we shall limit our discussion to the case where this parameter equals two, but generalization to other integer values is straightforward. We shall not give all details of the proofs in this section. Note that as the Gamma distribution can be written as a limit of combinations of exponential distributions, no new situations are added when an explicit expression for $g(u, y)$ can be found.

Consider the following density function $p(x)$:

$$p(x) = \sum_{j=1}^{n} A_j \beta_j^2 x e^{-\beta_j x}, \qquad x > 0, \tag{20}$$

with β_j positive and

$$A_1 + A_2 + \cdots + A_n = 1. \tag{21}$$

From (8) one obtains after some calculation:

$$\gamma(r, y) = (\lambda/c) \sum_{j=1}^{n} A_j \frac{e^{-\beta_j y}}{(\beta_j - r)^2} (\beta_j^2 y - \beta_j y r + 2\beta_j - r) \Big/ \left[1 - (\lambda/c) \sum_{j=1}^{n} A_j \frac{2\beta_j - r}{(\beta_j - r)^2}\right]. \tag{22}$$

Again applying the method of partial fractions, we can rewrite expression (22) in the following form:

$$\gamma(r, y) = \sum_{j=1}^{n} \sum_{k=1}^{2n} C'_{jk}(y)\ e^{-\beta_j y}/(r_k - r). \tag{23}$$

Note that in this case we obtain coefficients C'_{jk} depending on y. Here $r_1, r_2, \ldots, r_{2n}$ are the zeros of the denominator of (22), i.e. the solutions of the equation:

$$1 - (\lambda/c) \sum_{j=1}^{n} A_j \frac{2\beta_j - r}{(\beta_j - r)^2} = 0. \tag{24}$$

Once more we assume all these roots to be distinct, although the more general case presents no insuperable difficulties. One of the roots equals the adjustment coefficient. In the same way (15) was derived, one has:

$$C'_{jm}(y) = \frac{A_j}{(\beta_j - r_m)^2} (\beta_j^2 y - \beta_j y r_m + 2\beta_j - r_m) \Big/ \sum_{l=1}^{n} A_l \frac{3\beta_l - r_m}{(\beta_l - r_m)^3}. \tag{25}$$

Inversion of (23) leads to

$$g(u, y) = \sum_{j=1}^{n} \sum_{k=1}^{2n} C'_{jk}(y)\ e^{-\beta_j y} e^{-r_k u}. \tag{26}$$

Again the probability of ruin can be obtained as:

$$\psi(u) = \sum_{k=1}^{2n} C'_k\ e^{-r_k u}, \tag{27}$$

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where

$$(28)\quad C_k' = \sum_{j=1}^{n} \int_0^{\infty} C_{jk}'(y)\, e^{-\beta_j y}\, dy = \sum_{j=1}^{n} A_j \frac{3 - 2r_k/\beta_j}{(\beta_j - r_k)^2} \Bigg/ \sum_{j=1}^{n} A_j \frac{3\beta_j - r_k}{(\beta_j - r_k)^3}.$$

The above exposition can be generalized to other integer values of the non-scale parameter, and also to arbitrary positive real values.

5. ILLUSTRATIONS

EXAMPLE 1. Suppose that $n = 2$, $A_1 = A_2 = \frac{1}{2}$, $\beta_1 = 3$, $\beta_2 = 7$, $\lambda = 1$ and $c = 1/3$; these are the specifications of example 12.10 of BOWERS *et al.* (1987). The roots of equation (14) are $r_1 = R = 1$ and $r_2 = 6$. Then we obtain from (15) the following coefficients:

$$C_{11} = 9/5 \qquad C_{12} = -3/10.$$
$$C_{21} = 3/5 \qquad C_{22} = 9/10.$$

Thus

$$g(u, y) = \frac{9}{5}\, e^{-3y-u} + \frac{3}{5}\, e^{-7y-u} - \frac{3}{10}\, e^{-3y-6u} + \frac{9}{10}\, e^{-7y-6u}.$$

Integration over y gives

$$\psi(u) = \frac{24}{35}\, e^{-u} + \frac{1}{35}\, e^{-6u},$$

which is the result found by BOWERS *et al.* (1987).

EXAMPLE 2. Let $\lambda = 1, c = 1$ and

$$p(x) = 12e^{-3x} - 12e^{-4x}, \qquad x \geqslant 0.$$

This distribution has non-zero mode $\ln(4/3) = 0.288$, and mean $7/12 = 0.583$. In this example $\beta_1 = 3$, $\beta_2 = 4$, $A_1 = 4$, $A_2 = -3$. The roots of (14) are $r_1 = R = 1$ and $r_2 = 5$. This leads to the following coefficients:

$$C_{11} = 3 \qquad C_{12} = 1$$
$$C_{21} = -3/2 \qquad C_{22} = -3/2.$$

Thus

$$g(u, y) = 3e^{-3y-u} - \frac{3}{2} e^{-4y-u} + e^{-3y-5u} - \frac{3}{2} e^{-4y-5u},$$

which we can use to obtain

$$\psi(u) = \frac{5}{8}\, \mathrm{e}^{-u} - \frac{1}{24}\, e^{-5u}.$$

EXAMPLE 3. Let $\lambda = 1$, $c = 1$, and

$$p(x) = \frac{1}{2}(5e^{-2x} - 12e^{-4x} + 15e^{-6x}), \qquad x \geqslant 0.$$

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Thus $n=3$, $\beta_1=2, \beta_2=4, \beta_3=6$, $A_1=A_3=5/4$, $A_2=-3/2$. From (14) we find $r_1=R=1$ and a pair of conjugate complex roots: $r_2=5+i, r_3=5-i$. From (15) we get

$$C_{11}=\frac{75}{68};\qquad C_{12}=\frac{5}{68}+\frac{20}{68}i;\qquad C_{13}=\frac{5}{68}-\frac{20}{68}i$$

$$C_{21}=-\frac{30}{68};\qquad C_{22}=-\frac{36}{68}-\frac{42}{68}i;\qquad C_{23}=-\frac{36}{68}+\frac{42}{68}i$$

$$C_{31}=\frac{15}{68};\qquad C_{32}=\frac{35}{68}-\frac{30}{68}i;\qquad C_{33}=\frac{35}{68}+\frac{30}{68}i$$

Substituting all these parameters into (16), we get an answer that is quite acceptable, if the calculations are done in complex mode. Alternatively, we remember that

$$e^{iu}=\cos u+i\sin u;\qquad e^{-iu}=\cos u-i\sin u$$

and observe that the coefficients C_{j2} and C_{j3} are conjugate complex. This way we see that

$$C_{j2}e^{-r_2u}+C_{j3}e^{-r_3u}=2e^{-5u}\{\mathrm{Re}(C_{j2})\cos u+\mathrm{Im}(C_{j2})\sin u\}$$

and the answer can be written in the following form:

$$\begin{aligned} g(u,y)=&(75/68)e^{-2y-u}-(30/68)e^{-4y-u}+(15/68)e^{-6y-u}\\ &+e^{-2y-5u}\{(5/34)\cos u+(20/34)\sin u\}\\ &-e^{-4y-4u}\{(36/34)\cos u+(42/34)\sin u\}\\ &+\mathrm{e}^{-6y-5u}\{(35/34)\cos u-(30/34)\sin u\}. \end{aligned}$$

From this and (17) we get

$$\psi(u)=\frac{65}{136}e^{-u}-\mathrm{e}^{-5u}\left\{\frac{1}{51}\cos u+\frac{11}{68}\sin u\right\}.$$

EXAMPLE 4. Suppose we have good estimates of the first three moments of a claim distribution. We want to estimate the distribution of the severity of ruin using a combination of two Gamma $(2,\beta)$ densities, i.e. a distribution with density:

$$p(x)=A_1\beta_1^2e^{-\beta_1x}x+A_2\beta_2^2e^{-\beta_2x}x.$$

To determine the unknown parameters of $p(x)$ by the method of moments, we have to get A_1, A_2, β_1 and β_2 from the following set of equations:

$$A_1+A_2=1,\qquad A_1\frac{2}{\beta_1}+A_2\frac{2}{\beta_2}=E[X],$$

$$A_1\frac{6}{\beta_1^2}+A_2\frac{6}{\beta_2^2}=E[X^2],\qquad A_1\frac{24}{\beta_1^3}+A_2\frac{24}{\beta_2^3}=E[X^3].$$

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Writing $A = A_1$, $b_j = 1/\beta_j$ and $q_j = E[X^j]/(j+1)!$, the equations can be rewritten in a simpler form as:

$$Ab_1 + (1-A)b_2 = q_1, \qquad Ab_1^2 + (1-A)b_2^2 = q_2, \qquad Ab_1^3 + (1-A)b_2^3 = q_3 .$$

The first two equations yield

$$A = \frac{q_1 - b_2}{b_1 - b_2}, \qquad b_1 = \frac{q_2 - q_1 b_2}{q_1 - b_2}.$$

Assuming without loss of generality that $\beta_1 \leqslant \beta_2$, we must have $A \geqslant 0$, otherwise $p(x)$ is negative for large values of x. By the above equations, this implies $q_1 \leqslant 1/\beta_2$. Note that a value $A \in [0, 1]$ is obtained if and only if the ratio $\mathrm{Var}[X]/(E[X])^2$ exceeds the value $\frac{1}{2}$ corresponding to a Gamma $(2, \beta)$ density.

Substituting the above expressions in the third equation, we obtain a quadratic function of b_2, with the following roots:

$$b_2 = \frac{(q_1 q_2 - q_3) \pm \sqrt{[(q_1 q_2 - q_3)^2 - 4(q_2 - q_1^2)(q_1 q_3 - q_2^2)]}}{2(q_1^2 - q_2)}.$$

A similar system of equations must be solved if one wishes to fit a combination of two exponential distributions to three given moments, or to a given mean, mode and variance.

Another necessary condition for $p(x)$ to be non-negative is that either $p(0) > 0$, or $p(0) = 0$ and $p'(0) \geqslant 0$ must hold. By fitting moments, this condition is sometimes violated, as can be seen by taking a distribution with mean and variance 6, and third central moment 36.

To make comparison possible with results previously obtained, assume that the moments of the distribution to be estimated are those of an exp(1) distribution, so mean and variance are 1, and the third central moment equals 2. We obtain the following values for the parameters of $p(x)$:

$$A_1 = A_2 = \tfrac{1}{2}, \qquad \beta_1 = 3 - \sqrt{3} = 1.268, \qquad \beta_2 = 3 + \sqrt{3} = 4.732.$$

The mode of this distribution equals 0.235. Taking $\lambda = 1$, and $c = 2$, we find the following roots for (24):

$r_1 = 0.506$ (This is the adjustment coefficient. For this value of the premium rate c, the adjustment coefficient of an exp(1) distribution equals 0.5.)

$r_2 = 1.765, \qquad r_3 = 3.544, \qquad r_4 = 5.685.$

The coefficients $C'_{jk}(y)$ for use in (26) can be obtained as:

$$\begin{aligned}
C'_{11}(y) &= 0.147y + 0.066, & C'_{21}(y) &= 0.218y + 0.458,\\
C'_{12}(y) &= -0.099y - 0.054, & C'_{22}(y) &= 0.158y - 0.193,\\
C'_{13}(y) &= 0.629y + 0.663, & C'_{23}(y) &= -0.088y - 0.031,\\
C'_{14}(y) &= 0.506y - 0.424, & C'_{24}(y) &= 0.029y + 0.016.
\end{aligned}$$

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The probability of ruin, obtained from (27) and (28) is:

$$\psi(u) = 0.517e^{-0.506u} - 0.070e^{-1.765u} + 0.089e^{-3.544u} - 0.036e^{-5.685u}.$$

The probability of ruin corresponding to the exp(1) distribution and this value of c equals

$$\psi(u) = 0.5e^{-0.5u}.$$

The maximum deviation of the ruin probability obtained with the approximating combination of Gammas and the exponential(1) ruin probability is 0.004.

6. A DIRECT METHOD

Equations (5) and (6) are defective renewal equations and can be solved (at least in principle) without the use of transforms. With the notation

$$h(x) = \frac{\lambda}{c}\,[1 - P(x)] \tag{29}$$

we can write equation (5) as

$$G(u, y) = \int_0^u G(u-x, y)\, h(x)\, dx + \int_u^{u+y} h(x)\, \mathrm{d}x. \tag{5'}$$

By successive substitution we obtain first the following formal solution:

$$G(u, y) = \int_0^u \sum_{n=0}^{\infty} h^{*n}(x) \int_{u-x}^{u-x+y} h(z)\, dz\, dx. \tag{30}$$

A rigorous proof follows from the following interpretation (combined with the law of total probability):

$$h^{*n}(x) \int_{u-x}^{u-x+y} h(z)\, dz\, dx$$

is the probability of the event that the nth record low of the surplus process is between $u - x$ and $u - x + dx$ and that ruin occurs with the following record low, such that the deficit is less than y; see theorem 12.2 of BOWERS *et al.* (1987).

Expression (30) shows that $G(u, y)$ has indeed a density $g(u, y)$. Taking derivatives we obtain

$$g(u, y) = \int_0^u \sum_{n=0}^{\infty} h^{*n}(x) h(u - x + y)\, dx. \tag{31}$$

In the following section we shall illustrate the application of (31) in a particular case.

REMARK. If we set $y = \infty$ in (30), we obtain a well-known representation for the probability of ruin (the so-called "convolution formula").

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7. UNIT CLAIM AMOUNTS

Suppose that all claims are of size one. Thus, by (29),

$$h(x) = \lambda/c \qquad \text{if } 0 < x < 1; \qquad 0 \text{ otherwise.}$$

We can write this defective probability density as

$$h(x) = \alpha f(x),$$

where $\alpha = \lambda/c$ and $f(x)$ is the uniform $(0, 1)$ density. There is an explicit expression for the n-fold convolution of f:

$$(32) \qquad f^{*n}(x) = \frac{1}{(n-1)!} \sum_{j=0}^{n} \binom{n}{j} (-1)^j (x-j)_+^{n-1}.$$

This formula can be found in FELLER (1966, theorem 1, I.9), and a very elegant derivation is given by SHIU (1985). We prefer to write it as

$$f^{*n}(x) = \frac{d}{dx} \sum_{j=0}^{n} \frac{1}{(n-j)!\,j!} (-1)^j (x-j)_+^n.$$

Then

$$\sum_{n=0}^{\infty} h^{*n}(x) = \sum_{n=0}^{\infty} \alpha^n f^{*n}(x)$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{(n-j)!\,j!} (-1)^j \alpha^n (x-j)_+^n.$$

Interchanging the order of summation, we obtain more simply:

$$(33) \qquad \sum_{n=0}^{\infty} h^{*n}(x) = \frac{d}{dx} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} (x-j)_+^j \sum_{n=j}^{\infty} \frac{\alpha^{n-j}}{(n-j)!} (x-j)_+^{n-j}$$

$$= \frac{d}{dx} \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} (x-j)_+^j e^{\alpha(x-j)_+}.$$

Note that this is in fact a finite sum, as terms with $j > x$ vanish. If we substitute this expression in (31), the integration can be limited from $x = (u+y-1)_+$ to $x = u$, where $h(u-x+y) = \alpha$. The resulting integral is trivial; for $u \geqslant 0$ and $0 < y < 1$, we obtain

$$(34) \qquad g(u, y) = \alpha \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} (u-j)_+^j e^{\alpha(u-j)_+}$$

$$- \alpha \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} (u+y-1-j)_+^j e^{\alpha(u+y-1-j)_+}.$$

APPENDIX: COMPUTER IMPLEMENTATION

Implementation of the algorithm suggested in Section 3 on a computer involves mainly elementary operations on polynomials. To solve (14), however, we must

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have a routine to compute all roots, real as well as complex, of a real polynomial. Any textbook on numerical mathematics contains material on this; see for instance STOER (1972). Also, any library of numerical routines such as the NAG or the IMSL library provides adequate software. One may also consult the ACM algorithms. Note that only for $n \geqslant 5$ may the need to iteratively compute complex roots arise. One of the roots is the adjustment coefficient, so at least one of the roots is real. In case all coefficients A_i are positive, one may show that all roots of (14) are real and non-negative. In this case, simpler algorithms will suffice, for instance the Newton–Maehly algorithm described in STOER (1972, pp. 220–221).

An algorithm to compute complex roots of real polynomials that can be programmed easily, even using an electronic spreadsheet, is the method of Bairstow. For a motivation of the method, see STOER (1972, pp. 226–227). Its main advantage is that no complex arithmetic is involved. A disadvantage is that convergence cannot be guaranteed. PRESS *et al.* (1986) recommend a two-step procedure: first find approximations to all roots and then "polish" the roots found using Bairtow's method.

This method works as follows. First write (14) as the following polynomial equation:

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0. \tag{A1}$$

Next determine a quadratic divisor $r^2 + pr + q$, where $p^2 - 4q < 0$, as follows. Choose a starting point (q, p) and calculate the vector $(B_0, B_1, \ldots, B_n)$ by means of the following recursive scheme:

$$\begin{aligned} B_0 &= a_0, \\ B_1 &= a_1 - pB_0, \\ B_2 &= a_2 - pB_1 - qB_0, \\ &\vdots \\ B_{n-1} &= a_{n-1} - pB_{n-2} - qB_{n-3}, \\ B_n &= a_n - pB_{n-1} - qB_{n-2}. \end{aligned} \tag{A2}$$

Similarly, compute the vector $(C_0, C_1, \ldots, C_{n-1})$ as follows:

$$\begin{aligned} C_0 &= B_0, \\ C_1 &= B_1 - pC_0, \\ C_2 &= B_2 - pC_1 - qC_0, \\ &\vdots \\ C_{n-2} &= B_{n-2} - pC_{n-3} - qC_{n-4}, \\ C_{n-1} &= \qquad - pC_{n-2} - qC_{n-3}. \end{aligned} \tag{A3}$$

With the auxiliary quantities

$$\begin{aligned} D &= C_{n-2}^2 - C_{n-1}C_{n-3}, \\ P &= B_{n-1}C_{n-2} - B_nC_{n-3}, \\ Q &= B_nC_{n-2} - B_{n-1}C_{n-1}, \end{aligned} \tag{A4}$$

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we find the next approximation (q, p) as:

(A5) $$p := p + P/D, \qquad q := q + Q/D.$$

Now restart the algorithm with these values of q and p until the old and new values of q and p differ by less than the prescribed precision. A divisor $r^2 + pr + q$ of the left-hand side of (14) gives two complex conjugate roots. Having divided out this factor, run the algorithm again to determine the other roots.

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