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Hilbert domains that admit a quasi-isometric embedding into Euclidean space

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Abstract. We prove that a Hilbert domain which admits a quasi-isometric embedding into a finite-dimensional normed vector space is actually a convex polytope.

Key words. Primary: global Finsler geometry, Secondary: convexity.

1 Introduction

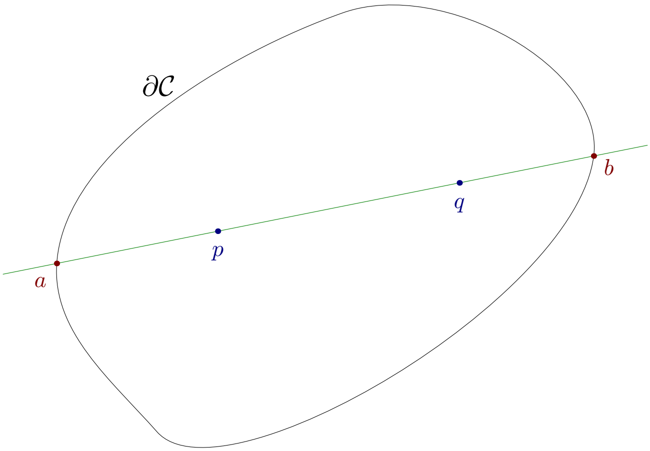
A *Hilbert domain* in $\mathbb{R}^m$ is a metric space $(\mathcal{C}, d_{\mathcal{C}})$, where $\mathcal{C}$ is an *open bounded convex* set in $\mathbb{R}^m$ and $d_{\mathcal{C}}$ is the distance function on $\mathcal{C}$ — called the *Hilbert metric* — defined as follows. Given two distinct points p and q in $\mathcal{C}$, let a and b be the intersection points of the straight line defined by p and q with $\partial\mathcal{C}$ so that $p = (1-s)a + sb$ and $q = (1-t)a + tb$ with $0 < s < t < 1$. Then

$$d_{\mathcal{C}}(p,q) := \frac{1}{2}\ln\,[a,p,q,b], \quad \text{where}$$
$$[a,p,q,b] := \frac{1-s}{s} \times \frac{t}{1-t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points (a,p,q,b). We complete the definition by setting $d_{\mathcal{C}}(p,p) := 0$.

The metric space $(\mathcal{C}, d_{\mathcal{C}})$ thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of $\mathbb{R}^m$ and in which the affine open segments joining two points of the boundary $\partial\mathcal{C}$ are geodesic lines. For further information about Hilbert geometry, we refer to [4, 5, 9, 11] and the excellent introduction [15] by Socié-Méthou.

The two fundamental examples of Hilbert domains $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbb{R}^m$ correspond to the case when $\mathcal{C}$ is an ellipsoid, which gives the Klein model of m-dimensional hyperbolic



geometry (see for example [15, first chapter]), and the case when the closure $\overline{\mathcal{C}}$ is a m-simplex for which there exists a norm $\|\cdot\|_{\mathcal{C}}$ on $\mathbb{R}^m$ such that $(\mathcal{C}, d_{\mathcal{C}})$ is isometric to the normed vector space $(\mathbb{R}^m, \|\cdot\|_{\mathcal{C}})$ (see [8, pages 110–113] or [14, pages 22–23]).

Much has been done to study the similarities between Hilbert and hyperbolic geometries (see for example [7], [16] or [1]), but little literature deals with the question of knowing to what extend a Hilbert geometry is close to that of a normed vector space in terms of *quasi-isometric embeddings* which are defined as follows (see Definition 8.14 in [3], page 138):

Definition. Given real numbers $A \geq 1$ and $B \geq 0$, a metric space (S, d), and a normed vector space $(V, \|\cdot\|)$, a map $f : S \to V$ is said to be a (A, B)-*quasi-isometric embedding* if

$$\frac{1}{A} d(p, q) - B \leq \|f(p) - f(q)\| \leq A d(p, q) + B$$

for all $p, q \in S$.

Definition. With this definition, a map $f : S \to V$ is called a *Lipschitz equivalence* if it is a bijection that is a $(A, 0)$-quasi-isometric embedding for some $A \geq 1$.

Let us then mention three results which are relevant for our present work.

Theorem 1.1 ([10], Theorem 2). *A Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbb{R}^m$ is isometric to some normed vector space if and only if $\mathcal{C}$ is the interior of a m-simplex.*

Theorem 1.2 ([6], Theorem 3.1). *If $\mathcal{C}$ is an open convex polygonal set in $\mathbb{R}^2$, then $(\mathcal{C}, d_{\mathcal{C}})$ is Lipschitz equivalent to the Euclidean plane.*

Theorem 1.3 ([2], Theorem 1.1. See also [17]). *If $\mathcal{C}$ is an open set in $\mathbb{R}^m$ whose closure $\overline{\mathcal{C}}$ is a convex polytope, then $(\mathcal{C}, d_{\mathcal{C}})$ is Lipschitz equivalent to Euclidean m-space.*

Recall that a convex *polytope* in $\mathbb{R}^m$ (called a convex *polygon* when $m := 2$) is the convex hull of a finite set of points whose affine span is the whole space $\mathbb{R}^m$. In the light of these facts, it is natural to ask whether the converse of Theorem 1.3 — which generalizes Theorem 1.2 in higher dimensions — holds. In other words, if a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbb{R}^m$ admits a quasi-isometric embedding into a normed vector space, what can be said about $\mathcal{C}$? The answer to that question is given by the following result which asserts that the converse of Theorem 1.3 is actually true:

Theorem 1.4. *If a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbb{R}^m$ admits a quasi-isometric embedding into a finite-dimensional normed vector space $(V, \|\cdot\|)$, then $\mathcal{C}$ is the interior of a convex polytope.*

2 Proof of Theorem 1.4

The proof of Theorem 1.4 is based on an idea developed by Foertsch and Karlsson in their paper [10]. For this purpose, let us first introduce a definition:

Definition. Given a convex set $\mathcal{C}$ in $\mathbb{R}^m$ and $x, y \in \partial\mathcal{C}$, we will say that the points x and y are *neighbors* if the line segment $[x, y]$ between them is contained in the boundary $\partial\mathcal{C}$.

To establish Theorem 1.4, we need the following fact due to Karlsson and Noskov:

Theorem 2.1 ([12], Theorem 5.2). *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in $\mathbb{R}^m$ and $x, y \in \partial\mathcal{C}$ such that x and y are* not *neighbors. Then, fixing any point $p_0 \in \mathcal{C}$, there exists a constant $K(p_0, x, y) > 0$ such that for any sequences $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ in $\mathcal{C}$ that converge respectively to x and y in $\mathbb{R}^m$ one has*

$$d_{\mathcal{C}}(x_n, y_n) \geq d_{\mathcal{C}}(x_n, p_0) + d_{\mathcal{C}}(y_n, p_0) - K(p_0, x, y)$$

for sufficiently large n.

Now, here is the key result which gives the proof of Theorem 1.4:

Proposition 2.1. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in $\mathbb{R}^m$ which admits a (A, B)-quasi-isometric embedding into a finite-dimensional normed vector space $(V, \|\cdot\|)$ for some real constants $A \geq 1$ and $B \geq 0$. Then, if $N = N(A, \|\cdot\|)$ denotes the maximum number of points in the ball $\{v \in V \mid \|v\| \leq 2A\}$ whose pairwise distances with respect to $\|\cdot\|$ are greater than or equal to $1/(2A)$, and if $X \subseteq \partial\mathcal{C}$ consists of pairwise non-neighboring points, we have $\operatorname{card}(X) \leq N$.*

Definition. The finiteness of the dimension of V is essential here since it yields the compactness of the ball $\{v \in V \mid \|v\| \leq 2A\}$, insuring that N is well defined (indeed, if there were an infinite number of points in this ball whose pairwise distances with respect to $\|\cdot\|$ are greater than or equal to $1/(2A)$, then these points would have an accumulation point in this ball, which is impossible). On the other hand, it is worth remembering that *any* metric space (S,d) can be *isometrically* embedded into the *infinite*-dimensional space of bounded real functions on S endowed with the uniform norm.

Proof of Proposition 2.1. Let $f : \mathcal{C} \to V$ such that

$$\frac{1}{A} d_{\mathcal{C}}(p,q) - B \leq \|f(p) - f(q)\| \leq A d_{\mathcal{C}}(p,q) + B \tag{2.1}$$

for all $p, q \in \mathcal{C}$. First of all, up to translations, we may assume that $0 \in \mathcal{C}$ and $f(0) = 0$.

Then suppose that there exists a subset X of the boundary $\partial\mathcal{C}$ such that $[x,y] \not\subseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y$ and $\mathrm{card}(X) \geq N+1$. So, pick $N+1$ distinct points $x_1, \ldots, x_{N+1}$ in X, and for each $k \in \{1, \ldots, N+1\}$, let $\gamma_k : [0, +\infty) \to \mathcal{C}$ be a geodesic of $(\mathcal{C}, d_{\mathcal{C}})$ that satisfies $\gamma_k(0) = 0$, $\lim_{t \to +\infty} \gamma_k(t) = x_k$ in $\mathbb{R}^m$ and $d_{\mathcal{C}}(0, \gamma_k(t)) = t$ for all $t \geq 0$.

This implies that for all integer $n \geq 1$ and every $k \in \{1, \ldots, N+1\}$, we have

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leq A + \frac{B}{n} \tag{2.2}$$

from the second inequality in Equation 2.1 with $p := \gamma_k(n)$ and $q := 0$. On the other hand, Theorem 2.1 yields the existence of some integer $n_0 \geq 1$ such that

$$d_{\mathcal{C}}(\gamma_i(n), \gamma_j(n)) \geq 2n - K(0, x_i, x_j)$$

for all integer $n \geq n_0$ and every $i, j \in \{1, \ldots, N+1\}$ with $i \neq j$, and hence

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{2}{A} - \frac{1}{n} \left(\frac{K(0, x_i, x_j)}{A} + B \right) \tag{2.3}$$

from the first inequality in Equation 2.1 with $p := \gamma_i(n)$ and $q := \gamma_j(n)$.

Now, fixing an integer $n \geq n_0 + AB + \max\{K(0, x_i, x_j) \mid i, j \in \{1, \ldots, N+1\}\}$, we get

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leq 2A$$

for all $k \in \{1, \ldots, N+1\}$ by Equation 2.2 together with

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{1}{2A}$$

for all $i, j \in \{1, \ldots, N+1\}$ with $i \neq j$ by Equation 2.3. But this contradicts the definition of $N = N(A, \|\cdot\|)$. □

Owing to Proposition 2.1, Theorem 1.4 is then a straightforward consequence of the following:

Proposition 2.2. *Let $\mathcal{C}$ be an open bounded convex set in $\mathbb{R}^m$. Then $\overline{\mathcal{C}}$ is a convex polytope if and only if there exists an integer $N \geq 1$ such that every set $X \subseteq \partial\mathcal{C}$ made up of pairwise non-neighboring points satisfies* $\mathrm{card}(X) \leq N$.

The proof of Proposition 2.2 will use two useful results.

Lemma 2.1. *In a convex set $\mathcal{C}$ in $\mathbb{R}^2$, any extreme point (hence in $\partial\mathcal{C}$) has at most two other extreme points as neighbors.*

Theorem 2.2 ([13], Theorem 4.7). *Let P be a convex set in $\mathbb{R}^m$ and $p \in \mathring{P}$. Then P is a convex polyhedron if and only if all its plane sections containing p are convex polyhedra.*

Here, as usual, a convex *polyhedron* in $\mathbb{R}^m$ is the intersection of a finite number of closed half-spaces.

Proof of Lemma 2.1. Suppose there exists an extreme point p_0 of $\mathcal{C}$ that has three other extreme points m, p and q in $\mathcal{C}$ as neighbors. Up to translations, we may assume that $p_0 = 0$. Since $[0,p]$ and $[0,q]$ are contained in the boundary $\partial\mathcal{C}$, the straight lines $(0p)$ and $(0q)$ are supporting lines to $\mathcal{C}$, and hence $\mathcal{C}$ lies in the half-cone $\{\lambda p + \mu q \mid \lambda \geq 0 \text{ and } \mu \geq 0\}$.

In particular, we can write $m = \lambda_0 p + \mu_0 q$ for some $\lambda_0 > 0$ and $\mu_0 > 0$ (being extreme in $\mathcal{C}$, the points m and p can neither satisfy $m \in [0,p]$, nor $p \in [0,m]$; therefore λ_0 cannot vanish, and the same holds for μ_0).

Now the open convex hull $\{\lambda p + \mu q \mid \lambda > 0, \mu > 0 \text{ and } \lambda + \mu = 1\}$ of the points $0, p, q \in \mathcal{C}$ is contained in $\mathring{\mathcal{C}}$, which yields $m/(\lambda_0 + \mu_0) \in \mathring{\mathcal{C}}$. Therefore, since $m/(\lambda_0 + \mu_0) \in [0,m]$, we get $[0,m] \not\subseteq \partial\mathcal{C}$, contradicting the fact that 0 and m are neighbors. □

Proof of Proposition 2.2. If $\mathcal{C}$ is a convex polytope, then any two points in $\partial\mathcal{C}$ that are not neighbors belong to two different 2-dimensional faces of $\mathcal{C}$. Thus, every set $X \subseteq \partial\mathcal{C}$ consisting of pairwise non-neighboring points satisfies $\operatorname{card}(X) \leq N$, where N denotes the number of 2-dimensional faces of $\mathcal{C}$. On the other hand, if $\mathcal{C}$ is not a convex polytope, then Theorem 2.2 implies that there exists a plane section of $\mathcal{C}$ which is not a convex polygon. So, we may assume that $m = 2$.

Since any compact convex set in a finite-dimensional real vector space is the convex hull of its extreme points (Minkowski's Theorem), $\mathcal{C}$ has infinitely many extreme points. Then, using Lemma 2.1, one can construct by induction a sequence $(x_n)_{n\in\mathbf{N}}$ of extreme points in $\mathcal{C}$ such that for all $n \in \mathbf{N}$ the point x_{n+1} has no neighbors in $\{x_0, \dots, x_n\}$, and hence the set $X := \{x_n \mid n \in \mathbf{N}\} \subseteq \partial\mathcal{C}$ is infinite and is made up of pairwise non-neighboring points. This proves Proposition 2.2. □

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