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EFFICIENCY IN LARGE DYNAMIC PANEL MODELS WITH COMMON FACTORS

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This paper deals with asymptotically efficient estimation in exchangeable nonlinear dynamic panel models with common unobservable factors. These models are relevant for applications to large portfolios of credits, corporate bonds, or life insurance contracts. For instance, the Asymptotic Risk Factor (ARF) model is recommended in the current regulation in Finance (Basel II and Basel III) and Insurance (Solvency II) for risk prediction and computation of the required capital. The specification accounts for both micro- and macrodynamics, induced by the lagged individual observations and the common stochastic factors, respectively. For large cross-sectional and time dimensions n and T, we derive the efficiency bound and introduce computationally simple efficient estimators for both the micro- and macroparameters. The results are based on an asymptotic expansion of the log-likelihood function in powers of $1/n$, and are linked to granularity theory. The results are illustrated with the stochastic migration model for credit risk analysis.

1. INTRODUCTION

This paper considers the asymptotically efficient estimation of nonlinear dynamic panel models with common unobservable factors. We focus on exchangeable specifications that are appropriate to analyze the set of histories of a large homogeneous population of individuals featuring serial and cross-sectional dependence. Such a framework is often encountered in credit risk applications. For instance, for the risk analysis in portfolios of corporate debt, the panel data

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are the default, loss given default, and rating migration histories of a large pool of firms in a given industrial sector and country. The common factors represent latent macrovariables, such as the sector and country specific business cycle, that introduce dependence across the nonlinear individual risks, such as default, loss given default, or migration correlations. The purpose of the analysis is to predict the future risk in a large portfolio of corporate bonds or credit derivatives issued by the firms in the pool. The panel data may also correspond to other risk characteristics in a pool of corporate loans, household mortgages, or life insurance contracts, such as prepayment, lapse, or mortality.

The model considered in this paper involves both micro- and macrodynamics. Conditional on a given factor path, the individuals are assumed independent and identically distributed (i.i.d.), with the histories of observations $y_{i,t}$, t varying, following the same time-inhomogeneous Markov process for any individual i. The transition density $h(y_{i,t}|y_{i,t-1}, f_t; \beta)$ between dates $t-1$ and t depends on the (multivariate) factor value f_t and the unknown parameter β. The microdynamics is captured by the lagged individual observation $y_{i,t-1}$ and unknown parameter β. The macrodynamics is driven by the time-varying stochastic common factor f_t. The latter follows a Markov process with transition density $g(f_t|f_{t-1}; \theta)$, which depends on the unknown parameter θ. In credit risk applications, the common factor f_t has to be considered unobservable in order to account for systematic risk. When this common factor is integrated out, it introduces both non-Markovian serial dependence within the individual histories, and cross-sectional dependence between individuals. The variables $y_{i,t}$ are either real-valued or discrete (as for default and rating histories in the credit risk application), while the components of the vector f_t are real valued (corresponding to a continuum of latent states). The model is potentially nonlinear in both micro- and macrodynamics.

When the cross-sectional dimension n is fixed and the time dimension T tends to infinity, the Maximum Likelihood (ML) estimators of microparameter β and macroparameter θ are asymptotically normal and efficient.[1] However, this asymptotic scheme is not appropriate for a setting involving very large n and moderately large T, as in credit risk applications. For instance, for corporate rating data the number of firms is typically of order $n \simeq 10,000$, while the number of dates is about $T \simeq 20$ with yearly data. In applications to mortgage or life insurance, we typically have $n \simeq 100,000 - 1,000,000$ contracts and $T \simeq 200$ months. Moreover, the numerical computation of the ML estimate is complicated, since the likelihood function involves a large dimensional integral with respect to (w.r.t.) the unobservable factor path.

The aim of this paper is to derive the asymptotic efficiency bound for estimating both the microparameter β and the macroparameter θ and to introduce asymptotically efficient estimators of β and θ that are easier to compute than the ML estimator. We consider the double asymptotics $n, T \to \infty$, such that $T^\nu/n = O(1)$, with either $\nu > 1$, for estimators maximizing a first-order expansion of the log-likelihood function w.r.t. $1/n$, or $\nu > 3/2$, for estimators maximizing a more accurate second-order expansion. We summarize our theoretical contributions as follows. First, we show that the efficiency bound for the microparameter β does not depend on the parametric model defining the macrodynamics. In particular, this bound coincides with the parametric efficiency bound with known transition of the factor, and also with the semiparametric efficiency bound when the transition of the factor is left unspecified. Second, the efficiency bound for the macroparameter θ is the same as if the factor values were observable. These findings correspond to oracle properties w.r.t. the factor dynamics for the microparameter, and w.r.t. the factor values for the macroparameter. Third, the asymptotic efficiency bound can be reached by optimizing approximated likelihood functions which do not involve integrals w.r.t. the factor path.

In Section 2 we introduce the nonlinear dynamic panel model with common factors. To provide motivation and grounding on potential applications, we first describe the Asymptotic Single Risk Factor (ASRF) model, which is the simplest benchmark model suggested for the regulation of credit risk in Basel II [Basel Committee on Banking Supervision (BCBS, 2001, 2003)]. Then, we present the general specification and discuss the stationarity and ergodicity assumptions needed for the asymptotic analysis. Our theoretical results are mainly based on a second-order asymptotic expansion of the log-likelihood function in powers of $1/n$ given in Section 3. The basic idea behind this expansion is that the integration of the latent factor path is performed along the lines of the Laplace approximation. In Section 4 we introduce estimators of both micro- and macroparameters that do not involve numerical integration w.r.t. the unobservable factor. These estimators are obtained by maximizing approximations of the log-likelihood function at order $1/n$, and $1/n^2$, respectively. They are called Cross-Sectional Asymptotic (CSA) and Granularity Adjusted (GA) maximum likelihood estimators, respectively. We study the asymptotic properties of these estimators under suitable identification conditions and prove their asymptotic efficiency. In Section 5 we introduce an asymptotically efficient estimation approach, in which the estimators of the micro- and macroparameters can be computed in two steps. The estimator of the microcomponent is a fixed effects estimator, which considers the factor values as nuisance parameters. The estimator of the macroparameter is obtained by maximizing the likelihood function of the macrodynamics, in which the unobservable factor values are replaced by suitable cross-sectional factor approximations. In Section 6, the results of the paper are applied to the stochastic migration model used for credit risk analysis. In this model, the observable endogenous variable corresponds to the rating and the common stochastic factors account for migration correlation. The patterns of the efficiency bound and the computation of the efficient estimators are illustrated for this example. We also investigate the finite-sample properties of the estimators in a Monte-Carlo experiment. Section 7 concludes. Appendix A.1 provides the regularity conditions for the large sample properties of the estimators. The proofs of the results are gathered in Appendices A.2 and A.3. The proofs rely on some Limit Theorems for uniform stochastic convergence with panel data and technical Lemmas. The details of these Theorems and Lemmas are provided online at Cambridge

Journals Online in supplementary material to this article. Readers may refer to the supplementary material associated with this article, available at Cambridge Journals Online (journals.cambridge.org/ect).

2. EXCHANGEABLE NONLINEAR PANEL MODEL WITH COMMON FACTORS

Exchangeable nonlinear panel models with common factors are the basis for risk analysis of homogeneous retail portfolios encountered in Finance and Insurance. Before describing the general specification, we review as an illustration the Asymptotic Single Risk Factor (ASRF) model introduced for default risk analysis by Vasicek (1987, 1991).

2.1. The Asymptotic (Single) Risk Factor (ASRF) model for default

The general specification considered in Section 2.2 is motivated by the ASRF model introduced by Vasicek (1987, 1991) and based on the Value of the Firm model (Merton, 1974). This model, possibly extended to include more factors, is recommended for the analysis of credit risk in Pillar 1 of Basel II regulation, concerning the minimum required capital, and in Pillar 2, concerning internal risk models (BCBS, 2001, 2003). The objective is to analyze the risk of a portfolio of loans or credit derivatives, included in the balance sheet of a bank or credit institution. These portfolios may contain several millions of individual contracts (assets) and have to be segmented into subportfolios, which are homogeneous by the type of contract (asset) and by the type of borrowers, including at least their ratings among their characteristics. The ASRF model is applied to these homogeneous subportfolios separately (or jointly), with parameters and factors which can depend on the segment. The sizes of these subportfolios may still be rather large including some ten thousands of individual loans for mortgages and credit cards, for instance.

The basic Vasicek model is written for firms, but the same approach is applicable to household borrowers. Let us consider a given subpopulation and a single-factor model. This model introduces the asset $A_{i,t}$ and liability $L_{i,t}$ as latent variables. Then, the latent model is written on the log-ratio of asset to liability $y^*_{i,t} = \log(A_{i,t}/L_{i,t})$ as:

$$y^*_{i,t} = \alpha + \gamma F_t + \sigma u_{i,t}, \qquad i \in PaR_t, \quad t = 1, \ldots, T,$$

where PaR_t denotes the Population-at-Risk, that is the set of firms in the portfolio which are still alive at time t, and where the common factor (F_t) and the errors $(u_{i,t})$ are independent standard Gaussian white noise processes. This specification distinguishes the idiosyncratic risks $u_{i,t}$, which can be diversified, and the undiversifiable systematic risk F_t. The latter component is introduced to represent risk dependence. It is especially important for financial stability analysis.

Indeed, the standard stress testing methodology corresponds to assessing the impact of extreme shocks on some components of the systematic risk factor. The coefficients α, γ, σ are independent of the individuals, according to the definition of a homogeneous portfolio. The parameters and factors depend on the segment, but the index of the segment is omitted for expository purpose.[2] The observed endogenous variable is the indicator for the default event that occurs when the asset is below liability:

$$y_{i,t} = \mathbb{1}_{A_{i,t}<L_{i,t}} = \mathbb{1}_{y^*_{i,t}<0}.$$

We deduce the Probability of Default (PD) at date t conditional on the common factor:

$$PD_t = \mathbb{P}\left[y_{i,t} = 1|y_{i,t-1} = 0, F_t\right] = \Phi\left[-(\alpha/\sigma) - (\gamma/\sigma)F_t\right], \tag{2.1}$$

where Φ denotes the cumulative distribution function (c.d.f.) of the standard normal distribution. Thus, the conditional probability of default is time-varying and driven by the common stochastic factor F_t. To summarize, the qualitative observations $y_{i,t}$ are independent conditional on the factor path with Bernoulli distribution:

$$y_{i,t}|F_t \sim \mathcal{B}(1, PD_t). \tag{2.2}$$

We get a probit model in which the explanatory variable F_t is unobservable and captures the systematic default risk. This basic static model can be extended by allowing for several factors in the given subpopulation, for dynamics of the common factors (e.g., Duffie and Singleton, 1998; Loeffler, 2003; Dembo, Deuschel, and Duffie, 2004; McNeil and Wendin, 2007; Duffie, Eckner, Horel, and Saita, 2009), and for a joint analysis of more than two rating levels by means of stochastic migration models describing the transitions between rating classes AAA, AA, ..., C, D, say (see Section 6 and references therein).

The unconditional probability of default is $PD = \mathbb{P}[y_{i,t} = 1] = \Phi\left(-\alpha/\sqrt{\gamma^2+\sigma^2}\right)$, whereas the unconditional default correlation between any two firms i and j is:

$$\rho = Corr\left(y_{i,t}, y_{j,t}\right) = \frac{\Psi\left(-\alpha/\sqrt{\gamma^2+\sigma^2}, -\alpha/\sqrt{\gamma^2+\sigma^2}; \rho^*\right) - PD^2}{PD(1-PD)}, \tag{2.3}$$

where $\rho^* = \gamma^2/\left(\gamma^2+\sigma^2\right)$ is the asset correlation, that is the correlation between the log asset/liability ratios of any two firms, and $\Psi(., .; \rho^*)$ denotes the joint cdf of the bivariate standard Gaussian distribution with correlation coefficient ρ^*. In the new regulation for credit risk, the required capital depends on the values of PD and ρ^*, that is, indirectly on the values α/σ and γ/σ, and is especially sensitive to the asset correlation parameter ρ^*. In standard implementations of the above risk factor model, the unknown parameters PD and ρ^* are replaced by their empirical counterparts, which are close to the true values when the subpopulation sizes are large. This explains the term "asymptotic" appearing in the usual methodology. However, it is important to check if not only consistency, but also efficiency can be attained by computationally simple estimators of the structural parameters.[3]

The above ASRF model assumes that the individual fixed effects depend on the segment only, that is, the individual fixed effects α_i, γ_i, σ_i, say, are identical for two individuals in the same segment. This model assumption is compatible with the two-step approach considered in credit risk applications. First, models with individual fixed effects are used to get the homogeneous subportfolios; then the ASRF model is written for each homogeneous subportfolio to derive the distribution of the future portfolio value and the corresponding 1% quantile, called CreditVaR. Such a two-step procedure has been preferred in the current regulation for at least the following reasons: first, in the standard regulation approach that applies for the banks with the least advanced risk management systems, a common segmentation can be proposed by the regulator itself. Thus, the risk analysis is performed by the banks with the same segmentation, which facilitates the aggregation of bank portfolios when analyzing the global risk of the system. Second, and more importantly, the introduction of several millions of individual fixed effects beyond segment effects would diminish the estimated magnitude of idiosyncratic risks. In a regulatory perspective, this would yield a significantly lower level of required capital. Indeed, the reserves for credit risk are typically computed with unknown parameters directly replaced by their estimates.[4] Finally, models without individual fixed effects are common in the credit risk literature on bankruptcy prediction (e.g., Shumway, 2001; Chava and Jarrow, 2004; Campbell, Hilscher, and Szilagyi, 2008), where individual heterogeneity is accounted for by observable characteristics. Duffie et al. (2009) estimate their model on US corporate default data and find that the inclusion of individual fixed effects does not lead to a significant improvement of the results.

2.2. The general specification

The basic ASRF model can be extended to include any number of factors and any type of parametric nonlinear dynamics. This extended model is introduced in this section. Let us consider panel data $y_{i,t}$ for a large homogeneous population of individuals $i = 1, \ldots, n$ observed at dates $t = 1, \ldots, T$. We assume that there exists a common (multidimensional) factor such that[5]:

A.1: *Conditional on the factor path* (f_t)*, the individual histories* $\left(y_{i,t}, t = 1, 2, \ldots\right)$*, for* i *varying, are i.i.d. time-inhomogeneous Markov processes of order 1, with transition probability density function (p.d.f.)* $h\left(y_{i,t} | y_{i,t-1}, f_t; \beta\right)$ *and unknown parameter* $\beta \in \mathcal{B}$*, where* $\mathcal{B} \subset \mathbb{R}^q$.

A.2: *The factor (f_t) is an exogenous Markov process of order 1 in $\mathbb{R}^m$, that is, the conditional distribution of f_t given the past of the factor $\underline{f_{t-1}} = (f_{t-1}, f_{t-2}, \ldots)$ and of the individual histories $\underline{y_{i,t-1}} = (y_{i,t-1}, y_{i,t-2}, \ldots)$, $i = 1, \ldots, n$, depends on f_{t-1} only, with transition p.d.f. $g(f_t | f_{t-1}; \theta)$ and unknown parameter $\theta \in \Theta$, where $\Theta \subset \mathbb{R}^p$.*

We denote by β_0 and θ_0 the true values of parameters β and θ, respectively. Factor f_t is assumed unobservable.[6] Thus, it has to be integrated out to derive the joint density of observations $y_{i,t}$. The latent factor introduces both non-Markovian individual dynamics and dependence across individuals. The exogeneity assumption means that: (i) there is no feedback from one specific individual history on the future factor values and (ii) the lagged factor value includes all informative macrosummaries of the past. The distribution of the individual histories $(y_{i,t})$ is exchangeable, i.e., invariant by permutation of the individuals. The exchangeability property is equivalent to the existence of a factor representation (de Finetti, 1931; Hewitt and Savage, 1955).[7] Such exchangeability assumptions have been introduced in the literature on linear dynamics (see, e.g., Andrews, 2005; Hjellwig and Tjostheim, 1999). The focus of our paper is on the efficient estimation of both microparameter β and macroparameter θ in the nonlinear exchangeable panel model A.1 and A.2.

Without Assumption A.2 on the parametric factor dynamics, the model introduced in Assumption A.1 might be seen as a model with time fixed effects instead of individual fixed effects. Thus, we might expect to derive the asymptotic results from the nonlinear panel literature with individual fixed effects by simply interchanging the roles of individual and time indices i and t, and the sizes n and T. For instance, Hahn and Newey (2004) consider estimation of nonlinear panel models with fixed individual effects, but their results cannot be applied to our framework, because they assume independence across time as well as in the cross-section. Moreover, this intuition is not correct, since there are important differences between our setting and the ones considered by the individual fixed effects panel literature:

(i) In applications to credit risk the size n of the segment is much larger than the number T of dates, and, therefore, the incidental parameter problem (for the pioneering paper, see Neyman and Scott, 1948; for a review, see Lancaster, 2000) is much less pronounced with time fixed effects than with individual fixed effects. In particular, bias corrections in the first-order asymptotic distributions are not required in our setting, since we assume $T/n \to 0$.

(ii) Assumption A.2 shows that the nonlinear panel model with common factor is a time series model introduced for prediction purpose. This fact is illustrated in Section 2.1 on default risk analysis, in which the final aim is the computation of reserves by means of a quantile of the conditional distribution of the future portfolio value, that is, the CreditVaR. Therefore, we are interested not only in the microparameter β, but also in the macroparameter θ.

(iii) The parametric Assumption A.2 on the factor dynamics provides additional information, which might allow for a more efficient estimation of the microparameter β.

To establish the large sample properties of the estimators, we introduce the next Assumptions A.3, A.4, and A.5. Assumptions A.3 and A.4 concern the stationarity and mixing properties of the factor process, and of the individual histories conditional on the factor process, respectively.

A.3: *The process* (f_t) *is strictly stationary and geometrically strong mixing, that is,* $\alpha(s) = O(\rho^s)$ *as* $s \to \infty$, *for some* $\rho \in (0,1)$, *where* $\alpha(s) = \sup_{A \in \mathcal{H}_{-\infty}^t, B \in \mathcal{H}_{t+s}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|$ *denotes the alpha mixing coefficient at lag* $s \in \mathbb{N}$, *and* $\mathcal{H}_{-\infty}^t = \sigma(f_t, f_{t-1}, \ldots)$ *and* $\mathcal{H}_{t+s}^\infty = \sigma(f_{t+s}, f_{t+s+1}, \ldots)$ *denote the sigma-fields generated by process* (f_t) *up to time* t, *and from time* $t+s$ *onward, respectively.*

A.4: *Conditional on the factor path* (f_t), *the individual process* $\left(y_{i,t}\right)$ *is beta mixing, such that the conditional beta mixing coefficients:*

$$\beta_t(s) \equiv \sup_{A \in \mathscr{B}(\mathbb{R})} \int \left| \mathbb{P}\left[y_{i,t} \in A | y_{i,t-s} = \eta, f_t, f_{t-1}, \ldots, f_{t-s+1}\right] \right.$$
$$\left. - \mathbb{P}\left[y_{i,t} \in A | \underline{f_t}\right] \right| \lambda(\eta) d\eta, \quad s \in \mathbb{N},$$

are measurable functions of $\underline{f_t}$ *and satisfy* $\beta_t(s) \to 0$ *as* $s \to \infty$, *for any* t *and* $\mathbb{P}$*-a.s., where* $\mathscr{B}(\mathbb{R})$ *denotes the Borel sigma-field on* $\mathbb{R}$, λ *is a strictly positive p.d.f. on* $\mathbb{R}$, *and* $\underline{f_t} = (f_t, f_{t-1}, \cdots)$.

Assumption A.4 requires that the Markov transition distribution of $y_{i,t}$ conditional on $y_{i,t-s}$ and the factor path converges to the long-run conditional distribution of $y_{i,t}$, denoted $\mathbb{P}[\cdot|\underline{f_t}]$, as the lag s tends to ∞. The conditional long-run distribution $\mathbb{P}[\cdot|\underline{f_t}]$ at date t, and the conditional beta mixing coefficients $\beta_t(s)$ at date t, depend on the factor path $\underline{f_t}$, and thus are stochastic. The beta mixing coefficients $\beta_t(s)$ are assumed to converge to zero as lag s increases, for any factor path, implying the irrelevance of the initial values of the $y_{i,t}$'s in the long-run conditional on the factor path. The convergence rate can be geometric, for instance. The integration w.r.t. the factor path is expected to decrease the decay rate of the mixing coefficients (Granger and Joyeux, 1980). However, by the Lebesgue Theorem, under Assumption A.4 the integrated mixing coefficients $E_0[\beta_t(s)]$ are such that $E_0[\beta_t(s)] \to 0$ as $s \to \infty$. The decay of the integrated mixing coefficients implies that the initial values of the $y_{i,t}$'s have no effect in the long run even after integrating out the factors. As usual, it is convenient for expository purpose to disregard the short run effect of the initial observations by introducing a suitable assumption on their distribution.

A.5: *The initial observations $y_{i,0}$, with $i = 1,\dots,n$, are i.i.d. conditional on the factor path (f_t), with distribution corresponding to the long-run distribution $\mathbb{P}[\cdot|\underline{f_0}]$ at time $t = 0$.*

Assumption A.5 implies that at each date t the distribution of $y_{i,t}$ conditional on the factor path is the long-run distribution $\mathbb{P}[\cdot|\underline{f_t}]$. We get a time homogeneity property conditional on the factor path, as the conditional distribution of $y_{i,t}$ depends on date t by means of the factor path $\underline{f_t}$ only.

3. THE LIKELIHOOD EXPANSION

The joint density of $\underline{y_T} = \left(y_{i,t}, t = 1,\dots,T, i = 1,\dots,n\right)$ and $\underline{f_T} = (f_t, t = 1,\dots,T)$ (conditional on the initial values) is given by:

$$l\left(\underline{y_T},\underline{f_T};\beta,\theta\right) = \prod_{i=1}^{n}\prod_{t=1}^{T} h\left(y_{i,t}|y_{i,t-1}, f_t;\beta\right)\prod_{t=1}^{T} g(f_t|f_{t-1};\theta) \qquad \textbf{(3.1)}$$
$$= l_{micro}\left(\underline{y_T}|\underline{f_T};\beta\right) l_{macro}\left(\underline{f_T};\theta\right), \text{ (say).}$$

If the factors were observable, the terms $l_{micro}\left(\underline{y_T}|\underline{f_T};\beta\right)$ and $l_{macro}\left(\underline{f_T};\theta\right)$ would correspond to the conditional microdensity of the endogeneous variables, and the macrodensity of the factors, respectively. Since the factors are unobservable, the density of observations $\underline{y_T}$ is obtained by integrating out the factor path $\underline{f_T}$:

$$l\left(\underline{y_T};\beta,\theta\right) = \int\cdots\int \prod_{t=1}^{T}\prod_{i=1}^{n} h\left(y_{i,t}|y_{i,t-1}, f_t;\beta\right)\prod_{t=1}^{T} g(f_t|f_{t-1};\theta)\prod_{t=1}^{T} df_t$$
$$= \int\cdots\int \exp\left\{n\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i=1}^{n}\log h\left(y_{i,t}|y_{i,t-1}, f_t;\beta\right)\right)\right\}$$
$$\times \prod_{t=1}^{T} g(f_t|f_{t-1};\theta)\prod_{t=1}^{T} df_t. \qquad \textbf{(3.2)}$$

This likelihood function involves an integral with a large dimension increasing with T, which complicates the analytical study of the Maximum Likelihood (ML) estimators and the numerical computation of the ML estimates.[8] However, for large n, this integral can be approximated along the lines of the Laplace approximation (Laplace, 1774). Laplace approximations can be found in the econometric literature as early as Holly and Phillips (1979) and Phillips (1983) for the derivation of the marginal distribution of instrumental variable estimators. Tierney and Kadane (1986) used this device to derive the posterior distribution in Bayesian statistics. More recently, the Laplace approximation has been used in Arellano and Bonhomme (2009) to derive the bias of the integrated likelihood in nonlinear panel models with individual fixed effects. Huber, Scaillet, and Victoria-Feser (2009) use the Laplace approximation to develop a tractable estimator for a multivariate logit model in a latent factor framework in finance. In our setting with serially dependent factors, the Laplace approximation is applied to an integral w.r.t. the full path of time effects. Specifically, we start by defining for any parameter value $\beta \in \mathcal{B}$ and date $t = 1, \dots, T$ the cross-sectional ML estimator of the factor value:

$$\hat{f}_{n,t}(\beta) = \arg\max_{f_t \in \mathcal{F}_n} \sum_{i=1}^{n} \log h\left(y_{i,t} | y_{i,t-1}, f_t; \beta\right), \tag{3.3}$$

where the compact set $\mathcal{F}_n \subset \mathbb{R}^m$ grows when $n \to \infty$ as described by Assumption H.6 in Appendix A.1. Then, by a Taylor expansion of the integrand in the RHS of equation (3.2) around $\left(\hat{f}_{n,1}(\beta)', \dots, \hat{f}_{n,T}(\beta)'\right)'$ that is the maximizer of $\sum_{t=1}^{T}\sum_{i=1}^{n} \log h(y_{i,t}|y_{i,t-1}, f_t; \beta)$ w.r.t. the factor path, we get:

$$\begin{aligned} l\left(\underline{y_T}; \beta, \theta\right) = & \prod_{t=1}^{T}\prod_{i=1}^{n} h\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta); \beta\right) \prod_{t=1}^{T} g\left(\hat{f}_{n,t}(\beta) | \hat{f}_{n,t-1}(\beta); \theta\right) \\ & \times \int \cdots \int \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \sqrt{n}\left(f_t - \hat{f}_{n,t}(\beta)\right)' I_{n,t}(\beta) \sqrt{n}\left(f_t - \hat{f}_{n,t}(\beta)\right)\right\} \\ & \times \exp\left\{\sum_{t=1}^{T} \psi_{n,t}(f_t, f_{t-1}; \beta, \theta)\right\} \prod_{t=1}^{T} df_t, \end{aligned}$$

where

$$I_{n,t}(\beta) = -\frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 \log h}{\partial f_t \partial f_t'}\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta); \beta\right), \tag{3.4}$$

and the term $\psi_{n,t}$ is defined in (A.3) in Appendix A.2.1. By introducing the change of variables $z_t = \sqrt{n}\left[I_{n,t}(\beta)\right]^{1/2}\left(f_t - \hat{f}_{n,t}(\beta)\right) \Longleftrightarrow f_t = \hat{f}_{n,t}(\beta) + \frac{1}{\sqrt{n}}\left[I_{n,t}(\beta)\right]^{-1/2} z_t$, for $t = 1, \dots, T$, and expanding function $\exp\left(\sum_t \psi_{n,t}\right)$ in a power series of the $n^{-1/2} z_t$, the multivariate integral in the expression of the likelihood can be written as a linear combination of power moments of the standard Gaussian distribution, with coefficients depending on the observations. The next proposition gives the expansion for the (nT-standardized) log-likelihood function of the sample:

$$\mathcal{L}_{nT}(\beta, \theta) = \frac{1}{nT} \log l\left(\underline{y_T}; \beta, \theta\right), \tag{3.5}$$

as a power series of $1/n$, and controls the stochastic order of the remainder term.

PROPOSITION 1. *Let Assumptions A.1–A.5 and H.1–H.12 in Appendix A.1 be satisfied.*

(i) *If $n, T \to \infty$ such that $T^{\nu}/n = O(1)$, for a value $\nu > 1$, we have:*

$$\mathcal{L}_{nT}(\beta,\theta) = \mathcal{L}^*_{nT}(\beta) + \frac{1}{n}\mathcal{L}_{1,nT}(\beta,\theta) + \Psi_{nT}(\beta,\theta), \tag{3.6}$$

where

$$\mathcal{L}^*_{nT}(\beta) = \frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n}\log h\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta);\beta\right), \tag{3.7}$$

$$\begin{aligned}\mathcal{L}_{1,nT}(\beta,\theta) = &-\frac{1}{2}\frac{1}{T}\sum_{t=1}^{T}\log\det I_{n,t}(\beta)\\ &+\frac{1}{T}\sum_{t=1}^{T}\log g\left(\hat{f}_{n,t}(\beta)\,|\,\hat{f}_{n,t-1}(\beta);\theta\right),\end{aligned} \tag{3.8}$$

with $I_{n,t}(\beta)$ defined as in (3.4), and the term $\Psi_{nT}(\beta,\theta)$ is such that $\sup_{\beta\in\mathcal{B},\theta\in\Theta} |\Psi_{nT}(\beta,\theta)| = o_p(1/n)$ w.r.t. the true distribution.

(ii) *If $n, T \to \infty$ such that $T^{\nu}/n = O(1)$, for a value $\nu > 3/2$, we have:*

$$\mathcal{L}_{nT}(\beta,\theta) = \mathcal{L}^*_{nT}(\beta) + \frac{1}{n}\mathcal{L}_{1,nT}(\beta,\theta) + \frac{1}{n^2}\mathcal{L}_{2,nT}(\beta,\theta) + \tilde{\Psi}_{nT}(\beta,\theta), \tag{3.9}$$

where the term $\tilde{\Psi}_{nT}(\beta,\theta)$ is such that $\sup_{\beta\in\mathcal{B},\theta\in\Theta} |\tilde{\Psi}_{nT}(\beta,\theta)| = o_p\left(1/n^2\right)$. When the factor is one-dimensional, i.e., $m = 1$, the expression of term $\mathcal{L}_{2,nT}(\beta,\theta)$ is given by:

$$\begin{aligned}\mathcal{L}_{2,nT}(\beta,\theta) = &\frac{1}{8}\frac{1}{T}\sum_{t=1}^{T} J_{4,n,t}(\beta) + \frac{1}{2}\frac{1}{T}\sum_{t=1}^{T} D_{20,nt}(\beta,\theta)\\ &+\frac{1}{2}\frac{1}{T}\sum_{t=2}^{T} D_{02,nt}(\beta,\theta) + \frac{5}{24}\frac{1}{T}\sum_{t=1}^{T}\left[J_{3,nt}(\beta)\right]^2\\ &+\frac{1}{2}\frac{1}{T}\sum_{t=1}^{T}\left[D_{10,nt}(\beta,\theta)\right]^2 + \frac{1}{2}\frac{1}{T}\sum_{t=2}^{T}\left[D_{01,nt}(\beta,\theta)\right]^2\\ &+\frac{1}{2}\frac{1}{T}\sum_{t=1}^{T} J_{3,n,t}(\beta) D_{10,nt}(\beta,\theta)\\ &+\frac{1}{2}\frac{1}{T}\sum_{t=2}^{T} J_{3,n,t-1}(\beta) D_{01,nt}(\beta,\theta)\\ &+\frac{1}{T}\sum_{t=2}^{T} D_{10,n,t-1}(\beta,\theta) D_{01,nt}(\beta,\theta),\end{aligned} \tag{3.10}$$

with $J_{p,nt}(\beta) = \dfrac{1}{n}\sum_{i=1}^{n} \dfrac{\partial^p \log h}{\partial f_t^p}\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta);\beta\right)[I_{n,t}(\beta)]^{-p/2}$, *for* $p=3,4$, *and* $D_{pq,nt}(\beta,\theta) = \dfrac{\partial^{p+q}\log g}{\partial f_t^p \partial f_{t-1}^q}\left(\hat{f}_{nt}(\beta)|\hat{f}_{n,t-1}(\beta);\theta\right)[I_{n,t}(\beta)]^{-p/2}$ $[I_{n,t-1}(\beta)]^{-q/2}$, *for* $p,q=0,1,2.$

Proof. See Appendix A.2.1. ∎

Function $\mathcal{L}^*_{nT}(\beta)$, called profile log-likelihood function, is the micro log-likelihood of β concentrated w.r.t. the factor values, as if the latter ones were nuisance parameters. It contains the information on β which is independent of the factor dynamics. Proposition 1 shows that the leading term in the asymptotic expansion of the log-likelihood function $\mathcal{L}_{nT}(\beta,\theta)$ in powers of $1/n$ involves parameter β only and is equal to $\mathcal{L}^*_{nT}(\beta)$. The next term $\mathcal{L}_{1,nT}(\beta,\theta)$ at order $1/n$ is the first to provide information on parameter θ characterizing the factor dynamics. It corresponds to the macro log-likelihood after replacing the unobservable factor values with cross-sectional approximations depending on β. The log-det component comes from the Jacobian in the change of variable for Laplace approximation. The term of order $1/n^2$ involves first- and second-order derivatives of the macro log-density function, and third- and fourth-order derivatives of the micro log-density w.r.t. the factor value. Its specific expression seems difficult to interpret in the general framework. It is possible to derive $\mathcal{L}_{2,nT}(\beta,\theta)$ also in the multiple factor case ($m \geq 2$), but its expression is notationally cumbersome and is not provided here. Functions $\mathcal{L}^*_{nT}(\beta)$, $\mathcal{L}_{1,nT}(\beta,\theta)$, and $\mathcal{L}_{2,nT}(\beta,\theta)$ do not involve integrals w.r.t. the factor path, but only nonlinear aggregates of sample observations. In fact, all multidimensional integrals are included in the residual terms $o_p(1/n)$, or $o_p(1/n^2)$. Thus, Propositions 1 (i) and (ii) provide closed-form approximations of the log-likelihood function at order $o_p(1/n)$, and $o_p(1/n^2)$, respectively. The condition $T^\nu/n = O(1)$, $\nu > 1$, is used in Appendix A.2.1 to control the stochastic remainder term in the Laplace approximation at order $o_p(1/n)$. This condition constrains the growth rate of the dimension Tm of the integral in equation (3.2) relatively to the cross-sectional size n, which plays the role of the parameter tending to infinity in our application of the Laplace approximation method. The more restrictive condition $T^\nu/n = O(1)$, $\nu > 3/2$, is used to derive the more accurate log-likelihood approximation at order $o_p(1/n^2)$.

The true log-likelihood function $\mathcal{L}_{nT}(\beta,\theta)$ is invariant to one-to-one transformations of the factor vector $f \to \phi(f)$, say, where ϕ is any invertible mapping in $\mathbb{R}^m$. The leading term $\mathcal{L}^*_{nT}(\beta)$ in the log-likelihood expansion is invariant to such transformations, since it corresponds to the concentrated micro log-likelihood. As a consequence, also the terms $\mathcal{L}_{1,nT}(\beta,\theta)$ and $\mathcal{L}_{2,nT}(\beta,\theta)$ at order $1/n$ and $1/n^2$ are invariant to one-to-one factor transformations, as can be directly verified from their expressions in (3.8) and (3.10) (for $m=1$). In particular, the invariance of $\mathcal{L}_{1,nT}(\beta,\theta)$ explains the log-det component $-\dfrac{1}{2}\dfrac{1}{T}\sum_{t=1}^{T}\log\det I_{n,t}(\beta)$.

This component corresponds to the term introduced by Cox and Reid (1987) in their modified profile likelihood (see also Sweeting, 1987).[9]

We can interpret the leading term in the expansions given in Proposition 1 as an example of the asymptotic equivalence of frequentist and Bayesian methods in large sample (see, e.g., Bickel and Yahav, 1969; Ibragimov and Has'minskii, 1981). To get the intuition, let time dimension T be fixed and parameter θ be given for a moment. Then, our specification with stochastic common factor can be seen as a Bayesian approach w.r.t. parameter β and time effects $\underline{f_T}$. The prior distribution is such that the density of $\underline{f_T}$ given β is $\prod_{t=1}^{T} g\left(f_t | f_{t-1}; \theta\right)$, independent of β, and the prior distribution of β is diffuse. Then, the posterior density of $(\beta, \underline{f_T})$ corresponds to the RHS of equation (3.1), while the posterior density of β corresponds to the RHS of equation (3.2), up to multiplicative constants. Thus, as $n \to \infty$, the "Bayesian" log posterior density $\mathcal{L}_{nT}(\beta, \theta)$ approaches the log-likelihood $\mathcal{L}^*_{nT}(\beta)$, which is the "frequentist" log-likelihood for β concentrated w.r.t. parameters f_t, $t = 1, \ldots, T$. The asymptotic irrelevance of the second term in the RHS of (3.6), or (3.9), involving the transition density of the factor corresponds to the irrelevance of the prior distribution in large samples. Our results show that this asymptotic equivalence is still valid when the number of time effects parameters tends to infinity: $T \to \infty$, such that $T^\nu / n \to 0$, $\nu > 1$.[10]

4. MAXIMUM LIKELIHOOD AND MAXIMUM APPROXIMATED LIKELIHOOD ESTIMATORS

4.1. The estimators of micro- and macroparameters

The ML estimator of (β, θ) is derived by maximizing the log-likelihood function $\mathcal{L}_{nT}(\beta, \theta)$ defined in equation (3.5). Alternative estimators can be defined by maximizing jointly w.r.t. β and θ approximations of the log-likelihood function at probability order $1/n$, and $1/n^2$, respectively. From Proposition 1(i), an approximation at order $o_p(1/n)$ is given by:

$$\mathcal{L}^{\mathrm{CSA}}_{nT}(\beta, \theta) = \mathcal{L}^*_{nT}(\beta) + \frac{1}{n} \mathcal{L}_{1,nT}(\beta, \theta). \tag{4.1}$$

This approximation defines the cross-sectional asymptotic (CSA) log-likelihood function. Similarly, from Proposition 1(ii) an approximation valid up to order $o_p(1/n^2)$ is:

$$\mathcal{L}^{\mathrm{GA}}_{nT}(\beta, \theta) = \mathcal{L}^*_{nT}(\beta) + \frac{1}{n} \mathcal{L}_{1,nT}(\beta, \theta) + \frac{1}{n^2} \mathcal{L}_{2,nT}(\beta, \theta). \tag{4.2}$$

This approximated log-likelihood function defines the granularity adjusted (GA) log-likelihood function. Then, we define the maximum likelihood and maximum approximated likelihood estimators as follows:

DEFINITION 1.

(i) *The maximum likelihood estimator is* $\left(\tilde{\beta}_{nT}, \tilde{\theta}_{nT}\right) = \underset{\beta\in\mathcal{B},\theta\in\Theta}{\arg\max}\ \mathcal{L}_{nT}(\beta,\theta)$.

(ii) *The CSA maximum likelihood estimator is* $\left(\tilde{\beta}_{nT}^{CSA}, \tilde{\theta}_{nT}^{CSA}\right) = \underset{\beta\in\mathcal{B},\theta\in\Theta}{\arg\max}\ \mathcal{L}_{nT}^{CSA}(\beta,\theta)$.

(iii) *The GA maximum likelihood estimator is* $\left(\tilde{\beta}_{nT}^{GA}, \tilde{\theta}_{nT}^{GA}\right) = \underset{\beta\in\mathcal{B},\theta\in\Theta}{\arg\max}\ \mathcal{L}_{nT}^{GA}(\beta,\theta)$.

The CSA and GA maximum likelihood estimators are computationally more convenient than the standard ML estimator, since the CSA and GA log-likelihood functions do not involve integrals w.r.t. the factor path. The difference between the GA and CSA maximum likelihood estimators is called the granularity adjustment. This terminology is explained by the link with the recent literature on granularity adjustment in credit risk (see, e.g., BCBS, 2001; Gordy, 2003). This literature focuses on the computation of risk measures, such as the Value-at-Risk, for large homogeneous portfolios of n assets, whose values are affected by systematic risk factors. The basic idea is to expand the risk measure around the cross-sectional asymptotic limit of an infinitely fine grained portfolio ($n = \infty$), and compute the adjustment at order $1/n$ (for a general presentation of granularity for risk measures, see Gagliardini, Gouriéroux, and Monfort, 2012, Section 5). A similar approach is applied here on the likelihood function and ML estimators instead of being applied on the future portfolio value distribution and its quantiles.

4.2. Identification

To analyze the asymptotic properties of the estimators in Definition 1, we introduce suitable identification assumptions for the micro- and macroparameters. Identification is ensured by the global and local behavior of the large sample limit of the likelihood function around the true parameter value. We exploit the asymptotic expansion of the log-likelihood function in Proposition 1 and consider the case in which the next two conditions are satisfied: (i) the microparameter β is identifiable from the leading term $\mathcal{L}_{nT}^{*}(\beta)$ and (ii) the full parameter vector (β,θ) is identifiable from the log-likelihood approximation at first-order in $1/n$, that is, the CSA log-likelihood $\mathcal{L}_{nT}^{CSA}(\beta,\theta)$. The cases in which the identification of the microparameter requires the first-order term $n^{-1}\mathcal{L}_{1,nT}(\beta,\theta)$, or the identification of some parameters requires the second-order term $n^{-2}\mathcal{L}_{2,nT}(\beta,\theta)$, lead to different asymptotic behaviors of the estimators and are not considered in this paper. Let us now derive the identification assumptions, starting from the microparameter.

(i) Let us first define the population counterpart of the cross-sectional estimate of the factor value:

$$f_t(\beta) = \arg\max_{f\in\mathbb{R}^m} \mathrm{E}_0\left[\log h\left(y_{i,t}|y_{i,t-1}, f;\beta\right)|\underline{f_t}\right], \tag{4.3}$$

where $E_0\left[.|\underline{f_t}\right]$ denotes the expectation w.r.t. the true conditional distribution of $\left(y_{i,t}, y_{i,t-1}\right)$ given $\underline{f_t} = (f_t, f_{t-1}, \ldots)$. The pseudo-true factor value $f_t(\beta)$ maximizes the limiting cross-sectional log-likelihood at date t for given parameter value β. It is a function of both parameter β and factor path $\underline{f_t}$. Thus, $f_t(\beta)$ is a stochastic process, for any $\beta \in \mathcal{B}$. We assume that the pseudo-true factor value is globally and locally identified (see Assumption H.2 in Appendix A.1). By the properties of the Kullback-Leibler discrepancy, at true parameter value β_0, the pseudo-true factor value $f_t(\beta_0)$ coincides with the true factor value f_t, $\mathbb{P}$-a.s., for any t.

Let us now define the function:

$$\begin{aligned}\mathcal{L}^*(\beta) &= \operatorname*{plim}_{n,T\to\infty} \mathcal{L}^*_{nT}(\beta) = \operatorname*{plim}_{n,T\to\infty} \frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n} \log h\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta); \beta\right) \\ &= E_0\left[\log h\left(y_{i,t}|y_{i,t-1}, f_t(\beta); \beta\right)\right], \end{aligned} \tag{4.4}$$

where the convergence is uniform w.r.t. $\beta \in \mathcal{B}$ and is proved in Lemma 1(i) (see supplementary material). Intuitively, function $\mathcal{L}^*(\beta)$ is the asymptotic micro log-likelihood concentrated w.r.t. the stochastic process (f_t). The assumptions below concern the identification of parameter β.

A.6 (Global identification assumption for β**):** *The mapping* $\beta \to \mathcal{L}^*(\beta)$ *is uniquely maximized at the true parameter value* β_0.

A.7 (Local identification assumption for β**):** *The matrix* $I_0^* = -\dfrac{\partial^2 \mathcal{L}^*(\beta_0)}{\partial\beta\partial\beta'}$ *is positive definite.*

The matrix I_0^* is given by:

$$I_0^* = E_0\left[I_{\beta\beta}(t) - I_{\beta f}(t) I_{ff}(t)^{-1} I_{f\beta}(t)\right] = E_0\left[U_{it}U_{it}'\right], \tag{4.5}$$

where $I_{\beta\beta}(t)$, $I_{ff}(t)$, $I_{\beta f}(t)$, and $I_{f\beta}(t) = I_{\beta f}(t)'$ denote the blocks of the conditional information matrix at date t:

$$I(t) = E_0\left[-\frac{\partial^2 \log h\left(y_{i,t}|y_{i,t-1}, f_t; \beta_0\right)}{\partial\left(\beta', f_t'\right)'\partial\left(\beta', f_t'\right)}|\underline{f_t}\right], \tag{4.6}$$

and $U_{it} = \dfrac{\partial \log h(y_{i,t}|y_{i,t-1}; f_t; \beta_0)}{\partial\beta} - I_{\beta f}(t) I_{ff}(t)^{-1} \dfrac{\partial \log h(y_{i,t}|y_{i,t-1}; f_t; \beta_0)}{\partial f_t}$. Thus, I_0^* is the variance-covariance matrix of the residual U_{it} in the orthogonal conditional projection of the score w.r.t. the microparameter on the score w.r.t the factor value given $\underline{f_t}$.

(ii) Let us now consider the macrocomponent of the log-likelihood. Under Assumptions A.6 and A.7, parameter β can be estimated at a rate faster than the rate for parameter θ. Hence, the relevant criterion for identification of θ is the mapping $\theta \to \mathcal{L}_1(\beta_0, \theta)$, where $\mathcal{L}_1(\beta, \theta)$ is the large sample limit of $\mathcal{L}_{1,nT}(\beta,\theta)$ in equation (3.8). We have $\mathcal{L}_1(\beta_0,\theta) = \mathrm{E}_0\left[\log g(f_t|f_{t-1};\theta)\right]$, up to a term constant in θ [see Lemma 1(ii) in the supplementary material]. Thus, the identification assumptions for the macroparameter are the following:

A.8 (Global identification assumption for θ): *The mapping $\theta \to \mathrm{E}_0\left[\log g(f_t|f_{t-1};\theta)\right]$ is uniquely maximized at the true parameter value θ_0.*

A.9 (Local identification assumption for θ): *The matrix $I_{1,\theta\theta} = \mathrm{E}_0\left[-\dfrac{\partial^2 \log g(f_t|f_{t-1};\theta_0)}{\partial\theta\partial\theta'}\right]$ is positive definite.*

Assumptions A.8 and A.9 are the standard global and local identification conditions for estimating parameter θ in a model with observable factor values.

4.3. Asymptotic properties of the estimators

We consider the asymptotic properties of the CSA, GA, and true ML estimators in Definition 1 under Assumptions A.1–A.9 and H.1–H.14 in Appendix A.1. Assumptions A.1–A.9 are invariant to one-to-one transformations of the factor vector (if the transformation is independent of the parameters β,θ), whereas some of the Assumptions H.1–H.14 are not. Moreover, the CSA, GA, and true ML estimators are numerically invariant to one-to-one transformations of the factor. Thus, in order to establish their asymptotic properties it is enough that the regularity conditions H.1–H.14 in Appendix A.1 are satisfied for a suitable choice of the factor representation.

Let us first study the probability order of the difference between the CSA and GA ML estimators on the one hand, and the true ML estimators on the other hand.

PROPOSITION 2. *Under Assumptions A.1–A.9 and H.1–H.14, the CSA, GA, and true infeasible ML estimators in Definition 1 are such that:*

$$\tilde{\beta}_{nT}^{CSA} - \tilde{\beta}_{nT} = o_p(1/n), \qquad \tilde{\theta}_{nT}^{CSA} - \tilde{\theta}_{nT} = O_p\left(\frac{(\log n)^{\delta_1}}{\sqrt{n}}\right), \tag{4.7}$$

$$\tilde{\beta}_{nT}^{GA} - \tilde{\beta}_{nT} = o_p(1/n), \qquad \tilde{\theta}_{nT}^{GA} - \tilde{\theta}_{nT} = O_p\left(\frac{(\log n)^{\delta_1}}{\sqrt{n}}\right), \tag{4.8}$$

for a constant $\delta_1 > 0$, if $n,T \to \infty$ such that $T^\nu/n = O(1)$, $\nu > 1$, and:

$$\tilde{\beta}_{nT}^{CSA} - \tilde{\beta}_{nT} = O_p(1/n^2), \qquad \tilde{\theta}_{nT}^{CSA} - \tilde{\theta}_{nT} = O_p(1/n), \tag{4.9}$$

$$\tilde{\beta}_{nT}^{GA} - \tilde{\beta}_{nT} = o_p(1/n^2), \qquad \tilde{\theta}_{nT}^{GA} - \tilde{\theta}_{nT} = o_p(1/n), \tag{4.10}$$

if $n,T \to \infty$ such that $T^\nu/n = O(1)$, $\nu > 3/2$.

Proof. See Appendix A.2.2. ■

Proposition 2 states that the CSA, GA, and true ML estimators are asymptotically equivalent and provides the probability orders of this equivalence. If $T^{\nu}/n = O(1)$, $\nu > 3/2$, from equations (4.9) and (4.10) the GA maximum likelihood estimator provides a more accurate approximation of the true ML estimator compared to the CSA maximum likelihood estimator. The accuracy of the approximation is superior for the micro- than for the macroparameters. Under the less restrictive condition $T^{\nu}/n = O(1)$, $\nu > 1$, from equations (4.7) and (4.8) the CSA and GA ML estimators have the same order of accuracy in approximating the true ML estimator, and this accuracy is again superior for the microparameters.

The joint asymptotic distribution of the estimators of the micro- and macroparameters is given in the next proposition.

PROPOSITION 3. *Let Assumptions A.1–A.9 and H.1–H.14 be satisfied, and let* $(\hat{\beta}_{nT}, \hat{\theta}_{nT})$ *be either the CSA, GA, or true ML estimator in Definition 1. Then, if* $n, T \to \infty$ *such that* $T^{\nu}/n = O(1)$, $\nu > 1$, *estimator* $(\hat{\beta}_{nT}, \hat{\theta}_{nT})$ *is consistent and asymptotically normal:*

$$\begin{bmatrix} \sqrt{nT}\left(\hat{\beta}_{nT} - \beta_0\right) \\ \sqrt{T}\left(\hat{\theta}_{nT} - \theta_0\right) \end{bmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} B^*_{\beta\beta} & B^*_{\beta\theta} \\ B^*_{\theta\beta} & B^*_{\theta\theta} \end{pmatrix}\right), \tag{4.11}$$

with asymptotic variance-covariance matrix

$$B^* = \begin{pmatrix} B^*_{\beta\beta} & B^*_{\beta\theta} \\ B^*_{\theta\beta} & B^*_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \left(I_0^*\right)^{-1} & 0 \\ 0 & I_{1,\theta\theta}^{-1} \end{pmatrix},$$

where $I_0^* = \mathrm{E}_0\left[I_{\beta\beta}(t) - I_{\beta f}(t) I_{ff}(t)^{-1} I_{f\beta}(t)\right]$ *and* $I_{1,\theta\theta} = \mathrm{E}_0\left[-\dfrac{\partial^2 \log g\left(f_t | f_{t-1}; \theta_0\right)}{\partial\theta\partial\theta'}\right]$.

Proof. See Appendix A.2.3. ■

Proposition 3 states that the CSA, GA, and ML estimators are asymptotically normal with different rates of convergence for the micro- and macrocomponent that are root-nT and root-T, respectively, if $T^{\nu}/n = O(1)$, $\nu > 1$. The asymptotic variance-covariance matrix B^* defines the joint efficiency bound for estimating both micro- and macroparameters (β, θ). Matrix B^* is block-diagonal for the micro- and macrocomponents, with the diagonal blocks corresponding to the Hessian matrices $I_0^* = -\dfrac{\partial^2 \mathcal{L}^*\left(\beta_0\right)}{\partial\beta\partial\beta'}$ and $I_{1,\theta\theta} = -\dfrac{\partial^2 \mathcal{L}_1\left(\beta_0, \theta_0\right)}{\partial\theta\partial\theta'}$. The zero out-of-diagonal blocks in the efficiency bound imply that parameters β and θ can be considered independently for estimation purpose. This justifies ex-post their interpretation as micro- and macroparameters, respectively, since parameter β (respectively θ) contains no macro-information (respectively no micro-information) under identification Assumptions A.6–A.9. The condition $T^{\nu}/n = O(1)$, $\nu > 1$, implies that the asymptotic distributions of the estimators are centered. Thus, in our framework there is no incidental parameter bias (see, e.g., Neyman et al., 1948; Lancaster, 2000).

The result in Proposition 3 is a consequence of the expansion of the likelihood function in Proposition 1. Indeed, under identification Assumptions A.6–A.7 and the regularity conditions in Appendix A.1, for large n and T the relevant term for estimation of parameter β is $\mathcal{L}^*_{nT}(\beta)$. The corresponding limit log-likelihood function is $\mathcal{L}^*(\beta)$, and the efficiency bound $B^*_{\beta\beta}$ for β is the inverse of the Hessian I^*_0. Similarly, the efficiency bound $B^*_{\theta\theta}$ for θ is the inverse of the Hessian $I_{1,\theta\theta}$. Moreover, the (standardized) ML estimators of β and θ are asymptotically independent. Therefore, the efficiency bound $B^*_{\beta\beta}$ for β given in Proposition 3 is the same as the efficiency bound for β with known transition density of the factor. Finally, the information matrix I^*_0 is smaller than the information matrix $I^{**}_0 = \mathrm{E}_0\left[I_{\beta\beta}(t)\right]$ corresponding to the case of observable factor, while matrix $I_{1,\theta\theta}$ is equal to the information for θ with observable factor. Estimator $\hat{\theta}_{nT}$ is asymptotically equivalent to the infeasible ML estimator $\hat{\theta}^{**}_T = \arg\max_{\theta} \sum_{t=1}^{T} \log g\left(f_t | f_{t-1};\theta\right)$ that uses the true factor values. Therefore, the unobservability of the factor has no efficiency impact asymptotically for estimating θ, but has an impact for estimating β. Indeed, the factor values can be estimated at a rate close to $1/\sqrt{n}$ (see Proposition 5 below), a rate which is faster than the rate $1/\sqrt{T}$ for estimating θ, if $T^{\nu}/n = O(1)$, $\nu > 1$, and slower than the rate $1/\sqrt{nT}$ for estimating β.

Proposition 3 shows that the computationally convenient CSA and GA ML estimators are asymptotically efficient estimators of parameters β and θ (see also Section 5 for other asymptotically efficient estimators). This result concerns first-order asymptotics only. It is out of the scope of the present paper to get the higher-order expansion of the asymptotic distribution of the standardized estimators $\left[\sqrt{nT}\left(\hat{\beta}_{nT} - \beta_0\right)', \sqrt{T}\left(\hat{\theta}_{nT} - \theta_0\right)'\right]'$ in the sense of Ghosh and Subramanyam (1974) and Pfanzagl and Wefelmeyer (1978), for instance to correct for the higher-order bias in n and/or T. It is likely difficult to derive the higher-order expansions due to the double asymptotics and the different rates of convergence of the estimators of micro- and macroparameters. The GA ML estimator is closer to the infeasible ML estimator than the CSA ML estimator is, if $T^{\nu}/n = O(1)$, $\nu > 3/2$, and likely inherits its finite-sample properties. In some applications to credit risk, the ML and GA ML estimators can feature worse finite-sample properties than the CSA ML estimator (see Gourieroux and Jasiak, 2012). Therefore, we may expect different higher-order expansions for the CSA and GA ML estimators.

4.4. Semiparametric efficiency

The efficiency bound $B^*_{\beta\beta}$ for parameter β in Proposition 3 is independent of the parametric model $g(f_t|f_{t-1};\theta)$, $\theta \in \mathbb{R}^p$, for the transition density of the factor, that is, factor distribution free. This suggests that the efficiency result extends to a semiparametric setting. Specifically, the asymptotic semiparametric efficiency bound B for β is the efficiency bound for estimating β in the semiparametric model in which the transition $g(f_t|f_{t-1})$ of the factor is a functional parameter. The semiparametric efficiency bound B can be computed by using Stein's heuristic (Stein, 1956; Severini and Tripathi, 2001). More precisely, let $g_\theta = g(f_t|f_{t-1};\theta)$ be a well-specified parametric model for the transition of f_t with parameter $\theta \in \mathbb{R}^p$ that satisfies Assumptions A.8 and A.9 and the regularity conditions H.11–H.14 in Appendix A.1, and let $B^*_{\beta\beta}(g_\theta)$ be the corresponding parametric efficiency bound for estimating β.

DEFINITION 2. *The semiparametric efficiency bound B is defined by:*

$$B = \max_{g_\theta} B^*_{\beta\beta}(g_\theta),$$

where the maximization is performed w.r.t. the well-specified parametric models g_θ for the transition of f_t that satisfy Assumptions A.8 and A.9 and H.11–H.14.

The result in Proposition 3 shows that $B^*_{\beta\beta}(g_\theta)$ is independent of g_θ. Therefore, we deduce:

COROLLARY 4. *Under Assumptions A.1–A.7 and H.1–H.10, and if $n,T \to \infty$ such that $T^\nu/n = O(1)$, $\nu > 1$, the semiparametric efficiency bound for β is equal to the parametric efficiency bound: $B = B^*_{\beta\beta} = E_0\left[I_{\beta\beta}(t) - I_{\beta f}(t) I_{ff}(t)^{-1} I_{f\beta}(t)\right]^{-1}$.*

Thus, any well-specified parametric model g_θ is the least-favorable one in the sense of Chamberlain (1987). Proposition 3 and Corollary 4 show that the knowledge of the parametric model for the transition of the factor, and even the knowledge of the transition itself, are irrelevant for the asymptotically efficient estimation of microparameter β.[11]

4.5. Approximation of the factor values

Given a consistent estimator of the microparameter β, we can use the cross-sectional aggregate $\hat{f}_{n,t}(\beta)$ to get consistent approximations of the factor value f_t.[12]

DEFINITION 3. *Let $\hat{\beta}_{nT}$ denote either the CSA, GA, or true ML estimator of the microparameter β in Definition 1. Then a cross-sectional approximation of the factor value at date t is:*

$$\hat{f}_{nT,t} = \hat{f}_{n,t}\left(\hat{\beta}_{nT}\right),$$

for $t = 1, \ldots, T$, *where* $\hat{f}_{n,t}(\beta)$ *is defined in equation (3.3).*

For any given date t, the factor approximation $\hat{f}_{nT,t}$ depends on the whole individual histories and is a kind of smoothed factor value. Its asymptotic properties are given in the next proposition.

PROPOSITION 5. *Suppose Assumptions A.1–A.9 and H.1–H.14 hold, and let* $n, T \to \infty$ *such that* $T^{\nu}/n = O(1)$, $\nu > 1$. *Then:*

(i) *For any date* t, *conditional on* $\underline{f_t}$ *we have:* $\sqrt{n}\left(\hat{f}_{nT,t} - f_t\right) \xrightarrow{d} N\left(0, I_{ff}(t)^{-1}\right)$.

(ii) $\sup_{1 \le t \le T} \left\| \hat{f}_{nT,t} - f_t \right\| = O_p\left(\frac{(\log n)^{\delta_2}}{\sqrt{n}}\right)$, *where* $\delta_2 = \gamma_2 + \gamma_3/2 + 2/d_3 + 1/2$ *and constants* $\gamma_2, \gamma_3 \ge 0$, $d_3 > 0$ *are defined in Assumptions H.7–H.9 in Appendix A.1.*

Proof. See Appendix A.2.4. ∎

Conditionally on the factor path, the factor approximation converges to the true factor value f_t at rate $1/\sqrt{n}$. Since $\hat{\beta}_{nT}$ is root-nT consistent, estimator $\hat{f}_{nT,t}$ is asymptotically equivalent to the infeasible ML estimator $\hat{f}_{n,t}(\beta_0)$ for known microparameter β_0. The asymptotic variance $I_{ff}(t)^{-1}$ of $\hat{f}_{nT,t}$ is the inverse of the Fisher information for estimating f_t in the cross-section at date t with known β_0. The uniform convergence in Proposition 5(ii) follows from the convergence of $\hat{f}_{n,t}(\beta)$ to $f_t(\beta)$ uniformly in $\beta \in \mathcal{B}$ and $t = 1, \ldots, T$ (see Limit Theorem 1 in the supplementary materials) and the root-nT consistency of estimator $\hat{\beta}_{nT}$ (see Proposition 3). Proposition 5(ii) is not invariant to one-to-one transformations of the factor, since the regularity assumptions include tail conditions on the factor distribution (see Assumptions H.7–H.9).

5. TWO-STEP EFFICIENT ESTIMATORS

In this section we introduce another asymptotically efficient estimation approach, in which the estimators of the micro- and macroparameters can be computed in two steps and are easy to interpret.

DEFINITION 4. *The two-step estimator is defined by:*

$$\hat{\beta}^*_{nT} = \arg\max_{\beta \in \mathcal{B}} \sum_{t=1}^{T} \sum_{i=1}^{n} \log h\left(y_{i,t} | y_{i,t-1}, \hat{f}_{n,t}(\beta); \beta\right),$$

and:

$$\hat{\theta}_{nT}^{*} = \underset{\theta \in \Theta}{\arg\max} \sum_{t=1}^{T} \log g\left(\hat{f}_{nT,t}^{*} | \hat{f}_{nT,t-1}^{*}; \theta\right),$$

where $\hat{f}_{n,t}(\beta)$ *is defined in equation (3.3) and* $\hat{f}_{nT,t}^{*} = \hat{f}_{n,t}\left(\hat{\beta}_{nT}^{*}\right)$ *for* $t = 1, \ldots, T$.

In the first step, the estimator $\hat{\beta}_{nT}^{*}$ of the microparameter is obtained by maximizing the profile likelihood function $\mathcal{L}_{nT}^{*}(\beta)$ defined in equation (3.7). Thus, $\hat{\beta}_{nT}^{*}$ is the time fixed effects estimator of β which considers the f_t values as additional unknown parameters. Since the function $\mathcal{L}_{nT}^{*}(\beta)$ does not involve the transition p.d.f. of the factor, the estimator $\hat{\beta}_{nT}^{*}$ does not depend on the specification of the factor dynamics. In this sense, $\hat{\beta}_{nT}^{*}$ is a semiparametric estimator, which is not the case for the CSA and GA ML estimators. Estimator $\hat{\beta}_{nT}^{*}$ is used to derive cross-sectional approximations $\hat{f}_{nT,t}^{*}$ of the factor values. These cross-sectional factor approximations correspond to the ML estimates of the time fixed effects. In the second step, the approximations of the factor values are used to derive the approximation of the macrolikelihood function $\sum_{t=1}^{T} \log g\left(\hat{f}_{nT,t}^{*} | \hat{f}_{nT,t-1}^{*}; \theta\right)$. By maximizing this approximate likelihood w.r.t. θ, we get an estimator of the macroparameter.

The asymptotic distribution of the two-step estimator is given in the next proposition.

PROPOSITION 6. *Suppose Assumptions A.1–A.9 and H.1–H.14 hold, and let* $n, T \to \infty$ *such that* $T^{\nu}/n = O(1)$, $\nu > 1$. *Then the estimators in Definition 4 are such that:*

(i) $\hat{\beta}_{nT}^{*} - \tilde{\beta}_{nT} = O_p(1/n)$, $\hat{\theta}_{nT}^{*} - \tilde{\theta}_{nT} = O_p\left(\frac{(\log n)^{\delta_1}}{\sqrt{n}}\right)$, *for* $\delta_1 > 0$ *as in Proposition 2.*

(ii) *The estimator* $(\hat{\beta}_{nT}^{*\prime}, \hat{\theta}_{nT}^{*\prime})'$ *is consistent and asymptotically normal such that:*

$$\begin{bmatrix} \sqrt{nT}\left(\hat{\beta}_{nT}^{*} - \beta_0\right) \\ \sqrt{T}\left(\hat{\theta}_{nT}^{*} - \theta_0\right) \end{bmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (I_0^{*})^{-1} & 0 \\ 0 & I_{1,\theta\theta}^{-1} \end{pmatrix}\right),$$

where matrices I_0^{*} *and* $I_{1,\theta\theta}$ *are given in Proposition 3.*

Proof. See Appendix A.2.5. ■

From Propositions 2 and 6(i), the two-step estimator of the microparameter provides a less accurate approximation of the true ML estimator compared with the CSA and GA ML estimators. However, the semiparametric estimator $\hat{\beta}_{nT}^{*}$ still achieves asymptotically the (semi)parametric efficiency bound. In other words, the conditional likelihood estimator of β (based on concentrating out the f_t) is first-order asymptotically equivalent to the full likelihood estimator of β.

The first-order asymptotic distribution of the fixed effects estimator $\hat{\beta}^*_{nT}$ in Proposition 6(ii) is not surprising in view of Theorem 1 in Hahn et al. (2004), who consider a nonlinear setting with microdensity $h(y_{i,t}|\alpha_i;\beta)$ and individual fixed effects α_i. In particular, the interpretation of the asymptotic variance I_0^* in equation (4.5) as the outer product of the residual in the orthogonal projection of the score w.r.t. the microparameter on the score w.r.t. the fixed effect is the same as in Theorem 1 in Hahn et al. (2004). However, Proposition 6 cannot be obtained by interchanging the individual and time indices, and also the sizes n and T, and by letting $\rho \to 0$ in Hahn et al. (2004), where their parameter $\rho > 0$ is such that $n/T \to \rho$. Indeed, in our paper the microdensity $h(h_{i,t}|y_{i,t-1}, f_t;\beta)$ depends on the lagged variable $y_{i,t-1}$, the latent factor f_t features a dynamic, and our asymptotic results are under a probability measure such that the time effects f_t define a stochastic process with parametric dynamics and are not a sequence of fixed constants. Hahn and Kuersteiner (2002) consider a linear dynamic panel model with individual fixed effects[13] and prove that the (bias-corrected) fixed effects estimator is asymptotically efficient in the sense of Hayek's convolution theorem. Proposition 6 differs from Hahn et al. (2002), since we define the efficiency bound as the asymptotic variance of the ML estimator under a parametric dynamics of the stochastic time effects.

6. STOCHASTIC MIGRATION MODEL

In this section we illustrate the finite sample properties of the two-step estimators in Definition 4 with a stochastic migration model.

6.1. The model

The stochastic migration model has been introduced to analyze the dynamics of corporate ratings and is a basic element for the prediction of future credit risk in a homogeneous pool of credits (e.g., Gupton, Finger, and Bhatia, 1997; Gordy and Heitfield, 2002; Gagliardini and Gouriéroux, 2005a, 2005b; Feng, Gouriéroux, and Jasiak, 2008; Koopman, Lucas, and Monteiro, 2008). A basic stochastic migration model is the ordered qualitative model with one factor, which extends the ASRF model of Section 2.1 to more than two alternatives. Let us denote by $y_{i,t}$, with t varying, the sequence of ratings for corporation i. The possible ratings are $k = 1, 2, \ldots, K$, say.[14] The microdynamics is deduced from the latent model for the log asset/liability ratios:

$$y^*_{i,t} = \alpha_l + \gamma_l f_t + \sigma_l u_{i,t}, \quad i \in \mathscr{R}_{l,t-1}, \quad t = 1, \ldots, T,$$

where $\mathscr{R}_{l,t-1}$ denotes the set of companies with rating $y_{i,t-1} = l$ at time $t-1$. The idiosyncratic shocks $u_{i,t}$ are i.i.d. across companies and time with cdf G

that corresponds, for instance, to the standard normal distribution for the probit model, when $G(x) = \Phi(x)$, and to the logistic distribution for the logit model, when $G(x) = 1/\left(1+e^{-x}\right)$. The intercept α_l, the factor sensitivity γ_l, and the idiosyncratic volatility σ_l depend on the lagged rating l. Thus, in this model the homogeneous segments correspond to the rating classes. The current ratings are deduced by discretization of the log asset/liability ratios:

$$y_{i,t} = k, \quad \text{if } c_{k-1} < y^*_{i,t} \le c_k,$$

for $k = 1, \dots, K$, where $c_1 < c_2 < \dots < c_{K-1}$ are unknown thresholds and $c_0 = -\infty$, $c_K = +\infty$. Then, the microdynamics is defined by the rating transition matrices conditional on the factor value:

$$\begin{aligned}\pi_{lk,t} &= \mathbb{P}\left[y_{i,t} = k | y_{i,t-1} = l, f_t\right] \\ &= G\left(\frac{c_k - \gamma_l f_t - \alpha_l}{\sigma_l}\right) - G\left(\frac{c_{k-1} - \gamma_l f_t - \alpha_l}{\sigma_l}\right),\end{aligned}$$

where c_k, $\alpha_l, \gamma_l, \sigma_l$, for $k, l = 1, \dots, K$, are unknown microparameters. Thus, we have a set of ordered probit or logit models with latent factors and common parameters, since the thresholds c_k appear in each row of the transition matrix. The ratios $a_{l,k,t} = (c_k - \gamma_l f_t - \alpha_l)/\sigma_l$ in the above transition probabilities identify semiparametrically the microparameters and the factor values up to location and scale transformations. Assumptions A.6 and A.7 for semiparametric identification are satisfied if we impose the constraints $c_1 = 0$, $\sigma_1 = 1$, $\alpha_1 = 0$, $\gamma_1 = 1$ when $K > 2$, and additionally $\sigma_2 = 1$ when $K = 2$ (see Appendix A.3). For instance, the vector of free microparameters is $\beta = (\alpha_l, \gamma_l, \sigma_l, l = 2, \dots, K, c_k,$ $k = 2, \dots, K)$ when $K > 2$. Finally, in the microdynamics we assume for illustration a single common factor f_t, which follows a linear Gaussian autoregressive process:

$$f_t = \mu + \rho f_{t-1} + \sigma \eta_t, \tag{6.1}$$

where (η_t) is $IIN(0,1)$, and μ, ρ, and σ are unknown macroparameters.

6.2. Estimation of the microparameters

The micro log-density is given by:

$$\begin{aligned}\log h\left(y_{i,t} | y_{i,t-1}, f_t; \beta\right) = \sum_{k=1}^{K}\sum_{l=1}^{K} 1\left\{y_{i,t} = k, y_{i,t-1} = l\right\} \\ \log\left[G\left(\frac{c_k - \gamma_l f_t - \alpha_l}{\sigma_l}\right) - G\left(\frac{c_{k-1} - \gamma_l f_t - \alpha_l}{\sigma_l}\right)\right].\end{aligned}$$

The estimators of the factor values given β are:

$$\hat{f}_{n,t}(\beta) = \arg\max_{f_t} \sum_{k=1}^{K} \sum_{l=1}^{K} N_{lk,t} \log \left[G\left(\frac{c_k - \gamma_l f_t - \alpha_l}{\sigma_l} \right) - G\left(\frac{c_{k-1} - \gamma_l f_t - \alpha_l}{\sigma_l} \right) \right], \quad t = 1, \ldots, T, \tag{6.2}$$

and depend on the data through the aggregate counts $N_{lk,t}$ of transitions from rating l at time $t-1$ to rating k at time t, for $k, l = 1, \ldots, K$ and $t = 1, \ldots, T$. The two-step (semi)parametrically efficient estimator of the microparameter is:

$$\hat{\beta}^*_{nT} = \arg\max_{\beta} \sum_{k=1}^{K} \sum_{l=1}^{K} \sum_{t=1}^{T} N_{lk,t} \log \left[G\left(\frac{c_k - \gamma_l \hat{f}_{n,t}(\beta) - \alpha_l}{\sigma_l} \right) - G\left(\frac{c_{k-1} - \gamma_l \hat{f}_{n,t}(\beta) - \alpha_l}{\sigma_l} \right) \right]. \tag{6.3}$$

This estimator is computed from the aggregate data on rating transition counts $(N_{lk,t})$.

To compare the finite-sample distribution of estimator $\hat{\beta}^*_{nT}$ and the semiparametric efficiency bound, we perform a Monte-Carlo study. We consider the two-state case $K = 2$ and assume a logistic function G. Under the semiparametric identification constraints $c_1 = \alpha_1 = 0$ and $\gamma_1 = \sigma_1 = \sigma_2 = 1$, the microparameter to estimate is $\beta = (\gamma_2, \alpha_2)'$. The parameter values used in the Monte-Carlo study are displayed in Table 1.

In Figures 1 and 2, we consider the sample sizes $n = 200$, $T = 20$, and $n = 1000$, $T = 20$, respectively. In each figure, the two panels display the finite sample distributions of the estimators of the two microparameters (solid lines) that are the components of $\hat{\beta}^*_{nT}$. We also display for each microparameter the Gaussian distribution (dashed lines) with mean equal to the true parameter value and variance equal to the semiparametric efficiency bound divided by nT. The estimator $\hat{\beta}^*_{nT}$ is computed from equation (6.3) by numerical optimization. To evaluate the profile microloglikelihood function for any given β, the estimate $\hat{f}_{n,t}(\beta)$ in equation (6.2) is computed by grid search. As expected from the stochastic migration literature, the γ_2 parameter, which represents the sensitivity of the transition probabilities with respect to the systematic factor, is the most difficult to estimate. Its asymptotic variance is larger and the convergence of the finite sample distribution to the asymptotic one is slower. A comparison of

TABLE 1. Parameter values

$\alpha_1 = 0$	$\gamma_1 = 1$	$\sigma_1 = 1$	$\alpha_2 = -0.5$	$\gamma_2 = 1$	$\sigma_2 = 1$
$c_0 = -\infty$	$c_1 = 0$	$c_2 = +\infty$	$\mu = 0.1$	$\rho = 0.5$	$\sigma = 0.5$

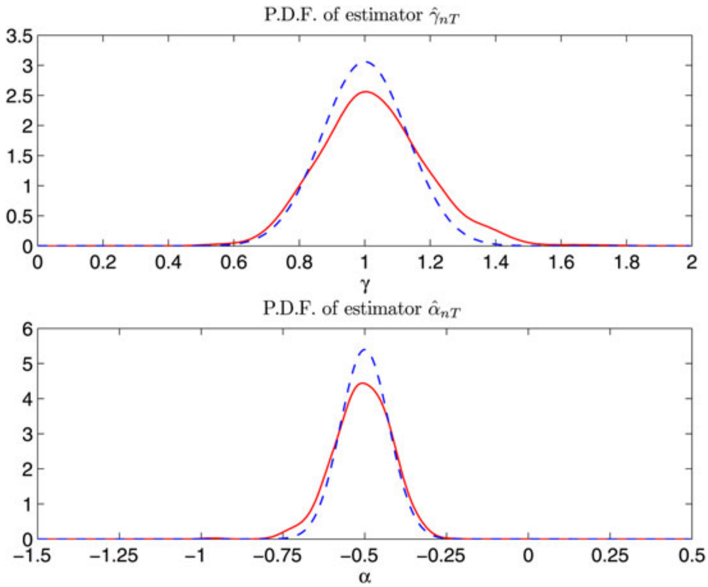

FIGURE 1. Distribution of the two-step semiparametrically efficient estimators of the microparameters, sample size $n = 200$ and $T = 20$.

The solid lines give the p.d.f. of the two-step semiparametrically efficient estimators of parameter γ (upper Panel, true value 1) and parameter α (lower Panel, true value -0.5). The p.d.f. is computed by a kernel density estimator. Sample sizes are $n = 200$ and $T = 20$. The dashed lines in the two Panels give the p.d.f. of a normal distribution centered at the true value of the parameter and with variance equal to the semiparametric efficiency bound divided by nT.

Figures 1 and 2 shows that the standard deviations of the estimators decrease by a factor of about 2 when passing from $n = 200$ to $n = 1000$, as suggested by the rate of convergence $\sqrt{nT}$ of the microparameters estimators. Finally, the latter estimators feature a rather small finite sample bias.

The semiparametric efficiency bound for $\beta = (\gamma_2, \alpha_2)'$ is easily derived from Proposition 3 and is given by:

$$B^*_{\beta\beta} = \mathrm{E}_0 \left[\frac{\mu_{1,t-1}\pi_{12,t}\left(1-\pi_{12,t}\right) \cdot \mu_{2,t-1}\pi_{22,t}\left(1-\pi_{22,t}\right)}{\mu_{1,t-1}\pi_{12,t}\left(1-\pi_{12,t}\right) + \mu_{2,t-1}\pi_{22,t}\left(1-\pi_{22,t}\right)\gamma_2^2} \begin{pmatrix} f_t^2 & f_t \\ f_t & 1 \end{pmatrix} \right]^{-1},$$

where $\pi_{12,t} = 1/(1+e^{-f_t})$, $\pi_{22,t} = 1/(1+e^{-\gamma_2 f_t - \alpha_2})$, and $\mu_{1,t-1} = \mathbb{P}\left[y_{i,t-1} = 1 | \underline{f_{t-1}}\right] = 1 - \mu_{2,t-1}$. The matrix $B^*_{\beta\beta}$ involves the probabilities $\mu_{1,t-1}$ and $\mu_{2,t-1}$ of the lagged states, conditional on the factor path, and the conditional variances of the indicator of state 2, that are $\pi_{21,t}(1-\pi_{21,t})$ and $\pi_{22,t}(1-\pi_{22,t})$, respectively, according to the previous state. The matrix $B^*_{\beta\beta}$ depends on macroparameters μ, ρ, σ^2 by means of the expectation E_0.

Let us now study the pattern of the semiparametric efficiency bound of parameter γ_2 as a function of the autoregressive coefficient ρ and the unconditional

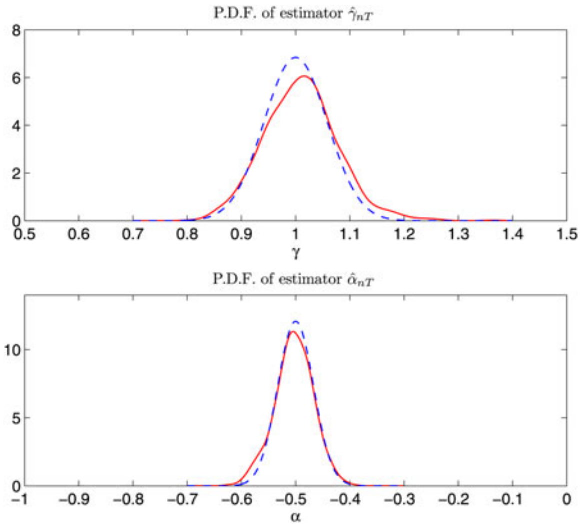

FIGURE 2. Distribution of the two-step semiparametrically efficient estimators of the microparameters, sample size $n = 1000$ and $T = 20$.

The solid lines give the p.d.f. of the two-step semiparametrically efficient estimators of parameter γ (upper Panel, true value 1) and parameter α (lower Panel, true value -0.5). The p.d.f. is computed by a kernel density estimator. Sample sizes are $n = 1000$ and $T = 20$. The dashed lines in the two Panels give the p.d.f. of a normal distribution centered at the true value of the parameter and with variance equal to the semiparametric efficiency bound divided by nT.

variance $\frac{\sigma^2}{1-\rho^2}$ of the factor process (f_t). Figure 3 displays the asymptotic standard deviation $\left(\frac{1}{nT} B^*_{\gamma_2\gamma_2}\right)^{1/2}$ as a function of these two macroparameters, where $n = 1000$ and $T = 20$, and the semiparametric efficiency bound $B^*_{\gamma_2\gamma_2}$ is approximated numerically by Monte-Carlo integration. The values of the microparameters and of μ are given in Table 1. The semiparametric efficiency bound is decreasing w.r.t. the factor variance. The pattern is almost flat w.r.t. the autoregressive coefficient ρ of the factor, except for values of ρ close to 1, where the semiparametric efficiency bound diverges to infinity.

6.3. Estimation of the macroparameters

Let us now consider the efficient estimation of the macroparameter $\theta = \left(\mu, \rho, \sigma^2\right)'$. The estimator is based on the cross-sectional approximations of the factor values $\hat{f}^*_{nT,t} = \hat{f}_{n,t}\left(\hat{\beta}^*_{nT}\right)$ from equations (6.2) and (6.3). The estimators $\hat{\mu}$ and $\hat{\rho}$ are obtained by OLS on the regression:

$$\hat{f}^*_{nT,t} = \mu + \rho \hat{f}^*_{nT,t-1} + u_t, \quad t = 2, \ldots, T.$$

The estimator of parameter σ^2 is given by $\hat{\sigma}^2 = \dfrac{1}{T-1} \displaystyle\sum_{t=2}^{T} \hat{u}_t^2$, where $\hat{u}_t = \hat{f}^*_{nT,t} - \hat{\mu} - \hat{\rho} \hat{f}^*_{nT,t-1}$ are the OLS residuals. The estimator $\hat{\theta}^* = \left(\hat{\mu}, \hat{\rho}, \hat{\sigma}^2\right)'$ achieves the

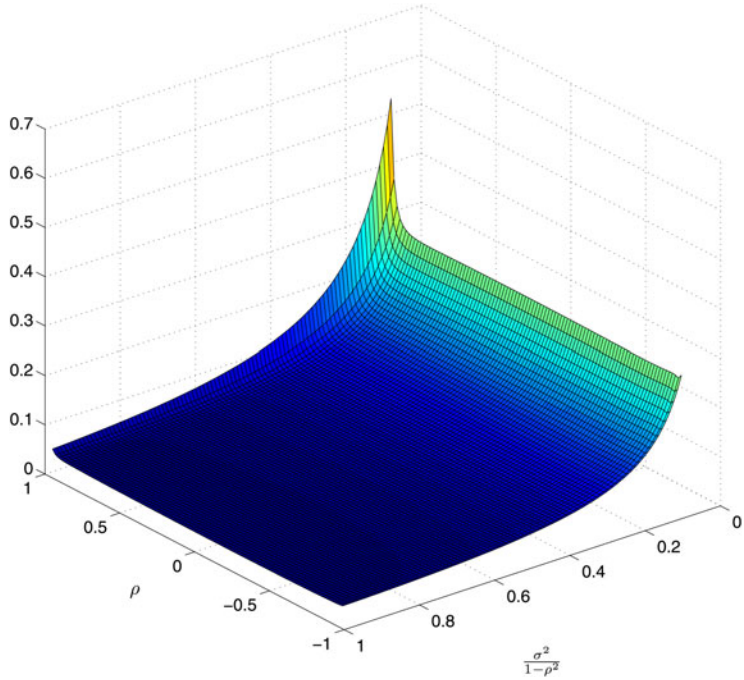

FIGURE 3. Semiparametric efficiency bound of the microparameter γ_2.

The figure displays $\left(\frac{1}{nT} B^*_{\gamma_2\gamma_2}\right)^{1/2}$, where $B^*_{\gamma_2\gamma_2}$ is the semiparametric efficiency bound for parameter γ_2 and $n = 1000, T = 20$, as a function of the autoregressive coefficient ρ and the variance $\frac{\sigma^2}{1-\rho^2}$ of the factor process (f_t).

asymptotic efficiency bound with observable factor, that is, the Cramer-Rao bound for θ in the linear Gaussian model (6.1). Thus, the asymptotic efficiency bound is such that the estimators of $(\mu, \rho)'$ and σ^2 are asymptotically independent, root-T consistent, with asymptotic variance:

$$B^*_{(\mu,\rho)} = \sigma_0^2 \mathrm{E}_0 \left[\begin{pmatrix} 1 & f_t \\ f_t & f_t^2 \end{pmatrix} \right]^{-1} = \begin{pmatrix} \sigma_0^2 + \mu_0^2 \frac{1+\rho_0}{1-\rho_0} & -\mu_0(1+\rho_0) \\ -\mu_0(1+\rho_0) & 1-\rho_0^2 \end{pmatrix},$$

for $(\mu, \rho)'$, and $B^*_{\sigma^2} = 2\sigma_0^4$, for σ^2.

Figures 4 and 5 display the distributions (solid lines) of the efficient estimators $\hat{\mu}$, $\hat{\rho}$, and $\hat{\sigma}^2$ in the Monte-Carlo study for sample sizes $n = 200$, $T = 20$, and $n = 1000$, $T = 20$, respectively. The parameter values are given in Table 1. We also display Gaussian distributions (dashed lines) centered at the true values of the parameters, with variances equal to the efficiency bounds divided by T. As expected, it is more difficult to estimate the autoregressive coefficient ρ and the variance σ^2 than to estimate the intercept μ. The estimators $\hat{\rho}$ and $\hat{\sigma}^2$ feature

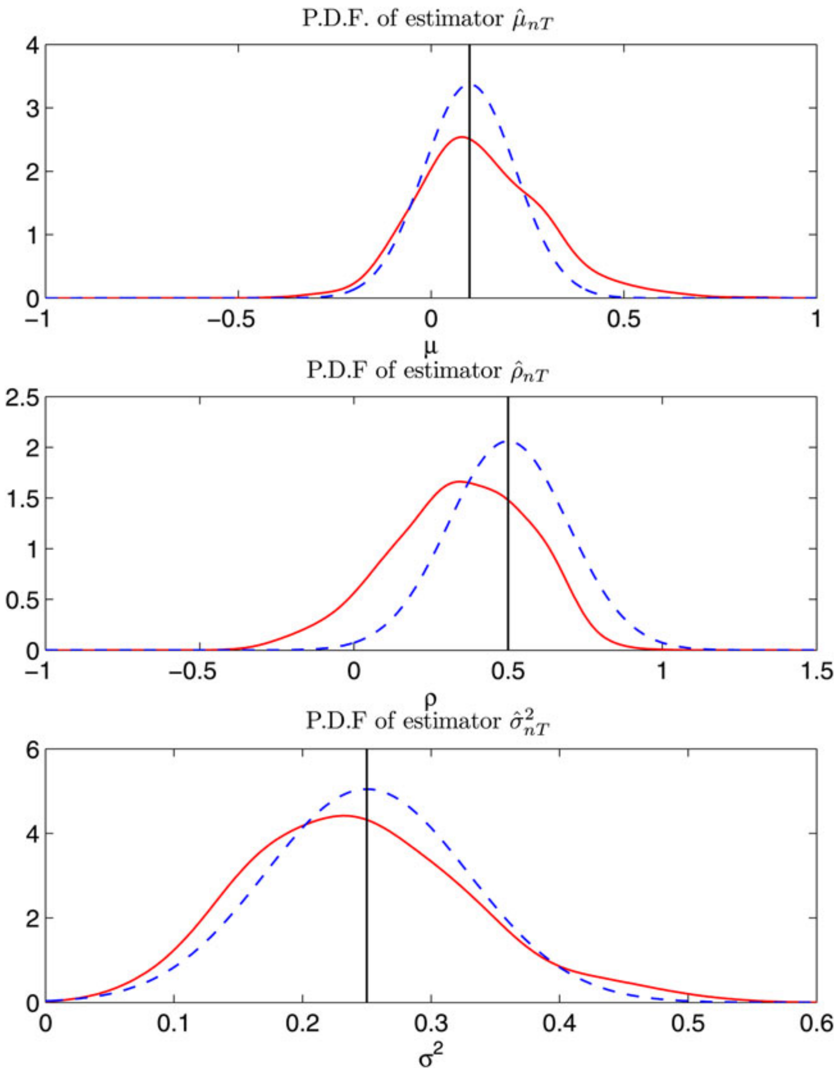

FIGURE 4. Distribution of the two-step efficient estimators of the macroparameters, sample size $n = 200$ and $T = 20$.

The solid lines give the p.d.f. of the two-step efficient estimators of parameter μ (upper Panel, true value 0.1), parameter ρ (central Panel, true value 0.5), and parameter σ^2 (lower Panel, true value 0.25). The p.d.f. is computed by a kernel density estimator. Sample sizes are $n = 200$ and $T = 20$. The dashed lines in the three Panels give the p.d.f. of a normal distribution centered at the true value of the parameter and with variance equal to the efficiency bound divided by T.

moderate downward biases. By comparing Figure 4 and Figure 5, we notice that the standard deviations of the estimators are rather similar for the two sample sizes and do not scale with n. Moreover, by comparing Figure 2 and Figure 5,

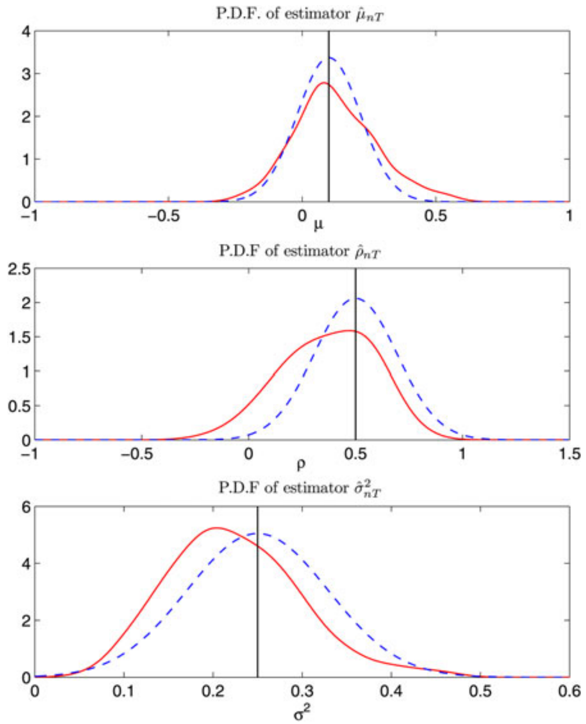

FIGURE 5. Distribution of the two-step efficient estimators of the macroparameters, sample size $n = 1000$ and $T = 20$.

The solid lines give the p.d.f. of the two-step efficient estimators of parameter μ (upper Panel, true value 0.1), parameter ρ (central Panel, true value 0.5), and parameter σ^2 (lower Panel, true value 0.25). The p.d.f. is computed by a kernel density estimator. Sample sizes are $n = 1000$ and $T = 20$. The dashed lines in the three Panels give the p.d.f. of a normal distribution centered at the true value of the parameter and with variance equal to the efficiency bound divided by T.

it is seen that the discrepancy between the finite-sample distribution and the asymptotic efficiency bound is more pronounced for the macroparameters than for the microparameters for our sample sizes. These findings are a consequence of the different convergence rates of the two types of estimators that are $\sqrt{T}$ and $\sqrt{nT}$, respectively.

7. CONCLUDING REMARKS

We consider nonlinear dynamic panel models with common unobservable factors, in which it is possible to disentangle the micro- and the macrodynamics, the latter ones being captured by the factor dynamics. Such models are often encountered in finance and insurance when the joint individual risk dynamics are followed in large homogeneous pools of individual contracts such as corporate loans, household mortgages, or life insurance contracts. In such applications the model allows to disentangle the dynamics of systematic and unsystematic risks. These models are also appropriate for extracting the business cycle from tendency surveys (Gouriéroux and Monfort, 2009), to disentangle inequality and mobility features in the dynamic analysis of income distributions, or to analyze longevity risk (e.g., Lee and Carter, 1972; Schrager, 2006; Gouriéroux and Monfort, 2008). The considered specifications include both segment fixed effects and dynamic factors, but no individual fixed effects. For large cross-sectional and time dimensions $n, T \to \infty$, such that $T^{\nu}/n = O(1)$, $\nu > 1$, we have derived the semiparametric efficiency bound of the parameter β characterizing the microdynamics. This semiparametric efficiency bound takes into account the factor unobservability and coincides with the bound for known factor transition. The efficiency bound for parameter θ characterizing the macrodynamics is the same as if the factor were observable. Moreover, we have shown that the efficiency bound for (β, θ) can be reached by estimators that do not involve numerical integration w.r.t. the factor path and thus are easy to implement. These results require a large cross-sectional dimension to approximate the likelihood function by a closed form expression. When $T^{\nu}/n = O(1)$, $\nu > 3/2$, the higher-order terms in this expansion around $n = \infty$ are the basis for granularity adjustments, which yield asymptotically efficient estimators that are more accurate approximations of the true ML estimator. For prediction purposes, it could be useful to include time-invariant observable individual characteristics x_i in the microdensity $h(y_{i,t}|y_{i,t-1}, x_i, f_t; \beta)$. The results in the paper can be easily extended to this case.

The condition $T^{\nu}/n = O(1)$, $\nu > 1$, implies that in our framework the incidental parameters problem does not induce a bias in the first-order asymptotic distribution of the estimators. An interesting venue for future research is to investigate the properties of the CSA, GA, and true ML estimators, as well as of the two-step estimators, when T/n converges to a nonzero constant. This asymptotic scheme is common in the panel literature with individual fixed effects, which focuses on bias correction of the fixed effects estimator (for analytical bias correction, see, e.g., Woutersen, 2002; Hahn et al., 2004; Arellano and Hahn, 2006; Bester and Hansen, 2009; Hahn et al., 2011 and for bias correction by jackknife and indirect inference, see, e.g., Hahn et al., 2004; Dhaene, Jochmans, and Thuysbaert, 2006; Gourieroux, Phillips, and Yu, 2010). When $n, T \to \infty$ such that $T/n \to c$ (say), $c > 0$, it is possible to prove that the fixed effects estimator $\hat{\beta}^*_{nT}$, as well as the CSA and GA ML estimators of β are asymptotically normal, with variance-covariance matrix $(I_0^*)^{-1}$, and feature an asymptotic bias in the general nonlinear case. In the case of a linear dynamic panel model with time effects, the results in Hahn and Moon (2006) imply that the fixed effects estimator of the autoregressive parameter is asymptotically unbiased. Moreover, since the true ML estimator of β admits an interpretation as a random effects estimator (see Section 3), the results in Hahn, Kuersteiner, and Cho (2005) and Arellano and Bonhomme (2009) suggest that the true ML estimator of parameter β could be first-order asymptotically unbiased when $T/n \to c$, $c > 0$. The proof of this conjecture is beyond the scope of the present paper.

NOTES

1. See, e.g., Douc, Moulines, and Rydèn (2004) for the asymptotic properties of the ML estimator in autoregressive models with Markov regimes.

2. When the subpopulation index k, with $k = 1, \ldots, K$, is introduced explicitly, the variables are triply indexed by k, i, t, and the latent model becomes $y^*_{k,i,t} = \alpha_k + \gamma_k F_{k,t} + \sigma_k u_{k,i,t}$, where $k = 1, \ldots, K$, $i \in PaR_{k,t}$ and $t = 1, \ldots, T$. The subpopulations fixed effects are α_k, γ_k, σ_k and the model allows for a crossing of fixed effects γ_k with time stochastic effects $F_{k,t}$. Moreover, we get a joint multifactor model, whenever the factors $F_{k,t}$ are different among classes.

3. The underestimation of the asset correlation parameter in 2007–2008 played a key role in the underpricing of Collateralized Debt Obligations (CDO) contracts and lead to severe losses during the recent subprime crisis.

4. In Basel II regulation, the lack of accuracy on estimated model parameters might be taken into account by means of reserves for estimation risk. However, in the current implementation, these reserves are usually set to zero. Moreover, the updating of the estimated individual fixed effects would induce a large volatility of the required capital for credit risk, with undesirable effects on financial market stability.

5. In an unobservable factor model, the factor process is usually defined up to some nonlinear dynamic transformation. Assumptions A.1 and A.2 have to be satisfied for an appropriate choice of factor f_t. As a consequence, the Markov assumption on factor f_t is rather mild. For instance, let us consider a dynamic model with a factor f_t satisfying Assumption A.1 and admitting a nonlinear moving average representation $f_t = a(\varepsilon_t, \varepsilon_{t-1}; \theta)$, say, with $\varepsilon_t \sim IIN(0,1)$. Then Assumptions A.1 and A.2 are satisfied with f_t replaced by $f_t^* = (\varepsilon_t, \varepsilon_{t-1})'$ and $h(y_{i,t}|y_{i,t-1}, f_t; \beta)$ replaced by $h^*(y_{i,t}|y_{i,t-1}, f_t^*; \beta^*) = h(y_{i,t}|y_{i,t-1}, a(\varepsilon_t, \varepsilon_{t-1}; \theta); \beta)$, where $\beta^* = (\beta', \theta')'$.

6. As in the ASRF model for default, we can introduce explicitly the fixed effects of the segments, that is, the factors $f_{k,t}$ can differ among the segments and parameters β_k, θ_k can depend on k, with $k = 1, \ldots, K$.

7. More precisely, by the de Finetti-Hewitt-Savage theorem, the infinite sequence of histories $y_i = (y_{i,t}, t = 1, \cdots, T)$, $i = 1, 2, \cdots$, is exchangeable if and only if there exists a sigma-field $\mathcal{F}$ such that y_i, $i = 1, 2, \cdots$, are i.i.d. conditional on $\mathcal{F}$ [see also Kingman, 1978]. Here, we assume that the sigma-field $\mathcal{F}$ is generated by the finite-dimensional Markov process (f_t).

8. In such a model with unobservable factors, the ML estimate could be computed numerically by means of an Expectation-Maximization (EM) algorithm (Dempster, Laird, and Rubin, 1977). The EM algorithm applies recursively the Expectation step, which computes the function:

$$Q\left[(\beta,\theta)|(\beta^{(p)},\theta^{(p)})\right] = \mathop{\mathrm{E}}_{(\beta^{(p)},\theta^{(p)})}\left[\log l\left(\underline{y_T}, \underline{f_T}; \beta, \theta\right)|\underline{y_T}\right],$$

and the Maximization step, providing the next value of the parameter as:

$$\left(\beta^{(p+1)}, \theta^{(p+1)}\right) = \mathop{\arg\max}_{(\beta,\theta)} Q\left[(\beta,\theta)|\left(\beta^{(p)}, \theta^{(p)}\right)\right].$$

In our nonlinear dynamic framework, the Expectation step requires the numerical approximation of function Q by means of a Gibbs sampler [see, e.g., Cappé, Moulines, and Rydén (2005) for general properties, and Fiorentini, Sentana, and Shephard (2004) and Duffie et al. (2009) for applications to credit and finance]. The closed form expression of the approximate likelihood function given in Proposition 1 avoids the numerically cumbersome expectation step, while controlling the approximation error.

9. When the microparameter β and the time effect f_t are information orthogonal, that is, $\mathrm{E}_0\left[-\frac{\partial^2 \log h(y_{i,t}|y_{i,t-1}, f_t; \beta_0)}{\partial\beta\partial f_t'}|\underline{f_t}\right] = 0$, $\mathbb{P}$-a.s., the score w.r.t. β of the approximated log-likelihood in Proposition 1(i) corresponds to the score of the profile log-likelihood in

Cox and Reid (1987) and to the score of the penalized log-likelihood in Bester and Hansen (2009), up to order $o_p(1/n)$. When information orthogonality does not apply, the scores of the three log-likelihoods differ at order $O_p(1/n)$.

10. See Belloni and Chernozhukov (2009) for another extension of the asymptotic normality of the (quasi-) posterior distribution when the number of parameters increases with the sample size. This extension is derived under different regularity conditions.

11. The proof of Proposition 3 shows that the CSA and GA ML estimators of parameter β based on a misspecified factor model remain consistent and first-order asymptotically efficient (but not the CSA and GA ML estimators of parameter θ).

12. Approximations of factor values in panel data with large cross-sectional and time dimensions have been proposed in, e.g., Forni and Reichlin (1998), Forni, Hallin, Lippi, and Reichlin (2000), Bai and Ng (2002), Stock and Watson (2002), and Connor, Hagmann, and Linton (2012). All these papers consider linear factor models for the microdynamics.

13. See also Hahn and Kuersteiner (2011) for the nonlinear case.

14. In practice, the alternative $k = K$ corresponds typically to default, which is an absorbing state. Then, the stationarity and mixing conditions in Assumptions A.3 and A.4 are not satisfied and the estimators might be inconsistent. A stationary and mixing framework can be recovered if we assume that the number n of operating firms in the portfolio is kept constant in time by replacing each defaulted firm by a new one, whose initial rating is randomly distributed across classes $k = 1, \ldots, K-1$ according to some distribution. This mechanism reflects the "static pool" definition of Standard & Poor's (see Brady and Bos, 2002). Then, the methodology can be applied considering the model for the transitions between rating classes $k = 1, \ldots, K-1$ (see Gagliardini and Gouriéroux 2005b). For expository purpose, we do not consider an absorbing state here and refer to Gagliardini and Gouriéroux (2005b, Section 4.2), for more details.

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APPENDIX A

In Appendix A.1 we provide the list of regularity conditions for the asymptotic analysis. The proofs of Propositions 1, 2, 3, 5, and 6 are given in Appendix A.2. They rely on Limit Theorems 1–3 and Lemmas 1–8, which are provided in the supplementary material. Appendix A.3 presents the proof of identification of the microparameters in the stochastic migration model. We denote by $\|A\|$ the Frobenius norm of matrix A. Moreover, b_i, c_i, d_i, and γ_i, for $i = 1, 2, \ldots$, denote constants in the regularity conditions, while $C_1, C_2, \ldots$ denote generic constants used in the proofs.

A.1. Regularity conditions

In addition to Assumptions A.1–A.9, we use the regularity conditions given below to derive the large sample properties of the estimators. Due to the invariance of the true and approximate log-likelihood functions under one-to-one factor transformations $f \to \phi(f)$, the validity of Propositions 1, 2, 3, and 6 only requires that the regularity conditions are satisfied for a suitable choice of the factor process.

H.1: *The parameter sets $\mathcal{B} \subset \mathbb{R}^q$ and $\Theta \subset \mathbb{R}^p$ are compact. The true parameter values β_0 and θ_0 are interior points of sets $\mathcal{B}$ and Θ, respectively.*

H.2: *For any date $t \in \mathbb{N}$, the mapping $(\beta, f) \to \mathcal{L}_t(\beta, f) = E_0\left[\log h\left(y_{i,t}|y_{i,t-1}, f; \beta\right)|\underline{f_t}\right]$ is continuous on the set $\mathcal{B} \times \mathbb{R}^m$, for any factor path $\underline{f_t}$ outside a set $\mathcal{N}$ with probability zero. For any given $\beta \in \mathcal{B}$, the mapping $f \to \mathcal{L}_t(\beta, f)$ admits a unique maximum, denoted by $f_t(\beta)$, for any factor path $\underline{f_t}$ outside $\mathcal{N}$. Moreover, $E_0\left[\frac{\partial \log h(y_{i,t}|y_{i,t-1}, f_t(\beta); \beta)}{\partial f_t}|\underline{f_t}\right] = 0$ and the matrix $E_0\left[-\frac{\partial^2 \log h(y_{i,t}|y_{i,t-1}, f_t(\beta); \beta)}{\partial f_t \partial f_t'}|\underline{f_t}\right]$ is positive definite, for any $\beta \in \mathcal{B}$ and any factor path $\underline{f_t}$ outside $\mathcal{N}$.*

H.3: *The microdensity is such that* (i) $\sup\{h(y_{i,t}|y_{i,t-1},f_t;\beta) : y_{i,t},y_{i,t-1}\in\mathbb{R}, f_t\in\mathbb{R}^m,\beta\in\mathcal{B}\}<\infty$, *and* (ii) $\mathrm{E}_0\left[\sup_{\beta\in\mathcal{B}}\left|\frac{\partial^{|\alpha|}\log h(y_{i,t}|y_{i,t-1},f_t(\beta);\beta)}{\partial^\alpha(\beta',f_t')'}\right|^8\right]<\infty$, *for any multi-index* $\alpha\in\mathbb{N}^{q+m}$ *with* $|\alpha|\le 3$.

H.4: *Let us define* $\xi_{t,1}=\max\{\xi^*_{t,1},\xi^{**}_{t,1}\}$, *where* $\xi^*_{t,1}=\left(\inf_{\beta\in\mathcal{B}}\inf_{f\in\mathbb{R}^m:\|f-f_t(\beta)\|\le\eta^*}\lambda_t(\beta,f)\right)^{-1}$, $\lambda_t(\beta,f)>0$ *is the smallest eigenvalue of the positive definite matrix* $I_t(\beta,f)\equiv\mathrm{E}_0\left[-\frac{\partial^2\log h(y_{i,t}|y_{i,t-1},f;\beta)}{\partial f\partial f'}|\underline{f_t}\right]$, $\eta^*>0$, *and*

$$\xi^{**}_{t,1}=\sup_{\alpha\in\mathbb{N}^{q+m}:|\alpha|\le 5}\ \sup_{\beta\in\mathcal{B}}\mathrm{E}_0\left[\sup_{f\in\mathbb{R}^m:\|f-f_t(\beta)\|\le\eta^*}\left|\frac{\partial^{|\alpha|}\log h(y_{i,t}|y_{i,t-1},f;\beta)}{\partial^\alpha(\beta',f')'}\right|^2|\underline{f_t}\right].$$

Then, $\sup_{t\in\mathbb{N}}\mathbb{P}\left[\xi_{t,1}\ge u\right]\le b_1\exp\left(-c_1u^{d_1}\right)$ *as* $u\to\infty$, *for some constants* $b_1,c_1,d_1>0$.

H.5: *The stationary process* $\xi_{t,2}=\sup_{\beta\in\mathcal{B}}\|f_t(\beta)\|$ *is such that* $\mathbb{P}\left[\xi_{t,2}\ge u\right]\le b_2\exp\left(-c_2u^{d_2}\right)$ *as* $u\to\infty$, *for some constants* $b_2,c_2,d_2>0$.

H.6: *The set* $\mathcal{F}_n\subset\mathbb{R}^m$ *is* (i) *compact and convex, for any* $n\in\mathbb{N}$, *and such that* (ii) $B_{r_n}(0)\subset\mathcal{F}_n$, *where* $B_{r_n}(0)$ *denotes the open ball in* $\mathbb{R}^m$ *centered at* 0 *and with radius* $r_n=\left[(2/c_2)\log(n)\right]^{1/d_2}$ *and* (iii) $\mathcal{F}_n\subset B_{R_n}(0)$, *where* $R_n=O([\log(n)]^{\gamma_1})$ *for a constant* γ_1 *with* $\gamma_1\ge 1/d_2$.

H.7: *There exists a constant* $\gamma_2\ge 0$ *such that:*

$$\mathcal{K}_t\equiv\inf_{n\ge 1}\inf_{\beta\in\mathcal{B}}\inf_{f\in\mathcal{F}_n:f\ne f_t(\beta)}[\log(n)]^{\gamma_2}\frac{2KL_t(f,f_t(\beta);\beta)}{\|f-f_t(\beta)\|^2}>0,$$

for any t, $\mathbb{P}$*-a.s., where* $KL_t(f,f_t(\beta);\beta)\equiv\mathrm{E}_0\left[\log\left(\frac{h(y_{i,t}|y_{i,t-1},f_t(\beta);\beta)}{h(y_{i,t}|y_{i,t-1},f;\beta)}\right)|\underline{f_t}\right]$.

H.8: *There exists a constant* $\gamma_3\ge 0$ *such that:*

$$\mathcal{R}_t\equiv\sup_{n\ge 1}[\log(n)]^{-\gamma_3}\mathrm{E}_0\left[\sup_{\beta\in\mathcal{B}}\sup_{f\in\mathcal{F}_n}\left\|\frac{\partial\log h(y_{i,t}|y_{i,t-1},f;\beta)}{\partial(\beta',f')'}\right\|^4|\underline{f_t}\right]<\infty,$$

for any t, $\mathbb{P}$*-a.s. Moreover,* $\mathrm{E}_0\left[\mathcal{R}_t^2\right]<\infty$.

H.9: *Let* $\xi_{t,3}=\max\{\Gamma_t,\mathcal{K}_t^{-1}\}$, *where*
$\Gamma_t\equiv\sup_{n\ge 1}\sup_{\beta\in\mathcal{B}}\sup_{f\in\mathcal{F}_n}[\log(n)]^{-\gamma_3}\ \mathrm{E}_0\left[\left\|\frac{\partial\log h(y_{i,t}|y_{i,t-1},f;\beta)}{\partial f}\right\|^2|\underline{f_t}\right]$ *and* $\mathcal{K}_t$ *is defined in Assumption H.7. Then,* $\sup_{t\in\mathbb{N}}\mathbb{P}\left[\xi_{t,3}\ge u\right]\le b_3\exp\left[-c_3u^{d_3}\right]$ *as* $u\to\infty$, *for some constants* $b_3,c_3,d_3>0$.

H.10: (i) *There exists a constant* $\gamma_4 \geq 0$ *such that* $\sup_{t\in\mathbb{N}} \mathbb{P}[\xi_{t,4} \geq u] \leq b_4 \exp\left(-c_4 u^{d_4}\right)$, *as* $u \to \infty$, *for some constants* $b_4, c_4, d_4 > 0$, *where* $\xi_{t,4} = \sup_{n\geq 1} \sup_{f\in\mathcal{F}_n} \sup_{\beta\in\mathcal{B}} [\log(n)]^{-\gamma_4} \mathrm{E}_0 \left[|\log h(y_{i,t}|y_{i,t-1}, f; \beta)| \,|\underline{f_t}\right]$. (ii) *There exists a constant* $\gamma_5 \geq 0$ *such that*

$$\mathrm{E}_0 \left[\sup_{\beta\in\mathcal{B}} \sup_{f\in\mathcal{F}_n} |\log h(y_{i,t}|y_{i,t-1}, f; \beta)|^4 \right] = O\left([\log(n)]^{\gamma_5}\right) \quad \textit{and}$$

$$\mathrm{E}_0 \left[\sup_{\beta\in\mathcal{B}} \sup_{f\in\mathcal{F}_n} \left\| \frac{\partial \log h(y_{i,t}|y_{i,t-1}, f; \beta)}{\partial(\beta', f')'} \right\| \right] = O\left([\log(n)]^{\gamma_5}\right).$$

(iii) *Conditions* (i) *and* (ii) *are satisfied when replacing* $\log h(y_{i,t}|y_{i,t-1}, f; \beta)$ *by* $\dfrac{\partial^{|\alpha|} \log h(y_{i,t}|y_{i,t-1}, f; \beta)}{\partial^{\alpha}(\beta', f')'}$, *for any multi-index* $\alpha \in \mathbb{N}^{q+m}$ *with* $|\alpha| \leq 5$.

H.11: *The stationary distribution* $\mathbb{P}_\theta$ *of Markov process* (f_t) *associated with the transition density* $g(f_t|f_{t-1};\theta)$ *is such that* $\sup_{\theta\in\Theta} \mathbb{P}_\theta[f_t \in \mathcal{F}_n^c] = O\left(e^{-\gamma_6 n^2}\right)$, *for a constant* $\gamma_6 > 0$.

H.12: *The function* $G(F_t;\theta) = \log g(f_t|f_{t-1};\theta)$, *where* $F_t = (f_t', f_{t-1}')'$, *is:* (i) *differentiable w.r.t.* $F_t \in \mathbb{R}^{2m}$ *and* $\theta \in \Theta$, *and such that* (ii) $\mathrm{E}_0 \left[\sup_{\theta\in\Theta} \sup_{\beta\in\mathcal{B}} \left\| \dfrac{\partial G(F_t(\beta);\theta)}{\partial\theta} \right\| \right] < \infty$ *and* (iii) $\sup_{t\in\mathbb{N}} \mathbb{P}\left[\xi_{t,5} \geq u\right] \leq b_5 \exp\left(-c_5 u^{d_5}\right)$, *as* $u \to \infty$, *for some constants* $b_5, c_5, d_5 > 0$, *where* $\xi_{t,5} = \sup_{\theta\in\Theta} \sup_{\beta\in\mathcal{B}} \sup_{F\in\mathbb{R}^{2m}:\|F-F_t(\beta)\|\leq\eta^*} \left\| \dfrac{\partial^{|\alpha|} G(F;\theta)}{\partial F^{\alpha}} \right\|$, $\eta^* > 0$, *for any multi-index* $\alpha \in \mathbb{N}^{2m}$ *such that* $|\alpha| \leq 3$.

H.13: *Assumption H.12 is satisfied for* $G(F_t;\theta) = \dfrac{\partial^2 \log g(f_t|f_{t-1};\theta)}{\partial\theta\partial\theta'}$, $= \dfrac{\partial^2 \log g(f_t|f_{t-1};\theta)}{\partial\theta\partial f_t'}$, *and* $= \dfrac{\partial^2 \log g(f_t|f_{t-1};\theta)}{\partial\theta\partial f_{t-1}'}$.

H.14: *The macroscore is such that* $\mathrm{E}_0 \left[\left\| \dfrac{\partial \log g(f_t|f_{t-1};\theta_0)}{\partial\theta} \right\|^4 \right] < \infty$.

Assumption H.1 is a standard condition on parameter sets and true parameter values. Assumptions H.2–H.4 concern the micro log-density and the pseudo-true factor values. Specifically, Assumption H.2 corresponds to the global and local identification conditions for the pseudo-true factor value $f_t(\beta)$ as the maximizer of the asymptotic cross-sectional likelihood function. The pseudo-true factor value $f_t(\beta)$ is well-defined for any factor path outside a negligible set $\mathcal{N}$, which can be selected uniformly w.r.t. the microparameter $\beta \in \mathcal{B}$. The definition of $f_t(\beta)$ is independent of the version of the conditional expectation $\mathrm{E}_0\left[\log h\left(y_{i,t}|y_{i,t-1}, f; \beta\right)|\underline{f_t}\right]$. Indeed, the functions $\mathcal{L}_t(\beta, f)$ corresponding to two versions of the conditional expectation $\mathrm{E}_0\left[\log h\left(y_{i,t}|y_{i,t-1}, f; \beta\right)|\underline{f_t}\right]$, for (β, f) varying, coincide on a countable dense subset of $\mathcal{B} \times \mathbb{R}^m$, for any factor path outside a negligible set $\mathcal{N}$. By the continuity condition is Assumption H.2, these two functions coincide on the entire set $\mathcal{B} \times \mathbb{R}^m$, for any factor path outside $\mathcal{N}$. By Lemma 2 in Jennrich (1969), the pseudo-true factor value $f_t(\beta)$ is a measurable function of the factor path $\underline{f_t}$ for any $\beta \in \mathcal{B}$ and is a continuous function of β for any factor path $\underline{f_t}$ outside a negligible set. Then, Assumption A.3 implies the strict stationarity and ergodicity of the pseudo-true factor value $f_t(\beta)$, for any given value of $\beta \in \mathcal{B}$, and of the process $\sup_{\beta \in \mathcal{B}} \| f_t(\beta) \|$. In Assumption H.3(i) the microdensity is upper bounded, uniformly w.r.t. the factor value and microparameter. Assumption H.3(ii) requires finite higher-order moments for $\log h(y_{i,t}|y_{i,t-1}, f; \beta)$ and its derivatives w.r.t. β and f, evaluated at $f = f_t(\beta)$, uniformly in $\beta \in \mathcal{B}$. Assumption H.4 strengthens the local identification condition of the pseudo-true factor value in Assumption H.2. It requires that matrix $I_t(\beta, f)$ is positive definite for any factor value f in a neighborhood of $f_t(\beta)$, uniformly w.r.t. the microparameter $\beta \in \mathcal{B}$, and for any factor path $\underline{f_t}$, $\mathbb{P}$-a.s. Moreover, Assumption H.4 implies a tail condition on the distribution of the positive process $\xi^*_{1,t}$. This condition is satisfied, when the factor paths associated with very small eigenvalues $\lambda_t(\beta, f)$, for some parameter value $\beta \in \mathcal{B}$ and factor value f close to $f_t(\beta)$, are sufficiently infrequent. Assumptions H.4 also implies a tail condition for the distribution of process $\xi^{**}_{t,1}$ involving higher-order derivatives of the micro log-density function. If $\xi^*_{t,1}$ and $\xi^{**}_{t,1}$ are measurable functions of the factor path, the process $\xi_{t,1}$ is strictly stationary from Assumption A.3, and we can dispense with the sup over t in the bound. Assumptions H.1–H.4 and their implications are used to show that the Regularity Conditions RC.2 and RC.3 in Limit Theorem 3 are satisfied when proving the uniform convergence of the profile log-likelihood function $\mathcal{L}^*_{nT}(\beta)$, and of its second-order derivative matrix w.r.t. β [see Lemmas 1(i) and 6(1i) in the supplementary material].

Assumptions H.5, H.6(i) and (ii), and H.7–H.9 are used in Limit Theorem 1 to derive the uniform rate of convergence of the factor approximations. Specifically, Assumption H.5 concerns the tail of the stationary distribution of process $\sup_{\beta \in \mathcal{B}} \| f_t(\beta) \|$. Assumptions H.6(ii) and (iii) introduce lower and upper bounds on the growth rate of set $\mathcal{F}_n$ as $n \to \infty$. These bounds are given in terms of expanding balls with radii of the order of powers of $\log(n)$. Under Assumptions H.5 and H.6(ii), the pseudo-true factor value $f_t(\beta)$ is in $\mathcal{F}_n$, for any $1 \leq t \leq T$ and $\beta \in \mathcal{B}$, with probability approaching (w.p.a.) 1 at rate $O(T/n^2)$. Assumption H.7 concerns the identifiability of the pseudo-true factor values from the asymptotic cross-sectional log-likelihood function. For any given microparameter value β and date t, the mapping $f \to KL_t(f, f_t(\beta); \beta)$ is a Kullback-Leibler divergence of the conditional p.d.f. $h(\cdot|\cdot, f; \beta)$ parametrized by $f \in \mathcal{F}_n$ from the pseudo-true conditional p.d.f. $h(\cdot|\cdot, f_t(\beta); \beta)$ given $\underline{f_t}$ under misspecification. From the global identification Assumption H.2, we have $KL_t(f, f_t(\beta); \beta) > 0$, for any factor value $f \neq f_t(\beta)$, parameter value β, and date t, $\mathbb{P}$-a.s. Assumption H.7 strengthens this condition by requiring that mapping $f \to KL_t(f, f_t(\beta); \beta)$ is bounded below by a quadratic function proportional to the squared distance $\| f - f_t(\beta) \|^2$, uniformly in $\beta \in \mathcal{B}$, $f \in \mathcal{F}_n$, and $n \in \mathbb{N}$. The scale factor is allowed to converge to zero at most at a logarithmic rate, as set $\mathcal{F}_n$ increases. Assumption H.8 introduces a uniform bound on the higher-order moments of the score of the log-density w.r.t. factor value $f \in \mathcal{F}_n$ and parameter $\beta \in \mathcal{B}$. The conditional moment of order 4 is allowed to diverge at a logarithmic rate as $\mathcal{F}_n$ increases. Assumption H.9 is a tail condition on the distribution of the processes $\mathcal{K}_t^{-1}$ and Γ_t. These processes correspond to the inverse of the measure $\mathcal{K}_t$ related to the conditional Kullback-Leibler discrepancy for cross-sectional factor approximation, and the measure Γ_t of second-order conditional moment of the score of the log-density w.r.t. f_t: they are both functions of the factor path

$\underline{f_t}$. Assumption H.9 is satisfied when the probability mass of $\mathcal{K}_t$ in a neighborhood of zero, and the probability mass for large values of Γ_t, are small.

Assumption H.10 introduces tail conditions and uniform bounds on conditional moments of the log microdensity, and of its derivatives w.r.t. factor f_t and parameter β. This assumption is used in Lemma 2 (see the supplementary material) to show the convergence in probability of the cross-sectional log-likelihood function, and of its derivatives w.r.t. the factor values, uniformly over the parameter value $\beta \in \mathcal{B}$, factor value $f \in \mathcal{F}_n$, and dates $1 \le t \le T$.

Assumptions H.11–H.14 concern the macro log-density and its derivatives w.r.t. factor values and macroparameter θ. Specifically, Assumption H.11 requires that the tail of the stationary distribution of the factor process is sufficiently thin, uniformly w.r.t. the macroparameter θ. This condition is used in Proposition A.2 (see Appendix A.2.1) to show that the contribution to the log-likelihood function coming from factor paths admitting some values outside set $\mathcal{F}_n$ is asymptotically negligible. Assumptions H.12(i) and (ii) require that function $\log g(f_t|f_{t-1};\theta)$ is differentiable w.r.t. the factor values and the macroparameter θ, with uniformly finite expectation of the first-order derivative w.r.t. θ. Assumption H.12(iii) is a condition on the tail of process $\xi_{t,5}$ involving the derivatives of $\log g(f_t|f_{t-1};\theta)$ w.r.t. the factor values. Assumption H.14 is a bound on the fourth-order moment of the macroscore $\dfrac{\partial \log g(f_t|f_{t-1};\theta_0)}{\partial\theta}$. Assumptions H.12–H.14 are used to show that Regularity Condition RC.1 in Limit Theorem 2 is satisfied when proving the convergence of $\mathcal{L}_{1,nT}(\beta,\theta)$, and of the Hessian $\dfrac{\partial^2 \mathcal{L}_{1,nT}(\beta,\theta)}{\partial\theta\partial\theta'}$, uniformly in $\beta \in \mathcal{B}, \theta \in \Theta$ [see Lemmas 1(ii) and 6(1ii) in the supplementary material].

A.2. Proofs of the asymptotic results

A.2.1. Proof of Proposition 1

(i) Preliminary expansions

Let us write the joint density in equation (3.2) as $l\left(\underline{y_T};\beta,\theta\right) = \int \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta)d\boldsymbol{f}_T$, where $\boldsymbol{f}_T = (f_1', f_2', \ldots, f_T')' \in \mathbb{R}^{Tm}$, function ϕ_{nT} is defined by $\phi_{nT}(\boldsymbol{f}_T;\beta) = \sum_{t=1}^{T} \mathcal{L}_{n,t}(f_t;\beta)$, with $\mathcal{L}_{n,t}(f_t;\beta) = \frac{1}{n}\sum_{i=1}^{n} \log h(y_{i,t}|y_{i,t-1}, f_t;\beta)$, and $g_T(\boldsymbol{f}_T;\theta) = \prod_{t=1}^{T} g(f_t|f_{t-1};\theta)$. Let $\varepsilon_n \downarrow 0$ be a sequence indexed by n, and let $B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))$ denote the open ball in $\mathbb{R}^{Tm}$ with radius ε_n centered in $\hat{\boldsymbol{f}}_{nT}(\beta) = (\hat{f}_{n,1}(\beta)', \ldots, \hat{f}_{n,T}(\beta)')'$. The integral in $l\left(\underline{y_T};\beta,\theta\right)$ can be decomposed as:

$$
\begin{aligned}
l\left(\underline{y_T};\beta,\theta\right) = &\int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta)d\boldsymbol{f}_T \\
&+ \int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))^c \cap \mathcal{F}_{nT}} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta)d\boldsymbol{f}_T \\
&+ \int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))^c \cap \mathcal{F}_{nT}^c} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta)d\boldsymbol{f}_T, \qquad \textbf{(A.1)}
\end{aligned}
$$

where $\mathcal{F}_{nT} = \mathcal{F}_n \times \cdots \times \mathcal{F}_n \subset \mathbb{R}^{Tm}$ and $\mathcal{F}_n \subset \mathbb{R}^m$ is the sequence of sets involved in the definition of estimator $\hat{f}_{n,t}(\beta)$ [see equation (3.3)].

Let us consider the first integral in the RHS of equation (A.1). We apply the Laplace approximation method with an explicit expression for the remainder term.

PROPOSITION A.1. *We have:*

$$\int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta) d\boldsymbol{f}_T$$
$$= \left(\frac{2\pi}{n}\right)^{Tm/2} \exp\left[nT\mathcal{L}^*_{nT}(\beta) + T\mathcal{L}_{1,nT}(\beta,\theta)\right] \Lambda_{nT}(\beta,\theta),$$

where

$$\Lambda_{nT}(\beta,\theta) = \frac{1}{(2\pi)^{Tm/2}} \int_{\mathcal{Z}_{nT}(\beta)} \exp\left(-\frac{1}{2}\sum_{t=1}^{T} z_t' z_t\right) \cdot \exp\left[\sum_{t=1}^{T} \psi_{n,t}\left(\hat{f}_{n,t}(\beta) + \frac{[I_{n,t}(\beta)]^{-1/2}}{\sqrt{n}} z_t, \hat{f}_{n,t-1}(\beta) + \frac{[I_{n,t-1}(\beta)]^{-1/2}}{\sqrt{n}} z_{t-1}; \beta, \theta\right)\right] dz, \tag{A.2}$$

the function $\psi_{n,t}$ is defined by:

$$\begin{aligned}\psi_{n,t}(f_t, f_{t-1};\beta,\theta) = {} & n\left[\mathcal{L}_{n,t}(f_t;\beta) - \mathcal{L}_{n,t}\left(\hat{f}_{n,t}(\beta);\beta\right)\right] \\ & + \frac{n}{2}\left[f_t - \hat{f}_{n,t}(\beta)\right]'\left[I_{n,t}(\beta)\right]\left[f_t - \hat{f}_{n,t}(\beta)\right] \\ & + \log g(f_t | f_{t-1};\theta) - \log g\left(\hat{f}_{n,t}(\beta) | \hat{f}_{n,t-1}(\beta);\theta\right),\end{aligned} \tag{A.3}$$

and the integration domain is $\mathcal{Z}_{nT}(\beta) = \left\{ z = (z_1', \ldots, z_T')' \in \mathbb{R}^{Tm} : \sum_{t=1}^{T} z_t' I_{n,t}(\beta)^{-1} z_t \leq n\varepsilon_n^2 \right\}$.

Proof of Proposition A.1. By the definition of function $\psi_{n,t}$ in equation (A.3), we have:

$$\int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta) d\boldsymbol{f}_T$$
$$= \prod_{t=1}^{T}\prod_{i=1}^{n} h\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta);\beta\right) \prod_{t=1}^{T} g\left(\hat{f}_{n,t}(\beta)|\hat{f}_{n,t-1}(\beta);\theta\right)$$
$$\cdot \int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))} \exp\left\{\sum_{t=1}^{T}\left(\psi_{n,t}(f_t, f_{t-1};\beta,\theta) - \frac{n}{2}[f_t - \hat{f}_{n,t}(\beta)]'[I_{n,t}(\beta)][f_t - \hat{f}_{n,t}(\beta)]\right)\right\} d\boldsymbol{f}_T.$$

Let us introduce the change of variable from f_t to $z_t = \sqrt{n}[I_{n,t}(\beta)]^{1/2}[f_t - \hat{f}_{n,t}(\beta)]$, for $t = 1, \ldots, T$. Then, we get:

$$\int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta)d\boldsymbol{f}_T$$
$$= \left(\frac{2\pi}{n}\right)^{Tm/2} \prod_{t=1}^{T} \left[\det I_{n,t}(\beta)\right]^{-1/2} \cdot \prod_{t=1}^{T}\prod_{i=1}^{n} h\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta);\beta\right)$$
$$\times \prod_{t=1}^{T} g\left(\hat{f}_{n,t}(\beta)\,|\,\hat{f}_{n,t-1}(\beta);\theta\right)\Lambda_{nT}(\beta,\theta). \tag{A.4}$$

By the definition of functions $\mathcal{L}^*_{nT}(\beta)$ and $\mathcal{L}_{1,nT}(\beta,\theta)$ in equations (3.7)–(3.8), the conclusion follows. ■

Let us now consider the next two terms in the RHS of equation (A.1). We bound these two terms at the beginning of the proof of Proposition A.2. The second integral in the RHS of equation (A.1) is asymptotically negligible for the expansion of the log-likelihood function in powers of $1/n$, if the sequence ε_n converges to zero slowly enough, namely if $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$ for some $\mu_1 > 0$. This condition on sequence $\varepsilon_n = o(1)$ can be satisfied if $T^\nu/n = O(1)$, with $\nu > 1$. The third integral in the RHS of equation (A.1) is asymptotically negligible if the set $\mathcal{F}_n$ expands fast enough as $n \to \infty$, whereas the tails of the factor distribution are not too heavy (see Assumption H.11).

PROPOSITION A.2. *Under Assumptions A.1–A.5 and H.1–H.12, if $T^\nu/n = O(1)$, for $\nu > 1$, and $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$, for $\mu_1 > 0$, then:*

$$\mathcal{L}_{nT}(\beta,\theta) = \mathcal{L}^*_{nT}(\beta) + \frac{1}{n}\mathcal{L}_{1,nT}(\beta,\theta) + \frac{1}{nT}\log\left[\Lambda_{nT}(\beta,\theta) + o_p\left(n^{-\mu_2}\right)\right],$$

for any $\mu_2 > 0$, where the term $o_p\left(n^{-\mu_2}\right)$ is uniform w.r.t. $\beta \in \mathcal{B}$, $\theta \in \Theta$, and function $\Lambda_{nT}(\beta,\theta)$ is defined in equation (A.2).

Proof of Proposition A.2 . (*) The second integral in the RHS of equation (A.1) is such that:

$$\int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))^c\cap\mathcal{F}_{nT}} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta)d\boldsymbol{f}_T$$
$$= \prod_{t=1}^{T}\prod_{i=1}^{n} h\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta);\beta\right)$$
$$\cdot\int_{B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))^c\cap\mathcal{F}_{nT}} \exp\left(-n\left[\phi_{nT}\left(\hat{\boldsymbol{f}}_{nT}(\beta);\beta\right) - \phi_{nT}(\boldsymbol{f}_T;\beta)\right]\right) g_T(\boldsymbol{f}_T;\theta)d\boldsymbol{f}_T$$
$$\leq \prod_{t=1}^{T}\prod_{i=1}^{n} h\left(y_{i,t}|y_{i,t-1}, \hat{f}_{n,t}(\beta);\beta\right)\exp\left(-n\tau_{nT}(\beta)\right)$$
$$= \exp\left[nT\mathcal{L}^*_{nT}(\beta) - n\tau_{nT}(\beta)\right], \tag{A.5}$$

where

$$\tau_{nT}(\beta) = \inf_{\boldsymbol{f}_T\in B_{\varepsilon_n}(\hat{\boldsymbol{f}}_{nT}(\beta))^c\cap\mathcal{F}_{nT}} [\phi_{nT}(\hat{\boldsymbol{f}}_{nT}(\beta);\beta) - \phi_{nT}(\boldsymbol{f}_T;\beta)]. \tag{A.6}$$

(**) The third integral in the RHS of equation (A.1) is such that:

$$\int_{B_{\varepsilon_n}(\hat{f}_{nT}(\beta))^c \cap \mathcal{F}_{nT}^c} \exp\left[n\phi_{nT}(\boldsymbol{f}_T;\beta)\right] g_T(\boldsymbol{f}_T;\theta) d\boldsymbol{f}_T$$
$$\leq \bar{H}^{nT}\mathbb{P}_\theta[\boldsymbol{f}_T \in \mathcal{F}_{nT}^c] \leq \bar{H}^{nT} T\mathbb{P}_\theta\left[f_t \in \mathcal{F}_n^c\right] = O\left(T\bar{H}^{nT}e^{-\gamma_6 n^2}\right), \quad \textbf{(A.7)}$$

uniformly in $\beta \in \mathcal{B}, \theta \in \Theta$, where $\bar{H} = \sup\{h(y_{i,t}|y_{i,t-1}, f_t;\beta) : y_{i,t}, y_{i,t-1} \in \mathbb{R}, f_t \in \mathbb{R}^m, \beta \in \mathcal{B}\} < \infty$ [Assumption H.3(i)] and $\gamma_6 > 0$ (Assumption H.11).

(***) Then, from equation (A.1), inequality (A.5), the bound in (A.7), and Proposition A.1, we get:

$$l\left(\underline{y_T};\beta,\theta\right) = \left(\frac{2\pi}{n}\right)^{Tm/2} \exp\left[nT\mathcal{L}_{nT}^*(\beta) + T\mathcal{L}_{1,nT}(\beta,\theta)\right]\left[\Lambda_{nT}(\beta,\theta) + \Delta_{nT}(\beta,\theta)\right], \quad \textbf{(A.8)}$$

where

$$0 \leq \Delta_{nT}(\beta,\theta) \leq \left(\frac{2\pi}{n}\right)^{-Tm/2} \exp\left[T|\mathcal{L}_{1,nT}(\beta,\theta)|\right]$$
$$\cdot\left\{\exp\left[-n\tau_{nT}(\beta)\right] + \exp\left[nT(|\mathcal{L}_{nT}^*(\beta)| + C_1) - \gamma_6 n^2\right]\right\}, \quad \textbf{(A.9)}$$

for a constant $C_1 > 0$. To bound the RHS of inequality (A.9) we need the uniform asymptotic behavior of functions $\mathcal{L}_{nT}^*(\beta)$ and $\mathcal{L}_{1,nT}(\beta,\theta)$. These functions involve mixtures of cross-sectional and time series aggregates. By using results in Bosq (1998) and Davidson (1994), we prove in Lemma 1 in the supplementary material that $\mathcal{L}_{nT}^*(\beta)$ and $\mathcal{L}_{1,nT}(\beta,\theta)$ converge in probability to the corresponding population quantities $\mathcal{L}^*(\beta) = E_0[\log h(y_{i,t}|y_{i,t-1}, f_t(\beta);\beta)]$ and:

$$\mathcal{L}_1(\beta,\theta) = -\frac{1}{2}E_0\left[\log\det I_{t,ff}(\beta)\right] + E_0[\log g(f_t(\beta)|f_{t-1}(\beta);\theta)], \quad \textbf{(A.10)}$$

where $I_{t,ff}(\beta) = E_0\left[-\frac{\partial^2 \log h\left(y_{i,t}|y_{i,t-1}, f_t(\beta);\beta\right)}{\partial f \partial f'}|\underline{f_t}\right]$, uniformly in $\beta \in \mathcal{B}$, $\theta \in \Theta$. Moreover, $\sup_{\beta\in\mathcal{B}} |\mathcal{L}^*(\beta)| < \infty$ and $\sup_{\beta\in\mathcal{B},\theta\in\Theta} |\mathcal{L}_1(\beta,\theta)| < \infty$ from Assumptions H.1, H.3, H.4, and H.12. We deduce that:

$$\sup_{\beta\in\mathcal{B}} \mathcal{L}_{nT}^*(\beta) = O_p(1), \qquad \sup_{\beta\in\mathcal{B},\theta\in\Theta} \mathcal{L}_{1,nT}(\beta,\theta) = O_p(1). \quad \textbf{(A.11)}$$

Let us now prove that:

$$\inf_{\beta\in\mathcal{B}} \tau_{nT}(\beta) \geq C_2 \frac{\varepsilon_n^2}{[\log(n)]^{C_3}}, \quad \textbf{(A.12)}$$

w.p.a. 1, for some constants $C_2, C_3 > 0$, where $\tau_{nT}(\beta)$ is defined in equation (A.6). We have:

$$\begin{aligned}
\inf_{\beta\in\mathcal{B}} \tau_{nT}(\beta) &= \inf_{\beta\in\mathcal{B}} \inf_{f_T\in B_{\varepsilon_n}(\hat{f}_{nT}(\beta))^c\cap\mathcal{F}_{nT}} \sum_{t=1}^{T} [\mathcal{L}_{n,t}(\hat{f}_{n,t}(\beta);\beta) - \mathcal{L}_{n,t}(f_t;\beta)] \\
&= \inf_{\beta\in\mathcal{B}} \inf_{f_T\in B_{\varepsilon_n}(\hat{f}_{nT}(\beta))^c\cap\mathcal{F}_{nT}} \sum_{t=1}^{T} \frac{\mathcal{L}_{n,t}(\hat{f}_{n,t}(\beta);\beta) - \mathcal{L}_{n,t}(f_t;\beta)}{\|\hat{f}_{n,t}(\beta) - f_t\|^2} \|\hat{f}_{n,t}(\beta) - f_t\|^2 \\
&\geq \left(\inf_{1\leq t\leq T} \inf_{\beta\in\mathcal{B}} \inf_{f_t\in\mathcal{F}_n} \frac{\mathcal{L}_{n,t}(\hat{f}_{n,t}(\beta);\beta) - \mathcal{L}_{n,t}(f_t;\beta)}{\|\hat{f}_{n,t}(\beta) - f_t\|^2} \right) \varepsilon_n^2.
\end{aligned}$$

In Lemma 2 in the supplementary material we prove that the term in the round brackets is lower bounded by $C_2[\log(n)]^{-C_3}$, w.p.a. 1, for some constants $C_2, C_3 > 0$. Then, the lower bound (A.12) follows.

From inequalities (A.9) and (A.12), the bounds in (A.11), and condition $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$, $\mu_1 > 0$, we get:

$$\begin{aligned}
&\sup_{\beta\in\mathcal{B}} \sup_{\theta\in\Theta} \Delta_{nT}(\beta,\theta) \\
&\quad \leq \exp\left\{ -\frac{C_2 n\varepsilon_n^2}{[\log(n)]^{C_3}} \left[1 + O_p\left(\frac{T[\log(n)]^{C_3}}{n\varepsilon_n^2} \right) + O\left(\frac{T[\log(n)]^{C_3+1}}{n\varepsilon_n^2} \right) \right] \right\} \\
&\qquad + \exp\left\{ -\gamma_6 n^2 \left[1 + O_p(T/n) \right] \right\} \\
&\quad = o_p(n^{-\mu_2}),
\end{aligned}$$

for any $\mu_2 > 0$. By taking the log on equation (A.8), Proposition A.2 follows. ■

(ii) CSA log-likelihood expansion [proof of Proposition 1(i)]

In order to derive an expansion of the log-likelihood function at order $o_p(1/n)$ from Proposition A.2, we have to control the term $\Lambda_{nT}(\beta,\theta)$ uniformly in $\beta \in \mathcal{B}$, $\theta \in \Theta$. Since $\Lambda_{nT}(\beta,\theta)$ can take values both above, or below, 1, we need a uniform upper bound on the absolute value of $\log \Lambda_{nT}(\beta,\theta)$. Such a bound is provided next.

PROPOSITION A.3. *Under Assumptions A.1–A.5 and H.1–H.12, if $T^{\nu}/n = O(1)$, for $\nu > 1$, and $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$, for $\mu_1 > 0$, then:*

$$\mathcal{L}_{nT}(\beta,\theta) = \mathcal{L}^*_{nT}(\beta) + \frac{1}{n}\mathcal{L}_{1,nT}(\beta,\theta) + \frac{1}{nT}\log\left[\Lambda_{nT}(\beta,\theta) + o_p\left(n^{-\mu_2}\right)\right],$$

and:

$$|\log(\Lambda_{nT}(\beta,\theta))| \leq C_4 T\varepsilon_n[\log(n)]^{C_5}, \tag{A.13}$$

uniformly in $\beta \in \mathcal{B}, \theta \in \Theta$, w.p.a. 1, for any $\mu_2 > 0$ and some constants $C_4, C_5 > 0$.

Proof of Proposition A.3. (*) Let us perform a Taylor expansion of function $\psi_{n,t}$ defined in equation (A.3) around $(f_t, f_{t-1}) = (\hat{f}_{n,t}(\beta), \hat{f}_{n,t-1}(\beta))$, and then use this expansion to derive an upper bound for term $\psi_{n,t}\big(\hat{f}_{n,t}(\beta) + \frac{1}{\sqrt{n}}[I_{n,t}(\beta)]^{-1/2}z_t, \hat{f}_{n,t-1}(\beta) + \frac{1}{\sqrt{n}}\left[I_{n,t-1}(\beta)\right]^{-1/2}z_{t-1};\beta,\theta\big)$ in the RHS of equation

(A.2). To simplify the notation, we consider the case $m = 1$. We get for $z \in \mathcal{Z}_{nT}(\beta)$:

$$\left|\psi_{n,t}\left(\hat{f}_{n,t}(\beta)+\frac{1}{\sqrt{n}}\left[I_{n,t}(\beta)\right]^{-1/2}z_t,\hat{f}_{n,t-1}(\beta)+\frac{1}{\sqrt{n}}\left[I_{n,t-1}(\beta)\right]^{-1/2}z_{t-1};\beta,\theta\right)\right|$$
$$\leq \frac{1}{3!\sqrt{n}}\tilde{J}_{3,nt}(\beta)|z_t|^3+\frac{1}{\sqrt{n}}\tilde{D}_{10,nt}(\beta,\theta)|z_t|+\frac{1}{\sqrt{n}}\tilde{D}_{01,nt}(\beta,\theta)|z_{t-1}|, \qquad \textbf{(A.14)}$$

where $\tilde{J}_{3,nt}(\beta) = \left|I_{n,t}(\beta)\right|^{-3/2} \sup_{f_t\in\mathcal{S}_{n,t}(\beta)}\left|\frac{\partial^3\mathcal{L}_{n,t}(f_t;\beta)}{\partial f_t^3}\right|$, set $\mathcal{S}_{n,t}(\beta)$ is defined by:

$$\mathcal{S}_{n,t}(\beta)=\left\{f\in\mathbb{R}^m:|f-\hat{f}_{n,t}(\beta)|\leq\varepsilon_n\right\}, \qquad \textbf{(A.15)}$$

and $\tilde{D}_{pq,nt}(\beta,\theta) = \left|I_{n,t}(\beta)\right|^{-p/2}\left|I_{n,t-1}(\beta)\right|^{-q/2} \sup_{f_t\in\mathcal{S}_{n,t}(\beta),\,f_{t-1}\in\mathcal{S}_{n,t-1}(\beta)}$ $\left|\frac{\partial^{p+q}\log g}{\partial f_t^p\partial f_{t-1}^q}(f_t|f_{t-1};\theta)\right|$, for $p+q=1$. We use Lemma 3 in the supplementary material to get upper bounds for the coefficients $\tilde{J}_{3,nt}(\beta)$, $\tilde{D}_{10,nt}(\beta,\theta)$, and $\tilde{D}_{01,nt}(\beta,\theta)$ in the RHS of inequality (A.14), uniformly in $\beta\in\mathcal{B}$, $\theta\in\Theta$ and $1\leq t\leq T$. By exploiting the tail conditions in Assumptions H.4 and H.12(iii), and $T^\nu/n = O(1)$, $\nu>1$, the bounds diverge slowly with sample sizes n, T, namely as powers of $\log(n)$. More precisely, let us define the sequence:

$$\kappa_n = 2[\log(n)/C_6]^{C_7}, \quad n\in\mathbb{N}, \qquad \textbf{(A.16)}$$

where constants $C_6, C_7 > 0$ are such that $C_6 \leq \min\{c_1, c_5\}$ and $C_7 \geq \max\{3/d_1, 2/d_5\}$, for c_1, d_1 and c_5, d_5 defined in Assumptions H.4 and H.12(iii), respectively. If $z\in\mathcal{Z}_{nT}(\beta)$, we have $\|z\|^2 \leq \left[\sup_{1\leq t\leq T}\sup_{\beta\in\mathcal{B}} I_{n,t}(\beta)\right]n\varepsilon_n^2$. This implies $|z_t|\leq\|z\|\leq\sqrt{n}\varepsilon_n\kappa_n^{1/2}$ for any t, w.p.a. 1, since $\sup_{1\leq t\leq T}\sup_{\beta\in\mathcal{B}} I_{n,t}(\beta)\leq\kappa_n$ w.p.a. 1 from Lemma 3(ii). Then, by Lemma 3(iii) and (iv) and inequality (A.14), we get:

$$\left|\sum_{t=1}^T\psi_{n,t}\left(\hat{f}_{n,t}(\beta)+\frac{1}{\sqrt{n}}\left[I_{n,t}(\beta)\right]^{-1/2}z_t,\hat{f}_{n,t-1}(\beta)+\frac{1}{\sqrt{n}}\left[I_{n,t-1}(\beta)\right]^{-1/2}z_{t-1};\beta,\theta\right)\right|$$
$$\leq\frac{1}{3!}\kappa_n^{3/2}\varepsilon_n\|z\|^2+2T\kappa_n^{3/2}\varepsilon_n, \qquad \textbf{(A.17)}$$

uniformly in $\beta\in\mathcal{B}$, $\theta\in\Theta$, w.p.a. 1.

(**) Let us now use inequality (A.17) to derive uniform upper and lower bounds for $\Lambda_{nT}(\beta,\theta)$, whose expression is given in equation (A.2).

(a) *Uniform upper bound.* From inequality (A.17) we have (for $m=1$):

$$\Lambda_{nT}(\beta,\theta)\leq\frac{e^{2T\kappa_n^{3/2}\varepsilon_n}}{(2\pi)^{T/2}}\int_{\mathbb{R}^T}\exp\left(-\frac{1}{2}\left(1-\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right)\|z\|^2\right)dz=\frac{e^{2T\kappa_n^{3/2}\varepsilon_n}}{\left(1-\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right)^{T/2}}$$
$$\sim\exp\left(\frac{13}{6}T\kappa_n^{3/2}\varepsilon_n\right), \qquad \textbf{(A.18)}$$

uniformly in $\beta\in\mathcal{B}$, $\theta\in\Theta$.

(b) *Uniform lower bound.* If $\|z\|^2 \leq n\varepsilon_n^2 \inf_{1\leq t\leq T} \inf_{\beta\in\mathcal{B}} I_{n,t}(\beta)$, then $z \in \mathcal{Z}_{nT}(\beta)$. Moreover, from Lemma 3(i) we have $\inf_{1\leq t\leq T} \inf_{\beta\in\mathcal{B}} I_{n,t}(\beta) \geq \kappa_n^{-1}$, w.p.a. 1. Thus, from (A.2) and (A.17) we get:

$$\begin{aligned}\Lambda_{nT}(\beta,\theta) &\geq \frac{e^{-2T\kappa_n^{3/2}\varepsilon_n}}{(2\pi)^{T/2}} \int_{\|z\|^2\leq n\varepsilon_n^2/\kappa_n} \exp\left(-\frac{1}{2}\left(1+\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right)\|z\|^2\right) dz\\ &= \frac{e^{-2T\kappa_n^{3/2}\varepsilon_n}}{(2\pi)^{T/2}} \int_0^{\sqrt{n\varepsilon_n^2/\kappa_n}} \int_{S^{T-1}} \exp\left(-\frac{1}{2}\left(1+\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right) r^2\right) r^{T-1} dz' dr,\end{aligned}$$

w.p.a. 1, where $r^{T-1}dz'dr$ is the integration element in spherical coordinates in dimension T and S^{T-1} denotes the unit sphere in dimension T. By using $\int_{S^{T-1}} dz' = \frac{2\pi^{T/2}}{\Gamma(T/2)}$ and the change of variable from r to $u = \frac{1}{2}\left(1+\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right) r^2$, we get:

$$\Lambda_{nT}(\beta,\theta) \geq \frac{e^{-2T\kappa_n^{3/2}\varepsilon_n}}{\left(1+\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right)^{T/2}} \frac{1}{\Gamma(T/2)} \int_0^{a_n} u^{T/2-1} \exp(-u)\, du,$$

where $a_n = \frac{1}{2}n\varepsilon_n^2\kappa_n^{-1}\left(1+\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right)$. The quantity $q_{nT} = \frac{1}{\Gamma(T/2)}\int_0^{a_n} u^{T/2-1} \exp(-u)\, du$ is the value at a_n of the cumulative distribution function (cdf) of a Gamma distribution $\gamma(T/2)$ with parameter $T/2$. Equivalently, $q_{nT} = \mathbb{P}[X_{nT} \leq 1]$, where the random variable X_{nT} is such that $a_n X_{nT} \sim \gamma(T/2)$. The moment generating function of X_{nT} is $M_{nT}(s) = E\left[\exp(-sX_{nT})\right] = \left(1+\frac{s}{a_n}\right)^{-T/2}$, for $s \in \mathbb{R}_+$. Thus, $M_{nT}(s) \sim \exp\left(-\frac{Ts}{2a_n}\right) \to 1$, as $n,T \to \infty$, for any $s \in \mathbb{R}_+$, since $T/a_n = o(1)$ from the condition $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$, $\mu_1 > 0$. Thus, X_{nT} converges in distribution to the constant 1, as $n,T \to \infty$. This implies $q_{nT} = 1 + o(1)$. Thus, we get:

$$\Lambda_{nT}(\beta,\theta) \geq \frac{e^{-2T\kappa_n^{3/2}\varepsilon_n}}{\left(1+\frac{1}{3}\kappa_n^{3/2}\varepsilon_n\right)^{T/2}} (1+o(1)) \sim \exp\left(-\frac{13}{6}T\kappa_n^{3/2}\varepsilon_n\right), \tag{A.19}$$

uniformly in $\beta \in \mathcal{B}$, $\theta \in \Theta$, w.p.a. 1. From bounds (A.18)-(A.19), and the expression of κ_n in (A.16), the upper bound (A.13) in Proposition A.3 follows. ∎

To prove the CSA expansion in Proposition 1(i), we use Proposition A.3. If $n,T \to \infty$ such that $T^\nu/n = O(1)$, for $\nu > 1$, then there exists a sequence $\varepsilon_n \downarrow 0$ such that $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$, for some $\mu_1 > 0$, and $\varepsilon_n[\log(n)]^{C_5} = o(1)$, for constant C_5 of Proposition A.3. Thus, from Proposition A.3 and equation (A.8), we deduce that equation (3.6) holds with $\Psi_{nT}(\beta,\theta) = \frac{1}{nT}\log\left[\Lambda_{nT}(\beta,\theta) + \Delta_{nT}(\beta,\theta)\right]$, where $\Delta_{nT}(\beta,\theta) \geq 0$, $\Delta_{nT}(\beta,\theta) = o_p(n^{-\mu_2})$, for any $\mu_2 > 0$, and $|\log\Lambda_{nT}(\beta,\theta)| = o_p(T)$, uniformly in $\beta \in \mathcal{B}$, $\theta \in \Theta$.

Then, from the monotonicity of the logarithm, we have w.p.a. 1:

$$\begin{aligned}\Psi_{nT}(\beta,\theta) &\leq \frac{1}{nT}\max\{\log[2\Lambda_{nT}(\beta,\theta)],\log[2\Delta_{nT}(\beta,\theta)]\}\\ &\leq O\left(\frac{1}{nT}\right)+\frac{1}{nT}\max\{\log[\Lambda_{nT}(\beta,\theta)],0\}=o_p(1/n),\end{aligned}$$

and $\Psi_{nT}(\beta,\theta) \geq \frac{1}{nT}\log[\Lambda_{nT}(\beta,\theta)] = o_p(1/n)$, uniformly in $\beta \in \mathcal{B}$ and $\theta \in \Theta$. Proposition 1(i) follows.

(iii) GA log-likelihood expansion [proof of Proposition 1(ii)]

In order to derive an expansion of the log-likelihood function at order $o_p(1/n^2)$, we need a more accurate analysis of the term $\Lambda_{nT}(\beta,\theta)$ compared to Proposition A.3. A uniform asymptotic expansion for $\Lambda_{nT}(\beta,\theta)$ at order $o_p(T/n)$ is provided in Proposition A.4 below under an additional condition on the convergence rate of sequence ε_n, namely $\sqrt{T}\varepsilon_n^2 = O(n^{-\mu_3})$, with $\mu_3 > 0$. This condition is compatible with condition $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$, with $\mu_1 > 0$, if $n,T \to \infty$ such that $T^\nu/n = O(1)$, with $\nu > 3/2$.

PROPOSITION A.4. *Under Assumptions A.1–A.5 and H.1–H.12, if* $T^\nu/n = O(1)$, *with* $\nu > 3/2$, *and if* ε_n *is such that* $\frac{T}{n\varepsilon_n^2} = O(n^{-\mu_1})$ *and* $\sqrt{T}\varepsilon_n^2 = O(n^{-\mu_3})$, *for some* $\mu_1,\mu_3 > 0$, *then:*

$$\mathcal{L}_{nT}(\beta,\theta) = \mathcal{L}^*_{nT}(\beta)+\frac{1}{n}\mathcal{L}_{1,nT}(\beta,\theta)+\frac{1}{nT}\log\left[\Lambda_{nT}(\beta,\theta)+o_p\left(n^{-\mu_2}\right)\right],$$

for any $\mu_2 > 0$, *and:*

$$\Lambda_{nT}(\beta,\theta) = 1+\frac{T}{n}\mathcal{L}_{2,nT}(\beta,\theta)+o_p(T/n), \tag{A.20}$$

uniformly in $\beta \in \mathcal{B}$, $\theta \in \Theta$.

Proof of Proposition A.4. We perform a Taylor expansion of function $\psi_{n,t}$ in (A.3) around $(f_t, f_{t-1}) = (\hat{f}_{n,t}(\beta), \hat{f}_{n,t-1}(\beta))$. The expansion is of fifth-order for the part of the function in the RHS of the first line in equation (A.3), and of third order for the part of the function in the second line in equation (A.3), so that the remainder term involves a power $n^{-3/2}$. To simplify the notation, we consider the case $m = 1$. We get:

$$\begin{aligned}&\psi_{n,t}\left(\hat{f}_{n,t}(\beta)+\frac{1}{\sqrt{n}}\left[I_{n,t}(\beta)\right]^{-1/2}z_t,\hat{f}_{n,t-1}(\beta)+\frac{1}{\sqrt{n}}\left[I_{n,t-1}(\beta)\right]^{-1/2}z_{t-1};\beta,\theta\right)\\ &=\frac{1}{3!\sqrt{n}}J_{3,nt}(\beta)z_t^3+\frac{1}{4!n}J_{4,nt}(\beta)z_t^4+\frac{1}{\sqrt{n}}D_{10,nt}(\beta,\theta)z_t+\frac{1}{\sqrt{n}}D_{01,nt}(\beta,\theta)z_{t-1}\\ &\quad+\frac{1}{2n}D_{20,nt}(\beta,\theta)z_t^2+\frac{1}{2n}D_{02,nt}(\beta,\theta)z_{t-1}^2+\frac{1}{n}D_{11,nt}(\beta,\theta)z_tz_{t-1}\\ &\quad+R_{n,t}(z_t,z_{t-1};\beta,\theta),\end{aligned} \tag{A.21}$$

where the remainder term is such that:

$$\begin{aligned}|R_{n,t}(z_t,z_{t-1};\beta,\theta)| &\leq \frac{1}{5!n^{3/2}}\tilde{J}_{5,nt}(\beta)|z_t|^5\\ &\quad+\frac{1}{3!n^{3/2}}\sum_{j=0}^{3}\binom{3}{j}\tilde{D}_{3-j,j,nt}(\beta,\theta)|z_t|^{3-j}|z_{t-1}|^j,\end{aligned} \tag{A.22}$$

with $\tilde{J}_{5,nt}(\beta) = \left|I_{n,t}(\beta)\right|^{-5/2} \sup_{f_t \in \mathcal{S}_{n,t}(\beta)} \left|\frac{\partial^5 \mathcal{L}_{n,t}(f_t;\beta)}{\partial f_t^5}\right|$, set $\mathcal{S}_{n,t}(\beta)$ is defined in equation (A.15), and $\tilde{D}_{pq,nt}(\beta,\theta) = \left|I_{n,t}(\beta)\right|^{-p/2}\left|I_{n,t-1}(\beta)\right|^{-q/2} \sup_{f_t \in \mathcal{S}_{n,t}(\beta),\, f_{t-1} \in \mathcal{S}_{n,t-1}(\beta)} \left|\frac{\partial^{p+q} \log g}{\partial f_t^p \partial f_{t-1}^q}(f_t|f_{t-1};\theta)\right|$, for $p+q=3$. Let us write the exponential $\exp\left(\sum_{t=1}^T \psi_{n,t}\right)$ in equation (A.2) as a series, and interchange the series and the integral by applying the Lebesgue theorem on the bounded domain $\mathcal{Z}_{nT}(\beta)$. We get $\Lambda_{nT}(\beta,\theta) = \sum_{j=0}^{\infty} \frac{1}{j!}\Lambda_{j,nT}(\beta,\theta)$, where

$$\Lambda_{j,nT}(\beta,\theta) = \frac{1}{(2\pi)^{T/2}} \int_{\mathcal{Z}_{nT}(\beta)} \exp\left(-\frac{1}{2}\|z\|^2\right) \cdot \left[\sum_{t=1}^T \psi_{n,t}\left(\hat{f}_{n,t}(\beta) + \frac{[I_{n,t}(\beta)]^{-1/2}}{\sqrt{n}} z_t, \hat{f}_{n,t-1}(\beta) + \frac{[I_{n,t-1}(\beta)]^{-1/2}}{\sqrt{n}} z_{t-1}; \beta, \theta\right)\right]^j dz. \tag{A.23}$$

We analyze the terms $\Lambda_{j,nT}(\beta,\theta)$, for $j=0,1,\ldots$, separately. By replacing expansion (A.21) into equation (A.23), we show below that:

$$\Lambda_{0,nT}(\beta,\theta) = 1 + o_p(T/n), \tag{A.24}$$

$$\Lambda_{1,nT}(\beta,\theta) = \frac{1}{8n}\sum_{t=1}^T J_{4,nt}(\beta) + \frac{1}{2n}\sum_{t=1}^T D_{20,nt}(\beta,\theta) + \frac{1}{2n}\sum_{t=2}^T D_{02,nt}(\beta,\theta) + o_p(T/n), \tag{A.25}$$

$$\begin{aligned}\Lambda_{2,nT}(\beta,\theta) = {} & \frac{5}{12n}\sum_{t=1}^T J_{3,nt}(\beta)^2 + \frac{1}{n}\sum_{t=1}^T D_{10,nt}(\beta,\theta)^2 + \frac{1}{n}\sum_{t=2}^T D_{01,nt}(\beta,\theta)^2 \\ & + \frac{1}{n}\sum_{t=1}^T J_{3,nt}(\beta) D_{10,nt}(\beta,\theta) + \frac{1}{n}\sum_{t=2}^T J_{3,n,t-1}(\beta) D_{01,nt}(\beta,\theta) \\ & + \frac{2}{n}\sum_{t=2}^T D_{10,n,t-1}(\beta,\theta) D_{01,nt}(\beta,\theta) + o_p(T/n),\end{aligned} \tag{A.26}$$

and:

$$\sum_{j=3}^{\infty} \frac{1}{j!}|\Lambda_{j,nT}(\beta,\theta)| = o_p(T/n), \tag{A.27}$$

uniformly in $\beta \in \mathcal{B}, \theta \in \Theta$. By combining equations (A.24)-(A.27), equation (A.20) follows.

(a) *Proof of equivalence (A.24).* We have $\Lambda_{0,nT}(\beta,\theta) = 1 - \frac{1}{(2\pi)^{T/2}} \int_{\mathcal{Z}_{nT}(\beta)^c} \exp\left(-\frac{1}{2}\|z\|^2\right) dz$. Let us derive an upper bound for the integral $\frac{1}{(2\pi)^{T/2}} \int_{\mathcal{Z}_{nT}(\beta)^c} \exp\left(-\frac{1}{2}\|z\|^2\right) z_t^{2k} dz$, with $k \in \mathbb{N}$ and $t=1,\ldots,T$. If $z \in \mathcal{Z}_{nT}(\beta)^c$, we have $\|z\|^2 \geq \left[\inf_{1\leq t\leq T}\inf_{\beta\in\mathcal{B}} I_{n,t}(\beta)\right] n\varepsilon_n^2 \geq n\varepsilon_n^2\kappa_n^{-1}$, w.p.a. 1, from Lemma 3(i). We get:

$$\begin{aligned}
&\frac{1}{(2\pi)^{T/2}}\int_{\mathcal{Z}_{nT}(\beta)^c}\exp\left(-\frac{1}{2}\|z\|^2\right)z_t^{2k}dz \\
&\quad\leq \frac{1}{(2\pi)^{T/2}}\int_{\|z\|^2\geq n\varepsilon_n^2\kappa_n^{-1}}\exp\left(-\frac{1}{2}\|z\|^2\right)z_t^{2k}dz \\
&\quad\leq \frac{1}{(2\pi)^{T/2}T}\int_{\sqrt{n\varepsilon_n^2\kappa_n^{-1}}}^{\infty}\int_{S^{T-1}}\exp\left(-r^2/2\right)r^{T+2k-1}dz'dr \\
&\quad= \frac{1}{T2^{T/2-1}\Gamma(T/2)}\int_{\sqrt{n\varepsilon_n^2\kappa_n^{-1}}}^{\infty}\exp\left(-r^2/2\right)r^{T+2k-1}dr, \qquad \textbf{(A.28)}
\end{aligned}$$

uniformly in $1\leq t\leq T$ and $\beta\in\mathcal{B}$, where we have used spherical coordinates as in the proof of Proposition A.3. By the change of variable from r to $u=\frac{1}{2}r^2$, we have:

$$\begin{aligned}
&\frac{1}{2^{T/2-1}\Gamma(T/2)}\int_{\sqrt{n\varepsilon_n^2\kappa_n^{-1}}}^{\infty}\exp\left(-r^2/2\right)r^{T+2k-1}dr \\
&\quad= \frac{2^k\Gamma(T/2+k)}{\Gamma(T/2)}\frac{1}{\Gamma(T/2+k)}\int_{\bar{a}_n}^{\infty}e^{-u}u^{T/2+k-1}du, \qquad \textbf{(A.29)}
\end{aligned}$$

where $\bar{a}_n=\frac{1}{2}n\varepsilon_n^2\kappa_n^{-1}$. The RHS involves the survivor function of the Gamma distribution $\gamma(T/2+k)$ evaluated at $\bar{a}_n$. Since $\bar{a}_n\to\infty$ as $n\to\infty$, to upper bound the RHS of equation (A.29) it is enough to bound the right tail of the cdf of the Gamma distribution from above. By repeated partial integration, we get for any $s\geq 1$ and $\delta\geq 1$:

$$\begin{aligned}
\frac{1}{\Gamma(\delta)}\int_s^{\infty}e^{-u}u^{\delta-1}du &= \frac{e^{-s}s^{\delta-1}}{\Gamma(\delta)}+\frac{e^{-s}s^{\delta-2}}{\Gamma(\delta-1)}+\cdots+\frac{e^{-s}s^{l+1}}{\Gamma(l+2)}+\frac{1}{\Gamma(l+1)}\int_s^{\infty}e^{-u}u^l du \\
&\leq e^{-s}(s^{\delta-1}+s^{\delta-2}+\cdots+s^{l+1})+\int_s^{\infty}e^{-u}u\,du \\
&\leq (\lfloor\delta\rfloor+1)e^{-s}s^{\delta-1}, \qquad \textbf{(A.30)}
\end{aligned}$$

where $\lfloor\delta\rfloor$ denotes the integer part of δ and $l=\delta-\lfloor\delta\rfloor$ is the decimal part of δ. From inequality (A.28) and equation (A.29), and by using bound (A.30) with $s=\bar{a}_n$ and $\delta=T/2+k$, we get:

$$\begin{aligned}
&\sup_{1\leq t\leq T}\sup_{\beta\in\mathcal{B}}\frac{1}{(2\pi)^{T/2}}\int_{\mathcal{Z}_{nT}(\beta)^c}\exp\left(-\frac{1}{2}\|z\|^2\right)z_t^{2k}dz \\
&\quad\leq \frac{2^k\Gamma(T/2+k)}{\Gamma(T/2)}\frac{T/2+k+1}{T}e^{-\bar{a}_n}\bar{a}_n^{T/2+k-1}.
\end{aligned}$$

By the Stirling's formula, we have $\frac{\Gamma(T/2+k)}{\Gamma(T/2)}=O(T^k)$ for large T. Moreover, from condition $\frac{T}{n\varepsilon_n^2}=O(n^{-\mu_1})$, $\mu_1>0$, we have:

$$e^{-\bar{a}_n}\bar{a}_n^{T/2+k-1}=\exp\left\{-\frac{n\varepsilon_n^2}{2\kappa_n}\left[1+o\left(\frac{T\kappa_n\log(n)}{n\varepsilon_n^2}\right)\right]\right\}\leq\exp\left(-\frac{n\varepsilon_n^2}{4\kappa_n}\right)=o(n^{-\mu_4}),$$

for any $\mu_4 > 0$. Thus, we get for any $k \in \mathbb{N}$:

$$\sup_{1\le t\le T}\sup_{\beta\in\mathcal{B}} \frac{1}{(2\pi)^{T/2}}\int_{\mathcal{Z}_{nT}(\beta)^c} \exp\left(-\frac{1}{2}\|z\|^2\right) z_t^{2k} dz = o_p\left(n^{-\mu_4}\right), \qquad \textbf{(A.31)}$$

for any $\mu_4 > 0$. In particular, equivalence (A.24) follows.

(b) *Proof of equivalence (A.25).* By the symmetry of the domain of integration $\mathcal{Z}_{nT}(\beta)$ we have:

$$\begin{aligned}\Lambda_{1,nT}(\beta,\theta) &= \frac{1}{4!n}\sum_{t=1}^{T} J_{4,nt}(\beta)a_{2,nT,t}(\beta) + \frac{1}{2n}\sum_{t=1}^{T} D_{20,nt}(\beta,\theta)a_{1,nT,t}(\beta)\\ &+ \frac{1}{2n}\sum_{t=2}^{T} D_{02,nt}(\beta,\theta)a_{1,nT,t-1}(\beta)\\ &+ \sum_{t=1}^{T}\frac{1}{(2\pi)^{T/2}}\int_{\mathcal{Z}_{nT}(\beta)} \exp\left(-\frac{1}{2}\|z\|^2\right) R_{n,t}(z_t,z_{t-1};\beta,\theta)dz,\end{aligned} \qquad \textbf{(A.32)}$$

where we use the notation $a_{k,nT,t}(\beta) = \frac{1}{(2\pi)^{T/2}}\int_{\mathcal{Z}_{nT}(\beta)} \exp\left(-\frac{1}{2}\|z\|^2\right) z_t^{2k} dz$. To control the RHS of equation (A.32), we use Lemma 4 in the supplementary material, which provides uniform upper bounds for terms $J_{p,nt}(\beta)$ and $D_{pq,nt}(\beta,\theta)$ involving higher-order partial derivatives w.r.t. the factor values. From inequality (A.22) and Lemma 4, the last term in the RHS of equation (A.32) is $O_p\left(\frac{T\kappa_n}{n^{3/2}}\right) = o_p(T/n)$, uniformly in $\beta \in \mathcal{B}, \theta \in \Theta$, where sequence κ_n is defined in (A.16). By using the bound in (A.31), we have $a_{2,nT,t} = 3 + o_p(n^{-\mu_5})$ and $a_{1,nT,t} = 1 + o_p(n^{-\mu_5})$, uniformly in $t = 1,\ldots,T$ and $\beta \in \mathcal{B}$, for any $\mu_5 > 0$. Then, from equation (A.32) and Lemma 4 in the supplementary material, we get equivalence (A.25).

(c) *Proof of equivalence (A.26).* By the symmetry of domain $\mathcal{Z}_{nT}(\beta)$, we have:

$$\begin{aligned}\Lambda_{2,nT}(\beta,\theta) &= \frac{1}{(3!)^2 n}\sum_{t=1}^{T} J_{3,nt}(\beta)^2 a_{3,nT,t}(\beta) + \frac{1}{n}\sum_{t=1}^{T} D_{10,nt}(\beta,\theta)^2 a_{1,nT,t}(\beta)\\ &+ \frac{1}{n}\sum_{t=2}^{T} D_{01,nt}(\beta,\theta)^2 a_{1,nT,t-1}(\beta)\\ &+ \frac{2}{3!n}\sum_{t=1}^{T} J_{3,nt}(\beta)D_{10,nt}(\beta,\theta)a_{2,nT,t}(\beta)\\ &+ \frac{2}{3!n}\sum_{t=2}^{T} J_{3,n,t-1}(\beta)D_{01,nt}(\beta,\theta)a_{2,nT,t-1}(\beta)\\ &+ \frac{2}{n}\sum_{t=2}^{T} D_{10,n,t-1}(\beta,\theta)D_{01,nt}(\beta,\theta)a_{1,nT,t-1}(\beta) + O_p\left(\frac{T^2\kappa_n^2}{n^2}\right).\end{aligned}$$

From equation (A.31) we get $a_{3,nT,t}(\beta) = 15 + o_p(n^{-\mu_6})$ uniformly, for any $\mu_6 > 0$. Then, from Lemmas 3 and 4 in the supplementary material, equivalence (A.26) follows.

(d) *Proof of equivalence (A.27).* We use Lemma 5 in the supplementary material, which provides the following uniform upper bounds for $\Lambda_{j,nT}(\beta,\theta)$, for any integer $j \geq 3$:

$$\Lambda_{j,nT}(\beta,\theta) \leq C_j^* \left(\frac{T^2 \kappa_n^j}{n^2} \right), \tag{A.33}$$

and:

$$\Lambda_{j,nT}(\beta,\theta) \leq C_8 \kappa_n^{2j} j! \left(\frac{T}{n} + \sqrt{T} \varepsilon_n^2 \right)^j, \tag{A.34}$$

uniformly in $\beta \in \mathcal{B}, \theta \in \Theta$, w.p.a. 1, for some constants $C_j^* > 0$, $j = 3, 4, \ldots$, and $C_8 > 0$, and where sequence κ_n is defined in (A.16). The sequence of constants C_j^* in bound (A.33) diverges rapidly as j increases, and the sequence $C_j^* \kappa_n^j / j!$ might not be summable. This explains why, for any given $J \geq 3$ independent of n and T, we use the bound in (A.33) for $j \leq J$ and the bound in (A.34) for $j > J$, to get w.p.a. 1:

$$\begin{aligned}
\sum_{j=3}^{\infty} \frac{1}{j!} |\Lambda_{j,nT}(\beta,\theta)| &\leq \sum_{j=3}^{J} C_j^* \frac{T^2 \kappa_n^j}{j! n^2} + \sum_{j=J+1}^{\infty} C_8 \kappa_n^{2j} \left(\frac{T}{n} + \sqrt{T} \varepsilon_n^2 \right)^j \\
&= \sum_{j=3}^{J} C_j^* \frac{T^2 \kappa_n^j}{j! n^2} + C_8 \frac{\rho_{nT}^{J+1}}{1 - \rho_{nT}} = o_p(T/n) + O_p\left(\rho_{nT}^{J+1} \right),
\end{aligned}$$

uniformly in $\beta \in \mathcal{B}$, $\theta \in \Theta$, where $\rho_{nT} = \kappa_n^2 \left(\frac{T}{n} + \sqrt{T} \varepsilon_n^2 \right) = o(n^{-\mu_7})$, for any μ_7 such that $0 < \mu_7 < \min\{\mu_3, 1 - 1/\nu\}$, if $T^\nu / n = O(1)$, for $\nu > 3/2$, and $\sqrt{T} \varepsilon_n^2 = O(n^{-\mu_3})$, $\mu_3 > 0$. If we choose $J \geq \max\{3, 1/\mu_7 - 1\}$, we get $\rho_{nT}^{J+1} = o\left(n^{-1}\right)$, which implies equation (A.27). ■

From Lemmas 3 and 4 in the supplementary material, we have that $\frac{T}{n} \mathcal{L}_{2,nT}(\beta,\theta) = o_p(1)$, uniformly in $\beta \in \mathcal{B}$, $\theta \in \Theta$. Then, from Proposition A.4 and the expansion of the logarithm in a neighborhood of 1, Proposition 1(ii) follows.

A.2.2. Proof of Proposition 2. The proof is in two steps. We first show the consistency of the estimators, which is then used to derive the stochastic difference between the estimators.

(i) Consistency of the estimators

Let us prove the consistency of the estimators when $n, T \to \infty$ such that $T^\nu / n = O(1)$, $\nu > 1$. We start with the ML estimator $(\tilde{\beta}_{nT}, \tilde{\theta}_{nT})$. Let us first prove that $\tilde{\beta}_{nT}$ is consistent. For any $\varepsilon > 0$ we have:

$$\begin{aligned}
\mathbb{P}\left[\|\tilde{\beta}_{nT} - \beta_0\| \geq \varepsilon \right] &\leq \mathbb{P}\left[\sup_{\beta \in \mathcal{B}: \|\beta - \beta_0\| \geq \varepsilon} \mathcal{L}_{nT}(\beta, \tilde{\theta}_{nT}) \geq \mathcal{L}_{nT}(\tilde{\beta}_{nT}, \tilde{\theta}_{nT}) \right] \\
&\leq \mathbb{P}\left[\sup_{\beta \in \mathcal{B}: \|\beta - \beta_0\| \geq \varepsilon} \mathcal{L}_{nT}(\beta, \tilde{\theta}_{nT}) \geq \mathcal{L}_{nT}(\beta_0, \theta_0) \right].
\end{aligned}$$

By using Proposition 1(i), Lemma 1(i) in the supplementary material, and the second bound in (A.11), we get:

$$\mathbb{P}\left[\|\tilde{\beta}_{nT}-\beta_0\| \geq \varepsilon\right] \leq \mathbb{P}\left[\sup_{\beta\in\mathcal{B}:\|\beta-\beta_0\|\geq\varepsilon} \mathcal{L}^*(\beta)-\mathcal{L}^*(\beta_0) \geq o_p(1)\right], \tag{A.35}$$

where $\mathcal{L}^*(\beta)$ is the probability limit of $\mathcal{L}^*_{nT}(\beta)$ defined in equation (4.4). The probability in the RHS of inequality (A.35) is $o(1)$, since $\sup_{\beta\in\mathcal{B}:\|\beta-\beta_0\|\geq\varepsilon} \mathcal{L}^*(\beta)-\mathcal{L}^*(\beta_0) < 0$ by global identification Assumption A.6, continuity of function $\mathcal{L}^*(\beta)$, and compactness of set $\mathcal{B}$.

Let us now show that $\tilde{\theta}_{nT}$ is consistent. For any $\varepsilon > 0$ we have:

$$\begin{aligned}\mathbb{P}\left[\|\tilde{\theta}_{nT}-\theta_0\| \geq \varepsilon\right] &\leq \mathbb{P}\left[\sup_{\theta\in\Theta:\|\theta-\theta_0\|\geq\varepsilon} \mathcal{L}_{nT}(\tilde{\beta}_{nT},\theta) \geq \mathcal{L}_{nT}(\tilde{\beta}_{nT},\tilde{\theta}_{nT})\right]\\ &\leq \mathbb{P}\left[\sup_{\theta\in\Theta:\|\theta-\theta_0\|\geq\varepsilon} \mathcal{L}_{nT}(\tilde{\beta}_{nT},\theta) \geq \mathcal{L}_{nT}(\tilde{\beta}_{nT},\theta_0)\right].\end{aligned}$$

Using Proposition 1(i), Lemma 1(ii), and the consistency of $\tilde{\beta}_{nT}$, the RHS probability is such that:

$$\begin{aligned}&\mathbb{P}\left[\sup_{\theta\in\Theta:\|\theta-\theta_0\|\geq\varepsilon} \mathcal{L}_{nT}(\tilde{\beta}_{nT},\theta) \geq \mathcal{L}_{nT}(\tilde{\beta}_{nT},\theta_0)\right]\\ &\quad= \mathbb{P}\left[\sup_{\theta\in\Theta:\|\theta-\theta_0\|\geq\varepsilon} \frac{1}{n}[\mathcal{L}_{1,nT}(\tilde{\beta}_{nT},\theta)-\mathcal{L}_{1,nT}(\tilde{\beta}_{nT},\theta_0)] \geq o_p(1/n)\right]\\ &\quad= \mathbb{P}\left[\sup_{\theta\in\Theta:\|\theta-\theta_0\|\geq\varepsilon} \mathcal{L}_1(\beta_0,\theta)-\mathcal{L}_1(\beta_0,\theta_0) \geq o_p(1)\right],\end{aligned}$$

where $\mathcal{L}_1(\beta,\theta)$ is the probability limit of $\mathcal{L}_{1,nT}(\beta,\theta)$ defined in equation (A.10). Therefore, we get:

$$\mathbb{P}\left[\|\tilde{\theta}_{nT}-\theta_0\| \geq \varepsilon\right] \leq \mathbb{P}\left[\sup_{\theta\in\Theta:\|\theta-\theta_0\|\geq\varepsilon} \mathcal{L}_1(\beta_0,\theta)-\mathcal{L}_1(\beta_0,\theta_0) \geq o_p(1)\right].$$

The RHS probability is $o(1)$, since $\sup_{\theta\in\Theta:\|\theta-\theta_0\|\geq\varepsilon} \mathcal{L}_1(\beta_0,\theta)-\mathcal{L}_1(\beta_0,\theta_0) < 0$ from global identification Assumption A.8, continuity of mapping $\theta \to \mathcal{L}_1(\beta_0,\theta)$, and the compactness of set Θ. The consistency of $\tilde{\theta}_{nT}$ follows.

The proof of the consistency of $\left(\tilde{\beta}^{CSA}_{nT},\tilde{\theta}^{CSA}_{nT}\right)$ and $\left(\tilde{\beta}^{GA}_{nT},\tilde{\theta}^{GA}_{nT}\right)$ is similar, by replacing criterion $\mathcal{L}_{nT}(\beta,\theta)$ with $\mathcal{L}^{CSA}_{nT}(\beta,\theta)$, and $\mathcal{L}^{GA}_{nT}(\beta,\theta)$, respectively, in the above arguments.

(ii) Stochastic difference between estimators (proof of Proposition 2)

Since the CSA, GA, and true ML estimators are consistent, the stochastic difference between these estimators can be derived along the lines of Robinson (1988, Theorem 1). However, we have to carefully take into account the double asymptotics in n and T. We provide the proof for $n, T \to \infty$ such that $T^{\nu}/n = O(1)$ with $\nu > 1$ (the proof for $\nu > 3/2$ is similar).

Let us first prove the stochastic difference between $\left(\tilde{\beta}_{nT}^{CSA},\tilde{\theta}_{nT}^{CSA}\right)$ and $\left(\tilde{\beta}_{nT},\tilde{\theta}_{nT}\right)$ [equivalence (4.7) in Proposition 2]. From the first-order conditions of the true and CSA ML estimators, Proposition 1(i) and the mean value Theorem, we have:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}_{nT}\left(\tilde{\beta}_{nT},\tilde{\theta}_{nT}\right)}{\partial\left(\beta',\theta'\right)'} = \frac{\partial \mathcal{L}_{nT}^{CSA}\left(\tilde{\beta}_{nT},\tilde{\theta}_{nT}\right)}{\partial\left(\beta',\theta'\right)'} + \frac{\partial \Psi_{nT}\left(\tilde{\beta}_{nT},\tilde{\theta}_{nT}\right)}{\partial(\beta',\theta')'} \\
&= \frac{\partial^2 \mathcal{L}_{nT}^{CSA}\left(\bar{\beta}_{nT},\bar{\theta}_{nT}\right)}{\partial\left(\beta',\theta'\right)'\partial\left(\beta',\theta'\right)}\begin{pmatrix}\tilde{\beta}_{nT}-\tilde{\beta}_{nT}^{CSA}\\ \tilde{\theta}_{nT}-\tilde{\theta}_{nT}^{CSA}\end{pmatrix} + \frac{\partial \Psi_{nT}\left(\tilde{\beta}_{nT},\tilde{\theta}_{nT}\right)}{\partial(\beta',\theta')'},
\end{aligned}
\tag{A.36}
$$

where $\bar{\beta}_{nT}$ is between $\tilde{\beta}_{nT}$ and $\tilde{\beta}_{nT}^{CSA}$ (componentwise), and similarly for $\bar{\theta}_{nT}$. From section (i) above, $\left(\bar{\beta}_{nT},\bar{\theta}_{nT}\right)$ converges to (β_0,θ_0) in probability. Let us now use Lemma 6 in the supplementary material, which provides the uniform convergence of functions $\mathcal{L}_{nT}^{*}$, $\mathcal{L}_{1,nT}$, $\mathcal{L}_{2,nT}$, Ψ_{nT}, $\tilde{\Psi}_{nT}$ in the asymptotic expansion of the log-likelihood function, and of their partial derivatives. From Lemmas 6(1), (2iii) and (iv) we get $\dfrac{\partial^2 \mathcal{L}_{nT}^{CSA}\left(\bar{\beta}_{nT},\bar{\theta}_{nT}\right)}{\partial\beta\partial\beta'} = -I_0^* + o_p(1)$, $\dfrac{\partial^2 \mathcal{L}_{nT}^{CSA}\left(\bar{\beta}_{nT},\bar{\theta}_{nT}\right)}{\partial\theta\partial\theta'} = -\dfrac{1}{n}I_{1,\theta\theta} + o_p(1/n)$ and $\dfrac{\partial^2 \mathcal{L}_{nT}^{CSA}\left(\bar{\beta}_{nT},\bar{\theta}_{nT}\right)}{\partial\beta\partial\theta'} = O_p(1/n)$, where matrices I_0^* and $I_{1,\theta\theta}$ are defined in Assumptions A.7 and A.9. Moreover, from Lemma 6(3), we have $\dfrac{\partial \Psi_{nT}\left(\tilde{\beta}_{nT},\tilde{\theta}_{nT}\right)}{\partial(\beta',\theta')'} = \left[o_p(1/n), O_p\left(\dfrac{[\log(n)]^{C_9}}{n^{3/2}}\right)\right]'$, for a constant $C_9 > 0$. From equation (A.36) we deduce:

$$
-I_0^*\left(\tilde{\beta}_{nT}-\tilde{\beta}_{nT}^{CSA}\right) + o_p\left(\tilde{\beta}_{nT}-\tilde{\beta}_{nT}^{CSA}\right) + O_p\left(\frac{1}{n}\left(\tilde{\theta}_{nT}-\tilde{\theta}_{nT}^{CSA}\right)\right) = o_p(1/n),
\tag{A.37}
$$

$$
-I_{1,\theta\theta}\left(\tilde{\theta}_{nT}-\tilde{\theta}_{nT}^{CSA}\right) + o_p\left(\tilde{\theta}_{nT}-\tilde{\theta}_{nT}^{CSA}\right) + O_p\left(\tilde{\beta}_{nT}-\tilde{\beta}_{nT}^{CSA}\right) = O_p\left(\frac{[\log(n)]^{C_9}}{n^{1/2}}\right).
\tag{A.38}
$$

Since matrix I_0^* is positive definite, and $\tilde{\theta}_{nT}-\tilde{\theta}_{nT}^{CSA} = o_p(1)$ by consistency of the estimators, equation (A.37) implies $\tilde{\beta}_{nT}-\tilde{\beta}_{nT}^{CSA} = o_p(1/n)$, that is the first equation in the equivalence (4.7) in Proposition 2. Then, since $I_{1,\theta\theta}$ is a positive definite matrix, equation (A.38) implies the second equation in the equivalence (4.7) in Proposition 2 (with $\delta_1 = C_9$).

To derive the stochastic difference between the true and GA ML estimators [equivalence (4.8) in Proposition 2], we use that $\mathcal{L}_{nT}(\beta,\theta) = \mathcal{L}_{nT}^{GA}(\beta,\theta) + \tilde{\Psi}_{nT}(\beta,\theta)$, where $\tilde{\Psi}_{nT}(\beta,\theta) = \Psi_{nT}(\beta,\theta) - \dfrac{1}{n^2}\mathcal{L}_{2,nT}(\beta,\theta)$. From Lemma 6(2ii), (3), we get $\sup_{\beta\in\mathcal{B},\theta\in\Theta}\left\|\dfrac{\partial\tilde{\Psi}_{nT}(\beta,\theta)}{\partial\beta}\right\| = o_p(1/n)$ and $\sup_{\beta\in\mathcal{B},\theta\in\Theta}\left\|\dfrac{\partial\tilde{\Psi}_{nT}(\beta,\theta)}{\partial\theta}\right\| = O_p\left(\dfrac{[\log(n)]^{C_9}}{n^{3/2}}\right)$ when $T^{\nu}/n = O(1)$, $\nu > 1$. Then, by similar arguments as above, the equivalence (4.8) follows.

A.2.3. Proof of Proposition 3. The proof is in three steps. We first derive the asymptotic expansion of the standardized CSA ML estimator in terms of the standardized score. Then, we prove the asymptotic normality of the standardized score. Finally, this asymptotic normality and the asymptotic equivalences (Proposition 2) are used to deduce the asymptotic normality of the different estimators.

(i) Asymptotic expansion of the CSA ML estimator

The first-order conditions for $\left(\hat{\beta}_{nT},\hat{\theta}_{nT}\right)=\left(\tilde{\beta}_{nT}^{CSA},\tilde{\theta}_{nT}^{CSA}\right)$ are:

$$0=\frac{\partial\mathcal{L}_{nT}^{CSA}}{\partial\beta}\left(\hat{\beta}_{nT},\hat{\theta}_{nT}\right)=\frac{\partial\mathcal{L}_{nT}^{*}}{\partial\beta}\left(\hat{\beta}_{nT}\right)+\frac{1}{n}\frac{\partial\mathcal{L}_{1,nT}}{\partial\beta}\left(\hat{\beta}_{nT},\hat{\theta}_{nT}\right),$$

$$0=\frac{\partial\mathcal{L}_{nT}^{CSA}}{\partial\theta}\left(\hat{\beta}_{nT},\hat{\theta}_{nT}\right)\quad\Leftrightarrow\quad 0=\frac{\partial\mathcal{L}_{1,nT}}{\partial\theta}\left(\hat{\beta}_{nT},\hat{\theta}_{nT}\right),$$

where the factor $1/n$ in the second equation cancels. Let us multiply the first equation by $\sqrt{nT}$, the second equation by $\sqrt{T}$, and use the mean value Theorem to get:

$$0=\sqrt{nT}\frac{\partial\mathcal{L}_{nT}^{*}(\beta_0)}{\partial\beta}+\frac{\partial^2\mathcal{L}_{nT}^{*}(\bar{\beta}_{nT})}{\partial\beta\partial\beta'}\sqrt{nT}\left(\hat{\beta}_{nT}-\beta_0\right)+\sqrt{\frac{T}{n}}\frac{\partial\mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial\beta}$$
$$+\frac{1}{n}\frac{\partial^2\mathcal{L}_{1,nT}(\bar{\beta}_{nT},\bar{\theta}_{nT})}{\partial\beta\partial\beta'}\sqrt{nT}\left(\hat{\beta}_{nT}-\beta_0\right)+\frac{1}{\sqrt{n}}\frac{\partial^2\mathcal{L}_{1,nT}(\bar{\beta}_{nT},\bar{\theta}_{nT})}{\partial\beta\partial\theta'}\sqrt{T}\left(\hat{\theta}_{nT}-\theta_0\right),$$

and:

$$0=\sqrt{T}\frac{\partial\mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial\theta}+\frac{1}{\sqrt{n}}\frac{\partial^2\mathcal{L}_{1,nT}\left(\bar{\beta}_{nT},\bar{\theta}_{nT}\right)}{\partial\theta\partial\beta'}\sqrt{nT}\left(\hat{\beta}_{nT}-\beta_0\right)$$
$$+\frac{\partial^2\mathcal{L}_{1,nT}\left(\bar{\beta}_{nT},\bar{\theta}_{nT}\right)}{\partial\theta\partial\theta'}\sqrt{T}\left(\hat{\theta}_{nT}-\theta_0\right),$$

where $\bar{\beta}_{nT}$ and $\bar{\theta}_{nT}$ are mean values. In matrix form we have:

$$-\begin{bmatrix}\frac{\partial^2\mathcal{L}_{nT}^{*}(\bar{\beta}_{nT})}{\partial\beta\partial\beta'}+\frac{1}{n}\frac{\partial^2\mathcal{L}_{1,nT}(\bar{\beta}_{nT},\bar{\theta}_{nT})}{\partial\beta\partial\beta'} & \frac{1}{\sqrt{n}}\frac{\partial^2\mathcal{L}_{1,nT}(\bar{\beta}_{nT},\bar{\theta}_{nT})}{\partial\beta\partial\theta'}\\ \frac{1}{\sqrt{n}}\frac{\partial^2\mathcal{L}_{1,nT}(\bar{\beta}_{nT},\bar{\theta}_{nT})}{\partial\theta\partial\beta'} & \frac{\partial^2\mathcal{L}_{1,nT}(\bar{\beta}_{nT},\bar{\theta}_{nT})}{\partial\theta\partial\theta'}\end{bmatrix}\begin{bmatrix}\sqrt{nT}\left(\hat{\beta}_{nT}-\beta_0\right)\\ \sqrt{T}\left(\hat{\theta}_{nT}-\theta_0\right)\end{bmatrix}$$
$$=\begin{bmatrix}\sqrt{nT}\frac{\partial\mathcal{L}_{nT}^{*}(\beta_0)}{\partial\beta}\\ \sqrt{T}\frac{\partial\mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial\theta}\end{bmatrix}+\begin{bmatrix}\sqrt{\frac{T}{n}}\frac{\partial\mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial\beta}\\ 0\end{bmatrix}+o_p(1). \tag{A.39}$$

The second term in the RHS of (A.39) contributes to the asymptotic bias. From Lemma 6(2i) in the supplementary material, and since $T/n\to 0$, this term is $o_p(1)$. From Lemma 6(1), (2iii) and (iv) , we get:

$$\begin{bmatrix}\sqrt{nT}\left(\hat{\beta}_{nT}-\beta_0\right)\\ \sqrt{T}\left(\hat{\theta}_{nT}-\theta_0\right)\end{bmatrix}=\left[\begin{pmatrix}(I_0^{*})^{-1} & 0\\ 0 & I_{1,\theta\theta}^{-1}\end{pmatrix}+o_p(1)\right]\begin{bmatrix}\sqrt{nT}\frac{\partial\mathcal{L}_{nT}^{*}(\beta_0)}{\partial\beta}\\ \sqrt{T}\frac{\partial\mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial\theta}\end{bmatrix}+o_p(1). \tag{A.40}$$

(ii) Asymptotic normality of the standardized score vector

PROPOSITION A.5. *Let Assumptions A.1–A.5 and H.1–H.14 be satisfied. If $n,T \to \infty$ such that $T^{\nu}/n = O(1)$, $\nu > 1$, the standardized approximate score vector of the partial derivatives of functions $\mathcal{L}^*_{nT}(\beta)$ and $\mathcal{L}_{1,nT}(\beta,\theta)$ w.r.t. β and θ, respectively, is such that:*

$$\begin{bmatrix} \sqrt{nT}\dfrac{\partial \mathcal{L}^*_{nT}(\beta_0)}{\partial \beta} \\ \sqrt{T}\dfrac{\partial \mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial \theta} \end{bmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_0^* & 0 \\ 0 & I_{1,\theta\theta} \end{pmatrix} \right),$$

where $I_0^* = \mathrm{E}_0\left[I_{\beta\beta}(t) - I_{\beta f}(t) I_{ff}(t)^{-1} I_{f\beta}(t) \right]$ *and*
$I_{1,\theta\theta} = \mathrm{E}_0\left[-\dfrac{\partial^2 \log g\left(f_t | f_{t-1};\theta_0\right)}{\partial\theta\partial\theta'} \right].$

Proof of Proposition A.5. Let us first consider the approximate score w.r.t. β. By the envelope Theorem (e.g., Dixit, 1990) we have $\sqrt{nT}\dfrac{\partial \mathcal{L}^*_{nT}(\beta_0)}{\partial \beta} = \dfrac{1}{\sqrt{nT}} \sum_{t=1}^{T} \sum_{i=1}^{n} \dfrac{\partial \log h}{\partial \beta} \left(y_{i,t} | y_{i,t-1}, \hat{f}_{n,t}(\beta_0);\beta_0 \right)$. By the mean value Theorem we get:

$$\sqrt{nT}\frac{\partial \mathcal{L}^*_{nT}(\beta_0)}{\partial \beta} = \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{\partial \log h}{\partial \beta} \left(y_{i,t} | y_{i,t-1}, f_t;\beta_0 \right) + \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{\partial^2 \log h}{\partial \beta \partial f_t'} \left(y_{i,t} | y_{i,t-1}, \tilde{f}_t;\beta_0 \right) \left(\hat{f}_{n,t}(\beta_0) - f_t \right),$$

where $\tilde{f}_t$ are mean values. By Assumption H.10, Limit Theorem 1 in the supplementary material and condition $T^{\nu}/n = O(1)$, $\nu > 1$, we can show that $\dfrac{1}{n}\sum_{i=1}^{n} \dfrac{\partial^2 \log h}{\partial \beta \partial f_t'} \left(y_{i,t} | y_{i,t-1}, \tilde{f}_t;\beta_0 \right) = -I_{\beta f}(t) + O_p\left(\dfrac{(\log n)^{C_{10}}}{\sqrt{n}} \right)$, uniformly in $1 \le t \le T$, for some constant $C_{10} > 0$, where $I_{\beta f}(t)$ is the (β, f) block of the matrix $I(t)$ defined in equation (4.6). Then, by Limit Theorem 1 and the condition $T^{\nu}/n = O(1)$, $\nu > 1$, we have:

$$\sqrt{nT}\frac{\partial \mathcal{L}^*_{nT}(\beta_0)}{\partial \beta} = \frac{1}{\sqrt{nT}} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{\partial \log h}{\partial \beta} \left(y_{i,t} | y_{i,t-1}, f_t;\beta_0 \right) - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} I_{\beta f}(t) \sqrt{n} \left(\hat{f}_{n,t}(\beta_0) - f_t \right) + o_p(1). \tag{A.41}$$

Let us now derive an asymptotic expansion for $\sqrt{n}\left(\hat{f}_{n,t}(\beta_0) - f_t \right)$. Since f_t is in the interior of set $\mathcal{F}_n$ w.p.a. 1 from Assumptions H.5 and H.6(i) and (ii), and $\hat{f}_{n,t}(\beta_0)$ converges in probability to f_t by Limit Theorem 1, the first-order condition $\dfrac{1}{\sqrt{n}} \sum_{i=1}^{n} \dfrac{\partial \log h\left(y_{i,t} | y_{i,t-1}, \hat{f}_{n,t}(\beta_0);\beta_0 \right)}{\partial f_t} = 0$ holds w.p.a. 1. Then, by the mean value

Theorem, we have:

$$0 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \log h}{\partial f_t}(y_{i,t}|y_{i,t-1}, f_t; \beta_0) + \left(\frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2 \log h}{\partial f_t \partial f_t'}(y_{i,t}|y_{i,t-1}, \bar{f}_t; \beta_0) \right) \times \sqrt{n}\left(\hat{f}_{n,t}(\beta_0) - f_t \right),$$

where $\bar{f}_t$ is a mean value. Similarly to above, by Assumption H.10, Limit Theorem 1, and condition $T^\nu/n = O(1)$, $\nu > 1$, we have $\frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2 \log h}{\partial f_t \partial f_t'} \left(y_{i,t}|y_{i,t-1}, \bar{f}_t; \beta_0\right) = -I_{ff}(t) + O_p\left(\frac{(\log n)^{C_{11}}}{\sqrt{n}} \right)$, uniformly in $1 \le t \le T$, for some constant $C_{11} > 0$, where $I_{ff}(t)$ is the (f, f) block of the matrix $I(t)$ defined in equation (4.6). Then, by Limit Theorem 1 and Assumption H.4 we get:

$$\sqrt{n}\left(\hat{f}_{n,t}(\beta_0) - f_t \right) = I_{ff}(t)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \log h}{\partial f_t} \left(y_{i,t}|y_{i,t-1}, f_t; \beta_0\right) + O_p\left(\frac{(\log n)^{C_{12}}}{\sqrt{n}} \right), \quad \textbf{(A.42)}$$

uniformly in $1 \le t \le T$, for some constant $C_{12} > 0$. By replacing expansion (A.42) into expansion (A.41), and by using the condition $T^\nu/n = O(1)$, $\nu > 1$, and Assumption H.4, we get:

$$\sqrt{nT} \frac{\partial \mathcal{L}_{nT}^*(\beta_0)}{\partial \beta} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\psi_{n,\beta}(t) - I_{\beta f}(t) I_{ff}(t)^{-1} \psi_{n,f}(t) \right] + o_p(1), \quad \textbf{(A.43)}$$

where

$$\psi_{n,\beta}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \log h}{\partial \beta} \left(y_{i,t}|y_{i,t-1}, f_t; \beta_0\right),$$

$$\psi_{n,f}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \log h}{\partial f_t} \left(y_{i,t}|y_{i,t-1}, f_t; \beta_0\right). \quad \textbf{(A.44)}$$

Let us now consider the approximated score w.r.t. θ. By the mean value Theorem, we have:

$$\begin{aligned}
\sqrt{T} \frac{\partial \mathcal{L}_{1,nT}(\beta_0, \theta_0)}{\partial \theta} &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial \log g}{\partial \theta} \left(\hat{f}_{n,t}(\beta_0) | \hat{f}_{n,t-1}(\beta_0); \theta_0 \right) \\
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial \log g}{\partial \theta} \left(f_t | f_{t-1}; \theta_0\right) \\
&\quad + \sqrt{\frac{T}{n}} \left(\frac{1}{T} \sum_{t=1}^{T} \frac{\partial^2 \log g}{\partial \theta \partial f_t'} \left(\tilde{f}_t | \tilde{f}_{t-1}; \theta_0 \right) \sqrt{n} \left(\hat{f}_{n,t}(\beta_0) - f_t \right) \right. \\
&\quad \left. + \frac{1}{T} \sum_{t=1}^{T} \frac{\partial^2 \log g}{\partial \theta \partial f_{t-1}'} \left(\tilde{f}_t | \tilde{f}_{t-1}; \theta_0 \right) \sqrt{n} \left(\hat{f}_{n,t-1}(\beta_0) - f_{t-1} \right) \right).
\end{aligned}$$

By using $T^{\nu}/n = O(1)$, $\nu > 1$, Assumption H.13, and Limit Theorem 1, it follows that:

$$\sqrt{T}\frac{\partial \mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial \theta} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial \log g}{\partial \theta}\left(f_t|f_{t-1};\theta_0\right) + o_p(1). \tag{A.45}$$

Thus, from equations (A.43) and (A.45) we deduce:

$$\begin{bmatrix} \sqrt{nT}\frac{\partial \mathcal{L}^{*}_{nT}(\beta_0)}{\partial \beta} \\ \sqrt{T}\frac{\partial \mathcal{L}_{1,nT}(\beta_0,\theta_0)}{\partial \theta}\end{bmatrix} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta_{n,t} + o_p(1),$$

$$\zeta_{n,t} \equiv \begin{bmatrix} \psi_{n,\beta}(t) - I_{\beta f}(t) I_{ff}(t)^{-1}\psi_{n,f}(t) \\ \frac{\partial \log g}{\partial \theta}\left(f_t|f_{t-1};\theta_0\right)\end{bmatrix}, \tag{A.46}$$

where $\psi_{n,\beta}(t)$ and $\psi_{n,f}(t)$ are defined in (A.44). Proposition A.5 follows if we prove that $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta_{n,t} \xrightarrow{d} N(0,\Omega)$ as $n,T\to\infty$, where $\Omega = \begin{pmatrix} I_0^* & 0 \\ 0 & I_{1,\theta\theta}\end{pmatrix}$. Since $\{\zeta_{n,t},\mathcal{G}_{n,t}, 1\le t\le T;\ n\in\mathbb{N}\}$ is a martingale difference array w.r.t. the filtration $\mathcal{G}_{n,t} = \left(\underline{y_{i,t}},\ 1\le i\le n,\ \underline{f_t}\right)$, t varying, namely $\zeta_{n,t}$ is measurable w.r.t. $\mathcal{G}_{n,t}$ and $E_0[\zeta_{n,t}|\mathcal{G}_{n,t-1}] = 0$ for any $t\le T$ and $n\in\mathbb{N}$, we can apply a multivariate version of Theorem 3.2 in Hall and Heyde (1980). Thus, Proposition A.5 follows if we prove the next three conditions:

(a) $\frac{1}{\sqrt{T}}\max_{1\le t\le T}\|\zeta_{n,t}\| \xrightarrow{p} 0$; (b) $\frac{1}{T}\sum_{t=1}^{T}\zeta_{n,t}\zeta_{n,t}' \xrightarrow{p} E_0\left[\zeta_{n,t}\zeta_{n,t}'\right] = \Omega$;

(c) $\frac{1}{T}E_0\left(\max_{1\le t\le T}\|\zeta_{n,t}\|^2\right) = O(1)$.

These conditions are checked in Lemma 7 in the supplementary material when $n,T\to\infty$ such that $T^{\nu}/n = O(1)$ with $\nu > 0$. In particular, the variance-covariance matrix Ω of the random vector $\zeta_{n,t}$ in (A.46) is block-diagonal, since the microcomponent $\psi_{n,\beta}(t) - I_{\beta f}(t)I_{ff}(t)^{-1}\psi_{n,f}(t)$ is zero-mean conditional on the factor path, while the macrocomponent $\partial \log g\left(f_t|f_{t-1};\theta_0\right)/\partial\theta$ depends on the factor path only. ■

(iii) Asymptotic normality of the estimators (proof of Proposition 3)

The joint asymptotic normality of the CSA ML estimator $\left(\hat{\beta}_{nT},\hat{\theta}_{nT}\right)$ follows from the asymptotic expansion (A.40) and Proposition A.5. The asymptotic normality of the GA and true ML estimators is implied by the asymptotic normality of the CSA ML estimator and the asymptotic equivalences (4.7)-(4.8) in Proposition 2 when $T^{\nu}/n = O(1)$, $\nu > 1$.

A.2.4. Proof of Proposition 5

(i) Proof of Proposition 5(i)

By the mean value Theorem we have:

$$\sqrt{n}\left(\hat{f}_{nT,t} - f_t\right) = \sqrt{n}\left(\hat{f}_{n,t}(\beta_0) - f_t\right) + \frac{\partial \hat{f}_{n,t}\left(\dot{\beta}_{nT}\right)}{\partial \beta'}\sqrt{n}\left(\hat{\beta}_{nT} - \beta_0\right), \tag{A.47}$$

where $\dot{\beta}_{nT}$ is a mean value. Let us consider the first term in the RHS. By the proof of Limit Theorem 1 in the supplementary material, we get that $\hat{f}_{n,t}(\beta_0)$ converges in probability to f_t, conditional on $\underline{f_t}$, for $\mathbb{P}$-almost every (a.e.) $\underline{f_t}$. Thus, $\hat{f}_{n,t}(\beta_0)$ coincides with the maximizer of the cross-sectional log-likelihood function $\sum_{i=1}^{n} \log h(y_{i,t}|y_{i,t-1}, f; \beta_0)$ w.r.t. f in set $\{f \in \mathbb{R}^m : \|f - f_t\| \leq r\}$, w.p.a. 1, conditional on $\underline{f_t}$, for any $r > 0$. From Assumptions A.1, A.2, and H.2, we get $\sqrt{n}\left(\hat{f}_{n,t}(\beta_0) - f_t\right) \xrightarrow{d} N\left(0, I_{ff}(t)^{-1}\right)$, conditionally on $\underline{f_t}$, by applying Theorem 4.2.4 of Amemiya (1985) on the asymptotic normality of ML estimators. In checking the conditions of Theorem 4.2.4 of Amemiya (1985), we use that observations $(y_{i,t}, y_{i,t-1})$, for $i = 1, \ldots, n$, are i.i.d. conditional on the factor path $\underline{f_t}$ from Assumptions A.1 and A.2, and that Assumption H.2 implies the global and local identification conditions of f_t. Moreover, the score $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \log h(y_{i,t}|y_{i,t-1}, f_t; \beta_0)}{\partial f_t}$ is asymptotically $N(0, I_{ff}(t))$ distributed, conditional on $\underline{f_t}$, by applying a standard CLT and using Assumption H.2.

Let us now consider the second term in the RHS of equation (A.47). We use Lemma 8 in the supplementary material, which provides a probability bound for $\partial \hat{f}_{n,t}(\beta)/\partial \beta'$, uniformly in $\beta \in \mathcal{B}$, conditionally on $\underline{f_t}$. Then, from Lemma 8 and Proposition 3, the second term in the RHS of equation (A.47) is $o_p(1)$, conditionally on $\underline{f_t}$. The asymptotic normality in Proposition 5(i) follows.

(ii) Proof of Proposition 5(ii)

We have $\|\hat{f}_{nT,t} - f_t\| \leq \|\hat{f}_{n,t}\left(\hat{\beta}_{nT}\right) - f_t\left(\hat{\beta}_{nT}\right)\| + \|f_t\left(\hat{\beta}_{nT}\right) - f_t(\beta_0)\|$ and thus:

$$\sup_{1 \leq t \leq T} \left\|\hat{f}_{nT,t} - f_t\right\| \leq \sup_{1 \leq t \leq T} \sup_{\beta \in \mathcal{B}} \left\|\hat{f}_{n,t}(\beta) - f_t(\beta)\right\| + \sup_{1 \leq t \leq T} \sup_{\beta \in \mathcal{B}} \left\|\frac{\partial f_t(\beta)}{\partial \beta'}\right\| \left\|\hat{\beta}_{nT} - \beta_0\right\|. \tag{A.48}$$

From Limit Theorem 1, the first term in the RHS of inequality (A.48) is $O_p\left(n^{-1/2}[\log(n)]^{\delta_2}\right)$. Let us consider the second term. By differentiating the first-order condition $E_0\left[\frac{\partial \log h(y_{i,t}|y_{i,t-1}, f_t(\beta); \beta)}{\partial f_t} |\underline{f_t}\right] = 0$ w.r.t. β, we deduce $\frac{\partial f_t(\beta)}{\partial \beta'} = -I_{t,ff}(\beta)^{-1} I_{t,f\beta}(\beta)$, where $I_{t,ff}(\beta)$ and $I_{t,f\beta}(\beta)$ are the blocks of the Hessian matrix $I_t(\beta)$ defined by:

$$I_t(\beta) = E_0\left[-\frac{\partial^2 \log h(y_{i,t}|y_{i,t-1}, f_t(\beta); \beta)}{\partial(\beta', f_t')'\partial(\beta', f_t')} |\underline{f_t}\right]. \tag{A.49}$$

From Assumption H.4, we get $\sup_{1 \leq t \leq T} \sup_{\beta \in \mathcal{B}} \left\|\frac{\partial f_t(\beta)}{\partial \beta'}\right\| = O_p\left([\log(n)]^{C_{13}}\right)$, for some $C_{13} > 0$. Then, from Proposition 3, the second term in RHS of inequality (A.48) is $O_p\left((nT)^{-1/2}[\log(n)]^{C_{13}}\right)$. The uniform convergence rate in Proposition 5(ii) follows

A.2.5. Proof of Proposition 6

(i) Consistency

Let us first show that the estimator $\left(\hat{\beta}^*_{nT}, \hat{\theta}^*_{nT}\right)$ is consistent. The consistency of $\hat{\beta}^*_{nT}$ follows by similar arguments as in Section A.2.2(i), by setting functions $\mathcal{L}_{1,nT}(\beta,\theta)$ and $\Psi_{nT}(\beta,\theta)$ equal to zero. To prove the consistency of $\hat{\theta}^*_{nT}$, we use that $\hat{\theta}^*_{nT}$ is the maximizer of criterion $Q_T(\theta) = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} \log g\left[\hat{f}_{n,t}\left(\hat{\beta}^*_{nT}\right)|\hat{f}_{n,t-1}\left(\hat{\beta}^*_{nT}\right);\theta\right]$ over the set Θ. We have $Q_T(\theta) = \mathcal{L}_{1,nT}\left(\hat{\beta}^*_{nT},\theta\right)$, up to a constant independent of θ. By a slight modification of Lemma 1(ii) and the consistency of $\hat{\beta}^*_{nT}$, criterion $Q_T(\theta)$ converges in probability to $Q_\infty(\theta) = \mathrm{E}_0\left[\log g(f_t|f_{t-1};\theta)\right]$ uniformly in $\theta \in \Theta$. Since function Q_T is continuous, set Θ is compact, and θ_0 is the unique maximizer of function Q_∞ by the global identification Assumption A.8, we can apply the standard consistency theorem for extremum estimators [e.g., Amemiya (1985), Theorem 4.1.1]; it follows that $\hat{\theta}^*_{nT}$ converges to θ_0 in probability.

(ii) Stochastic difference between estimators [proof of Proposition 6(i)]

The first-order conditions of estimators $\left(\tilde{\beta}^{CSA}_{nT}, \tilde{\theta}^{CSA}_{nT}\right)$ and $\left(\hat{\beta}^*_{nT}, \hat{\theta}^*_{nT}\right)$ are given by:

$$\begin{cases} \dfrac{\partial \mathcal{L}^*_{nT}\left(\tilde{\beta}^{CSA}_{nT}\right)}{\partial \beta} + \dfrac{1}{n}\dfrac{\partial \mathcal{L}_{1,nT}\left(\tilde{\beta}^{CSA}_{nT}, \tilde{\theta}^{CSA}_{nT}\right)}{\partial \beta} = 0, \\ \dfrac{\mathcal{L}_{1,nT}\left(\tilde{\beta}^{CSA}_{nT}, \tilde{\theta}^{CSA}_{nT}\right)}{\partial \theta} = 0, \end{cases} \qquad \begin{cases} \dfrac{\partial \mathcal{L}^*_{nT}\left(\hat{\beta}^*_{nT}\right)}{\partial \beta} = 0, \\ \dfrac{\partial \mathcal{L}_{1,nT}\left(\hat{\beta}^*_{nT}, \hat{\theta}^*_{nT}\right)}{\partial \theta} = 0, \end{cases}$$

respectively. Let us expand the first-order conditions of $\left(\tilde{\beta}^{CSA}_{nT}, \tilde{\theta}^{CSA}_{nT}\right)$ around $\left(\hat{\beta}^*_{nT}, \hat{\theta}^*_{nT}\right)$. By the mean value Theorem, and the first-order conditions of $\left(\hat{\beta}^*_{nT}, \hat{\theta}^*_{nT}\right)$, we get:

$$0 = \frac{\partial^2 \mathcal{L}^*_{nT}(\bar{\beta}_{nT})}{\partial \beta \partial \beta'}\left(\tilde{\beta}^{CSA}_{nT} - \hat{\beta}^*_{nT}\right) + \frac{1}{n}\frac{\partial \mathcal{L}_{1,nT}\left(\tilde{\beta}^{CSA}_{nT}, \tilde{\theta}^{CSA}_{nT}\right)}{\partial \beta}, \tag{A.50}$$

and:

$$0 = \frac{\partial^2 \mathcal{L}_{1,nT}(\bar{\beta}_{nT}, \bar{\theta}_{nT})}{\partial \theta \partial \beta'}\left(\tilde{\beta}^{CSA}_{nT} - \hat{\beta}^*_{nT}\right) + \frac{\partial^2 \mathcal{L}_{1,nT}(\bar{\beta}_{nT}, \bar{\theta}_{nT})}{\partial \theta \partial \theta'}\left(\tilde{\theta}^{CSA}_{nT} - \hat{\theta}^*_{nT}\right), \tag{A.51}$$

where $(\bar{\beta}_{nT}, \bar{\theta}_{nT})$ are mean values. Since the CSA and two-step estimators are consistent by Proposition 3 and section (i) above, the mean values $\left(\bar{\beta}_{nT}, \bar{\theta}_{nT}\right)$ are consistent as well. From Lemmas 6(1), (2i), and (2iii) we get $\dfrac{\partial^2 \mathcal{L}^*_{nT}(\bar{\beta}_{nT})}{\partial \beta \partial \beta'} = -I^*_0 + o_p(1)$, $\dfrac{\partial \mathcal{L}_{1,nT}\left(\tilde{\beta}^{CSA}_{nT}, \tilde{\theta}^{CSA}_{nT}\right)}{\partial \beta} = O_p(1)$, $\dfrac{\partial^2 \mathcal{L}_{1,nT}\left(\bar{\beta}_{nT}, \bar{\theta}_{nT}\right)}{\partial \theta \partial \beta'} = O_p(1)$, and $\dfrac{\partial^2 \mathcal{L}_{1,nT}(\bar{\beta}_{nT}, \bar{\theta}_{nT})}{\partial \theta \partial \theta'} = -I_{1,\theta\theta} + o_p(1)$. Then, equation (A.50) implies $\tilde{\beta}^{CSA}_{nT} - \hat{\beta}^*_{nT} = O_p(1/n)$, and equation (A.51) implies $\tilde{\theta}^{CSA}_{nT} - \hat{\theta}^*_{nT} = O_p\left(\tilde{\beta}^{CSA}_{nT} - \hat{\beta}^*_{nT}\right) = O_p(1/n)$. Then, from equivalence (4.7) in Proposition 2, we get $\hat{\beta}^*_{nT} - \tilde{\beta}_{nT} = O_p(1/n)$ and $\hat{\theta}^*_{nT} - \tilde{\theta}_{nT} = O_p\left(\dfrac{[\log(n)]^{\delta_1}}{\sqrt{n}}\right)$.

(iii) Asymptotic normality [proof of Proposition 6(ii)]

From the condition $T^{\nu}/n = O(1)$, $\nu > 1$, and Proposition 6(i), we get $\left(\sqrt{nT}(\hat{\beta}^*_{nT} - \beta_0)', \sqrt{T}(\hat{\theta}^*_{nT} - \theta_0)'\right)' = \left(\sqrt{nT}(\tilde{\beta}_{nT} - \beta_0)', \sqrt{T}(\tilde{\theta}_{nT} - \theta_0)'\right)' + o_p(1)$. Then, Proposition 6(ii) follows from Proposition 3.

A.3. Identification in the stochastic migration model

The stochastic migration model is a set of ordered qualitative models, with an unobservable stochastic factor and a common vector of threshold parameters c_k, $k = 1, .., K-1$. This explains why the identification conditions have to be derived carefully.

(i) Let us first consider the two-state case, $K = 2$. The transition matrix $\pi_t = [\pi_{lk,t}]$ is:

$$\pi_t = \begin{bmatrix} G\left(\frac{c_1 - \gamma_1 f_t - \alpha_1}{\sigma_1}\right) & 1 - G\left(\frac{c_1 - \gamma_1 f_t - \alpha_1}{\sigma_1}\right) \\ G\left(\frac{c_1 - \gamma_2 f_t - \alpha_2}{\sigma_2}\right) & 1 - G\left(\frac{c_1 - \gamma_2 f_t - \alpha_2}{\sigma_2}\right) \end{bmatrix}.$$

By reparametrizing coefficients α_1 and α_2, we can assume $c_1 = 0$. The transition matrix becomes:

$$\pi_t = \begin{bmatrix} G\left(-\frac{\gamma_1 f_t + \alpha_1}{\sigma_1}\right) & 1 - G\left(-\frac{\gamma_1 f_t + \alpha_1}{\sigma_1}\right) \\ G\left(-\frac{\gamma_2 f_t + \alpha_2}{\sigma_2}\right) & 1 - G\left(-\frac{\gamma_2 f_t + \alpha_2}{\sigma_2}\right) \end{bmatrix}.$$

We can also scale the parameters to get $\sigma_1 = \sigma_2 = 1$:

$$\pi_t = \begin{bmatrix} G(-\gamma_1 f_t - \alpha_1) & 1 - G(-\gamma_1 f_t - \alpha_1) \\ G(-\gamma_2 f_t - \alpha_2) & 1 - G(-\gamma_2 f_t - \alpha_2) \end{bmatrix}.$$

Finally, by standardizing the factor, we can set $\gamma_1 = 1$ and $\alpha_1 = 0$:

$$\pi_t = \begin{bmatrix} G(-f_t) & 1 - G(-f_t) \\ G(-\gamma_2 f_t - \alpha_2) & 1 - G(-\gamma_2 f_t - \alpha_2) \end{bmatrix}.$$

Then, the values of the factor f_t are identified by the first row of the transition matrix, $t = 1, \ldots, T$. The values of γ_2, α_2 are identified by the second row, when $T \geq 2$.

(ii) Let us now consider the case $K > 2$. The l-th row of the transition matrix is:

$$\left[G\left(\frac{c_1 - \gamma_l f_t - \alpha_l}{\sigma_l}\right), G\left(\frac{c_2 - \gamma_l f_t - \alpha_l}{\sigma_l}\right) - G\left(\frac{c_1 - \gamma_l f_t - \alpha_l}{\sigma_l}\right), \ldots, \right.$$
$$\left. 1 - G\left(\frac{c_{K-1} - \gamma_l f_t - \alpha_l}{\sigma_l}\right)\right],$$

for $l = 1, \ldots, K$. As above, we can first set $c_1 = 0$:

$$\left[G\left(-\frac{\gamma_l f_t + \alpha_l}{\sigma_l}\right), G\left(\frac{c_2 - \gamma_l f_t - \alpha_l}{\sigma_l}\right) - G\left(-\frac{\gamma_l f_t + \alpha_l}{\sigma_l}\right), \ldots, \right.$$
$$\left. 1 - G\left(\frac{c_{K-1} - \gamma_l f_t - \alpha_l}{\sigma_l}\right)\right]. \quad \textbf{(A.52)}$$

Second, by normalizing the factor values and the thresholds, we can set $\gamma_1 = \sigma_1 = 1$ and $\alpha_1 = 0$ in the first row. Then, the transition matrix has a first row given by:

$$\left[G(-f_t), G(c_2 - f_t) - G(-f_t), \ldots, 1 - G\left(c_{K-1} - f_t\right)\right],$$

and row l is given by equation (A.52) for $l \geq 2$. From the first row, we identify the factor value f_t and the $K-2$ thresholds $c_2, \ldots, c_K$. Then, the values of $\gamma_l, \alpha_l, \sigma_l$ are identified by the row l, for $l = 2, \ldots, K$, when $(K-1)T \geq 3$.